Price dynamics in equilibrium models

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Chapter 3

Perfect Foresight Cycles in OG Models

3.1 The overlapping generations model

Static general equilibrium models of the type discussed in the previous chapter are not particularly well suited for the analysis of dynamical issues. An alternative is the overlapping generations model, which has become a popular tool to study issues in monetary economics, public finance, business cycle theory and other fields in economics. The overlapping generations model is one of the first models in which it was shown that cyclical and chaotic behaviour can occur in equilibrium models with utility maximizing agents, market clearing in each period and agents that have perfect foresight. These perfect foresight cycles in overlapping generations models correspond to asymmetric equilibria of cyclical general equilibrium models of the type studied in the previous chapter. This equivalence is the subject of this chapter. But first, as a reminder, we will briefly discuss a simple example of an overlapping generations model with three generations.[1]

In each period t a new generation of agents that live for three periods is born. For simplicity we assume that all members of the generation are identical and that their number is constant. Agents of different generations have the same preferences and endowments and there is only one commodity, which cannot be stored. In particular, an agent born in period t has initial endowment $\mathbf{w} = (w_0, w_1, w_2)$, where w_s is his endowment in period $t+s$, and a lifetime utility function $U\left(c_0^t, c_1^t, c_2^t\right)$, where c_s^t is consumption in period $t+s$. An agent born in period t faces prices p_t, p_{t+1} and p_{t+2}, when "young", "middle-aged" and

[1]Although most of the applications of the overlapping generations model deal with two generations a number of properties only can be seen clearly in a model with at least three generations.

"old" respectively and maximizes his lifetime utility $U(c_0^t, c_1^t, c_2^t)$ subject to his lifetime budget constraint

$$p_t c_0^t + p_{t+1} c_1^t + p_{t+2} c_2^t \leq p_t w_0 + p_{t+1} w_1 + p_{t+2} w_2.$$

This gives demand functions

$$d_0(p_t, p_{t+1}, p_{t+2}, \mathbf{w}), \; d_1(p_t, p_{t+1}, p_{t+2}, \mathbf{w}) \text{ and } d_2(p_t, p_{t+1}, p_{t+2}, \mathbf{w}).$$

Notice that agents of different generations have the same demand functions but face different prices during their lifetime. In each period t there are a young, a middle-aged and an old generation alive. Equilibrium on the market for the good in period t then requires

$$d_0(p_t, p_{t+1}, p_{t+2}, \mathbf{w}) + d_1(p_{t-1}, p_t, p_{t+1}, \mathbf{w}) + d_2(p_{t-2}, p_{t-1}, p_t, \mathbf{w}) = w_0 + w_1 + w_2. \quad (3.1)$$

An equilibrium path under perfect foresight is a price path that satisfies (3.1) for all t. Since only relative prices matter this equilibrium condition could also be formulated in terms of interest factors $\rho_t \equiv \frac{p_t}{p_{t+1}}$ as

$$d_0(\rho_t, \rho_{t+1}, \mathbf{w}) + d_1(\rho_{t-1}, \rho_t, \mathbf{w}) + d_2(\rho_{t-2}, \rho_{t-1}, \mathbf{w}) = w_0 + w_1 + w_2. \quad (3.2)$$

Given ρ_t, ρ_{t-1} and ρ_{t-2} (3.2) determines the interest factor ρ_{t+1}. We are dealing with a so-called *temporary equilibrium model*, where in each period a new price or interest factor emerges that clears the goods market in that period, given that past prices or interest factors are such that the markets cleared in previous periods. If the interest factor ρ_{t+1} determined by condition (3.2) exists and is unique (which is not always the case) it is implicitly given by the temporal equilibrium map

$$\rho_{t+1} = F(\rho_t, \rho_{t-1}, \rho_{t-2}). \quad (3.3)$$

Often it will be difficult or impossible to find an explicit expression for F. All paths generated by (3.2) or (3.3) are equilibrium paths. If almost all of them are feasible, that is, a solution exists for all t, then we have a continuum of equilibria. The overlapping generations model is then called *indeterminate* in the sense that its equilibria are not even locally unique (for a comprehensive discussion of determinateness of dynamic economic models see Farmer (1993)). In the literature the main focus has been on stationary equilibrium paths. It can be easily seen that there is always a solution with a constant price level p (and a corresponding interest factor $\rho^* = 1$). Condition (3.1) then becomes

$$d_0(p, p, p, \mathbf{w}) + d_1(p, p, p, \mathbf{w}) + d_2(p, p, p, \mathbf{w}) = w_0 + w_1 + w_2.$$

Multiplying both sides with p shows that this condition coincides with the budget constraint for an individual agent and therefore is always satisfied. Notice that $\rho^* = 1$ implies an interest rate of 0.[2] Now we have to address the following issue. Implicit in the construction of the model is the assumption that agents can trade commodities with each other. This is not the case. Take for example the situation where an agent receives all of his endowment when he is young and none when he is middle-aged or old. If he has convex preferences he wants to trade some of his endowment when he is young for consumption in the later periods of his life. The only agents he could trade with are the presently middle-aged and old. But they have nothing to offer him, so these trades cannot take place. The young agent wants some of the good in the future from the (unborn) young generations of the next two periods in order to give some of his endowment to the present middle-aged and old in this period. Therefore there has to be some institution (for example money or debt) that can realize trades between agents that are not able to trade directly with each other. The steady state discussed above therefore is called the *monetary* steady state. Aggregate savings in the monetary steady state can be positive or negative. Gale (1973) calls overlapping generations models where aggregate savings are positive in the monetary steady state *Samuelson* and overlapping generations models where they are negative *classical*. There are also steady states where aggregate savings are 0 and trade can be executed directly between agents. These steady states are called *balanced*. For the model with two generations it is clear that the only balanced steady state corresponds to *autarchy* where each generation consumes its own endowment and no trade takes place.[3] For overlapping generations models with more than two generations (or several commodities per period and different agents per generation) balanced steady states do in general not correspond to no-trade equilibria. As an example consider the following specification of the three generations model with loglinear utility functions

$$U(c_0, c_1, c_2) = \ln c_0 + \ln c_1 + \ln c_2.$$

[2]If we would allow the population to grow at a constant rate β, then in this equilibrium prices would decrease at rate β^{-1} and the steady state interest factor would be β. The interest rate of an overlapping generations economy therefore only depends upon the biological growth rate and is not determined by economic factors, such as investment possibilities or discounting of the future by young consumers.

[3]The autarkic steady state interest factor ρ^a is then such that it is optimal for an agent to consume his own endowment. It is easily seen to be

$$\rho^a = \frac{p_t}{p_{t+1}} = \frac{U_1(w_0, w_1)}{U_2(w_0, w_1)},$$

where $U_i(.,.)$ is the partial derivative of the utility function with respect to the i'th variable.

The demand functions of a generation born in period t are

$$\begin{aligned} d_0(\rho_t,\rho_{t+1},\mathbf{w}) &= \frac{1}{3}\left[w_0+\frac{1}{\rho_t}w_1+\frac{1}{\rho_t\rho_{t+1}}w_2\right], \\ d_1(\rho_t,\rho_{t+1},\mathbf{w}) &= \frac{1}{3}\left[\rho_t w_0+w_1+\frac{1}{\rho_{t+1}}w_2\right], \\ d_2(\rho_t,\rho_{t+1},\mathbf{w}) &= \frac{1}{3}\left[\rho_t\rho_{t+1}w_0+\rho_{t+1}w_1+w_2\right], \end{aligned}$$

Condition (3.2) can then be written as[4]

$$(2-\rho_{t-1}-\rho_{t-1}\rho_{t-2})\,w_0+\left(2-\rho_{t-1}-\frac{1}{\rho_t}\right)w_1+\left(2-\frac{1}{\rho_t}-\frac{1}{\rho_t\rho_{t+1}}\right)w_2=0.$$

The steady state equilibria are the solutions of a fourth order polynomial. One of the roots is $\rho^*=1$, which corresponds to the monetary steady state. The other steady state equilibria are solutions of

$$w_0\rho^3+(2w_0+w_1)\,\rho^2-(w_1+2w_2)\,\rho-w_2=0. \tag{3.4}$$

It can easily be checked that (3.4) has at most one positive root. Notice that if an agent only has endowment when young, or only when old, a balanced steady state interest factor does not exist. Now we briefly consider two examples of balanced equilibria.

First let endowments be $\mathbf{w}=(8,12,1)$. The unique balanced steady state then corresponds to $\rho^b=\frac{1}{2}$. Individual excess demands are

$$\begin{aligned} d_0(\rho^b,\rho^b,\mathbf{w})-w_0 &= \frac{1}{3}\left[w_0+2w_1+4w_2\right]-w_0=12-8=4 \\ d_1(\rho^b,\rho^b,\mathbf{w})-w_1 &= \frac{1}{3}\left[\frac{1}{2}w_0+w_1+2w_2\right]-w_1=6-12=-6 \\ d_2(\rho^b,\rho^b,\mathbf{w})-w_2 &= \frac{1}{3}\left[\frac{1}{4}w_0+\frac{1}{2}w_1+w_2\right]-w_2=3-1=2 \end{aligned}$$

Trade proceeds as follows: the young generation buys 4 units from the middle aged and pays $\frac{1}{2}\times 4=2$ back when middle aged itself to the then old generation.

[4] For this model the temporal equilibrium map F can be explicitly derived as

$$\rho_{t+1}=\frac{w_2}{\rho_t\left[(2-\rho_{t-1}-\rho_{t-1}\rho_{t-2})\,w_0+\left(2-\rho_{t-1}-\frac{1}{\rho_t}\right)w_1+\left(2-\frac{1}{\rho_t}\right)w_2\right]}.$$

As another example consider the endowment vector $\mathbf{w} = (5, 0, 16)$. The steady state interest factor is $\rho^b = 2$. Individual excess demands are

$$\begin{aligned} d_0\left(\rho^b, \rho^b, \mathbf{w}\right) - w_0 &= \frac{1}{3}\left[w_0 + \frac{1}{2}w_1 + \frac{1}{4}w_2\right] - w_0 = 3 - 5 = -2 \\ d_1\left(\rho^b, \rho^b, \mathbf{w}\right) - w_1 &= \frac{1}{3}\left[2w_0 + w_1 + \frac{1}{2}w_2\right] - w_1 = 6 - 0 = 6 \\ d_2\left(\rho^b, \rho^b, \mathbf{w}\right) - w_2 &= \frac{1}{3}\left[4w_0 + 2w_1 + w_2\right] - w_2 = 12 - 16 = -4 \end{aligned}$$

The young generation trades 2 units of the commodity with the middle-aged consumer for $2 \times 2 = 4$ units in the next period.

For both examples consumption in the monetary steady state is $(c_0^*, c_1^*, c_2^*) = (7, 7, 7)$. It can easily be checked that for the first example aggregate savings are positive in the monetary steady state (the Samuelson case) and in the second example aggregate savings are negative in the monetary steady state (the classical case). Gale (1973) shows that for the Samuelson (classical) case one must have $\rho^b < \rho^*$ ($\rho^b > \rho^*$). The intuition is clear: in order to give up saving in the Samuelson case, interest rates have to be negative and in order to give up borrowing in the classical case interest rates have to be positive.

Optimality of the balanced equilibria has received a lot of attention in the literature. Balanced equilibria in the Samuelson case are not Pareto optimal. From the first example above this can be seen by letting each middle-aged agent transfer some endowment to the then old. Utility then increases for each generation. For the second example it is not possible to arrive at a Pareto improvement by letting each old agent transfer some of his income to the middle aged, because this makes the old agent of period 0 worse off.

Up to now we have considered only stationary equilibrium paths. We could also have equilibrium paths where prices move in a cycle, for example when a price path of the form $(\ldots, p_0, p_1, p_2, p_0, p_1, p_2, p_0, \ldots)$ satisfies condition (3.1) for all t. For some time it has been known that the overlapping generations model can exhibit these perfect foresight equilibrium cycles (see e.g. Benhabib and Day (1982) and Grandmont (1985)). There is a correspondence between these equilibrium cycles and equilibria in cyclical general equilibrium models. Different generations in the overlapping generations model have the same preferences and endowments but differ in the prices they face. Different agents in an exchange economy face the same prices but differ in their preferences and endowments. In this chapter we investigate the existence of (monetary) cycles of increasing length in overlapping generations models with a given number of generations, by utilizing the correspondence between these cycles and the equilibria in a sequence of cyclical general equilibrium models, with an increasing number of agents and commodities. Such a correspondence was, as far as we know, first mentioned in Balasko and Ghiglino (1995) for the case of 2-cycles in two generations models. We study standard overlapping generations models with n generations, one commodity per period and no bequest motive, while all generations have the same preferences and endowments. We look at cycles of length $m \geq n$, with the help of general equilibrium models with m goods and m agents having cyclically permuted preferences in only n goods (and eventually a permuted production structure). Symmetric and asymmetric equilibria correspond to steady states and equilibrium cycles, respectively. It seems to be impossible to extend the equivalence result to balanced steady states or balanced cycles in overlapping generations models.

The rest of this chapter is organized as follows. In Section 3.2 we define the cyclical exchange economy and show that the asymmetric equilibria of this economy correspond to an equilibrium cycle in the overlapping generations model. Furthermore we show that this approach can be applied to two generations models to study the existence of cycles of any period and we show that the model can be extended in different ways. In Section 3.3 we give some examples. The first example is a three generations model where cycles of different periods are shown to exist. The second example is a two generations model where cycles of any period can be shown to exist and the last example is a two generations model with capital, where also cycles of any period can be shown to exist.

3.2 Equivalence

3.2.1 A cyclical exchange economy

In Chapter 2 we constructed exchange economies with three commodities and three consumers which had different kinds of symmetries: cyclical, reflectional or both. In this section we construct an exchange economy with $n+1$ agents and commodities that has a cyclical symmetry. This construction is equivalent to the one used in Proposition 3 of Chapter 2. We let $U : I\!R^{n+1}_{+} \to I\!R$ be a strictly quasi concave, strictly monotonic utility function and $\mathbf{p} = (p_0, ..., p_n) \in I\!R^{n+1}_{+}$ and $\mathbf{w} = (w_0, ..., w_n) \in I\!R^{n+1}_{+}$ a vector of prices and a vector of endowments respectively. The utility maximizing consumption bundle of an agent with these preferences and endowments is

$$\mathbf{x} = \mathbf{d}(\mathbf{p}, \mathbf{w}) = \arg \max_{\mathbf{x} \in I\!R^{n+1}_{+}} \{U(\mathbf{x}) \mid \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}\}, \tag{3.5}$$

where individual excess demand is given by $z(\mathbf{p}, \mathbf{w}) = \mathbf{d}(\mathbf{p}, \mathbf{w}) - \mathbf{w}$. We want to construct an exchange economy that has symmetry group M (that is, the aggregate excess demand functions must commute with $M : M \circ \mathbf{z} = \mathbf{z} \circ M$), where M is the permutation that shifts all elements of a vector one position backward and the first element to the last position:[5]

$$(y_0, y_1, ..., y_n) \xrightarrow{M} (y_1, ..., y_n, y_0). \tag{3.6}$$

The inverse M^{-1} of this permutation shifts all elements one position forward and the last element to the first position; applying M $n+1$ times leaves a vector invariant. M can be viewed as the following $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

working on a column vector. Observe that $M' = M^{-1}$ and $M^{n+1} = I$.

Now we define a cyclical exchange economy with $n+1$ agents, by constructing agents from the utility function U and the vector of endowments $\mathbf{w}$ as follows.

Definition 3.1 *Given a utility function* $U : I\!R_+^{n+1} \to I\!R$ *and a vector of endowments* $w = (w_0, ..., w_n) \in I\!R_+^{n+1}$ *the corresponding cyclical exchange economy* $\mathcal{E} = \{U, \mathbf{w}, n+1\}$ *is the exchange economy consisting of* $n+1$ *agents where the i'th agent has utility function* $U(M^i\mathbf{x})$ *and vector of endowments* $M^{-i}w$, $i = 0, 1, ..., n$.

The following proposition then shows that this cyclical exchange economy has symmetry M, as required.

Proposition 3.1 *The vector of aggregate excess demand functions* $\mathbf{z}(\mathbf{p}) \equiv \sum_{i=0}^{n} z_i(\mathbf{p}, \mathbf{w})$ *of* $\mathcal{E}$ *has* symmetry M, *that is*

$$\mathbf{z}(M\mathbf{p}) = M\mathbf{z}(\mathbf{p}).$$

Furthermore we have

- $\widehat{\mathbf{p}} = (1, 1, \ldots, 1)$ *is an equilibrium price vector*
- *If* $\mathbf{p}^* \neq \widehat{\mathbf{p}}$ *is an equilibrium price vector then so is* $M^k\mathbf{p}^*$ *for* $k = 1, 2, \ldots, n$.

[5] In the notation of Chapter 2 this permutation would be denoted $M_{23\cdots(n+1)1}$.

Proof. First consider the problem of maximizing $U(M^i\mathbf{x})$ subject to the budget constraint $\mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot(M^{-i}\mathbf{w})$. This is equivalent to maximizing $U(M^i\mathbf{x})$ subject to $(M^i\mathbf{p})\cdot M^i\mathbf{x} \leq (M^i\mathbf{p})\cdot\mathbf{w}$. The solution of this last problem satisfies $M^i\mathbf{x} = \mathbf{d}(M^i\mathbf{p},\mathbf{w})$ so the utility maximizing consumption bundle for the i'th agent given prices and endowments is

$$\mathbf{x}^i(\mathbf{p}) = M^{-i}\mathbf{d}\left(M^i\mathbf{p},\mathbf{w}\right) = \arg\max_{\mathbf{x}\in R_+^{n+1}}\left\{U\left(M^i\mathbf{x}\right) \mid \mathbf{p}\cdot\mathbf{x} \leq \mathbf{p}\cdot\left(M^{-i}\mathbf{w}\right)\right\}, \tag{3.7}$$

where

$$x_j^i(\mathbf{p}) = d_{j-i}\left(M^i\mathbf{p},\mathbf{w}\right) \qquad\qquad i,j = 0,1,\ldots,n,$$

is the consumption of commodity j by agent i. The vector $\mathbf{x}(\mathbf{p}) = \sum_{i=0}^n \mathbf{x}^i(\mathbf{p})$ can be written as

$$\mathbf{x}(\mathbf{p}) = \begin{pmatrix} d_0(\mathbf{p},\mathbf{w}) \\ d_1(\mathbf{p},\mathbf{w}) \\ \vdots \\ d_n(\mathbf{p},\mathbf{w}) \end{pmatrix} + \begin{pmatrix} d_n(M\mathbf{p},\mathbf{w}) \\ d_0(M\mathbf{p},\mathbf{w}) \\ \vdots \\ d_{n-1}(M\mathbf{p},\mathbf{w}) \end{pmatrix} + \ldots + \begin{pmatrix} d_1(M^n\mathbf{p},\mathbf{w}) \\ d_2(M^n\mathbf{p},\mathbf{w}) \\ \vdots \\ d_0(M^n\mathbf{p},\mathbf{w}) \end{pmatrix}. \tag{3.8}$$

By (3.7)

$$M\mathbf{x}^i(\mathbf{p}) = M^{1-i}\mathbf{d}\left(M^i\mathbf{p},\mathbf{w}\right) = M^{1-i}\mathbf{d}\left(M^{i-1}(M\mathbf{p}),\mathbf{w}\right) = \mathbf{x}^{i-1}(M\mathbf{p})$$

that is: the demand of agent $i-1$ at prices $M\mathbf{p}$ equals $M\mathbf{x}^i(\mathbf{p})$. Hence, given the $n+1$ tuple of demand vectors $(\mathbf{x}^0(\mathbf{p}),\mathbf{x}^1(\mathbf{p}),\ldots,\mathbf{x}^n(\mathbf{p}))$ at $\mathbf{p}$, $(M\mathbf{x}^0(\mathbf{p}),M\mathbf{x}^1(\mathbf{p}),\ldots,M\mathbf{x}^n(\mathbf{p})) = (\mathbf{x}^n(M\mathbf{p}),\mathbf{x}^0(M\mathbf{p}),\ldots,\mathbf{x}^{n-1}(M\mathbf{p}))$ and it follows

$$\begin{aligned}\mathbf{z}(M\mathbf{p},\mathbf{w}) &= \sum_{i=0}^n \mathbf{z}^i(M\mathbf{p},\mathbf{w}) = \sum_{i=0}^n \left(\mathbf{x}^i(M\mathbf{p}) - M^{-i}\mathbf{w}\right) \\ &= \sum_{i=0}^n \left(M\mathbf{x}^{i+1}(\mathbf{p}) - M^{-i}\mathbf{w}\right) = M\sum_{i=0}^n \mathbf{z}^i(\mathbf{p},\mathbf{w}) = M\mathbf{z}(\mathbf{p},\mathbf{w})\end{aligned}$$

taking into account that $\sum_{i=1}^n M^{-i}\mathbf{w} = \left(\sum_{j=0}^n w_j, \sum_{j=0}^n w_j, \ldots, \sum_{j=0}^n w_j\right)' = \sum_{i=0}^n M^{1-i}\mathbf{w}$ and $\mathbf{x}^{n+1}(\mathbf{p}) = \mathbf{x}^0(\mathbf{p})$. Denote $\mathbf{e} = (1,1,\ldots,1)$. For all j,k $z_j(\mathbf{e}) = z_k(\mathbf{e})$ while

$$z_j(\mathbf{e}) = \sum_{i=0}^n d_i(\mathbf{e},\mathbf{w}) - \sum_{i=0}^n w_i = 0$$

coincides with the budget equation for each agent. Furthermore, if $\mathbf{p}^*$ is an equilibrium then by symmetry, so is $M^k\mathbf{p}^*$ since

$$\mathbf{z}\left(M^k\mathbf{p}^*\right) = M^k\mathbf{z}(\mathbf{p}^*) = 0.$$

■

The equilibrium price vector $\mathbf{p}^*$ might consist of a repetition of a shorter price vector with length m such that $km = n+1$, where k is an integer. In this case we have $M^{m+1}\mathbf{p}^* = \mathbf{p}^*$ and the number of asymmetric equilibria corresponding to $\mathbf{p}^*$ then is m. Given that generically the number of equilibria in an exchange economy is odd (Dierker (1972)), this implies that for m even (odd) there must be an even (including zero) (odd) number of other asymmetric equilibria. In particular, if the number of commodities is odd and there are multiple equilibria, then there have to be at least two different asymmetric price vectors not related to each other by symmetry (compare the first example of Section 3.3).

3.2.2 An overlapping generations model

We now present an overlapping generations model with $n+1$ generations, similar to the model with three generations discussed in Section 3.1. Consider an overlapping generations model (denoted $\mathcal{OG}$) with $n+1$ generations and stationary endowments and preferences. Let each generation consist of a single agent. A generation born in period t has endowments $\mathbf{w} = (w_0, w_1, \ldots, w_n)$ and utility function $U(\mathbf{c}^t)$ with $\mathbf{c}^t = (c_0^t, \ldots, c_n^t)$, where c_s^t denotes consumption in period $t+s$ of a person born in period t. The model is specified by $\mathcal{OG} = \{U, \mathbf{w}, n+1\}$. A generation born in period t faces prices $\mathbf{p}^t = (p_t, \ldots, p_{t+n})$ and maximizes $U(\mathbf{c}^t)$ subject to his budget constraint $\mathbf{p}^t \cdot \mathbf{c}^t \leq \mathbf{p}^t \cdot \mathbf{w}$. This gives a vector of demand functions

$$\mathbf{c}^t = \mathbf{d}\left(\mathbf{p}^t, \mathbf{w}\right), \tag{3.9}$$

or

$$c_s^t = d_s\left(\mathbf{p}^t, \mathbf{w}\right), \qquad s = 0, 1, ..., n.$$

In every period total demands of all living generations have to equal total endowments of all living generations. This gives the following set of equilibrium conditions (compare (3.1))

$$\sum_{s=0}^{n} c_s^{t-s} = \sum_{s=0}^{n} d_s\left(\mathbf{p}^{t-s}, \mathbf{w}\right) = \sum_{s=0}^{n} w_s \qquad \forall t. \tag{3.10}$$

A price path $\mathbf{p}^\infty = (\ldots, p_{-1}, p_0, p_1, p_2, \ldots)$ is an equilibrium path under perfect foresight if it satisfies conditions (3.10).

Consider equilibrium cycles of period $n+1$, that is price paths that repeat after $n+1$ periods: $\mathbf{p}^\infty = (\ldots, p_0, p_1, \ldots, p_n, p_0, p_1, \ldots, p_n, p_0, \ldots)$. For $\mathbf{p} \equiv \mathbf{p}^0 = (p_0, p_1, \ldots, p_{n-1}, p_n)$ we have $\mathbf{p}^t = M^t\mathbf{p}$, hence (3.10) becomes

$$\sum_{s=0}^{n} c_s^{t-s} = \sum_{s=0}^{n} d_s\left(M^{t-s}\mathbf{p}, \mathbf{w}\right) = \sum_{s=0}^{n} w_s \qquad t = 0, 1, \ldots, n. \tag{3.11}$$

We denote by $\mathcal{OG}(n+1)$ the model $\mathcal{OG}$ with equilibrium conditions (3.11).

3.2.3 Equivalence of cycles and asymmetric equilibria

In this section we will show that there is a one-to-one correspondence between the equilibrium cycles of the overlapping generations model and (asymmetric) equilibria of the cyclical exchange economy defined above.

Theorem 3.1 *$\mathcal{OG}(n+1)$ has equilibrium cycle p^* if and only if the cyclical exchange economy $\mathcal{E}$ has equilibrium price vector p^*.*

Proof. Using $M = M^{-n}, M^2 = M^{1-n}$ etc. we can rewrite (3.8) as

$$\mathbf{x}(\mathbf{p}) = \begin{pmatrix} d_0(\mathbf{p}, \mathbf{w}) + d_1(M^{-1}\mathbf{p}, \mathbf{w}) + \ldots + d_n(M^{-n}\mathbf{p}, \mathbf{w}) \\ d_0(M^{-n}\mathbf{p}, \mathbf{w}) + d_1(\mathbf{p}, \mathbf{w}) + \ldots + d_n(M^{1-n}\mathbf{p}, \mathbf{w}) \\ \vdots \\ d_0(M^{-1}\mathbf{p}, \mathbf{w}) + d_1(M^{-2}\mathbf{p}, \mathbf{w}) + \ldots + d_n(\mathbf{p}, \mathbf{w}) \end{pmatrix}.$$

From this it follows that an equilibrium price vector $\mathbf{p}^*$ of $\mathcal{E}$ has to satisfy

$$\sum_{i=0}^{n} d_i\left(M^{j-i}\mathbf{p}^*, \mathbf{w}\right) = \sum_{i=0}^{n} w_i, \qquad j = 0, 1, \ldots, n. \tag{3.12}$$

By (3.11) the equilibrium conditions in $\widehat{\mathcal{OG}}(n+1)$ are

$$\sum_{s=0}^{n} d_s\left(M^{t-s}\mathbf{p}^*, \mathbf{w}\right) = \sum_{s=0}^{n} w_s, \qquad t = 0, 1, \ldots, n. \tag{3.13}$$

Conditions (3.12) and (3.13) are exactly the same. So the existence of an equilibrium cycle $\mathbf{p}^*$ in $\widehat{\mathcal{OG}}(n+1)$ is equivalent with the existence of an equilibrium price vector $\mathbf{p}^*$ in $\mathcal{E}$. ■

By Proposition 3.1, $\mathbf{p} = (1, 1, \ldots, 1)$ is an equilibrium price in $\mathcal{E}$, hence by Theorem 3.1 it is an equilibrium in $\mathcal{OG}(n+1)$, the existence of which is well known and easily proved directly from (3.10) or (3.11). The corresponding equilibrium allocation in $\mathcal{OG}(n+1)$ is usually called the monetary steady state.

Theorem 3.1 also says that if $\mathcal{E}$ has an asymmetric equilibrium $\mathbf{p}$, then $\mathcal{OG}(n+1)$ has equilibrium cycle $\mathbf{p}$. Its period is equal to m (with $n+1=km$, where k is an integer), if $\mathbf{p}$ is a repetition of k identical m-cycles. Theorem 3.1 therefore only identifies cycles with a period which is equal to $\frac{n+1}{k}$ times the number of generations in $\mathcal{OG}$.

An extension to cycles with a period different from $n+1$, possibly consisting of sub-cycles, is however straightforward. Let

$$\widehat{U}(c_0, c_1, \ldots, c_n, c_{n+1}, \ldots, c_{n'}) \equiv U(c_0, \ldots, c_n),$$

and

$$\widehat{\mathbf{w}} \equiv (w_0, \ldots, w_n, 0, \ldots, 0),$$

then $\widehat{\mathcal{E}}(n'+1) = \left\{\widehat{U}, \widehat{\mathbf{w}}, n+1, n'+1\right\}$ is again a cyclical exchange economy. A consumer in $\widehat{\mathcal{E}}(n'+1)$ only consumes and owns the first $n+1$ commodities (in the order in which they appear in his own utility function).

Proposition 3.2 *$\mathcal{OG}(n+1)$ has equilibrium cycle p^* of length $n'+1$ if and only if $\widehat{\mathcal{E}}(n'+1)$ has an asymmetric equilibrium price vector.*

Proof. Demands for agent 0 are

$$\widetilde{d}_j(q_0, q_1 \ldots, q_{n'}, \widehat{\mathbf{w}}) = \begin{cases} d_j(q_0, \ldots, q_n, \mathbf{w}) & j = 0, 1, \ldots, n \\ 0 & j = n+1, \ldots, n' \end{cases},$$

where $\mathbf{q} \in I\!R_+^{n'+1}$ is a price vector. Agent i in $\widehat{\mathcal{E}}(n'+1)$ then has demand function

$$\widehat{d}_j(\mathbf{p}, \widehat{\mathbf{w}}) = \widetilde{d}_{j-i}\left(\widehat{M}^i \mathbf{p}, \widehat{\mathbf{w}}\right),$$

where $\widehat{M}$ is a permutation matrix of order $n'+1$. In $\mathcal{OG}$ it can be assumed without loss of generality, that every agent lives for $n'+1$ periods, but consumes and owns resources only in the first $n+1$ periods of his life. The demand functions of an agent born in period t are

$$\widehat{d}_s(p_t, \ldots, p_{t+n'}, \widehat{\mathbf{w}}) = \begin{cases} d_s(p_t, \ldots, p_{t+n}, \mathbf{w}) & s = 0, 1, \ldots, n \\ 0 & s = n+1, \ldots, n' \end{cases}$$

In every period there are $n'+1$ different generations, but only $n+1$ of these contribute to the economy in the sense that they may have a nonzero excess demand for consumption.

Equilibrium cycles with period $n'+1$ must satisfy (3.10) but with $t = 0, 1, \ldots, n'$, since they repeat after $n'+1$ periods. Denote by $\widehat{\mathcal{OG}}(n'+1) = \left\{\widehat{U}, \widehat{\mathbf{w}}, n+1, n'+1\right\}$ the overlapping generations model with equilibrium cycles specified by

$$\sum_{s=0}^{n'} \widehat{d}_s\left(\widehat{M^{t-s}}\mathbf{p}, \widehat{\mathbf{w}}\right) = \sum_{s=0}^{n'} \widehat{w}_s, \qquad t = 0, 1, \ldots, n'.$$

Note that $\mathcal{OG}(n+1)$ and $\widehat{\mathcal{OG}}(n'+1)$ are special cases of $\mathcal{OG}$ and that these three models are not qualitatively different, since equilibrium cycles in all three satisfy (3.10). By Theorem 3.1, $\mathbf{p}$ is an equilibrium cycle in $\widehat{\mathcal{OG}}(n'+1)$ if and only if $\mathbf{p}$ is an equilibrium in $\widehat{\mathcal{E}}(n'+1)$ .Now assume $\widehat{\mathcal{E}}(n'+1)$ has an asymmetric equilibrium. We consider two cases. If this asymmetric equilibrium price vector does not consist of a repetition of a shorter price vector, by Theorem 3.1, $\widehat{\mathcal{OG}}(n'+1)$ has a $n'+1$ cycle. Therefore also $\mathcal{OG}$ has a $n'+1$ cycle. If, on the other hand, the asymmetric equilibrium price vector consists of a repetition of subvectors of length m, where $m < n+1$ we find that $\widehat{\mathcal{OG}}(n'+1)$ and therefore also $\mathcal{OG}$ have a m-cycle. ■

By looking at a series of exchange economies $\widehat{\mathcal{E}}(n'+1)$ we can try to find equilibrium cycles of any period of the overlapping generations model. Notice that it is not the number of generations of the overlapping generations model, but the length of the equilibrium cycle that determines the number of agents and commodities in the exchange economy we need to study. If, for example, an overlapping generations model has two coexisting equilibrium cycles of different length, then these equilibrium cycles correspond to the asymmetric equilibria of two *different* cyclical exchange economies, $\widehat{\mathcal{E}}(n'+1)$ and $\widehat{\mathcal{E}}(n''+1)$.

3.2.4 A special case: the two generations overlapping generations model and Sarkovskii's theorem

We want to spend some extra space on the overlapping generations model with two generations. The equilibrium condition for this overlapping generations model is

$$(d_0(p_t, p_{t+1}, \mathbf{w}) - w_0) + (d_1(p_{t-1}, p_t, \mathbf{w}) - w_1) = 0,$$

which can be written as a dynamical system implicitly as

$$Z_0(\rho_t) = -Z_1(\rho_{t-1}) \tag{3.14}$$

where $\rho_t \equiv \frac{p_t}{p_{t+1}}$ is the interest factor in period t. As was discussed in Section 3.1, this dynamical system has generically two steady states, the monetary steady state $\rho^* = 1$ and the autarkic steady state ρ^a for which we have $Z_0(\rho^a) = Z_1(\rho^a) = 0$ and which does not correspond to an equilibrium in an exchange economy. We can distinguish two cases: the *classical case* where $\rho^a > \rho^*$ and where in the monetary equilibrium people borrow when they are young and pay back when they are old and the *Samuelson case* where $\rho^a < \rho^*$ and where in the monetary equilibrium people save when they are young and dissave when they are old. It is well-known that the autarkic steady state is unstable in the classical case and locally stable in the Samuelson case.[6] For the overlapping generations model with two generations we can use our methods in a special way. This is due to the following well-known result from the theory on nonlinear dynamical systems (Devaney (1989), p.62)

Theorem 3.2 ***(Sarkovskii's Theorem)*** *Consider the following ordering of all positive integers:*

$$3 \succ 5 \succ 7 \succ .... \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ ... \succ 2^k \cdot 3 \succ 2^k \cdot 5 \succ 2^k \cdot 7 \succ .... \succ 2^2 \succ 2 \succ 1$$

If f is a continuous map of an interval into itself with a periodic point of period p and $p \succ q$ in this ordering, then f has a periodic point of period q.

From this theorem it follows that if the overlapping generations model with two generations can be described as a continuous map that maps an interval into itself and that if this overlapping generations model has a period three cycle, it has cycles of every period.

Proposition 3.3 *The overlapping generations model with two generations can be represented by a uniquely defined, continuous function $f : I \to I$, if*

- *$\rho^a > 1$ and $Z_0'(\rho) \neq 0$, for all $\rho \in I = [0, \rho^a]$ or*
- *$\rho^a < 1$ and $Z_1'(\rho) \neq 0$ for all $\rho \in I = [0, \overline{\rho}]$, where $\overline{\rho} \in (\rho^a, 1]$ is the largest interest factor for which*

$$Z_0(\rho) = -Z_1(\overline{\rho})$$

has a solution in $(\rho^a, 1)$.

---

[6] That this is so can be easily seen by differentiating the budget restriction of the consumer at ρ^a to obtain

$$-\frac{Z_1'(\rho^a)}{Z_0'(\rho^a)} = \rho^a.$$

Then linearizing (3.14) around the autarkic steady state gives

$$(\rho_t - \rho) = -\frac{Z_1'(\rho^a)}{Z_0'(\rho^a)}(\rho_{t-1} - \rho) = \rho^a(\rho_{t-1} - \rho).$$

Furthermore, if the overlapping generations model has a monetary cycle of period three, it has monetary cycles of any period.

Proof. $Z_1(\rho)$ and $Z_0(\rho)$ are continuously differentiable by the assumptions on the utility functions. First consider the classical case where $\rho^a > 1$. Since ρ^a is the only solution to $Z_0(\rho^a) + Z_1(\rho) = 0$ we know that if a function of the overlapping generations model exists, it maps $[0, \rho^a]$ into $[0, \rho^a]$. Furthermore, if $Z_0'(\rho) \neq 0$ on $[0, \rho^a]$ then we can invert (3.14) and write

$$\rho_t = Z_0^{-1}(-Z_1(\rho_{t-1})) = f(\rho_{t-1}).$$

Now consider the Samuelson case where $\rho^a < 1$. In this case we have to study the backward dynamics. Clearly this backward dynamic map g maps $[0, \overline{\rho}]$ into $[0, \overline{\rho}]$. Furthermore, since the global inverse of $Z_1(\rho)$ exists, we have

$$\rho_{t-1} = Z_1^{-1}(-Z_0(\rho_t)) = g(\rho_t)$$

which is continuous. This completes the first statement. Application of Sarkovskii's theorem then gives the last part. ∎

This proposition also implies that if the overlapping generations model with two generations has a three cycle we can construct a series of exchange economies $\widehat{\mathcal{E}}(n'+1)$, having multiple equilibria for any $n' \geq 1$.

3.2.5 Extensions

We have considered, except for the number of generations, the most basic overlapping generations model with only one commodity, one agent per generation and a constant population. The model can be extended to allow for more commodities and more agents per generations and for a growing population. This section briefly deals with these extensions.

More goods or more consumers

The case with m goods, but with a single consumer in each period, is straightforward. The previous analysis can be repeated, but with $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ replaced by $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$, with $\mathbf{x}_s = (x_{s1}, x_{s2}, \ldots, x_{sm})$; similarly for prices $\mathbf{p}$ and resources $\mathbf{w}$. The transformation M now works on vectors: $M(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}_0)$.

From the utility function $U\left(\mathbf{x}_0^t, \mathbf{x}_1^t, \ldots, \mathbf{x}_n^t\right)$ and the resource vector $\mathbf{w}$, $(n+1)\times m$ demand functions are derived (compare (3.5))

$$x_{sj} = d_{sj}\left(\mathbf{p}, \mathbf{w}\right), \qquad s = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, m. \tag{3.15}$$

$\mathcal{OG}$ now is an overlapping generations economy with m goods in each period. The equilibrium conditions (3.11) for a cycle of period $n+1$ are replaced by

$$\sum_{s=0}^{n} \mathbf{c}_s^{t-s} = \sum_{s=0}^{n} \mathbf{d}_s\left(M^{t-s}\mathbf{p}, \mathbf{w}\right) = \sum_{s=0}^{n} \mathbf{w}_s, \qquad t = 0, 1, \ldots, n. \tag{3.16}$$

The agents of the corresponding symmetric exchange economy $\mathcal{E}\,(n+1)$ are specified by $U\left(M^i\mathbf{x}\right)$ and $M^{-i}\mathbf{w}$ $(i = 0, 1, \ldots n)$ (compare Definition 3.1). Theorem 3.1 remains valid: in the proof scalar functions d_s are replaced by vector functions $\mathbf{d}_s$.

Note that in this case there exists $\overline{\mathbf{p}} = (\overline{p}_1, \ldots, \overline{p}_m)$,such that $\mathbf{p}^0 = (\overline{\mathbf{p}}, \overline{\mathbf{p}}, \ldots, \overline{\mathbf{p}})$ is an equilibrium of the exchange economy. Maximize $U\left(\mathbf{x}\right)$ under the constraint $\sum_{t=0}^{n} \mathbf{x}^t \leq \sum_{t=0}^{n} \mathbf{w}^t \equiv \mathbf{W}$. Since the constraint set is convex and compact, a solution $\mathbf{x}^*$ exists. Substitution of the constraint for $\mathbf{x}^0$ in U gives

$$V\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n\right) = U\left(\mathbf{W} - \mathbf{x}^1 - \mathbf{x}^2 \cdots - \mathbf{x}^n, \mathbf{x}^1, \ldots, \mathbf{x}^n\right).$$

Differentiating in $\mathbf{x}^*$ gives the first order conditions

$$\frac{\partial V}{\partial x_i^k} = -\frac{\partial U}{\partial x_i^0} + \frac{\partial U}{\partial x_i^k} = 0, \qquad i = 1, 2, \ldots, m, \ k = 1, 2, \ldots, n.$$

Set $\overline{p}_i = \partial U / \partial x_i^0$, then $\mathbf{x}^* = \mathbf{d}\left(\overline{\mathbf{p}}, \overline{\mathbf{p}}, \ldots, \overline{\mathbf{p}}\right)$ maximizes U under $\overline{\mathbf{p}} \cdot \left(\sum_{t=0}^{n} \mathbf{x}^t - \sum_{t=0}^{n} \mathbf{w}^t\right) \leq 0$, which implies $\mathbf{x}^*$ is an equilibrium at $\mathbf{p}^0 = (\overline{\mathbf{p}}, \overline{\mathbf{p}}, \ldots, \overline{\mathbf{p}})$: it represents the "monetary" steady state in this economy.

With l consumers $h = 1, 2, \ldots, l$, in each period and a single good, we must have l utility functions $U^h\left(\mathbf{x}^h\right)$ and l resource vectors $\mathbf{w}^h = \left(w_0^h, w_1^h, \ldots, w_n^h\right)$ giving $n+1$ demand functions for each agent h, giving, instead of (3.15)

$$x_s^h = d_s^h\left(\mathbf{p}, \mathbf{w}^h\right), \qquad s = 0, 1, \ldots, n, \ h = 1, 2, \ldots, l.$$

$\mathcal{OG} = \left\{U^h, \mathbf{w}^h, n+1\right\}$ is now an economy with l consumers in each period. An equilibrium cycle of period $n+1$ is a solution of (instead of (3.10))

$$\sum_{h=1}^{l}\sum_{s=0}^{n} c_s^{h,t-s} = \sum_{h=1}^{l}\sum_{s=0}^{n} d_s^h\left(M^{t-s}\mathbf{p}, \mathbf{w}^h\right) = \sum_{h=1}^{l}\sum_{s=0}^{n} w_s^h, \qquad t = 0, 1, \ldots n. \tag{3.17}$$

The corresponding cyclical exchange economy $\mathcal{E}\left(U^h, \mathbf{w}^h, n+1\right)$, has $l \times (n+1)$ consumers with utilities $U^h\left(\mathbf{M}^i\mathbf{x}\right)$ and resources $M^{-i}\mathbf{w}^h$. It is equivalent to the $\mathcal{OG}-$model by Theorem 1: in the proof the individual demands d_s have to be replaced by aggregate demands $\sum_{h=1}^{l} d_s^h$ and w_s is replaced by $\sum_{h=1}^{l} w_s^h$. Again $\mathbf{p} = \mathbf{e}$ is an equilibrium price vector: (3.17) coincides with the *sum* of all budget equations.

For a model with m goods *and* l consumers we get similar results.

Growth

We study the model $\mathcal{OG}\left(n+1,\beta\right)$ which is the model $\mathcal{OG}\left(n+1\right)$ of Section 3.2.2, with a single good in each period, a lifetime resource vector $\mathbf{w}$ and with the utility function $U\left(\mathbf{x}\right)$ for all agents of all generations, but with a growing population: normalizing the number of agents born in $t=0$ at 1, β^t $(t \in (-\infty, \infty))$ are born in t, β being the rate of population growth.

It is well known that with population growth at a rate β the monetary steady state of $\mathcal{OG}\left(n+1\right)$ is replaced by a growth path with identical consumption of all generations, but with prices decreasing at the rate β^{-1}, where Samuelson (1958) called the rate $\beta - 1$ the biological interest rate.

The equilibrium condition (3.10) for period t is replaced by

$$\sum_{s=0}^{n} \beta^{-s} c_s^{t-s} = \sum_{s=0}^{n} \beta^{-s} d_s\left(\mathbf{p}_{t-s}, \mathbf{w}\right) = \sum_{s=0}^{n} \beta^{-s} w_s. \tag{3.18}$$

$\mathbf{p} = \left(1, \beta^{-1}, \beta^{-2}, \beta^{-3}, ...\right)$ is always a solution, since for every t condition (3.18) then reduces to the budget equation of the individual agent. This price sequence corresponds to the monetary steady state $\rho = \frac{p_t}{p_{t+1}} = \beta$. Cycles are also possible. A cycle then has the following structure $\left(\ldots, \beta^{n+1}p_0, \beta^{n+1}p_1, \ldots, \beta^{n+1}p_n, p_0, \ldots, p_n, \beta^{-n-1}p_0, \ldots\right)$. Prices decrease along the cycle but interest factors $\rho_t = \frac{p_t}{p_{t+1}}$ move within a cycle. In fact, we have $\prod_{i=0}^{n} \rho_i = \beta^{n+1}$.

Now we construct $n+1$ utility functions and vectors of endowments in the following way: $V_i\left(\mathbf{x}\right) = U\left(M_\beta^i \mathbf{x}\right)$ and $\mathbf{w}_i = M_\beta^{-i}\mathbf{w}$, and $N_\beta = \left(M_\beta^{-1}\right)'$ where M_β and N_β are the following $(n+1) \times (n+1)$ matrices

$$M_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & \beta^{-n} \\ \beta & 0 & \cdots & 0 & 0 \\ 0 & \beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta & 0 \end{pmatrix}, N_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & \beta^{n} \\ \beta^{-1} & 0 & \cdots & 0 & 0 \\ 0 & \beta^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta^{-1} & 0 \end{pmatrix}. \quad (3.19)$$

Notice that $M_\beta^{n+1} = I$ and $N_\beta = \left(M_\beta^{-1}\right)' = M_{\frac{1}{\beta}}$. The demand functions are

$$\mathbf{x}_i = M_\beta^{-i}\mathbf{d}\left(N_\beta \mathbf{p}, \mathbf{w}\right),$$

and the aggregate excess demand functions become

$$\mathbf{z}\left(\mathbf{p}\right) = \sum_{i=0}^{n} M_\beta^{-i}\mathbf{d}\left(N_\beta^i \mathbf{p}, \mathbf{w}\right) - \sum_{i=0}^{n} M_\beta^{-i}\mathbf{w}.$$

We have

$$\mathbf{z}\left(\mathbf{p}\right) = \begin{pmatrix} d_0\left(\mathbf{p},\mathbf{w}\right) + \beta^{-1}d_1\left(N_\beta \mathbf{p},\mathbf{w}\right) + \ldots + \beta^{-n}d_n\left(N_\beta^n \mathbf{p},\mathbf{w}\right) - \sum_{i=0}^{n}\beta^{-i}w_i \\ \beta d_0\left(N_\beta^n \mathbf{p},\mathbf{w}\right) + d_1\left(\mathbf{p},\mathbf{w}\right) + \ldots + \beta^{1-n}d_n\left(N_\beta^{n-1} \mathbf{p},\mathbf{w}\right) - \sum_{i=0}^{n}\beta^{1-i}w_i \\ \vdots \\ \beta^{n}d_0\left(N_\beta \mathbf{p},\mathbf{w}\right) + \beta^{n-1}d_1\left(N_\beta^2 \mathbf{p},\mathbf{w}\right) + \ldots + d_n\left(\mathbf{p},\mathbf{w}\right) - \sum_{i=0}^{n}\beta^{n-i}w_i \end{pmatrix}. \quad (3.20)$$

Multiplying the j'th element of $\mathbf{z}\left(\mathbf{p}\right)$ by β^{-j} (where $j = 0, 1, \ldots, n$) gives the following $n+1$ equilibrium conditions

$$\sum_{i=0}^{n} \beta^{-i}d_i\left(N_\beta^{i-j}\mathbf{p},\mathbf{w}\right) = \sum_{i=0}^{n}\beta^{-i}w_i, \qquad j = 0, .., n+1. \quad (3.21)$$

Conditions (3.18) can be written in exactly the same way. The price vector $\widehat{\mathbf{p}} = \left(1, \beta^{-1}, \beta^{-2}, ..., \beta^{-}\right.$ is always an equilibrium vector. Furthermore, from conditions (3.20) it follows that if $\mathbf{p}^*$ is an equilibrium, then so is $N_\beta^k\mathbf{p}^*$, for $k = 1, ..., n$. (Notice that $N_\beta\widehat{\mathbf{p}} = \widehat{\mathbf{p}}$.) Hence, if $\mathbf{p}^* = (p_0, p_1, \ldots, p_n)$ is an asymmetric equilibrium, then this exchange economy has n other asymmetric equilibria, namely, $\left(\beta^{n+1}p_n, p_0, \ldots, p_{n-1}\right), \left(\beta^{n+1}p_{n-1}, \beta^{n+1}p_n, p_0, \ldots, p_{n-2}\right), \ldots,$ $\left(\beta^{n+1}p_1, \ldots, \beta^{n+1}p_n, p_0\right)$.

3.3 Examples

We will now illustrate the results obtained from the previous section by a set of examples.

3.3.1 A three generations overlapping generations model with CES utility functions

As a first example we study the occurrence of cycles in the overlapping generations model with three generations. A consumer born in period t has the CES utility function

$$U(c_t, c_{t+1}, c_{t+2}) = \left(c_t^\theta + \alpha_1 c_{t+1}^\theta + \alpha_2 c_{t+2}^\theta\right)^{\frac{1}{\theta}} \qquad -\infty \leq \theta \leq 1, \alpha_1, \alpha_2 \geq 0, \tag{3.22}$$

and a vector of endowments $\mathbf{w} = (1, 0, 0)$. The individual demand functions are

$$\begin{aligned} d_0(p_t, p_{t+1}, p_{t+2}) &= \frac{p_t^{1-\sigma}}{p_t^{1-\sigma} + \alpha_1^\sigma p_{t+1}^{1-\sigma} + \alpha_2^\sigma p_{t+2}^{1-\sigma}}, \\ d_1(p_t, p_{t+1}, p_{t+2}) &= \frac{\alpha_1^\sigma p_t p_{t+1}^{-\sigma}}{p_t^{1-\sigma} + \alpha_1^\sigma p_{t+1}^{1-\sigma} + \alpha_2^\sigma p_{t+2}^{1-\sigma}}, \\ d_2(p_t, p_{t+1}, p_{t+2}) &= \frac{\alpha_2^\sigma p_t p_{t+2}^{-\sigma}}{p_t^{1-\sigma} + \alpha_1^\sigma p_{t+1}^{1-\sigma} + \alpha_2^\sigma p_{t+2}^{1-\sigma}}, \end{aligned} \tag{3.23}$$

where $\sigma = \frac{1}{1-\theta}$ is the elasticity of substitution. Recall that the corresponding cyclical exchange economy was also studied in Section 2.8 of Chapter 2. There we took $\sigma = \frac{1}{10}$ and $\alpha_1 = \alpha_2 = \alpha$ and studied the emergence of asymmetric equilibria as α decreased. It was found there that asymmetric equilibria appeared at about $\alpha \approx 0.051138$. Notice however that this exchange economy has two symmetries: a cyclical and a reflectional one. Here we want to study the occurrence of asymmetric equilibria when the exchange economy has only a cyclical symmetry. Therefore we take $\alpha_1 = \frac{1}{20}$ and $\alpha_2 = \frac{1}{25}$ and study the emergence of asymmetric equilibria as the elasticity of substitution decreases. This emergence of asymmetric equilibria occurs at $\sigma^* \approx 0.131$ (or $\theta^* \approx -6.634$). For $\sigma > \sigma^*$ the symmetric equilibrium is unique but for $\sigma < \sigma^*$ there are 7 different equilibrium price vectors. One of them is the symmetric equilibrium price vector, three of them correspond to one asymmetric equilibrium price vector and its permutations and three of them correspond to another asymmetric equilibrium price vector and its permutations. For example for $\sigma = \frac{1}{10}$ (or $\theta = -9$) the equilibria are $\widehat{\mathbf{p}} = (1, 1, 1)$, $\mathbf{p}^1 \approx (2.2637, 0.4121, 0.3242)$ and its permutations and $\mathbf{p}^2 \approx (2.2637, 0.4121, 0.3242)$ and its permutations. This means that for $\sigma = \frac{1}{10}$ $\mathcal{OG}$ has two 3-cycles in interest factors: $\boldsymbol{\rho}^1 \approx (5.493, 1.271, 0.143)$ and $\boldsymbol{\rho}^2 \approx (1.565, 1.261, 0.507)$.

We also study the occurrence of equilibrium cycles of period 2 and period 4 in the overlapping generations model $\widehat{\mathcal{OG}}(4)$. Let

$$V(x_0, x_1, x_2, x_3) = \left(x_0^\theta + \alpha_1 x_1^\theta + \alpha_2 x_2^\theta\right)^{\frac{1}{\theta}}, \qquad \widehat{\mathbf{w}} = (1, 0, 0, 0), \tag{3.24}$$

and construct the other three agents by permutation. Again consider $\alpha_1 = \frac{1}{20}$ and $\alpha_2 = \frac{1}{25}$. The symmetric equilibrium is unique for $\sigma \geq \sigma^0 \approx 0.374$. At σ^0 two new equilibria emerge of the form $\mathbf{p}^* = (p, q, p, q)$. Notice that such equilibria are not symmetric with respect to M, but they are symmetric with respect to M^2 (which is also a symmetry of $\widehat{\mathcal{E}}(4)$), since $M^2\mathbf{p}^* = \mathbf{p}^*$, $\mathbf{p}^*$ corresponds to a cycle of period 2 in $\mathcal{OG}$. As the elasticity of substitution decreases, 4 new equilibria of the form (p, q, r, s) emerge. These correspond to an equilibrium cycle of period 4 in $\mathcal{OG}$. This happens at $\sigma^1 \approx 0.190$.

3.3.2 A two generations overlapping generations model with CARA utility functions

Let an agent born in period t maximize the following *CARA* utility function[7]

$$U(x_0, x_1) = -e^{-x_0} - e^{-\beta x_1} \qquad 0 < \beta < 1, \tag{3.25}$$

where β is the *coefficient of absolute risk aversion* of the old generation. Furthermore we take $w_0 = w_1 = 1$. The demand functions of the corresponding (unpermuted) agent in the exchange economy are

$$d_0(p_0, p_1, \mathbf{w}) = \frac{\beta(p_0 + p_1) - p_1 \ln \beta \frac{p_0}{p_1}}{\beta p_0 + p_1}, d_1(p_0, p_1, \mathbf{w}) = \frac{p_0 + p_1 + p_0 \ln \beta \frac{p_0}{p_1}}{\beta p_0 + p_1}. \tag{3.26}$$

To study the existence of a 2-cycle we consider the corresponding economy $\mathcal{E}$. The excess demand function of the first commodity with commodity 2 as a numeraire equals

$$z_0(p, 1) = d_0(p, 1, \mathbf{w}) + d_1(1, p, \mathbf{w}) - w_0 - w_1 = \frac{\beta(p+1) - \ln \beta p}{\beta p + 1} + \frac{1 + p + \ln \beta \frac{1}{p}}{\beta + p} - 2.$$

Notice that $\lim_{p \to 0} z_0(p, 1) > 0$ and $\lim_{p \to \infty} z_0(p, 1) < 0$ and that $z_0(p, 1)$ is continuous. This implies that if the excess demand function is upward sloping at the symmetric

[7] This is the sum of two so called Constant Absolute Risk Aversion utility functions: for $U(c) = -e^{-\alpha c}$ the coefficient of absolute risk aversion $-\frac{U''(c)}{U'(c)}$ is equal to α and hence constant. Notice that this utility function is not homothetic.

equilibrium $p^* = 1$, there have to be at least two other equilibria. The slope of the excess demand function is

$$\left.\frac{\partial z_0(p,1)}{\partial p}\right|_{p=1} = \frac{(\beta-1)\ln\beta - (\beta^2+3)}{(\beta+1)^2}.$$

So at $\beta^* \approx 0.04337$ two new equilibria emerge.[8] The two asymmetric equilibria correspond to a 2-cycle in $\mathcal{OG}$. For example: for $\beta = \frac{1}{25}$ we have the asymmetric equilibrium price vector $\mathbf{p}^* = (1.3442, 0.6558)$, so $\mathcal{OG}$ has a 2-cycle in interest factors: $\boldsymbol{\rho}^* = (2.0497, 0.4879)$. Hence the overlapping generations model undergoes a period doubling bifurcation at $\beta = \beta^*$. We can also look for 4-cycles in $\mathcal{OG}$ by studying $\widehat{\mathcal{E}}(4)$. For $\beta \in (0, \beta^0)$ there are, besides the symmetric equilibrium and the two other asymmetric equilibria, four asymmetric equilibria of the form $\mathbf{p} = (p, q, r, s)$. These equilibrium price vectors correspond to a 4-cycle in the overlapping generations model. So at β^0 a second period doubling bifurcation occurs. Using the LOCBIF program (Khibnik, Kuznetsov, Levitin and Nikolaev (1992)) we constructed Figure 3.1 for $\widehat{\mathcal{E}}(4)$

Figure 3.1 shows what happens as β decreases: the cyclical exchange economy goes from a unique equilibrium to three equilibria to seven equilibria. The overlapping generations model with two generations therefore undergoes two period doubling bifurcations.

Equilibrium cycles with period three also occur. Let

$$W(x_0, x_1, x_2) = -e^{-x_0} - e^{-\beta x_1}, \quad \mathbf{w}^1 = (1, 1, 0)$$

and construct the other 2 agents by permutation. $\widehat{\mathcal{E}}(3)$ has asymmetric equilibria for $\beta < \beta^c \approx 0.01013$. Hence $\mathcal{OG}$ with two generations and CARA utility functions has a 3-cycle for $\beta < \beta^c$. Furthermore, it is easily established that the conditions of Proposition 3.3 are satisfied so we immediately find

Proposition 3.4 *The overlapping generations model with two generations and CARA utility functions has equilibrium cycles of any period, for $\beta < \beta^c$. Furthermore, if we construct a cyclical exchange economy with $n' + 1$ commodities and $n' + 1$ agents by expanding and permuting the CARA utility functions and vector of endowments, with $\beta < \beta^c$, then this exchange economy has multiple equilibria for all $n' \geq 1$.*

[8] We would also have found these two extra equilibria if we would have replaced the CARA utility functions (3.25) by CES utility functions. This is similar to the result by Bala (1997) who finds for a cyclical exchange economy with two agents and CES utility functions that two extra asymmetric equilibria might emerge. These equilibria correspond to a period two cycle in the corresponding overlapping generations model. The approach by Bala differs from ours in that he does not seem to be aware of this correspondence and the fact that he is mainly interested in the dynamics of the continuous tâtonnement adjustment process, $\frac{\partial p_i}{\partial t} = z_0(p, 1)$.

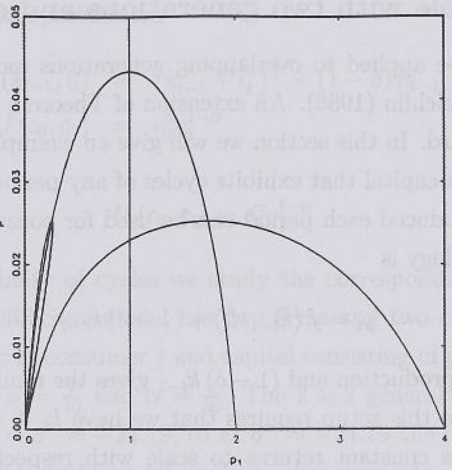

Figure 3.1: Equilibria in the cyclical exchange economy with four commodities and CARA utility functions as a function of β.

The results obtained in our example correspond to the example in Grandmont (1985). The utility function he uses in his example is (see Grandmont (1985), p. 1029)

$$U(x_0, x_1) = 2\sqrt{x_0} + \frac{1}{1-\alpha} x_1^{1-\alpha}.$$

Here α is the constant *relative rate of risk aversion* (which is $-x_1 U''/U'$) of the old generation. Grandmont shows that, as this relative rate of risk aversion increases to a certain point there is coexistence of equilibrium cycles of any period. This is due to a conflict between the income and substitution effects of a change in the inflation rate. This conflict may make the savings function of the young generation nonmonotonic, introducing the possibility of cycles. In our model the old generation has a very low *absolute* rate of risk aversion. This is equivalent with a high absolute rate of risk aversion of the young generation. (That this is so can easily be seen by multiplying the CARA utility function (3.25) by $e^{\frac{1}{\beta}}$, which does not change the preferences of the agents.) Coupling this with the fact that we are in the *classical* situation, where income is transferred from old to young agents, whereas Grandmont considers the *Samuelson* situation where the opposite occurs, leads to the result that the savings function of the old agents in our model may be nonmonotonic. Therefore our model is just the opposite of the Grandmont model.

3.3.3 An example with two generations and capital

Our results can also be applied to overlapping generations models with capital of the type studied by e.g. Reichlin (1986). An extension of Theorem 3.1 to these models with capital is straightforward. In this section we will give an example of such an overlapping generations model with capital that exhibits cycles of any period.

The commodity produced each period can be used for consumption and investment. The production technology is

$$y_t = f(k_{t-1}, l_t) + (1-\delta) k_{t-1} \tag{3.27}$$

where $f(.,.)$ gives net production and $(1-\delta)k_{t-1}$ gives the remaining capital stock after depreciation. Of course this setup requires that we have $k_t \geq (1-\delta)k_{t-1}$ for all t. We assume that f exhibits constant returns to scale with respect to capital and labour. Furthermore we assume labour is inelastically supplied and equal to 1. Profit for the firm in period t is then given by

$$p_t y_t - p_{t-1} k_{t-1} - q_t l_t$$

where q_t is the nominal wage rate. Capital demand follows from the first order condition

$$\frac{\partial f(k_{t-1}, 1)}{\partial k_{t-1}} + (1-\delta) = \frac{p_{t-1}}{p_t} = \rho_{t-1} \Rightarrow k^*_{t-1} = k_{t-1}(\rho_{t-1}), \tag{3.28}$$

production then equals

$$y_t(\rho_{t-1}) = f(k_{t-1}(\rho_{t-1}), 1) + (1-\delta) k_{t-1}(\rho_{t-1}), \tag{3.29}$$

and due to constant returns to scale the real wage income is

$$\omega(\rho_{t-1}) = y(\rho_{t-1}) - \rho_{t-1} k(\rho_{t-1}).$$

The consumers are assumed to have one unit of labour when they are old and no units of labour when they are young. They therefore solve the following problem

$$\max_{c_0, c_1} U(c_0, c_1) \qquad \text{s.t. } p_t c_0 + p_{t+1} c_{t+1} \leq q_{t+1}. \tag{3.30}$$

This gives consumption demand functions $c_0(\rho_t)$ and $c_1(\rho_t)$. The equilibrium condition for period t then is

$$c_0(\rho_t) + c_1(\rho_{t-1}) + k_t(\rho_t) = y_t(\rho_{t-1}), \tag{3.31}$$

which is an implicitly defined one-dimensional map $\rho_t = F(\rho_{t-1})$. If we can establish that this map has a period three cycle then it has cycles of all periods.

We study the following specification

$$\begin{aligned} f(k_{t-1}, l_t) &= \left(k_{t-1}^{\sigma} + l_t^{\sigma}\right)^{\frac{1}{\sigma}} + (1-\delta)\, k_{t-1}, \\ U(c_0, c_1) &= c_0^{\alpha} c_1^{1-\alpha}. \end{aligned}$$

$$U(c_0, c_1) = c_0^{\alpha} c_1^{1-\alpha}.$$

To investigate the possibility of cycles we study the corresponding general equilibrium model. This general equilibrium model has two firms and two consumers. Firm i produces good i using labour of consumer j and capital consisting of good j, $i \neq j, i, j = 0, 1$. We study the case with $\delta = \frac{1}{10}$ and $\alpha = \frac{1}{26}$. The 2×2 general equilibrium model has multiple equilibria for $\sigma < \sigma^* \approx -24.19$, so at $\sigma^* \approx -24.19$ the overlapping generations model undergoes a period doubling bifurcation. For example, for $\sigma = -25$ we have the following asymmetric equilibrium $\mathbf{p}^* = (1.0173138, 0.9826862)$ which corresponds to the following two-cycle in interest factors $\boldsymbol{\rho}^* = (0.9659617, 1.0352377)$. We also can study the extended model where there are three consumers and three firms. In that case asymmetric equilibria appear at $\sigma^c \approx -44.22$. For example for $\sigma = -45$ we have $\mathbf{p}^* \approx (1.0921670, 0.9037847, 1.0040483)$ corresponding to a three cycle in interest factors $\boldsymbol{\rho}^* \approx (1.2084371, 0.9001407, 0.9193176)$. Therefore, given the continuity of the system, we can apply Sarkovskii's theorem and find cycles of any period for $\sigma \leq \sigma^c$. For example, for $\sigma = -35$ we have a four cycle $\mathbf{p}^* \approx (1.0335635, 0.9792206, 1.0494497, 0.9377662)$ or equivalently $\boldsymbol{\rho}^* \approx (1.0554961, 0.9330801, 1.1190952, 0.9073136)$. Notice that for very low values of the elasticity of substitution the CES production function is in some sense close to the Leontief production function, which is the limit case for $\sigma \to -\infty$. In this limit case also $3-$cycles appear.

Note that the present example corresponds to the classical case for the overlapping generations model, because wage income is obtained in the second period. The three commodities general equilibrium model, corresponding to labour supplied in the first period, gives similar results, including asymmetric equilibrium price vectors. But the difference equation of the overlapping generations model for that case is two dimensional, hence Sarkovskii's theorem cannot be applied. However, de Vilder (1995,1996) has shown that chaotic dynamics can occur in this two-dimensional version of the model, when there is a Leontief production technology and the utility functions are of the constant relative risk aversion type, similar to the ones studied by Grandmont (1985).

We study the following specification:

$$f(k_{t-1}, l_t) = (\delta k_{t-1}^{\sigma} + (1-\delta) l_t^{\sigma})^{1/\sigma},$$
$$U(c_1, c_2) = c_1^a c_2^{1-a}.$$

$$\mathcal{E} = \frac{1}{1-\sigma}.$$

To investigate the possibility of cycles we study the corresponding general equilibrium model. This general equilibrium model has two firms and two consumers. Firm i produces good i using labour of consumer j and capital consisting of good j, $i \neq j$, $i, j = 0, 1$. We study the case with $\delta = \frac{1}{2}$ and $a = \frac{1}{2}$. The 2×2 general equilibrium model has multiple equilibria for $\sigma < \sigma^* \approx -24.18$, so at $\sigma^* \approx -24.18$ the overlapping generations model undergoes a period doubling bifurcation. For example, for $\sigma = -25$ we have the following asymmetric equilibrium $p^* = (1.0174185, 0.9828882)$, which corresponds to the following two-cycle in interest factors $\rho = (0.9829841, 1.0173227)$. We also can study the extended model where there are three consumers and three firms. In that case asymmetric equilibria appear at $\sigma^* \approx -34.22$. For example for $\sigma = -35$ we have $p^* = (1.0272016, 0.9808177, 0.9920808)$ corresponding to a three cycle in interest factors $\rho^* = (1.2084871, 0.9901407, 0.9125729)$. Therefore, given the continuity of the system, we can apply Sarkovskii's theorem and find cycles of any period for $\sigma \leq \sigma^*$. For example, for $\sigma = -35$ we have a four cycle $p^* = (1.0033648, 0.9977266, 1.0056406, 0.9937053)$ or equivalently $\rho^* = (1.0056051, 0.9922701, 1.0120032, 0.9078138)$. Notice that for very low values of the elasticity σ other than the CES production function is in some sense close to the Leontief production function, which is the limit case for $\sigma \to -\infty$. In this limit case also 3–cycles appear.

Note that the present example corresponds to the classical case for the overlapping generations model, because wage income is obtained in the young period. The three commodities general equilibrium model, corresponding to taking different income first period, gives similar results, including asymmetric equilibrium price vectors. But the difference equation of the overlapping generations model for this case is two dimensional, hence Sarkovskii's theorem cannot be applied. However, de Vilder (1995, 1996) has shown that chaotic dynamics can occur in this two-dimensional version of the model, when there is a Leontief production technology and the utility functions are of the constant elasticity of substitution type, similar to the case studied by Grandmont (1985).