

## Supplemental Material: Finite temperature disordered bosons in two dimensions

### Single particle localization in 2D.

Let us consider a weakly bound state with localization length  $\zeta$  in a short-range Gaussian random potential  $U(x)$ , with correlation length  $\sigma$  and amplitude  $U_0$ . Under the condition  $\zeta \gg \sigma$ , the kinetic energy is given by:

$$K \simeq \frac{\hbar^2}{2m\zeta^2}. \quad (1)$$

The potential energy contribution is computed noting that the contribution of a well in 2D is  $\sim U_0\sigma^2/\zeta^2$  and multiplying it by  $\zeta/\sigma$ , which is the square root of the number of wells on the length scale  $\zeta$ .

We now minimize the total energy with respect to  $\zeta$ :

$$\frac{\partial E}{\partial \zeta} = -\frac{\hbar^2}{m\zeta^3} + U_0 \frac{\sigma}{\zeta^2} = 0, \quad (2)$$

and obtain a characteristic length  $\zeta_* \sim \hbar^2/mU_0\sigma$ . This gives a characteristic energy  $\epsilon_* \sim mU_0^2\sigma^2/\hbar^2 \ll U_0$ . It is convenient to define the characteristic energy  $\epsilon_*$  and length  $\zeta_*$  as:

$$\epsilon_* = \frac{mU_0^2\sigma^2}{\pi\hbar^2}; \quad \zeta_* = \sqrt{\frac{2e^2}{\pi}} \frac{\hbar^2}{mU_0\sigma}. \quad (3)$$

Then, the single-particle localization length in two dimensions at  $\epsilon > \epsilon_*$  [1] can be written in the form:

$$\zeta(\epsilon) = \frac{\zeta_*}{e} \sqrt{\frac{\epsilon}{\epsilon_*}} \exp \frac{\epsilon}{\epsilon_*} \quad (4)$$

so that  $\zeta(\epsilon_*) = \zeta_*$ .

### Temperature dependence of the MBLDT.

As is already said in the main text, at zero temperature on the insulator side the occupation number is given by

$$N_\epsilon = \frac{\zeta^2(\epsilon) (\mu - \epsilon)}{g} \Theta(\mu - \epsilon), \quad (5)$$

where  $\Theta$  is the theta-function. Then Eq. (11) of the main text becomes:

$$n = \int_{-\epsilon_*}^{\epsilon_*} \rho_0 \frac{\zeta_*^2 (\mu - \epsilon)}{g} d\epsilon + \int_{\epsilon_*}^{\mu} \rho_0 \frac{\zeta^2(\epsilon) (\mu - \epsilon)}{g} d\epsilon, \quad (6)$$

and it yields

$$\frac{ng}{\epsilon_*} = f(\mu) \equiv \frac{\exp(2\mu/\epsilon_*)}{4\pi^3} (\mu/\epsilon_* - 1) + \frac{e^2}{4\pi^3} (7\mu/\epsilon_* + 1). \quad (7)$$

At the same time, Eq. (9) of the main text gives:

$$g\rho_0^2 \frac{1}{\zeta^2(\epsilon_\alpha)} \int_{-|\epsilon_*|}^{\epsilon_*} \frac{\zeta_*^2 (\mu - \epsilon)}{g} \zeta_*^4 d\epsilon + g\rho_0^2 \frac{1}{\zeta^2(\epsilon_\alpha)} \int_{\epsilon_*}^{\epsilon_\alpha} \frac{\zeta^2(\epsilon) (\mu - \epsilon)}{g} \zeta^4(\epsilon) d\epsilon + g\rho_0^2 \zeta^2(\epsilon_\alpha) \int_{\epsilon_\alpha}^{\mu} \frac{\zeta^2(\epsilon) (\mu - \epsilon)}{g} d\epsilon = C \quad (8)$$

where  $C$  is a coefficient of order 1. The resulting relation between  $\mu$  and  $\epsilon_\alpha$  is:

$$F(\epsilon_\alpha, \mu) \equiv \frac{e^4}{\pi^6} \left( \frac{\zeta(\epsilon_\alpha)^4}{\zeta_*^4} \left( \frac{\epsilon_\alpha}{\epsilon_*} \left( \frac{1}{3} - \frac{7\epsilon_*}{18\epsilon_\alpha} + \frac{7\epsilon_*^2}{36\epsilon_\alpha^2} + \frac{\epsilon_*^3}{54\epsilon_\alpha^3} - \frac{\epsilon_*^4}{324\epsilon_\alpha^4} \right) + \frac{\mu}{\epsilon_*} \left( -\frac{1}{3} + \frac{\epsilon_*}{6\epsilon_\alpha} + \frac{\epsilon_*^2}{36\epsilon_\alpha^2} - \frac{\epsilon_*^3}{216\epsilon_\alpha^3} \right) \right) \right. \\ \left. + \frac{\zeta(\epsilon_\alpha)^2 \zeta(\mu)^2}{4 \zeta_*^4} \left( 1 - \frac{\epsilon_*}{\mu} \right) + \frac{\zeta_*^2}{\zeta(\epsilon_\alpha)^2} \left( \frac{409\mu}{216\epsilon_*} + \frac{31}{324} \right) \right) = C. \quad (9)$$

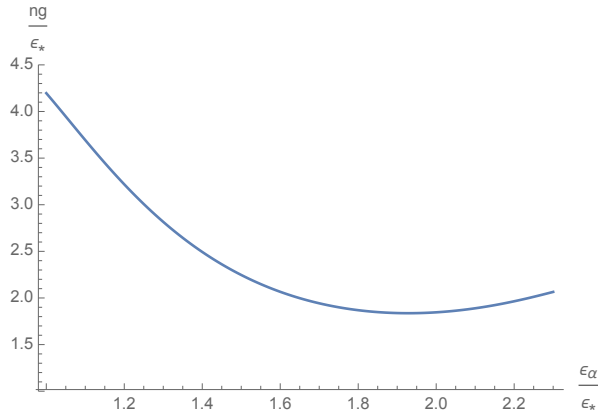


Figure 1: The dependence of the critical coupling  $ng$  on  $\epsilon_\alpha$  at zero temperature for  $C = 1$ . The minimum is at  $\epsilon_\alpha = 1.93\epsilon_*$ .

We found numerically from equations (7) and (9), with  $C = 1$ , that the coupling strength is minimal for  $\epsilon_{\alpha 0} = 1.93\epsilon_*$ . It is equal to  $ng_{c0} = 1.84\epsilon_*$ , and the related chemical potential is  $\mu_0 = 2.23\epsilon_*$ . This is seen from the obtained dependence of  $ng_{c0}$  on  $\epsilon_\alpha$  (Fig. 1).

It is instructive to look what happens for different values of the constant  $C$  in equation (9). Figure 2 shows the zero-temperature critical coupling  $ng_{c0}$  obtained for values of  $C$  between 0.3 and 3.

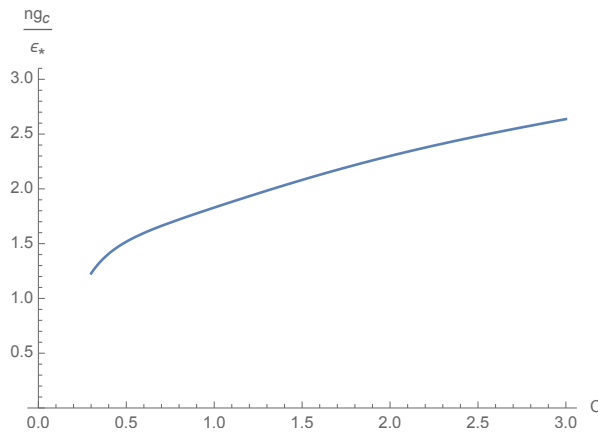


Figure 2: The critical coupling  $ng_{c0}/\epsilon_*$  as a function of the constant  $C$ .

At temperatures  $T \ll \epsilon_*^2/T_d$ , the corrections to the zero temperature result are negligible. For temperatures  $\epsilon_*^2/T_d \ll T \ll \epsilon_*$ , using equations (11) and (12) of the main text we have the following relations for the occupation numbers:

$$N_\epsilon = \begin{cases} \frac{\zeta^2(\epsilon)(\mu - \epsilon)}{g} + \frac{T}{\mu - \epsilon}; & -\epsilon_* < \epsilon < \mu - \delta \\ \sim \frac{\zeta^2(\mu)}{2g} \left( \mu - \epsilon + \sqrt{(\epsilon - \mu)^2 + 4Tg/\zeta^2(\mu)} \right); & \mu - \delta < \epsilon < \mu + \delta \\ (e^{(\epsilon - \mu)/T} - 1)^{-1}, & \mu + \delta < \epsilon \end{cases} \quad (10)$$

where  $\delta$  is a small quantity such that  $\sqrt{Tg/\zeta^2(\mu)} \ll \delta \ll T$ . Eq. (10) of the main text gives

$$\frac{ng_c}{\epsilon_*} = f(\mu) + \frac{T}{T_d} \frac{ng_{c0}}{\epsilon_*} \ln \left( \frac{(\mu_0 + \epsilon_*)\zeta^2(\mu_0)}{g_{c0}} \right). \quad (11)$$

As the correction to the chemical potential should be small, we can expand  $f(\mu)$  near  $\mu_0$  and obtain:

$$\frac{ng_c}{\epsilon_*} = f(\mu_0) + \frac{(\mu - \mu_0)}{\epsilon_*} f'_\mu(\mu_0) + \frac{T}{T_d} \frac{ng_{c0}}{\epsilon_*} \ln \left( \frac{(\mu_0 + \epsilon_*) \zeta^2(\mu_0)}{g_{c0}} \right). \quad (12)$$

Similarly, in the MBLDT criterion at finite temperatures we expand the function  $F(\epsilon_\alpha, \mu)$  near  $\mu_0$  and  $\epsilon_{\alpha 0}$ , which gives:

$$F(\epsilon_{\alpha 0}, \mu_0) + \frac{(\mu - \mu_0)}{\epsilon_*} F'_\mu(\epsilon_{\alpha 0}, \mu_0) + \frac{(\epsilon_\alpha - \epsilon_{\alpha 0})}{\epsilon_*} F'_{\epsilon_\alpha}(\epsilon_{\alpha 0}, \mu_0) + \frac{T}{T_d} \frac{ng_{c0}}{\epsilon_*} G(\epsilon_{\alpha 0}, \mu_0, g_{c0}) = C \quad (13)$$

with

$$G(\epsilon_{\alpha 0}, \mu_0, g_{c0}) \equiv \frac{e^2}{\pi^3} \left( \frac{\zeta(\alpha_0)^2}{\zeta_*^2} \left( \frac{\epsilon_*^2}{16\epsilon_{\alpha 0}^2} \left( 1 - \frac{4\epsilon_{\alpha 0}}{\epsilon_*} - 4\frac{\mu_0}{\epsilon_*} \right) + \ln \left( \frac{(\mu_0 - \epsilon_{\alpha 0}) \zeta^2(\mu_0)}{g_{c0}} \right) \right) \right. \\ \left. + \frac{\zeta(\mu_0)^4}{\zeta(\epsilon_{\alpha 0})^2 \zeta_*^2} \left( \text{Ei} \left( 4 - 4\frac{\mu_0}{\epsilon_*} \right) - \text{Ei} \left( 4\frac{\epsilon_{\alpha 0}}{\epsilon_*} - 4\frac{\mu_0}{\epsilon_*} \right) \right) + \frac{\zeta_*^2}{\zeta(\epsilon_{\alpha 0})^2} \left( \frac{3}{16} + \frac{\mu_0}{4\epsilon_*} + \ln \left( \frac{\mu + \epsilon_*}{\mu - \epsilon_*} \right) \right) \right), \quad (14)$$

where  $\text{Ei}(x)$  is the exponential integral function. We have inserted the zero-temperature values of  $\mu_0$  and  $g_{c0}$  in the temperature correction, as these corrections are of order  $T/T_d$ . By construction we have  $F'_{\epsilon_\alpha}(\epsilon_{\alpha 0}, \mu_0) = 0$  and  $F(\epsilon_{\alpha 0}, \mu_0) = C$ . This results in an expression for  $\mu - \mu_0$  which we substitute into Eq. (12). We then obtain:

$$\frac{ng_c}{\epsilon_*} = \frac{ng_{c0}}{\epsilon_*} \left( 1 - \frac{T}{T_d} \left( \frac{f'_\mu(\mu_0)}{F'_\mu(\epsilon_{\alpha 0}, \mu_0)} G(\epsilon_{\alpha 0}, \mu_0, g_{c0}) - \ln \left( \frac{(\mu_0 + \epsilon_*) \zeta^2(\mu_0)}{g_{c0}} \right) \right) \right). \quad (15)$$

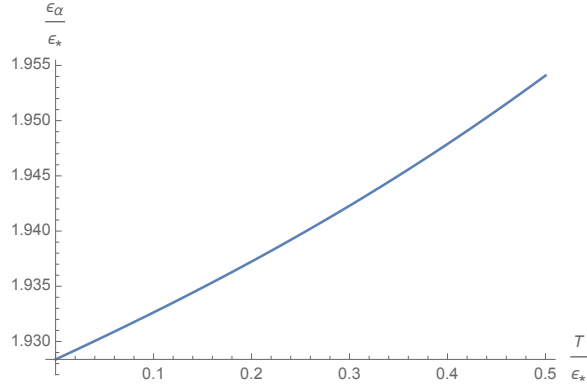


Figure 3: The temperature dependence of the value of  $\epsilon_\alpha$  that minimizes  $g_c$  at  $C = 1$ . We used here  $T_d/\epsilon_* = 50$ .

By setting  $C = 1$  and inserting the corresponding values of  $ng_{c0}$ ,  $\mu_0$ ,  $\epsilon_{\alpha 0}$  in the temperature corrections, we obtain Eq. (15) of the main text. The numerically obtained temperature dependence of  $\epsilon_\alpha$  that minimizes  $g_c$  is given in Fig. 3 for  $C = 1$ , and one sees that the optimal  $\epsilon_\alpha$  is very close to  $\epsilon_{\alpha 0}$ . For different values of  $C$  the numerical coefficients in Eq. (15) of the main text vary only slightly.

In the thermodynamic limit, the critical coupling tends to zero when  $T \rightarrow \epsilon_*/2$  from below. We may assume that the chemical potential decreases below  $\epsilon_*$ , so that  $\mu \rightarrow -|\epsilon_*|$  when  $g_c \rightarrow 0$ , and we expect  $\epsilon_\alpha$  to increase. Equation (9) of the main text takes the form:

$$g\rho_0^2 \left( \frac{1}{\zeta^2(\epsilon_\alpha)} \int_{-\epsilon_*}^{\epsilon_*} N_\epsilon \zeta_*^4 d\epsilon + \frac{1}{\zeta^2(\epsilon_\alpha)} \int_{\epsilon_*}^{\epsilon_\alpha} \frac{\zeta^4(\epsilon)}{e^{(\epsilon-\mu)/T} - 1} d\epsilon + \zeta^2(\epsilon_\alpha) \int_{\epsilon_\alpha}^{\infty} \frac{1}{e^{(\epsilon-\mu)/T} - 1} d\epsilon \right) \approx C. \quad (16)$$

As  $\epsilon_\alpha$  is large, we neglect the first integral and calculating the other integrals we keep only the highest power in  $\epsilon_\alpha$ . The coupling strength  $g$  is then minimized at  $\epsilon_\alpha = T\epsilon_*/(\epsilon_* - 2T)$ . Using  $\mu \simeq -|\epsilon_*|$  leads to the equation:

$$ng_c \simeq 4(\pi e)^3 \frac{T_d}{\epsilon_*} \left( \frac{\epsilon_*}{2} - T \right); \quad T \rightarrow \frac{\epsilon_*}{2}. \quad (17)$$

The results of exact numerics for  $g_c(T)$  and  $\mu(T)$  in the thermodynamic limit are shown in Figure 4  $C = 1$ . We can see that the values of both  $g_c(T)$  and  $\mu(T)$  are almost constant until we get to the vicinity of  $T = \epsilon_*/2$ . Here they both sharply drop with a finite but large derivative, as shown in the insets. This happens irrespective of the value of  $C$ . In the small intermediate region of  $T$  given by the insets, where the analytical approach is not possible, we numerically solved equations (9) and (10) of the main text, with the occupation numbers given by equation (10) in the Supplemental Material.

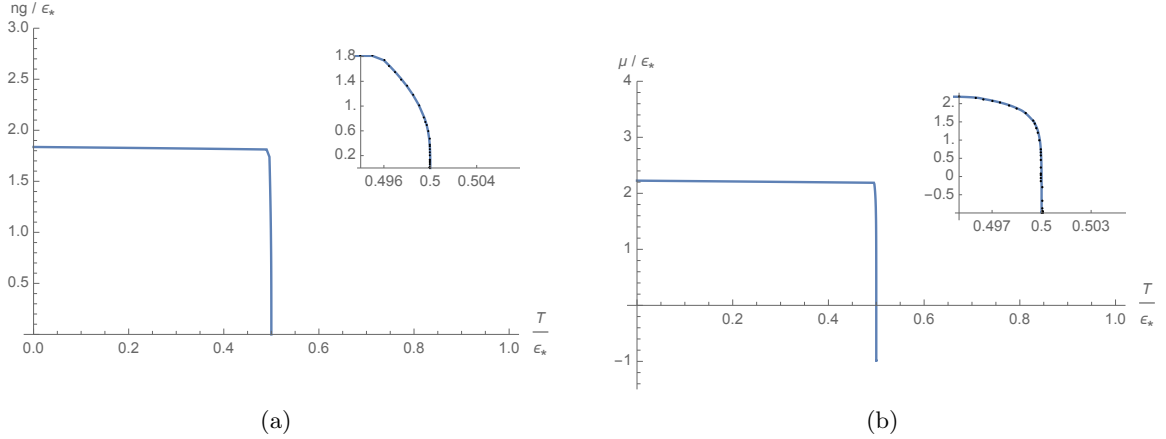


Figure 4: The critical coupling  $ng_{c0}$  in (a) and chemical potential  $\mu$  in (b) versus temperature for  $C = 1$ . In both insets we see that the derivative is always finite and becomes very large when we approach  $T = \epsilon_*/2$ .

### MBLDT for the truncated distribution function

In the main text we argued that in realistic systems the distribution function  $N_\epsilon$  is truncated at an energy  $\epsilon_b$ . We take  $\epsilon_b = 1.21ng + \eta T$ , where  $\eta$  ranges from 5 to 8. This means that the MBLDT criterion reads

$$C = g\rho_0^2 \left( \frac{1}{\zeta^2(\epsilon_\alpha)} \int_{-\epsilon_*}^{\epsilon_\alpha} N_\epsilon \zeta^4(\epsilon) d\epsilon + \zeta^2(\epsilon_\alpha) \int_{\epsilon_\alpha}^{\epsilon_b} N_\epsilon \right). \quad (18)$$

At  $T \ll \epsilon_*$  the truncation practically does not influence the results. However, at higher temperatures the influence is crucial. Considering  $T > \epsilon_*/2$  and setting  $\epsilon_\alpha \rightarrow \epsilon_b$ , with  $\mu + \delta < \epsilon_*$ , one gets

$$C = g\rho_0^2 \frac{1}{\zeta^2(\epsilon_b)} \left( \int_{-\epsilon_*}^{\mu+\delta} \frac{\zeta_*^2}{2g} \left( \mu - \epsilon + \sqrt{(\mu - \epsilon)^2 + 4Tg/\zeta_*^2} \right) \zeta_*^4 d\epsilon \right. \\ \left. + \int_{\mu+\delta}^{\epsilon_*} \frac{1}{e^{(\epsilon-\mu)/T} - 1} \zeta_*^4 d\epsilon + \int_{\epsilon_*}^{\epsilon_b} \frac{1}{e^{(\epsilon-\mu)/T} - 1} \zeta^4(\epsilon) d\epsilon \right). \quad (19)$$

The last integral dominates and gives

$$C = \pi^{-3} \frac{ng}{\epsilon_*} \frac{\epsilon_b}{\epsilon_*} \frac{T}{T_d} \frac{1}{4T/\epsilon_* - 1} \exp \left\{ \epsilon_b \left( \frac{2}{\epsilon_*} - \frac{1}{T} \right) + \frac{\mu}{T} \right\}. \quad (20)$$

From Eq. (11) of the main text one has:

$$\mu \simeq \frac{\pi^{3/2}}{\sqrt{2e^2}} ng - \left( 1 - \frac{\pi^{3/2}}{\sqrt{2e^2}} \right) \epsilon_*. \quad (21)$$

For the critical coupling Eq. (20) then yields:

$$ng_c \simeq C \sqrt{\frac{2e^2}{\pi^3}} T W \left( \sqrt{\frac{\pi^9}{2e^2}} \frac{\epsilon_*^2}{T^2} \frac{T_d}{\epsilon_b} \left( \frac{4T}{\epsilon_*} - 1 \right) e^{\epsilon_b \left( \frac{1}{T} - \frac{2}{\epsilon_*} \right)} \right), \quad (22)$$

where  $W(x)$  is the (main branch) Lambert  $W$ -function defined as  $x = W(x)e^x$ . For  $T > \epsilon_*/2$ , the argument of  $W(x)$  is small. We then use the approximation  $W(x) \approx x$  for small  $x$ . This gives:

$$ng_c \simeq C \pi^3 \epsilon_* \frac{T_d}{\epsilon_b} \left( 4 - \frac{\epsilon_*}{T} \right) e^{\epsilon_b \left( \frac{1}{T} - \frac{2}{\epsilon_*} \right)}. \quad (23)$$

The term  $(4 - \epsilon_*/T)$  takes values ranging from 2 to 4 as  $T$  is increased above  $\epsilon_*/2$ . We take  $(4 - \epsilon_*/T) \approx 2$  for simplicity, which yields:

$$\epsilon_*^{MBL} = \frac{2\epsilon_b}{W \left( 4\pi^3 \frac{T_d e^{\epsilon_b/T}}{ng} \right)}. \quad (24)$$

The argument of  $W(x)$  is now large and we can use  $W(x) \approx (\ln x - \ln \ln x)$  for large  $x$  to get Eq. (16) of the main text, using  $C = 1$ . The comparison of this analytical expression with exact numerics is given in Figure 5, where we used  $\epsilon_b = 1.21ng + 5T$ . In Fig. 6 we show the same quantities as in Fig. 5, but for the truncation of the energy distribution function at  $\epsilon_b = 1.21ng + 8T$ . As one sees, the increase of  $\beta$  from 5 to 8 does not significantly change the MBLDT transition line.

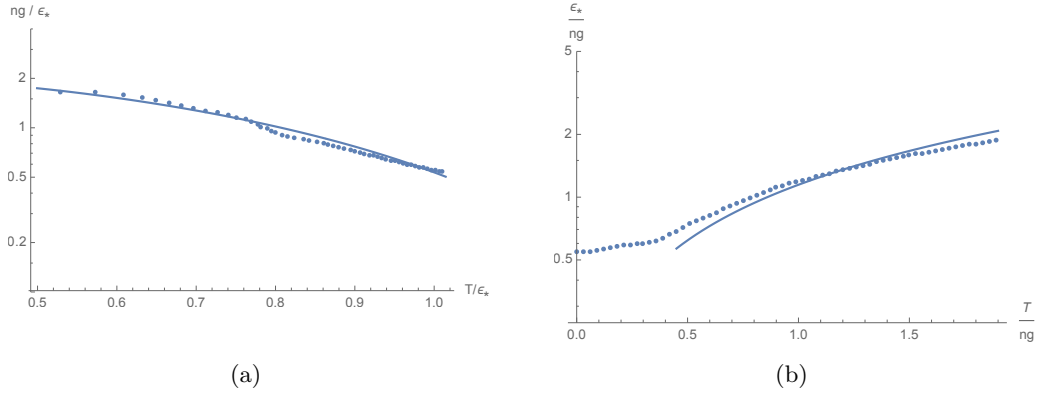


Figure 5: Comparison of numerical results with those from analytical expressions for  $\epsilon_b = 1.21ng + 5T$ . The dots are the results of numerically solving Eq.(19), and the solid curve is given by Eq. (22) in (a) and by Eq. (16) of the main text in (b). We used  $T_d/\epsilon_* = 20$  in (a) and  $T_d/ng = 11$  in (b).

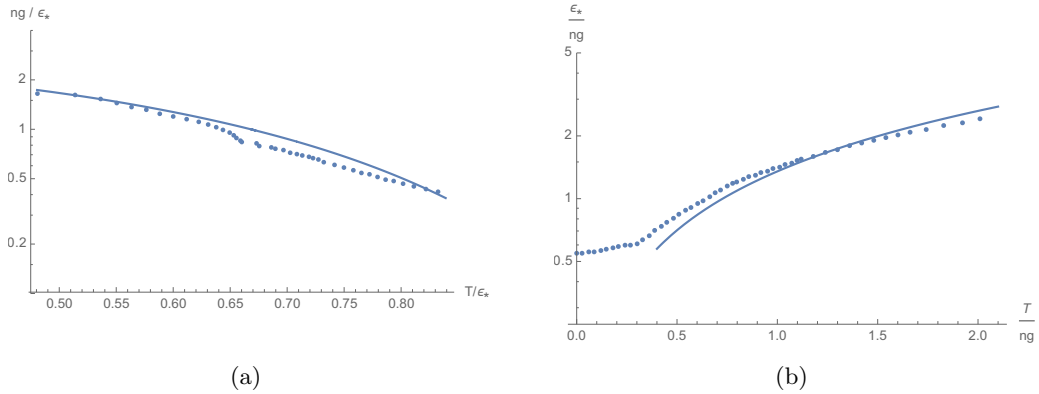


Figure 6: The same as in Fig.5, but for the truncation of the energy distribution function at  $\epsilon_b = 1.21ng + 8T$ .

### Tricritical point at $T = 0$

This subsection is dedicated to ruling out a direct transition from the insulator to superfluid phase. This transition would mean that one has a phase diagram either like Fig. (a) or Fig. (b).

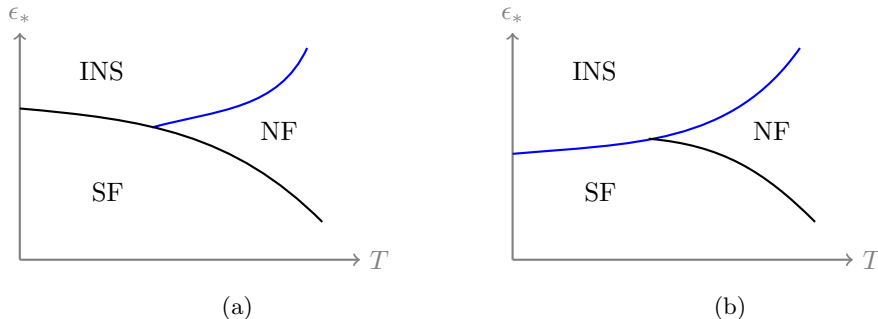


Fig. (a) is ruled out as follows. At the critical zero-temperature disorder and arbitrarily small finite temperatures, the system is unstable with respect to delocalization. This conclusion is supported by the quantitative analysis given in the text, where corrections to the zero-temperature value  $\epsilon_*^{MBL}(0)$  were found to be positive. The physical picture is that although excitations in the insulator at  $T = 0$  are always localized, their localization length at the critical disorder strength  $\epsilon_*^{MBL}(0)$  can be arbitrarily large. Therefore, at any fixed disorder  $\epsilon_* > \epsilon_*^{MBL}(0)$  the elementary excitations undergo many-body delocalization with increasing temperature. The critical temperature tends to zero as the localization length diverges, i.e. at arbitrary low finite temperatures there will be a range of disorder strengths corresponding to a normal fluid. Fig. (b) is ruled out by the following arguments. The zero-temperature insulator can be viewed as a composition of superfluid lakes with uncorrelated phases, which are separated from each other by a certain distance. Tunneling of particles between the lakes increases with decreasing the disorder strength, and at a critical disorder it establishes the phase coherence between the lakes, so that the whole system becomes superfluid. Consider a single lake at the disorder strength  $\epsilon_*^{MBL}(0) + 0$  and slightly increase the temperature. Then a certain fraction of particles in the lake will become non-superfluid. Assuming slowly varying density fluctuations, such that the Bogoliubov theory works, the non-superfluid fraction within the lake turns out to be

$$\frac{n'}{n} = 3\zeta(3) \frac{T^3}{T_d n^2 g^2},$$

where  $\zeta(3)$  is the Riemann zeta-function,  $g$  is the coupling strength of the interparticle interaction,  $n$  is the density, and  $T_d$  the degeneracy temperature. A decrease of the superfluid density  $n$  (which is equivalent to decreasing the coupling strength  $g$ ) reduces the probability of tunneling between neighbouring lakes, which behaves as (see e.g., [2])

$$t \sim \exp\left(-\sqrt{\epsilon_*^{MBL}(0)/ng}\right),$$

and the latter is unable to establish phase coherence between the lakes.

## References

- [1] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
- [2] G.M. Falco, T. Nattermann and V.L. Pokrovsky, Phys. Rev. B **80**, 104515 (2009).