Finite-Temperature Disordered Bosons in Two Dimensions

Bertoli, G.; Michal, V.P.; Altshuler, B.L.; Shlyapnikov, G.V.

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Almost 60 years since its discovery [1], the concept of Anderson localization (AL) of a quantum particle by a quenched disorder remains an extremely active direction of research [2]. To a large extent, this is due to a subtle problem of the effect of interaction between particles on the localization [3–6]. It has been demonstrated that interacting particles can undergo many-body localization-delocalization transition (MBLDT), that is the transition from insulator to the fluid state [6]. A new wave of interest in this problem was inspired by the observation of AL in dilute quasi-one-dimensional clouds of cold bosonic atoms with a negligible interaction [7,8]. Presently, the studies of ultracold interacting atoms form a rapidly growing domain in the physics of disordered quantum systems [9]. Although the first observations of MBLDT in these systems have been reported [10–12], many features of MBLDT remain unexplored, especially in higher than one dimension. In contrast to the one-dimensional (1D) case where at any nonzero temperature \( T > 0 \) only normal fluid and insulator (glass) phases are possible, in two dimensions (2D) the phase diagram contains one more phase. Two-dimensional bosons undergo the Berezinskii-Kosterlitz-Thouless (BKT) transition \([13,14]\) and form an algebraic superfluid below a critical temperature \( T_{\text{BKT}} \). While a number of studies \([15–17]\) were devoted to evaluating the critical disorder strength either for the MBLDT at zero temperature or for the BKT transition, the full finite-temperature phase diagram of such a system to the best of our knowledge has never been published \([18]\). 

In this Letter we construct the phase diagram of 2D weakly interacting bosons subject to a static random potential. The diagram is displayed in Fig. 1 in terms of \( T \) and \( \varepsilon_* \), where the energy scale \( \varepsilon_* \) characterizes the disorder strength. It turns out that there are two temperature dependent critical values of disorder: \( \varepsilon_{*\text{BKT}}(T) \) and \( \varepsilon_{*\text{MBL}}(T) \), i.e., two separatrices in Fig. 1 \([18]\). The first

![Phase diagram for 2D weakly interacting disordered bosons in terms of the dimensionless disorder strength \( \varepsilon_*/n_g \) and temperature \( T/n_g \).](image-url)
one separates the normal fluid from the superfluid phase and it shows the suppression of superfluidity by the disorder. Since superfluidity disappears at $T > T_{BKT}$ even without disorder, we have $e_{s}^{BKT}(T \geq T_{BKT}) = 0$. For sufficiently strong disorder, $e_{s} > e_{s}^{BKT}(0)$, the superfluid regime is absent even at $T = 0$. The second separatrix is the MBLDT curve. The region $e_{s} > e_{s}^{MBL}(T)$ corresponds to the insulator (glass) phase, which undergoes a transition to the normal fluid as the disorder is reduced to below $e_{s}^{MBL}(T)$.

The important property of 2D weakly interacting disordered bosons is the instability of the normal fluid at $T = 0$ with respect to a transition either to the superfluid or to the insulator regime. Accordingly, one has

$$e_{s}^{MBL}(0) = e_{s}^{BKT}(0).$$

(1)

This means that the point $T = 0$, $e_{s} = e_{s}^{MBL}(0)$ is a tricritical point, where the three phases coexist [18].

In terms of field operators $\hat{\Psi}(\mathbf{r})$, the Hamiltonian of 2D interacting disordered bosons reads:

$$\hat{H} = \int d^{2}r \left( -\frac{\hbar^{2}}{2m} \nabla^{2} \hat{\Psi}(\mathbf{r}) + g \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \hat{\Psi}(\mathbf{r}) + \hat{\Psi}(\mathbf{r}) U(\mathbf{r}) \hat{\Psi}(\mathbf{r}) \right).$$

(2)

The first term is the kinetic energy of particles ($m$ is the particle mass), and the second term (denoted below as $H_{int}$) describes a contact interaction between them, characterized by the coupling constant $g > 0$. The third term represents the effect of the random potential $U(\mathbf{r})$. We assume that $U(\mathbf{r})$ is a Gaussian short-range potential with zero mean, correlation length $\sigma$, and amplitude $U_{0}$ such that $U_{0} < \hbar^{2}/m\sigma^{2}$. The only disorder-related length and energy scales are known to be [19,20]

$$\zeta_{s} = \sqrt{\frac{3e^{2}}{\pi}} \frac{\hbar^{2}}{mU_{0}\sigma^{2}}; \quad e_{s} = \frac{mU_{0}\sigma^{2}}{\hbar^{2}}.$$

(3)

In the absence of disorder the density of states (DOS) for 2D bosons in the continuum is energy independent, $\rho_{0} = m/2\pi\hbar^{2}$. The random potential creates negative energy states, which form the so-called Lifshitz tails: the DOS decays exponentially as the absolute value of the energy increases [19,20]. Below we omit these states. For positive energies $\epsilon \gg e_{s}$ and even for $|\epsilon| \lesssim e_{s}$ the effect of the disorder is limited and $\rho(\epsilon) \approx \rho_{0}$ is a good approximation.

In two dimensions all single particle states are localized. The localization length $\zeta$ increases exponentially with the particle energy for $\epsilon > e_{s}$ [21]:

$$\zeta(\epsilon) = \frac{\zeta_{s}}{\epsilon} \sqrt{\frac{\epsilon}{e_{s}}} e^{\epsilon/e_{s}}; \quad \epsilon \gg e_{s}.$$

(4)

which was, in particular, observed in atomic kicked rotor experiments [22]. At energies $|\epsilon| \lesssim e_{s}$ one can neglect the energy dependence of $\zeta$ and approximate the localization length as $\zeta(\epsilon) \approx \zeta_{s}$.

We consider the weakly interacting regime, where the degeneracy temperature $T_{d} = 2\pi\hbar^{2}n/m$ greatly exceeds the mean interaction energy per particle $ng$, with $n$ being the mean density. Thus, there is a small parameter

$$\frac{ng}{T_{d}} = \frac{m \hbar}{2\pi^{2} e^{\epsilon_{s}}} \ll 1.$$  

(5)

We also assume that the disorder is weak, so that

$$e_{s} \ll T_{d}.$$  

(6)

In order to estimate the critical disorder $e_{s}^{MBL}$ at a given $g$, we employ the method developed in Refs. [6,23]. Namely, we consider a particular one-particle localized state $|\alpha\rangle$ and evaluate the probability $P_{\alpha}$ that there exist three other states $|\beta\rangle$, $|\alpha'\rangle$, $|\beta'\rangle$ such that the two-particle states $|\alpha, \beta\rangle$ and $|\alpha', \beta'\rangle$ are at resonance. This means that the matrix element of the interaction $\langle \alpha', \beta'|H_{int}|\alpha, \beta\rangle$ exceeds the energy mismatch $\Delta_{\alpha\beta}^{ab} = |\epsilon_{\alpha} + \epsilon_{\beta} - \epsilon_{\alpha'} - \epsilon_{\beta'}|$, where $\epsilon_{\alpha}$, $\epsilon_{\beta}$, $\epsilon_{\alpha'}$, $\epsilon_{\beta'}$ are one-particle energies. The MBLDT occurs when $P_{\alpha}$ becomes close to unity.

The matrix elements of the interaction are small unless the energies $\epsilon_{\alpha}$, $\epsilon_{\beta}$, $\epsilon_{\alpha'}$, $\epsilon_{\beta'}$ are almost equal pairwise, e.g., $\epsilon_{\alpha} \approx \epsilon_{\alpha'}$ and $\epsilon_{\beta} \approx \epsilon_{\beta'}$. Then we have (see Refs. [6,23]):

$$\langle \alpha', \beta'|H_{int}|\alpha, \beta\rangle \approx \frac{g N_{\beta}}{\max(\zeta_{\alpha,\beta}^{2}, \zeta_{\alpha'\beta'}^{2})},$$

(7)

where $\zeta_{\alpha,\beta} \equiv \zeta(\epsilon_{\alpha}, \epsilon_{\beta})$, and $N_{\beta}$ is the occupation number for the state $|\beta\rangle$.

For $|\alpha\rangle$ and $|\alpha'\rangle$ being nearest neighbors in energy the energy mismatch is $\Delta_{\alpha\beta}^{ab} = |\delta_{\alpha} + \delta_{\beta}|$, where $\delta_{\alpha}$ is the level spacing between the states on the length scale close to $\zeta_{s}$. The mismatch can thus be estimated as

$$\Delta_{\alpha\beta}^{ab} \approx \max(\delta_{\alpha}, \delta_{\beta}) = \frac{1}{\min(\rho_{\alpha}, \rho_{\beta})^{1/2}},$$

(8)

and $\langle \alpha', \beta'|H_{int}|\alpha, \beta\rangle$ exceeds $\Delta_{\alpha\beta}^{ab}$ for given $|\alpha\rangle$, $|\beta\rangle$, $|\alpha'\rangle$, $|\beta'\rangle$ with the probability

$$P_{\alpha\beta}^{ab} = \frac{\langle \alpha', \beta'|H_{int}|\alpha, \beta\rangle}{\Delta_{\alpha\beta}^{ab}}.$$  

(9)

The quantity $P_{\alpha}$ is the sum of $P_{\alpha\beta}^{ab}$ over $\beta$, $\alpha'$, $\beta'$, and the MBLDT criterion takes the form (see Refs. [6,23]):
with the corresponding chemical potential $\mu = 1.21ng$. The result of Eq. (16) is consistent with those obtained from the analysis of tunneling between bosonic lakes [15].

Corrections to the zero temperature result (16) are small as long as $T \ll \epsilon_s$. For calculating these corrections one integrates over $\epsilon$ in Eqs. (11) and (12). This gives the following critical disorder:

$$
\epsilon_{s}^{MBL}(T) = \epsilon_{s}^{MBL}(0) \left[ 1 + 0.66 \frac{T}{T_d} \ln \left( 0.09 \frac{T_d}{\epsilon_{s}^{MBL}(0)} \right) \right].
$$

(17)

Exponential increase of the localization length with the particle energy supports delocalization. In the thermodynamic limit, as discussed in Ref. [17], this leads to the disappearance of the insulating phase at temperatures $T > \epsilon_s / 2$. However, for realistic systems of cold bosonic atoms the energy distribution is truncated at sufficiently large energy. Indeed, in the process of evaporative cooling, atoms with energies above the trap barrier immediately leave the trap, and the distribution function $N_e$ is effectively truncated at a finite energy barrier $\epsilon_b$. Typical values of this energy for evaporative cooling to temperatures $T \gtrsim ng$ are equal to $\eta T$, where $\eta$ ranges from 5 to 8 (see, e.g., Refs. [26,27]). For cooling to temperatures $T \lesssim ng$ the value of the energy barrier can be written as $\epsilon_b = ng + \eta T$ [28]. Below we use $\eta = 5$ and, in order to match the zero temperature result, we truncate $N_e$ at $\epsilon_b = 1.21ng + \eta T$. Increasing $\eta$ up to 8 has little effect on the MBLDT transition line $\epsilon_{s}^{MBL}(T)$.

The truncation of the energy distribution practically does not influence the results at $T \ll \epsilon_s$ and thus Eq. (17) remains valid. However, at higher temperatures the truncation strongly limits the growth of the localization length, and the critical coupling $g_c$ remains finite even for $T > \epsilon_s / 2$; i.e., the insulator phase survives. In this case the expression for the critical disorder, valid for $T \ll \epsilon_b$, is

$$
\epsilon_{s}^{MBL}(T) = \frac{2\epsilon_b}{\ln(4\pi^3 T_d \epsilon_b^3T/ng) - \ln(4\pi^3 T_d \epsilon_b^{3s}T/ng)}.
$$

(18)

Equations (17)–(18) are in good agreement with the numerical solution of Eqs. (11)–(13).

Actually, the distribution function $N_e$ does not abruptly go to zero at $\epsilon = \epsilon_b$. It undergoes a smooth, although quite sharp, decrease to zero near $\epsilon_b$. [26,27]. The disorder potential introduces an additional smoothness of $N_e$. However, for a weak disorder, the disorder-induced increase of the energy interval near $\epsilon_b$, in which the distribution function goes to zero, is significantly smaller than $U_0^2/\epsilon_b$, and is only a fraction of $\epsilon_b$ for realistic parameters of the system. Our calculations show that this
does not change the result of Eqs. (17)–(18) by more than a few percent.

In the recent paper [29] it was claimed that many-body localization is prevented in continuum systems. The conclusion was based on the exchange of energy between highly energetic particles and states with typical energies. Without entering the discussion of collisional integrals, we simply note that the truncation of the distribution function (which should clearly emerge after several collision times [26]) means that such high-energy particles are not there to induce delocalization.

It is worth noting that MBLDT can be measured for typical values of disorder, temperature, and density of 2D trapped bosonic atoms. The most promising is the situation where all single-particle states are localized. For example, at densities \( n \approx 10^7 \text{ cm}^{-2} \) of \(^7\text{Li} \) atoms the degeneracy temperature is \( T_d \approx 50 \text{ nK} \). For the amplitude of the disorder potential, \( U_0 = 35 \text{ nK} \), and correlation length \( \sigma \approx 1.4 \mu \text{m} \), we have \( \xi_s \approx 3 \mu \text{m} \) and \( \epsilon_s \approx 11.5 \text{ nK} \). Considering temperatures \( T \sim 10 \text{ nK} \), for barrier energies \( e_b \approx 44 \text{ nK} \), the localization length at maximum particle energies can be estimated as \( \sim 100 \mu \text{m} \). The size of the system can be significantly larger, so that all single-particle states are really localized. The MBLDT can be identified by opening the trap. If most of the sample is in the insulator phase, then only a small fraction of particles will escape and the size of the remaining cloud will increase by an amount of the order of the localization length. On the contrary, if most of the sample is in the fluid phase, switching off the trap will lead to the expansion of the major part of the cloud. The MBLDT can be also identified \textit{in situ} by measuring the dynamical structure factor with the use of the Bragg spectroscopy, the method employed to distinguish between the superfluid and Mott insulator phases of lattice atomic systems (see, e.g., Refs. [30,31]).

We now start our discussion of the BKT transition between the normal fluid and superfluid phases with the high temperature regime, \( T \gg n g \). In the superfluid phase we assume that density fluctuations are small and the Bogoliubov approach remains valid in the presence of disorder. Following Refs. [32,33] we consider a weak disorder, \( \epsilon_s \ll n g \), and rely on the Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \int U(r) \delta \rho(r) d^2 r \), where \( \mathcal{H}_0 \) is the standard Bogoliubov Hamiltonian in the density-phase representation, while the second term describes the interaction of the density fluctuations \( \delta \rho(r) \) with disorder. Diagonalizing \( \mathcal{H}_0 \) and using the known relation for the density fluctuations we have,

\[
\mathcal{H} = \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_k n U_k (b_k + b_k^\dagger) \sqrt{E_k / \hbar \omega_k}. \tag{19}
\]

Here \( n \) is the mean density, \( b_k \) and \( \hbar \omega_k = \sqrt{E_k^2 + 2 n g E_k} \) are the operators and energies of Bogoliubov excitations with momentum \( \mathbf{k}, E_k = \hbar^2 k^2 / 2m \) is the free particle kinetic energy, and \( U_k \) is the Fourier transform of the disorder potential \( U(r) \). For the normal density we then have [33]

\[
n_f = \frac{1}{2} n \int \frac{\langle U_k^\dagger U_k \rangle}{(n g + E_k / 2)^2} \frac{d^2 k}{(2 \pi)^2} \int E_k \frac{dN_k}{d \hbar \omega_k (2 \pi)^2}, \tag{20}
\]

where we put the normalization volume equal to unity. The result of the integration in the first term of Eq. (20) depends on the correlation function of the disorder. For \( \langle U(r) U(r') \rangle = U_0 \delta(|r - r'| / \sigma) \), we have \( \langle U_k^\dagger U_k \rangle = U_0^2 \sigma^2 \)

and at temperatures \( T \gg n g \) Eq. (20) yields

\[
n_f = \frac{\epsilon_s}{2 g} + \frac{m T}{2 \pi \hbar^2} \ln \frac{T}{n g} \quad T \gg n g. \tag{21}
\]

The Bogoliubov approach works well in the superfluid phase, but it does not allow one to determine the exact value of the BKT transition temperature \( T_{\text{BKT}} \). At this temperature the superfluid density \( n_s \) undergoes a jump, and just below \( T_{\text{BKT}} \) the superfluid density satisfies the Nelson-Kosterlitz relation [34]:

\[
n_s (T_{\text{BKT}}) = \frac{2 m}{\pi \hbar^2} T_{\text{BKT}}. \tag{22}
\]

For \( \epsilon_s \ll n g \), the superfluid density \( n_s \) next to the BKT transition point is sufficiently large. Hence, it is possible to complement the Nelson-Kosterlitz relation with the expression for \( n_s \) from Bogoliubov theory. From Eqs. (21) and (22) we obtain a relation for the critical disorder of the BKT transition:

\[
\epsilon_s (T_{\text{BKT}}) = 2 n g \left[ 1 - \frac{T}{T_{\text{BKT}}} \ln \left( \frac{e^A T_{\text{BKT}}}{n g} \right) \right]. \tag{23}
\]

In the absence of disorder, the most precise value of \( T_{\text{BKT}} \) was obtained in Ref. [35] by Monte Carlo simulations: \( T_{\text{BKT}} = T_d / \ln (\xi T_d / n g) \), with \( \xi \approx 380 / 2 \pi \approx 60 \). In the limit \( \epsilon_s \rightarrow 0 \), Eq. (23) gives \( T_{\text{BKT}} \approx T_d / (\ln (e^A T_d / n g) + O(\ln \ln T_d / n g)) \). Therefore, \( T_{\text{BKT}} \) with \( n_s \) following from the Bogoliubov approach is close to the exact value of Ref. [35]. This justifies the validity of our method. For the Gaussian disorder correlation function, Eqs. (20) and (22) lead to critical values of the disorder versus \( (T_{\text{BKT}} - T) \), which for low disorder agree within 20\% with Monte Carlo calculations [16].

The employed Bogoliubov approach has to be corrected when \( n g \) is approaching \( \epsilon_s \). In this case the first term of Eqs. (20) and (21) should be complemented by the contribution of higher order diagrams. This can be done by keeping nonlinear (in \( b_k \)) interactions between atoms and random fields in the Hamiltonian (19), as it was done in the
three-dimensional case in Ref. [36]. Instead of Eq. (23) we then have

\[
\frac{e_{\text{BKT}}}{2n_g} = \left[ 1 - \frac{T}{T_d} \ln \left( e^{T} \frac{T}{n_g} \right) \right] f \left( \frac{e_{\text{MBL}}}{2n_g} \right),
\]

(24)

where the function \( f(x) \) is of order unity.

The BKT transition has been measured in ultracold atomic gases for clean harmonically trapped systems [37]. In the presence of disorder, coherence properties near the BKT superfluid transition [38] and the resistance for a strongly interacting gas [39] have been studied experimentally. We thus believe that an experimental validation of our results is possible in both harmonically trapped and uniform (box) confining potentials. The 2D Bose gas in a box potential has been created in a number of experiments [40], in particular with a tunable interaction strength [41], and realistic proposals of how to identify the BKT transition in this system have been made [42].

Returning to the phase diagram we should admit that close to the tricritical point Eqs. (16) and (24) can give only estimates rather than exact values of the critical disorder strengths \( e_{s_{\text{MBL}}} \) and \( e_{s_{\text{BKT}}} \) (because of not exactly known values of the constant \( C \) and function \( f \)). In particular, in Fig. 1 we took \( C = 1 \) and put \( f = 0.27 \) for \( e_{s} = 0.54n_g \). However, we argue that the identity (1) holds irrespective of the precision of our approximations and now we present the proof of this identity [18].

First of all, \( e_{s_{\text{BKT}}}(0) \) cannot exceed \( e_{s_{\text{MBL}}}(0) \). As it is explained in detail in the Supplemental Material, such a situation is not possible because the critical line for MBLDT is monotonically increasing, whereas the critical line for the BKT transition is monotonically decreasing. Whereas elementary excitations are extended in the superfluid, in the insulator they are localized by definition. Thus the localization length diverges when \( e_{s} \) approaches \( e_{s_{\text{MBL}}}(0) \). However, at any fixed disorder \( e_{s} \) greater than \( e_{s_{\text{MBL}}}(0) \), the elementary excitations undergo many-body delocalization with increasing temperature. The critical temperature tends to zero as the localization length diverges; i.e., at arbitrary low finite temperatures there will be a range of disorder strengths corresponding to a normal fluid.

On the other hand, \( e_{s_{\text{MBL}}}(0) \) cannot exceed \( e_{s_{\text{BKT}}}(0) \) either. Indeed, this would mean that the normal fluid is realized at \( T = 0 \) in a certain range of \( e_{s} \); i.e., elementary excitations are extended. However, as follows from the theory of weak localization (see, e.g., Ref. [21]) in two dimensions this is impossible for a nonsuperfluid state. At \( T = 0 \) the normal fluid is unstable with respect to the transition either to an insulator or to a superfluid, depending on the disorder.

We thus arrive at the phase diagram of Fig. 1 with \( e_{s_{\text{MBL}}}(0) = e_{s_{\text{BKT}}}(0) \), which should be valid as long as there exist only three phases: insulator, normal fluid, and superfluid. At low temperatures all phase transitions occur at the coupling strength \( ng \sim e_{s} \). In this respect it is worth noting that in the recent experiment on disordered 2D lattice bosons [12] it was observed that MBLDT happens when the interaction energy and the characteristic disorder are of the same order of magnitude.

One may think of a possible alternative to the phase diagram of Fig. 1. A phase with nonergodic but extended eigenstates (nonergodic phase; see Ref. [43] for a discussion of such states) can take place in the vicinity of the tricritical point. Detailed discussion of such a possibility goes beyond the scope of the present Letter.

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A preliminary discussion of this phase diagram and the proof of Eq. (1) are contained in the preprint of Ref. [23], I. L. Aleiner, B. L. Altshuler, and G. V. Shlyapnikov, arXiv:0910.4534v1.


I. L. Aleiner, B. L. Altshuler, and G. V. Shlyapnikov, Nat. Phys. 6, 900 (2010).

See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.121.030403 for details on the properties of single particle localization in two dimensions, calculations of the temperature dependence of the MBLDT, the effect of the truncated distribution on the MBLDT, and the discussion regarding the presence of the tricritical point at $T = 0$.


