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Josef Meixner: His life and his orthogonal polynomials

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Abstract

This paper starts with a biographical sketch of the life of Josef Meixner. Then his motivations to work on orthogonal polynomials and special functions are reviewed. Meixner's 1934 paper introducing the Meixner and Meixner–Pollaczek polynomials is discussed in detail. Truksa's forgotten 1931 paper, which already contains the Meixner polynomials, is mentioned. The paper ends with a survey of the reception of Meixner's 1934 paper.

Contents

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1. Introduction

This paper grew from the idea that certain names occurring in the *Askey scheme of hypergeometric orthogonal polynomials* [35, Appendix], [60, p. 184] deserve more historical explanation, both biographically and mathematically. We decided to start with Meixner, whose name occurs in the Meixner polynomials and the Meixner–Pollaczek polynomials. The German theoretical physicist Josef Meixner lived from 1908 until 1994. He spent most of his career in Aachen.

Meixner's paper [2] containing the polynomials named after him appeared in 1934 as one of his first papers, while he was still in München. In current terminology this paper introduced Sheffer polynomials by giving two equivalent definitions, one in terms of generating functions, and then classified all orthogonal polynomials which are Sheffer polynomials. Two new classes came up in this classification: the Meixner and Meixner–Pollaczek polynomials. Reception of this work was not quick, but after some decades the paper got the recognition it deserved.

Meixner did much further work on special functions, in particular on spheroidal functions and Mathieu functions, on which he published with Schäfke the research monograph [26]. In physics Meixner is best known for his work on the thermodynamics of irreversible processes, see his article [29] with Reik in *Handbuch der Physik*. In this area he was competing, in a certain sense, with Nobel prize winner Prigogine.

The contents of this paper are as follows. In Section 2 we describe Meixner's life. Section 3 contains quotations about his motivations to work on orthogonal polynomials and special functions. Next, in Section 4, we give a detailed sketch of his paper [2] introducing the Meixner polynomials. However, there was a predecessor on Meixner polynomials: Truksa [79]. This is discussed in Section 5, where we also pay attention to Pollaczek's 1950 paper [66], which rediscovered the Meixner–Pollaczek polynomials. Finally, in Section 6, we review the reception of Meixner's 1934 paper [2] over a period of more than 80 years. A concluding remark mentions the irony of history in fixing names for classes of polynomials and for theorems.

We start the References with a selection of Meixner's publications: [1–11] give papers on orthogonal polynomials and special functions, [12–25] papers on spheroidal wave functions and Mathieu functions, and [26–32] books and Handbuch articles.

2. Biographical notes

Josef Meixner, born 24 April 1908 in Percha (now part of Starnberg, Bayern), studied mathematics, physics and physical chemistry at the *Universität München* during 1926–1931. Shortly after receiving the upper level teaching certificate in February 1931 he obtained his doctorate under Arnold Sommerfeld[1] with the thesis *Die Greensche Funktion des wellenmechanischen Keplerproblems*. It was the first treatment of Green's function in quantum mechanics and the first example of the *S*-Matrix Theory of Wheeler (1937) and Heisenberg (1943). Ten pages of this thesis were devoted to mainly new investigations of confluent hypergeometric series needed, series which occur rather frequently in a good part of Meixner's oeuvre.

He was an assistant at the Institute of Theoretical Physics in München until 1934 when he became an assistant to Karl Bechert[2] at the *Universität Gießen*. In 1934 he also became a

[1] Arnold Sommerfeld (1868–1951), the internationally respected theoretical physicist, was counted "among the few German scientists who were untainted with regard to Nazi affiliation", see Eckert [47] and http://www.encyclopedia.com/topic/Arnold_Johannes_Wilhelm_Sommerfeld.aspx .

[2] Karl Bechert (1901–1981), a Sommerfeld student and Professor of Theoretical Physics at the *Universität Gießen* since 1933, was appointed Rector of this university by the Americans in 1945/46, he being one of the few who was not politically polluted. Thereafter he became Director of the Institute of Theoretical Physics at the *Universität Mainz*, refounded by the French military government in 1946. See http://www.regionalgeschichte.net/bibliothek/texte/biographien/bechert-karl.html .

member of the SA and in 1937 of the NSDAP (see [57, p. 325, footnote 1]). In Gießen he received the *Habilitation* degree together with a teaching assignment in theoretical physics in 1936/37. After lecturing at the universities of Marburg and Berlin in the period 1938–1942,[3] he was appointed as an extraordinary Professor of Theoretical Physics at the *Technische Hochschule Aachen* in December 1942, as successor to Prof. Wilhelm Seitz (1872–1945). But he could not take up this position as he was not released from the Armed Forces; he was stationed at the weather station at Vadsö, Kirkenes, Norway, since September 1941. Released from the forces in summer 1943, he was sent by Major Gottfried Eckart (Heidelberg), in charge of high frequency research, to Murnau (south of München) to do this research together with F. Sauter and A. W. Maue, who were also former students of Sommerfeld.[4] It was here that Meixner began his work on Mathieu and spheroidal functions.

Meixner was welcomed by Aachen's (provisional) Rector, Prof. Paul Röntgen[5] (1881–1968, born in Aachen) on 30 September 1945. However, his wartime appointment in Aachen could not be automatically continued, but for reemployment a *Persilschein* (denazification certificate) was needed which was written by Sommerfeld.[6] He writes that he has known Meixner for 17 years, that Meixner has never been a supporter of the Nazi system, but that in Meixner's given circumstances in Gießen in 1934 it would have been very difficult for him to avoid membership of the SA. He urges TH Aachen to try its best to keep "such an outstanding teacher and scholar" as Meixner in office. Meixner was appointed as an ordinary Professor on a personal title in 1948, and as a regular chairholder in 1951. In fact, he was not only a member of the Faculty of Natural Sciences, but later also of the Faculty of Electrical Engineering. The only physicist at the time in Aachen was Prof. Wilhelm Fucks (1902–1990), one of the politically active National Socialists at the TH [57, pp. 320–323]. Meixner stayed in Aachen despite several attractive offers from other universities in Germany and abroad. He was visiting or research professor in the USA at New York University (1954), Michigan State University (1955/56), University of Michigan (1961), University of Southern California (1964), Brown University (1965), Lehigh University (1969/70) and in Japan at Hokkaido University (1975). He received an honorary doctorate from the *Universität Köln* in 1968. He was a Fellow of the *Nordrhein-Westfälische Akademie der Wissenschaften und der Künste*. He is the author of 150 publications.

In 1954 Meixner wrote together with the mathematician Friedrich Schäfke the well-known book [27] on Mathieu functions and spheroidal functions. In 1956 he published an article on special functions in the *Handbuch der Physik* [28].

A major part of Meixner's research work concerned the thermodynamics of irreversible processes and he is counted as one of the founding fathers of that field. The fundamental principles of this theory he published together with G. Reik, in their famous article in the *Handbuch der Physik* [29] in 1959. The Nobel Prize for Chemistry for the year 1977[7] was awarded to Ilya Prigogine (1917–2003), see [71], especially for his work on the thermodynamics of irreversible

[3] These were the years that Meixner was also quite active as reviewer for *JFM* (do structured search in the reviewer field in https://zbmath.org/) on physics related papers; earlier he had reviewed a few papers on orthogonal polynomials for *Zentralblatt*.

[4] This move was possibly also intended as giving a counterweight to Sommerfeld's successor in office, Wilhelm Müller (born 1880), known to be a notorious Nazi.

[5] As to Röntgen: "during the Nazi era he remained neutral as far as possible ... non-Nazi elements speak of him as being one of the few respectable and acceptable members of the TH staff ... during the Nazi regime as well as now ..." [57]. See also [51] about the TH Aachen during the war.

[6] 14 December 1945, Deutsches Museum, Archiv, NL 89, 020, Mappe 8,3; included in questionnaire of the British Army of 30 July 1946, see [57, p. 325, footnote 1].

[7] See http://www.nobelprize.org/nobel_prizes/chemistry/laureates/1977/ .

processes. It was said the Prize Committee spent 4 overtime hours before reaching its decision, and there were rumors that Meixner was also a candidate. Meixner's first basic paper in the matter dates back to 1941, Prigogine's work started in 1947. In fact, D. Bedeaux and I. Oppenheim [37] in their obituary notice on Mazur write: "Josef Meixner in 1941 and, independently, Ilya Prigogine in 1947 set up a consistent phenomenological theory of irreversible processes, incorporating both Onsager's[8] reciprocity theorem and the explicit calculation for some systems of the so-called entropy source strength. Shortly thereafter, Mazur and de Groot joined this group as founding fathers of the new field of nonequilibrium thermodynamics". Furthermore, A. R. Vasconcellos et al. [80] write: "J. Meixner, over twenty years ago in papers that did not obtain a deserved diffusion gave some convincing arguments to show that it is unlikely that a nonequilibrium ...". In 1975 both Prigogine and Meixner lectured at an Academy session [31] in Düsseldorf. See also [70]. Another appraisal of Meixner's work on thermodynamics appeared in [55].

After Sommerfeld's death in 1951 Meixner and F. Bopp were entrusted by his last will to complete his manuscript on thermodynamics and statistics [26]. "Especially Meixner 's section on thermodynamics of irreversible process fits in admirably in Sommerfeld's masterpiece". (see [53] and [69]).

We could trace three doctorates with Meixner as *Referent* (first advisor): Ingo Müller[9] (born 1936, doctorate in 1966), Wolfgang Kern and Gunter Weiner[10] . Kern and Weiner, together with Meixner, are the authors of the booklet [30].

Meixner retired from his professorship in Aachen in 1974. He passed away in 1994. From an obituary note by Schlögl [72,73] we quote the following. "Josef Meixner applied severe but just standards in both research and teaching. His well-founded, not-always-comfortable judgments were highly respected by his colleagues".

3. Meixner's work in orthogonal polynomials and special functions: his motivations

One of us (PB) had access to Meixner's handwritten *Erinnerungen*, which were later typewritten (44 pp.). We quote a few parts of this document which give insight about Meixner's motivations for his work on orthogonal polynomials and special functions.

"The Hermite polynomials[11] have a generating function of the form

$$f(t)\exp(xt)=\sum_{n=0}^{\infty}H_n(x)\,\frac{t^n}{n!}.$$

The mathematician Salomon Bochner[12] and I pursued the question as to what functions $f(t)$ lead to other orthogonal polynomial systems. The result was: only the Hermite polynomials do have such a generating function. I broadened the question and sought for polynomial systems having a generating function

$$f(t)\exp(x\,u(t))=\sum_{n=0}^{\infty}p_n(x)\,\frac{t^n}{n!}\,, \tag{3.1}$$

[8] Lars Onsager (1903–1976) received the Nobel Prize for Chemistry in 1968. For a paper primarily based on an interview with Onsager shortly before his death see [63].

[9] See the Mathematics Genealogy Project, http://www.genealogy.math.ndsu.nodak.edu/id.php?id=48491; Meixner was only a *Korreferent* for the other three mentioned there.

[10] W. Kern confirmed to us that Weiner and he were Ph.D. students of Butzer.

[11] Meixner meant here the Hermite polynomials orthogonal with respect to the weight function $\exp(-x^2/2)$; these are nowadays often notated by $He_n(x)$.

[12] Meixner did not write a joint paper with Bochner but in a footnote to his paper in J. London Math. Soc. [2] he writes that in the simpler particular case $u(t)=t$ Bochner was involved in its investigation with him.

where $f(t)$ and $u(t)$ are formal power series in t, with $f(0) = 1$, $u(0) = 0$ and $u'(0) = 1$. The solution of this problem leads to interesting results (published 1934). Apart from the Hermite, Laguerre and Charlier polynomials there turned out to be two further classes of polynomials, one having discrete weights and one with a continuous weight distribution".

Later on he writes:

"These polynomial classes were rediscovered by Pollaczek and it lasted some years until someone realized that these polynomials were found by me. In the meantime they are called Meixner's polynomials, and especially the polynomial class having discrete weights has found numerous applications, see e.g. Chihara [42] . . .".

"With Bochner [1899–1982], who came circa 1929 to Munich, I was a close friend and I owe him not only much from the mathematical side; also his literary interests did indeed stimulate me. It was perhaps his most fruitful time, at the beginning of true functional analysis. His book Fourier Integrals, which in a hidden form already contains an essential part of the theory of distributions, and which also contains many other of his own results, testifies this".[13]

Elsewhere Meixner writes:

"Bochner foresaw the coming political development very clearly, and I recall when we, surely at the end of 1932, stood before a bulletin board of the *Völkischer Beobachter* and he said: 'Now it is almost time that I must depart'. When I [at age 24] replied that then I would also like to leave, he replied: You remain here; nothing will happen to you and for us there are too few places in the world".

On p. 40 of his *Erinnerungen* Meixner writes:

"The special functions of mathematical physics especially interested me already since my dissertation. In the last war year I occupied myself with a possible application of Mathieu functions and the spheroidal functions which are considerably more difficult to handle than most of the special functions studied until then. There existed many individual results but not even a logically consistent nomenclature. My contribution involved a rational terminology and the derivation of numerous new properties over a period of more than ten years. In this matter my later collaborator F. W. Schäfke[14] supported me and our association culminated in 1954 in a book (see [27]) concerning these functions which represent their first systematic theory".

Meixner made great impression by his lecture at the International Christoffel Symposium (Aachen, Monschau, November 1979), which was organized by one of us (PB) together with F. Fehér. The resulting paper [11] (1981), one of his last publications, discusses the *Christoffel–Darboux formula* [77, Theorem 3.2.2]: the way it was first derived by Christoffel (who was prior to Darboux) and its usage, in particular for approximation of functions by their partial sum expansions in terms of orthogonal polynomials. This paper, with its precise and stimulating presentation, is a nice testimony of Meixner's lasting interest in orthogonal polynomials and his keeping up with recent results in literature.

[13] Conversely, Bochner valued Meixner's work. Indeed, in October 1977 Meixner was invited to the birthday meeting of Prof. J. S. Frame who there recalled Bochner's lecture at Cambridge, UK, in 1933, in which he had placed special emphasis on Meixner's work on orthogonal polynomials having a generating function of special type; i.e. Meixner's paper [2].

[14] F. W. Schäfke in his letter of April 1978 addressed to Meixner writes that since at the end of the war he had neither a scientific advisor nor belonged to a "school" (his advisor Harald Geppert had died 1945) his formative years were essentially influenced through the collaboration with Meixner in their work on Mathieu functions. It formed the beginning of his own work of special functions . . . . Finally Schäfke adds: "You were for me fatherly-kind and encouraging, a shining scientific and human example".

4. Meixner's introduction of the Meixner and Meixner–Pollaczek polynomials

As could already be read in Section 3, Meixner introduced the orthogonal polynomials named after him in his 1934 paper [2] and this work arose from his contact with Bochner on a special case of the problem considered in [2]. Actually, this purely mathematical paper [2] was Meixner's first published paper after his paper [1], which was based on his more physically oriented dissertation.

Meixner classifies in [2] all orthogonal polynomial systems $\{p_n\}_{n=0,1,2,\ldots}$ which have a generating function (3.1) with f and u as specified after (3.1). The results of the classification are the cases I–V in [2, p. 11]. We want to observe that, on the one hand, (3.1) can be written more generally for not necessarily monic polynomials p_n as

$$f(t)\exp(x\,u(t)) = \sum_{n=0}^{\infty} c_n\, p_n(x)\, t^n,$$

for certain real non-zero coefficients c_n. With dilation of the argument x we can also replace the condition $u'(0) = 1$ by $u'(0) \neq 0$. This larger flexibility is useful when considering examples. On the other hand, with suitable translation of the argument x, we can make for (3.1) the further requirement that $f'(t) = 0$ or, equivalently, that $p_1(x) = x$. This is used by Meixner in the proof of his classification.

With names and notation as currently used (see [60, Chapter 9]), these five cases read as follows.

Case I. Hermite polynomials $H_n(x)$, [60, Section 9.15]:

$$\exp(xt)\, e^{-\frac{1}{4}t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n\, n!}\, t^n, \tag{4.1}$$

orthogonal on $(-\infty, \infty)$ with respect to the weight function e^{-x^2}.

Case II. Laguerre polynomials $L_n^{(\alpha)}(x)$, [60, Section 9.12]:

$$\exp\left(\frac{xt}{t+1}\right)(1+t)^{-\alpha-1} = \sum_{n=0}^{\infty} (-1)^n\, L_n^{(\alpha)}(x)\, t^n,$$

orthogonal on $[0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$ $(\alpha > -1)$.

Case III. Charlier polynomials $C_n(x; a)$, [60, Section 9.14]:

$$\exp\bigl(x\log(1+t)\bigr)\, e^{-at} = \sum_{n=0}^{\infty} \frac{(-a)^n\, C_n(x; a)}{n!}\, t^n,$$

orthogonal on $\{0, 1, 2, \ldots\}$ with respect to the weights $a^x/x!$ $(a > 0)$.

Case IV. Meixner polynomials $M_n(x; \beta, c)$, [60, Section 9.10]:

$$\exp\left(x\log\left(\frac{1-c+t}{1-c+ct}\right)\right)\left(1+\frac{ct}{1-c}\right)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!}\left(\frac{c}{c-1}\right)^n M_n(x; \beta, c)\, t^n,$$

orthogonal on $\{0, 1, 2, \ldots\}$ with respect to the weights $(\beta)_x/x!$ $(\beta > 0, 0 < c < 1)$.

Case V. Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$, [60, Section 9.7]:

$$\exp\left(ix\log\left(\frac{2\sin\phi-e^{i\phi}t}{2\sin\phi-e^{-i\phi}t}\right)\right)\left(\frac{4\sin^2\phi-4t\sin\phi\cos\phi+t^2}{4\sin^2\phi}\right)^{-\lambda}=\sum_{n=0}^{\infty}\frac{P_n^{(\lambda)}(x;\phi)}{(2\sin\phi)^n}\,t^n,$$

orthogonal on $(-\infty,\infty)$ with respect to the weight function $e^{-\pi x}|\Gamma(\lambda+ix)|^2$ $(\lambda>0,$ $0<\phi<\pi)$.

The cases I and II were well-known and Meixner refers for III to Charlier [40] (1905), Jordan [56] (1926) and Pollaczek–Geiringer [67] (1928). But Meixner did not give citations for IV and V. Still class IV had already appeared in a paper in 1931 by Truksa [79], see Section 5.

Meixner's starting point in [2] is a linear operator Λ on the space of polynomials such that

$$D\Lambda=\Lambda t(D), \tag{4.2}$$

where D is the differentiation operator and $t(D)$ is a formal power series in D of the form

$$t(D)=D+a_2D^2+a_3D^3+\cdots\,. \tag{4.3}$$

Meixner mentions as examples, for the case $t(D)=D$, the operators

$$\Lambda P(x):=\sum_{\nu=1}^{p}c_\nu P(x+d_\nu),$$

considered in 1927 by Bochner [38], and their continuous analogues

$$\Lambda P(x):=\int_{-\infty}^{\infty}P(x+y)\,d\psi(y). \tag{4.4}$$

Note the case $d\psi(y)=(2\pi)^{-1/2}e^{-y^2/2}\,dy$ of (4.4). Then

$$\Lambda P(x)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}P(x+y)\,e^{-y^2/2}\,dy.$$

With this choice of Λ we see, by the generating function (4.1) for Hermite polynomials, that

$$\Lambda P_n(x)=x^n\quad\text{if}\quad P_n(x)=H_n(2^{-1/2}x).$$

Meixner observes about the operator Λ in the general case that there is a nonnegative integer μ such that Λ sends any polynomial of degree n to a polynomial of degree $n-\mu$ if $n\ge\mu$ and to 0 if $n<\mu$. Normalize Λ such that $\Lambda x^\mu=1$. Let $u(s)$ be the formal power series

$$u(s)=s+b_2s^2+b_3s^3+\cdots\quad\text{such that}\quad t(u(s))=s. \tag{4.5}$$

Then, by (4.2),

$$\frac{d}{dx}(\Lambda e^{xu(s)})=s\Lambda e^{xu(s)},$$

which can be solved as

$$\Lambda e^{xu(s)}=\frac{s^\mu e^{xs}}{\mu!\,f(s)} \tag{4.6}$$

with $f(s)$ a formal power series with constant term equal to 1. We can expand

$$f(s)e^{xu(s)}=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}\,s^n \tag{4.7}$$

with $P_n(x)$ a monic polynomial of degree n. Then, by (4.6) and (4.7),

$$\Lambda P_n(x) = \begin{cases} \binom{n}{\mu} x^{n-\mu} & \text{if } n \geq \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

We have seen that the operator Λ, implying $t(D)$ and μ, determines $u(s)$ and $f(s)$. Conversely, $u(s)$ and $f(s)$ determine the $P_n(x)$ and next, together with μ, also Λ.

From now on assume that $\mu = 0$, so (4.8) takes the simple form

$$\Lambda P_n(x) = x^n.$$

In combination with (4.2) this shows that

$$t(D)P_n(x) = n P_{n-1}(x). \tag{4.9}$$

Also observe from (4.7) that

$$f(s) = \sum_{n=0}^{\infty} \frac{P_n(0)}{n!} s^n. \tag{4.10}$$

Meixner states that for a sequence of monic polynomials $P_n(x)$ the properties (4.7) and (4.9) are equivalent, where $t(D)$ of the form (4.3) is related to $u(s)$ by (4.5) and $f(s)$ is given by (4.10). Such polynomials $P_n(x)$ are nowadays called *Sheffer polynomials* because of Sheffer's paper [74] (1939), see the next section.

Next, in order to arrive at the classification I–V, Meixner assumes that the P_n are monic orthogonal polynomials,

$$\int_{-\infty}^{\infty} P_m(x)\, P_n(x)\, d\psi(x) = 0 \qquad (m \neq n). \tag{4.11}$$

Thus they have to satisfy a three-term recurrence relation

$$P_{n+1}(x) = (x + l_{n+1})P_n(x) + k_{n+1} P_{n-1}(x) \tag{4.12}$$

with $P_{-1}(x) := 0$ and with $k_{n+1} < 0$ for $n \geq 1$. Meixner observes that conversely (4.12) together with the negativity of the k_{n+1} implies the orthogonality (4.11) for some monotonic function ψ with infinitely many jumps and all moments existing. For this result he refers to Perron's book [65, p. 376] (1929). However, note that Meixner's statement cannot be found there explicitly. One has to combine several ingredients in Perron's book in order to get the result. In this way Meixner anticipated the formulation of what is usually called Favard's theorem, as given in Favard's 1935 paper [48]. Although Favard was criticized that his paper gave a well-known result (see for instance W. Hahn's review of [48] in JFM 61.0288.01), he was also praised that he was the first to formulate the results in terms of orthogonal polynomials, and thus Meixner can already be praised for this.

From (4.9) applied to (4.12) Meixner derives that the coefficients in (4.12) have the form

$$l_{n+1} = l_1 + n\lambda, \qquad k_{n+1} = n(k_2 + (n-1)\kappa) \quad (\lambda \in \mathbb{R},\ k_2 < 0,\ \kappa \leq 0), \tag{4.13}$$

and that $t(u)$ satisfies the differential equation

$$t'(u) = 1 - \lambda t(u) - \kappa t(u)^2. \tag{4.14}$$

In (4.12) with coefficients given by (4.13) l_1 can be put equal to zero without essential loss of generality (by suitable translation of the argument x). So we can work with the three-term recurrence relation

$$P_{n+1}(x) = (x + n\lambda)P_n(x) + n(k_2 + (n-1)\kappa)P_{n-1}(x). \tag{4.15}$$

From (4.10) together with (4.15) Meixner derives the differential equation

$$\frac{f'(t)}{f(t)} = \frac{k_2 t}{1 - \lambda t - \kappa t^2}\,. \tag{4.16}$$

Next he factorizes $1 - \lambda t - \kappa t^2 = (1 - \alpha t)(1 - \beta t)$, he distinguishes five different possibilities for α, β and he solves (4.14) and (4.16) for each of these possibilities. Thus he arrives at the five cases of the classification, for which he now explicitly has the functions $t(u)$, $u(t)$ (by functional inversion) and $f(t)$, and hence the generating function (3.1). In the case of generic α, β he obtains

$$t(D) = \frac{e^{(\alpha-\beta)D} - 1}{e^{(\alpha-\beta)D} - \beta}\,. \tag{4.17}$$

This degenerates for $\beta \to \alpha$ to $t(D) = \frac{D}{1+\alpha D}$, while in the generic case we observe that

$$\big(e^{(\alpha-\beta)D} - 1\big)g(x) = g(x + \alpha - \beta) - g(x).$$

He obtains the orthogonality measure $d\psi(x)$ by observing from (3.1) and (4.11) that

$$\int_{-\infty}^{\infty} e^{xu}\, d\psi(x) = \frac{1}{f(t(u))} \int_{-\infty}^{\infty} d\psi(x),$$

and next doing Fourier or Laplace inversion.

Meixner is also able to derive the second order differential or difference eigenvalue equation satisfied by the polynomials $P_n(x)$. For this purpose he applies $t(D)$ twice to (4.15) in order to obtain by (4.9) that

$$(n+2)P_n(x) = \big(x + (n+1)(\alpha+\beta)\big)t(D)P_n(x) + 2t'(D)P_n(x) + (k_2 - n\alpha\beta)t(D)^2 P_n(x).$$

The desired difference or differential equation follows by substitution of (4.17) (and some formal manipulations).

In general the paper is striking by its elegance. It is also a purely mathematical paper, without any sign that the author was educated as a theoretical physicist. Furthermore, the paper has a certain terseness. The proofs of many of the statements are silently left as short exercises for the reader.

5. A predecessor, a missed class of polynomials and a rediscovery

Meixner [3] does not give any citations for his classes IV and V, so the reader would think they are new. Still the class IV, i.e., the class of orthogonal polynomials nowadays called Meixner polynomials, already appears in 1931 in a paper by Truksa[15] [79]. This author starts with the 3-parameter class of orthogonal polynomials nowadays called Hahn polynomials [60, Section 9.5] which he can trace back to Chebyshev [41] (1873). In Part III Truksa [79] discusses all limit cases of the Hahn polynomials, as are now familiar from the Askey scheme [60, p. 184]. In particular, he obtains the generalized Kummer polynomials [79, p. 192], which are the polynomials nowadays called Meixner polynomials. He gives the weights, the hypergeometric

[15] See https://web.math.muni.cz/biografie/ladislav_truksa.html (in Czech). Ladislav Truksa, born 1891, studied actuarial science, mathematics and physics in Prague. His doctoral thesis in 1927 was on Legendre polynomials. He habilitated in actuarial and mathematical statistics in 1931. He played an important role in the early scientific and teaching activities of the Department of Probability and Mathematical Statistics, founded in 1952 within Charles University, Prague.

representation, the Rodrigues type formula, the three-term recurrence relation, and the second order difference equation. Curiously enough, both Truksa's paper [79] and Meixner's paper [2] were reviewed in *Jahrbuch über die Fortschritte der Mathematik* by W. Hahn (JFM 57.0413.01 and JFM 60.0293.01), but the reviewer did not observe in his review of [2] that the polynomials of class IV already appeared in Truksa's paper. There are hardly any citations of Truksa's paper in the literature. But Hahn mentions it in some of his JFM reviews. He also mentions Truksa's and Meixner's paper in his important paper [54, p. 32, footnote 1], but without giving details.

In addition to the five classes listed in the beginning of Section 4, there should have been a sixth class as the result of Meixner's classification:

Case VI. Krawtchouk polynomials $K_n(x; p, N)$ $(n = 0, 1, \dots, N)$, [60, Section 9.11]:

$$\exp\left(x \log\left(\frac{1 + (1 - p^{-1}t)}{1 + t}\right)\right)(1 + t)^N = \sum_{n=0}^{N} K_n(x; p, N)\, t^n \quad (x = 0, 1, \dots, N),$$

orthogonal on $\{0, 1, \dots, N\}$ with respect to the weights $\binom{N}{x} p^x (1 - p)^{N-x}$.

But Meixner cannot be blamed for this omission, since he only looked for infinite systems $\{p_n\}_{n=0,1,2,\dots}$ of orthogonal polynomials satisfying (3.1). The inclusion of Krawtchouk polynomials in Meixner's classification was probably first made by Lancaster [62] (1975).

The polynomials $P_n^{(\lambda)}(x; \phi)$ (case V of the classification) were rediscovered by Pollaczek [66] (1950). Quite remarkably, he obtained them as limit cases of a three-parameter class of orthogonal polynomials which he had introduced the year before and which are nowadays known as Pollaczek polynomials. These latter polynomials are of great importance but definitely outside the Askey scheme. For quite a while the polynomials $P_n^{(\lambda)}(x; \phi)$ were only attributed to Pollaczek in the literature. Around 1985, when the Askey scheme was formalized, the name Meixner–Pollaczek polynomials became common for these polynomials, see Andrews & Askey [34, p. 41] for the motivation of this name.

As observed in [34, p. 41], the Meixner, Krawtchouk and Meixner–Pollaczek polynomials are essentially the same polynomials, but for different parameter ranges and, in the Meixner–Pollaczek case, with a complex linear change of independent variable. Indeed, from the expressions as hypergeometric functions for the three classes, see [60, Ch. 9], it follows that

$$K_n(x; p, N) = M_n\left(x; -N, \frac{p}{p-1}\right) \quad (n = 0, 1, \dots, N),$$

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!}\, e^{in\phi} M_n(-\lambda - ix; 2\lambda, e^{2i\phi}).$$

We can group together the six cases of the Meixner classification into the "Meixner scheme", a subscheme of the Askey scheme [60, p. 184]. The arrows indicate limit transitions, both for the polynomials and for the orthogonality measures.

$$\begin{tikzcd}
\text{Meixner–Pollaczek} \arrow[dr] & & \text{Meixner} \arrow[dl] \arrow[dr] & & \text{Krawtchouk} \arrow[dl] \\
& \text{Laguerre} \arrow[dr] & & \text{Charlier} \arrow[dl] & \\
& & \text{Hermite} & &
\end{tikzcd}$$

One may also go directly from a family in the top row to Hermite in the bottom row by a limit.

6. Follow-up of Meixner's 1934 paper

We did some searches for texts which cite Meixner's paper [2] (1934), and in particular take up the new classes of orthogonal polynomials introduced in [2]. Some early citations to [2] concern the Charlier polynomials, see for instance Szegö's book [77, §2.81] (1939) and Meixner's own paper [3]. Early papers which continue work on the orthogonal polynomials with generating function (3.1) are Geronimus [52] (1938) and Sheffer [74] (1939). In particular, in [74, Section 2] a sequence of polynomials $P_n(x)$ of degree n satisfying the equivalent properties (4.7) and (4.9) (slightly renormalized) is called a set of type zero corresponding to the operator $t(D)$. Sheffer gives there many further properties of such sequences (although Meixner's operator Λ does not seem to figure in [74]).

Clearly, there are many systems of monic polynomials $P_n(x)$ satisfying (4.9) for given $t(D)$. However, as observed by Sheffer [74, Theorem 1.2], there exists a unique such sequence which moreover satisfies $P_n(0) = 0$ for $n > 0$. This is called the *basic sequence* associated with $t(D)$.[16] Steffensen [76] (1941) independently introduced these basic sequences, called *poweroids* by him.

Sheffer's sets of type zero [74] got much attention in later papers, without any reference to Meixner's paper [2]. They became generally known as *Sheffer polynomials*, see for instance [33]. In particular, Sheffer polynomials became an important topic in the so-called *umbral calculus*, pioneered by Rota, Kahaner & Odlyzko [68], see also the survey [45] by Di Bucchianico.

The probability distributions corresponding to the five classes of orthogonal polynomials resulting from Meixner's classification [2] got a lot of attention in papers on stochastics. See surveys by Lai [61] and Di Bucchianico [45]. These distributions also arise in the classification of the so-called *natural exponential families* with quadratic variance functions by Morris [64]. See also [44] on stochastic aspects of Meixner polynomials.

Meixner and Meixner–Pollaczek polynomials have group theoretic interpretations as matrix elements of discrete series representations of SL(2,$\mathbb{R}$) (or of its universal covering), see Basu and Wolf [36, Section 3] and Koornwinder [59, Section 7]. Foata and Labelle [49] gave combinatorial models for the Meixner polynomials.

It often happens with mathematical objects named after a person that a generalized object also keeps this name, although the generalization has been made by someone else. Thus happened also with Meixner polynomials. We meet for instance:

- *q-Meixner polynomials*, [60, Section 14.13]. These q-analogues of Meixner polynomials first occurred in Hahn's classification [54], but not yet bearing this name.
- *Multivariable Meixner polynomials*. These were introduced by Tratnik [78]. They involve (non-straightforward) products of one-variable Meixner polynomials. They could already have been, but were not, obtained as limits of the multivariable Hahn polynomials considered by Karlin and McGregor [58]. See Gasper and Rahman [50] for the multivariable q-case. In principle, another kind of multivariable Meixner polynomials, associated with root system BC_n, can be obtained as a limit case of the BC_n Hahn polynomials considered by van Diejen and Stokman [46, Section 5]. Possibly the papers by Chikuse [43] and Bryc and Letac [39], written from a probabilistic point of view, and the paper by Shibukawa [75] are related.

One also meets multiple Meixner polynomials, exceptional Meixner polynomials, d-orthogonal Meixner polynomials, matrix-valued Meixner polynomials and Meixner–Sobolev orthogonal

[16] Clearly, a basic sequence cannot be a sequence of orthogonal polynomials.

polynomials in the literature. In stochastics one finds Meixner distributions, Meixner ensembles and Meixner processes. On September 1, 2016 MathSciNet listed 188 publications having the word *Meixner* in its title. Similarly zbMATH found 182 documents.

As a concluding remark, the irony of history is very visible in connection with Meixner's 1934 paper [2]. Indeed, Meixner introduced the Sheffer polynomials, but they are called after Sheffer. Meixner formulated the Favard theorem, but it is called after Favard. Meixner gave the Meixner polynomials, but they should be called after Truksa. Meixner gave the Meixner–Pollaczek polynomials, but they were almost called just Pollaczek polynomials.

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Selected publications by Meixner

[1–11] are papers on orthogonal polynomials and hypergeometric functions, [12–25] are papers on spheroidal wave functions and Mathieu functions, and [26–32] are books and Handbuch articles.

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