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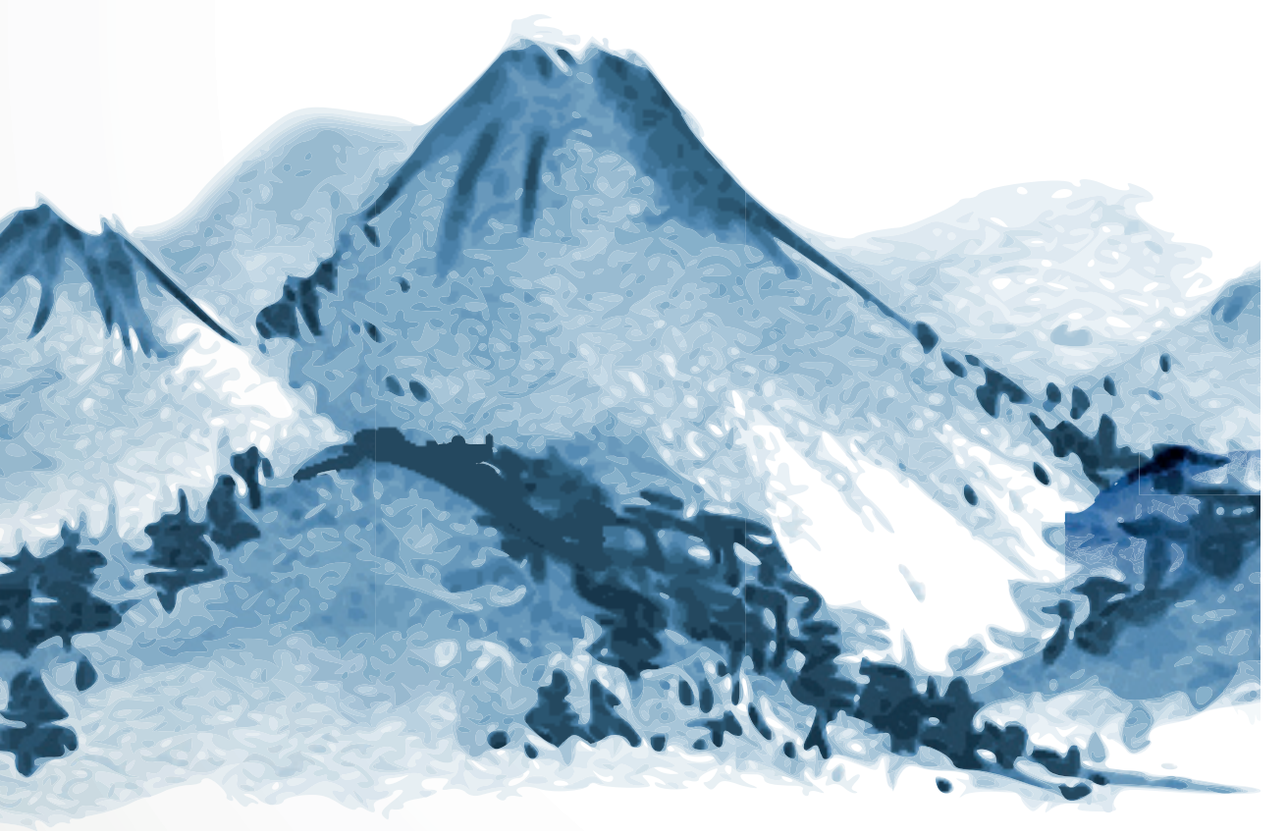

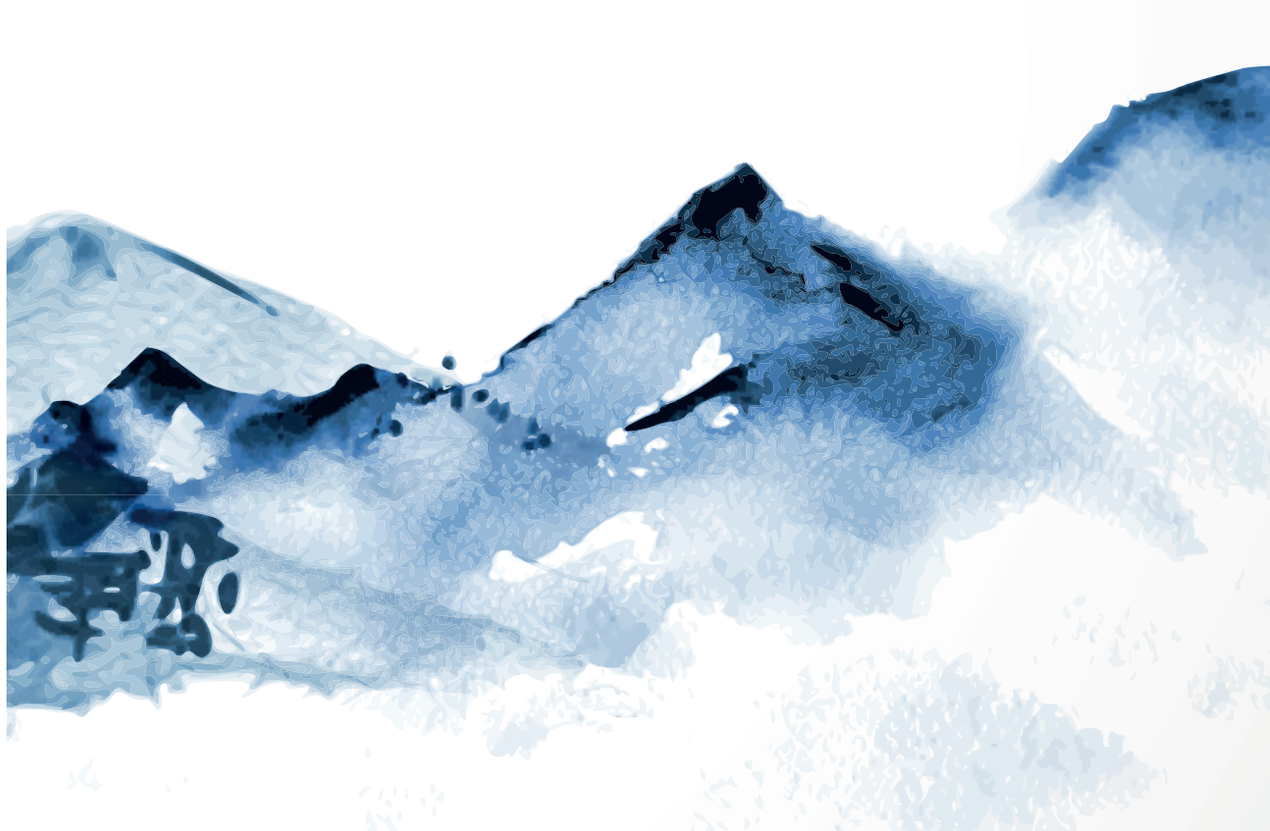

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ACADEMISCH PROEFSCHRIFT

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ten overstaan van een door
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in het openbaar te verdedigen in de Agnietenkapel
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Zijian Zhou

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Contents

Introduction

Foreword

The topic of this thesis is curves in characteristic $p > 0$. Curves and abelian varieties in positive characteristic behave quite differently from their counterparts in characteristic zero. A first example is the occurrence of supersingular elliptic curves. This different behaviour for abelian varieties is captured by the Ekedahl-Oort stratification which was introduced by Ekedahl and Oort [13, 37] in the 1990s. It is a stratification on the moduli of principally polarized abelian varieties in characteristic $p > 0$. Since each curve has a Jacobian, this induces via the Torelli map a stratification on the moduli of curves in characteristic $p > 0$ and one can ask what this stratification is. For the Ekedahl-Oort stratification on the moduli space of principally polarized abelian varieties of dimension g defined over an algebraically closed field k of characteristic $p > 0$, many facts are known, like dimensions, reducibility of strata and their cohomology classes. But for curves much less is known. It is tempting to think that there will be a lot of excess intersection of the Torelli locus and many of the Ekedahl-Oort strata.

There are results in the other direction, saying that certain Ekedahl-Oort strata intersect as expected, or do not intersect the Torelli locus. For example, there is a result of Ekedahl [12] for the smallest Ekedahl-Oort stratum. Re [45] generalized Ekedahl's result to the Ekedahl-Oort strata with p-rank 0. In this thesis, we generalize Re's bound on the genus of curves with the Cartier operator of rank 1. In the rest of the thesis we mostly look at the cases where the genus g and the characteristic p are "small" or where we are dealing with curves of a special type, like hyperelliptic or trigonal and try to get information on the induced Ekedahl-Oort strata on the moduli of curves. The main tool here is the use of Hasse-Witt matrices and de Rham cohomology.

Stratifications on moduli of abelian varieties and curves

It is known that elliptic curves in characteristic $p > 0$ come into two types: ordinary or supersingular. To see the distinction one looks at the p-rank of the elliptic curves, namely the ordinary elliptic curves have p-rank 1 and supersingular elliptic curves have p-rank 0. This gives a stratification of the moduli of elliptic curves. The general stratum parametrising ordinary elliptic curves is open and the other stratum consists of only finite many points.

One can extend this idea to the moduli of principally polarized abelian varieties of dimension $g \geq 2$ in positive characteristic. Let $\mathcal{A}_g \otimes k$ be the moduli space of principally polarized abelian varieties of dimension g defined over an algebraically closed field k of characteristic $p > 0$. More precisely, let us look at the p-rank of abelian varieties. By the work of Koblitz and Oort [25, 36], the p-rank defines a stratification. Each p-rank stratum is characterized by the isomorphism classes of abelian varieties with given p-rank and is invariant under isogeny. It was showed by Norman and Oort [34] that the p-rank stratum with p-rank f is non-empty of codimension $g - f$ in $\mathcal{A}_g \otimes k$.

Instead of looking just at the p-rank of abelian varieties, Ekedahl and Oort introduced a stratification on $\mathcal{A}_g \otimes k$ by looking at the isomorphism classes of the p-torsion subgroup scheme of abelian varieties [37]. Later, it was realized that this stratification can be defined using the de Rham cohomology of abelian varieties [13, 49]. The Ekedahl-Oort stratification consists of 2^g strata and each stratum is indexed by a Young diagram $\mu = [\mu_1, \ldots, \mu_n]$ with $0 \leq n \leq g$ and $\mu_1 > \mu_2 > \cdots > \mu_g > 0$, $\mu_i \in \mathbb{Z}$. The largest stratum is the locus of ordinary abelian varieties. The smallest stratum is of dimension zero and parametrises so-called superspecial abelian varieties. Oort showed that a superspecial abelian variety of dimension g is isomorphic to a product of g supersingular elliptic curves as a non-principally polarized abelian variety. There are many results about the Ekedahl-Oort stratification on $\mathcal{A}_g \otimes k$. To name a few, Oort showed that any positive dimensional stratum is connected and the Ekedahl-Oort stratum indexed by μ is of codimension $\sum_{i=1}^{n} \mu_i$ in $\mathcal{A}_g \otimes k$ by [37]. In [13], van der Geer and Ekedahl showed the irreducibility of certain Ekedahl-Oort strata. In the same paper, they also computed the Chow classes defined by the Ekedahl-Oort strata.

There is another generalization of the p-rank of abelian varieties, namely the Newton polygon which is associated to the formal type of the p-divisible group of a principally polarized abelian variety. The Newton polygon of an abelian variety

is a lower convex polygon consisting of finitely many line segments starting at $(0,0)$ and ending at $(2g,g)$ in $\mathbb{R}^2$. It is also symmetric, namely if slope $b \in [0,1]$ occurs the slope $1-b$ occurs with the same multiplicity. The Newton polygon defines a stratification on $\mathcal{A}_g \otimes k$ [24] where strata are given by the isomorphism classes of abelian varieties with a given Newton polygon. The largest stratum with the Newton polygon with slopes 0 and 1 corresponds to ordinary abelian varieties. The smallest stratum is the locus of supersingular abelian varieties with Newton polygon of slope $1/2$. Oort and Li showed that the supersingular stratum is of dimension $\lfloor g^2/4 \rfloor$, see [29]. Furthermore, Oort [36] proved that an abelian variety is supersingular if and only if it is isogenous to a product of supersingular elliptic curves. One can also ask how the Ekedahl-Oort stratification intersects the Newton polygon stratification. For example, it was shown by Oort that an Ekedahl-Oort stratum lies in the supersingular locus if the associated final type v satisfies $v(n) = 0$ for $n = \lfloor (g+1)/2 \rfloor$, cf. [8].

By associating to a curve its Jacobian, we have the Torelli map from the moduli of curves to the moduli of principally polarized abelian varieties. We write $\mathcal{M}_g \otimes k$ for the moduli of curves of genus g defined over k. Via the Torelli map, the Ekedahl-Oort stratification can be pulled back to $\mathcal{M}_g \otimes k$ and one can ask what the nature of this stratification is. Although we know a lot about the Ekedahl-Oort stratification on the moduli of principally polarized abelian varieties, there are much fewer results on the induced stratification on $\mathcal{M}_g \otimes k$. In general we do not know the dimensions of the induced Ekedahl-Oort strata. It is known that the smallest stratum is empty in $\mathcal{M}_g \otimes k$ with large g; actually it follows from an early result of Ekedahl [12], who showed that the genus g of a superspecial curve in characteristic $p > 0$ is bounded by $g \leq p(p-1)/2$. Recall that a curve is superspecial if its Jacobian is superspecial. For the p-rank strata, Faber and van der Geer [16] proved that every irreducible component of locus of curves with p-rank $\leq f$ has codimension $g - f$ in $\mathcal{M}_g \otimes k$ for $0 \leq f \leq g$, $g \geq 2$. In [40], Pries proved the existence of curves of genus $g \geq 2$ with p-rank $g-2$ with a-number 2.

One can look at the intersection of the induced Ekedahl-Oort strata with special loci in $\mathcal{M}_g \otimes k$, e.g. the hyperelliptic locus and the trigonal locus of trigonal curves. Some results are known. For the hyperelliptic locus $\mathcal{H}_g \otimes k$ in $\mathcal{M}_g \otimes k$, it is shown that every irreducible component of the locus of hyperelliptic curves with p-rank $\leq f$ is of codimension $g-f$ in $\mathcal{H}_g$, see [19]. Faber and van der Geer [49] showed that there exists a family of hyperelliptic curves of genus g in characteristic 2 with Ekedahl-Oort type $[g, g-2, g-4, \dots]$. As a generalization, Elkin and Pries [15] gave a complete description of Ekedahl-Oort types for hyperelliptic curves of genus g in characteristic 2. Also for other special curves, Pries and Zhu studied the p-rank stratification of the moduli space of Artin-Schreier curves of genus g over k, cf. [42]. In [43], the

Ekedahl-Oort types of all Hermitian curves in characteristic $p > 0$ were determined. For a given positive integer g, it is interesting to ask whether all Ekedahl-Oort types occur for curves of genus g in any positive characteristic. By Ekedahl's bound for the genus of superspecial curves in positive characteristic, it is not true for "small" characteristic. However one can still ask which Ekedahl-Oort types occur for curves and what the dimension of the corresponding stratum is.

One can also pull back the Newton stratification to $\mathcal{M}_g \otimes k$ via the Torelli map. There are many question to ask; here we are specially interested in the induced supersingular stratum in $\mathcal{M}_g \otimes k$. In this direction, van der Geer and van der Vlugt [51] showed that there exists a supersingular curve for every genus in characteristic 2. One may be tempted to think that for a given prime $p \geq 3$ there is a supersingular curve of genus g in characteristic p for any positive integer g. Re gave some evidence by showing in his thesis that there exists a supersingular curve of genus g in characteristic p for any $0 \leq g \leq 100$ and $3 \leq p \leq 23$ with a very few undecided exceptions. Furthermore, it was shown by Karemaker and Pries [41] that there exists a supersingular curve of genus $cp(p-1)^2/2$ in characteristic $p > 0$ for any positive integer c having only 0 and 1 as its coefficients in the base p expansion.

Results

First of all, we generalize a result of Re, who in his thesis (see also [45]) showed that the genus g of a curve in characteristic p with the Cartier operator of rank 1 is bounded by $g < p + p(p-1)$. We improve his bound [54].

Theorem 0.1. *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 1, we have $g \leq p + p(p-1)/2$.*

This is sharp for example for $p = 2$, see [40, Lemma 4.8]. In the case of higher rank we have the following result, see [54].

Theorem 0.2. *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 2, and if X possesses a point R such that linear system $|pR|$ is base point free, then $g \leq 2p + p(p-1)/2$, while if X does not have such a point, one has the bound $g \leq 2p + (4p^2 - 5p)/3$.*

We write Z_μ for the induced Ekedahl-Oort stratum in $\mathcal{M}_g \otimes k$ associated to the Young diagram μ. Recall that there is a partial order $\preceq$ on the set of Young diagrams

(see also in Section 1.3). We consider especially the case of hyperelliptic curves of genus 4 in characteristic 3.

Theorem 0.3. *Let k be an algebraically closed field with* $\mathrm{char}(k) = 3$. *A smooth hyperelliptic curve of genus* 4 *over k has a-number ≤ 2. In particular, $Z_\mu \cap \mathcal{H}_4$ is empty if $\mu \succeq [3,2,1]$. If $\mu \not\succeq [3,2,1]$, the codimension of $Z_\mu \cap \mathcal{H}_4$ in $\mathcal{H}_4$ equals the expected codimension $\sum_{i=1}^n \mu_i$. Moreover, in the cases $\mu = [4,1], [3,1], [3,2], [2,1]$ and* $[1]$ *the intersection $Z_\mu \cap \mathcal{H}_4$ is irreducible.*

Note that Frei [18] proved that hyperelliptic curves in odd characteristic can not have a-number $g-1$.

For general $p > 0$, we consider curves which are degree 5 covers of $\mathbb{P}^1$ branched at four points. We give a basis of the de Rham cohomology of the curves and show:

Theorem 0.4. *The locus $Z_{[4,2]}$ in $\mathcal{M}_4 \otimes k$ is non-empty of dimension at least* 1 *for any prime $p \equiv \pm 2 \,(\mathrm{mod}\, 5)$.*

For the existence of superspecial curves of genus 4, we have the following result.

Theorem 0.5. *Let k be an algebraically closed field of characteristic p. For any prime $p > 2$ with $p \equiv \pm 2 (\mathrm{mod}\, 5)$, the locus $Z_{[4,3]}$ in $\mathcal{M}_4 \otimes k$ is non-empty. For any prime $p \equiv -1 (\mathrm{mod}\, 5)$, there exists superspecial curves of genus* 4 *in characteristic p.*

Our computation is very explicit: we consider cyclic covers of the projective line branched at certain points, we give a basis of the space of holomorphic differential forms on the curves and calculate the rank of the Cartier operator.

Using a similar argument, we also reprove a result of Kudo on the existence of superspecial curves of genus 4 in characteristic $p \equiv 2 \,(\mathrm{mod}\, 3)$, he [27] uses the fact that a non-hyperelliptic curve of genus 4 is a complete intersection of an irreducible cubic surface and an irreducible quadric surface, while we use the de Rham cohomology.

By an Artin-Schreier curve we mean a smooth irreducible projective curve defined over an algebraically closed field k of characteristic $p > 0$ by an equation $y^p - y = f(x)$, where $f(x) \in k(x)$. We consider the locus of Artin-Schreier curves with prescribed a-number and have the following results.

Theorem 0.6. *Let k be an algebraically closed field with* $\mathrm{char}(k) = p > 0$. *Let X be an Artin-Schreier curve of genus $g > 0$ with equation $y^p - y = f(x)$, where $f(x) \in k(x)$. Assume that the a-number of X is $g-1$. Then*
(1) : *if $p = 2$, we have $g \leq 3$. Moreover, the curve X can be either written as*

$$y^2 + y = f(x)\,,$$

where $f(x) \in k[x]$ *and* $\deg f(x) = 5$ *or* 7*, or written as* $y^2 + y = f_0(x) + 1/x$ *with* $\deg f_0(x) = 1$ *or* 3 *and* $f_0(x) \in xk[x]$*;*
(2) : *if* $p \geq 3$*, then* $g \leq (p-1)p/2$*. More precisely,* X *is isomorphic to a curve with equation*

$$y^p - y = a_d x^d + a_{d-2} x^{d-2} + \cdots + a_1 x,$$

where $d|p+1$*. Moreover if* $d = p+1$*, then at least one of* a_i *with* $2 \leq i \leq d-2$ *is non-zero. If* $d < p+1, d \mid p+1$*, then at least one of* a_i *with* $1 \leq i \leq d-2$ *is non-zero.*

Theorem 0.7. *Let* k *be an algebraically closed field with* $\mathrm{char}(k) = p \geq 3$*. Let* X *be an Artin-Schreier curve of genus* $g > 0$ *with equation* $y^p - y = f(x)$*, where* $f(x) \in k[x]$ *and* $\deg f(x) = d$*. If* $d|p+1$ *and* $a(X) = g - 1$*, then either* $p = 5, d = 3$ *and* X *is isomorphic to a supersingular curve of genus* 4 *with equation:*

$$y^5 - y = x^3 + a_1 x,\ a_1 \in k^*\,, \tag{1}$$

or $p = 3, d = 4$ *and* X *is isomorphic to a supersingular curve of genus* 3 *with equation:*

$$y^3 - y = x^4 + a_2 x^2, a_2 \in k^*\,. \tag{2}$$

Moreover, the family of curves given by equation (1) *(resp.* (2)*) defines a component of* $Z_{[4,3,2]}$ *(resp.* $Z_{[3,2]}$*) in* $\mathcal{M}_4$ *(resp.* $\mathcal{M}_3$*) in characteristic* 5 *(resp.* 3*).*

We also bound the a-number of trigonal curves of genus 5 in small characteristics. Recall that a result of Re [45] states that if X is a non-hyperelliptic curve of genus g, then

$$a(X) \leq \frac{p-1}{p+1}\left(\frac{2g}{p} + g + 1\right).$$

Theorem 0.8. *Let* k *be an algebraically closed field of characteristic* 2 *(resp.* 3*). If* X *is a trigonal curve of genus 5 defined over* k*, then the* a*-number of* X *satisfies* $a(X) \leq 2$ *(resp.* $a(X) \leq 3$*).*

For $g = 5$ and $p = 2$, Re's bound indicates that $a(X) \leq 3$ while our result implies that $a(X) \leq 2$. Also for $g = 5$ and $p = 3$, Re's bound shows that $a(X) \leq 4$ while our result implies $a(X) \leq 3$.

In [45] Re also gave a bound for the a-number of hyperelliptic curves of genus g over k:

$$a < \frac{(p-1)g}{p} + \frac{p+1}{2p}\,.$$

In the following theorem we give a strengthening of this result.

Theorem 0.9. *A smooth projective hyperelliptic curve over k in characteristic $p \geq 3$ of genus $g \geq p$ has a-number*

$$a \leq \lceil (p-1)g/p - (3p-1)/2p \rceil .$$

A related result was given by Elkin [14, Corollary 1.2]; he showed that the a-number is bounded by $a \leq g(p-2)/p + 1$. In general we have

$$\lceil (p-1)g/p - (3p-1)/2p \rceil < (p-1)g/p + (p+1)/2p - 1 .$$

For example in the case $p = g = 3$, Re's bound shows that $a < 2 + 2/3$ and Elkin's bound shows that $a \leq 2$, while our bound shows that $a \leq 1$, which is sharp. Note also that for $p = g = 3$, Frei [18] ruled out the case $a = 2$.

Chapter 1

Preliminaries

In this chapter, we give definitions and review classical results related to them. Let k be an algebraically closed field of characteristic $p > 0$. Throughout the chapter, by a curve we mean a smooth irreducible projective curve defined over k. By abuse of notation, we write simply $\mathcal{A}_g$ (resp. $\mathcal{M}_g$, $\mathcal{H}_g$) for $\mathcal{A}_g \otimes k$ (resp. $\mathcal{M}_g \otimes k$, $\mathcal{H}_g \otimes k$). We say a scheme is of characteristic p if for any open subset of this scheme, the ring of regular functions on this subset is an $\mathbb{F}_p$-algebra.

1.1 The Cartier Operator

It is known that curves in positive characteristic have features that curves in characteristic zero do not have. One of these is that on curves in characteristic $p > 0$ we have the Frobenius morphism. In general, for a scheme Y in positive characteristic, there is an absolute Frobenius morphism on Y defined as follows.

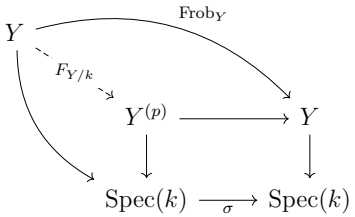
Definition 1.1. Let Y be a scheme of characteristic p. The absolute Frobenius morphism of Y is the morphism $\mathrm{Frob_Y} : Y \mapsto Y$ such that it is identity on the underlying topological space and the induced morphism $\mathrm{Frob_Y^{\#}} : \mathcal{O}_Y \mapsto \mathcal{O}_Y$ is given by $f \mapsto f^p$.

In particular, for $\mathrm{Spec}(k)$ the absolute Frobenius morphism is induced by the Frobenius automorphism $\sigma : t \mapsto t^p$ of the field k. By abuse of notation, we also denote the absolute Frobenius morphism of $\mathrm{Spec}(k)$ by σ.

The absolute Frobenius $\mathrm{Frob_Y}$ of a k-scheme Y is not a morphism of k-schemes. In the following, we define the relative Frobenius morphism, which is a k-scheme morphism.

Write $Y^{(p)}$ for the scheme over k which is the fibre product of $Y \times_{\sigma,k} \mathrm{Spec}(k)$, i.e. the pull back of $Y \mapsto \mathrm{Spec}(k)$ via the absolute Frobenius map $\sigma : \mathrm{Spec}(k) \mapsto \mathrm{Spec}(k)$.

Definition 1.2. The maps $\mathrm{Frob_Y} : Y \mapsto Y$ and $Y \mapsto \mathrm{Spec}(k)$, by the universal property of the fibre product, induces a unique morphism $F_{Y/k} : Y \mapsto Y^{(p)}$ fitting into the following commutative diagram:



The morphism $F_{Y/k} : Y \mapsto Y^{(p)}$ is called relative Frobenius of Y.

For example, suppose Y is a variety given by the equation $f = 0$ with $f = \sum_{i,j\geq 0} a_{i,j}x^i y^j$ in $k[x, y]$. Then the scheme $Y^{(p)}$ is given by equation $f^{(p)}$ obtained from f by raising all the coefficients to their p-th power, i.e $f^{(p)} = \sum_{i,j\geq 0} a_{i,j}^p x^i y^j$. The relative Frobenius $F_{Y/k} : Y \mapsto Y^{(p)}$ is given on rings by the homomorphism

$$k[x, y]/(f^{(p)}) \mapsto k[x, y]/(f)$$

with $x \mapsto x^p$ and $y \mapsto y^p$.

The relative Frobenius morphism of Y is purely inseparable.

Remark 1.3. Let (Y, λ) be a principally polarized abelian variety. Then the relative Frobenius morphism of Y is a homomorphism of k-group schemes of degree p^g. Note that the multiplication by p on Y is of degree p^{2g} and it factors through the relative Frobenius homomorphism, i.e. $[p]_Y = V_{Y/k} \circ F_{Y/k}$ with $V_{Y/k} : Y^{(p)} \mapsto Y$ the Verschiebung homomorphism, cf. [11, Definition 5.18]. Additionally, the map $F_{Y/k} \circ V_{Y/k}$ is multiplication by p on $Y^{(p)}$, i.e. $[p]_{Y^{(p)}} = F_{Y/k} \circ V_{Y/k}$.

Let X be a curve defined over k. The absolute Frobenius morphism of X induces a σ-linear map on the cohomology of structure sheaf on X:

$$F_X : H^1(X, \mathcal{O}_X) \mapsto H^1(X, \mathcal{O}_X)\,,$$

where the σ-linear means $F_X(cf) = c^p F_X(f)$ for any $c \in k, f \in H^1(X, \mathcal{O}_X)$. Given a basis of $H^1(X, \mathcal{O}_X)$, the *Hasse-Witt* matrix is defined to be the matrix representation of F_X on $H^1(X, \mathcal{O}_X)$ with respect to this basis.

Cartier [7] defined an operator $\mathcal{C}$ on the rational differential forms on a curve X with the following properties:

1) $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$,
2) $\mathcal{C}(f^p\omega) = f\,\mathcal{C}(\omega)$,
3) $\mathcal{C}(\mathrm{d}f) = 0$,
4) $\mathcal{C}(\mathrm{d}h/h) = \mathrm{d}h/h$,

where $k(X)$ is the function field of X and $f, h \in k(X)$ with $h \neq 0$. We call it the Cartier operator. Recall that if x is a separating variable of $k(X)$, any $f \in k(X)$ can be written as

$$f = f_0^p + f_1^p x + \ldots + f_{p-1}^p x^{p-1}, \text{ with } f_i \in k(X)\,. \tag{1.1}$$

For a rational differential form $\omega = f\,\mathrm{d}x$ with f as in (2.2), we have $\mathcal{C}(\omega) = f_{p-1}\,\mathrm{d}x$. In particular $\mathcal{C}^n(f^i\,\mathrm{d}f) = f^{(i+1)/p^n-1}\,\mathrm{d}f$ if $p^n|i+1$, and $\mathcal{C}^n(f^i\,\mathrm{d}f) = 0$ otherwise.

For a curve X, we define the *rank of the Cartier operator* to be the rank of the Cartier operator restricted to $H^0(X, \Omega^1_X)$. There is another matrix associated to the Cartier operator with respect to a basis of $H^0(X, \Omega^1_X)$, namely the Cartier-Manin matrix.

Definition 1.4. Given a basis $B = \{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega^1_X)$, the *Cartier-Manin* matrix is the matrix of the Cartier operator with respect to B, i.e $H = (h_{i,j})_{1\leq i,j\leq g}$ with $h_{i,j} \in k$ such that $\mathcal{C}(\omega_j) = \sum_{i=1}^g h_{i,j}\omega_i$.

If $\langle\,,\,\rangle_S$ denotes the Serre duality pairing $H^1(X, \mathcal{O}_X) \times H^0(X, \Omega^1_X) \mapsto k$, Serre [46, Proposition 9] showed that there is a duality relation between F_X and $\mathcal{C}$, i.e.

$$\langle f, \mathcal{C}(\omega)\rangle_S^\sigma = \langle F_X(f), \omega\rangle_S$$

for any $f \in H^1(X, \mathcal{O}_X)$ and $\omega \in H^0(X, \Omega^1_X)$. For a basis $B = \{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega^1_X)$, we have a dual basis $B^* = \{f_1, \ldots, f_g\}$ of $H^1(X, \mathcal{O}_X)$ with respect to $\langle\,,\,\rangle_S$, i.e. $\langle f_i, \omega_j\rangle = 1$ if $i = j$ and $\langle f_i, \omega_j\rangle = 0$ otherwise.

Then we have

$$\langle F_X(f_l), \omega_j\rangle_S = \langle f_l, \mathcal{C}(\omega_j)\rangle_S^\sigma = \langle f_l, \sum_{i=1}^g h_{i,j}\omega_i\rangle_S^\sigma = h_{l,j}^p\,.$$

This implies that $F_X(f_l) = \sum_{j=1}^g h_{l,j}^p f_j$ and the Hasse-Witt matrix is equal to $M = (m_{i,j})_{1\leq i,j\leq g}$ with $m_{i,j} = h_{j,i}^p$. Then we have the following corollary.

Corollary 1.5. *The rank of the Hasse-Witt matrix is equal to the rank of the Cartier-Manin matrix, i.e. the rank of the Cartier operator.*

1.2 The first de Rham cohomology

In this section we recall the first de Rham cohomology of a scheme Y using the Cěch cohomology.

Denote by $(\Omega_Y^{\cdot}, \mathrm{d})$ the complex of sheaves of Kähler differential forms on Y. The de Rham cohomology of Y is defined as the hypercohomology of the functor $H^0(Y, \cdot)$ with respect to $(\Omega_Y^{\cdot}, \mathrm{d})$, cf. [10, Section 11.4]. Since we mainly focus on curves and abelian varieties, we are only interested in the first de Rham cohomology, which by [35] can also be described in terms of Cěch cocycles as follows. Let $\mathcal{U} = \{U_i | i \in I\}$ be an open affine cover of Y. For a sheaf $\mathscr{F}$ of abelian groups, we denote the Cěch complex of abelian groups associated to $\mathscr{F}$ by $C^{\cdot}(\mathcal{U}, \mathscr{F})$ [21, Chapter III, Section 4]. We define the $H^1_{dR}(\mathcal{U})$ with respect to the covering $\mathcal{U}$ to be

$$H^1_{dR}(\mathcal{U}) = Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U}), \tag{1.2}$$

where $Z^1_{dR}(\mathcal{U})$ is defined as

$$\{(f, \omega) \,|\, f \in C^1(\mathcal{U}, \mathcal{O}_Y), \omega \in C^0(\mathcal{U}, \Omega^1_Y), f_{i,k} = f_{i,j} + f_{j,k}, \mathrm{d}f_{i,j} = \omega_i - \omega_j, \mathrm{d}\omega_i = 0\}$$

and $B^1_{dR}(\mathcal{U})$ is defined as

$$\{(f, \omega) \,|\, f \in C^1(\mathcal{U}, \mathcal{O}_Y), \omega \in C^0(\mathcal{U}, \Omega^1_Y), g \in C^0(\mathcal{U}, \mathcal{O}_Y), f_{i,j} = g_i - g_j, \omega_i = \mathrm{d}g_i\}\,.$$

The first de Rham cohomology $H^1_{dR}(Y)$ is independent of the choice of the open affine cover and we have

$$H^1_{dR}(Y) \cong H^1_{dR}(\mathcal{U})\,.$$

In particular for the case of a curve X, we can take an open affine cover $\mathcal{U} = \{U_1, U_2\}$ consisting of only two open parts. Then the de Rham cohomology $H^1_{dR}(Y)$ can be described using (1.2) with

$$Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2) |\ t \in \mathcal{O}_Y(U_1 \cap U_2), \omega_i \in \Omega^1_Y(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$$

and

$$B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2) |\ t_i \in \mathcal{O}_Y(U_i)\}\,.$$

Consider a curve X given as a branched cover of $\mathbb{P}^1$ with a morphism $\pi : X \mapsto \mathbb{P}^1$. We can take an open affine cover of our X with $U_1 = \pi^{-1}(\mathbb{P}^1_k - \{0\})$ and $U_2 = \pi^{-1}(\mathbb{P}^1_k - \{\infty\})$.

To give an example, let X be a hyperelliptic curve over k given by equation $y^2 = f(x) \in k[x]$ with $p > 2$. For $1 \leq i \leq g$, put $s_i(x) = xf'(x) - 2if(x)$ with $f'(x)$

the formal derivative of $f(x)$ and write $s_i(x) = s_i^{\leq i}(x) + s_i^{>i}(x)$ with $s_i^{\leq i}(x)$ the sum of monomials of degree $\leq i$. Köck and Tait [26] gave an explicit basis of $H^1_{dR}(X)$ of a hyperelliptic curve X in characteristic $p > 2$ with respect to the covering $\{U_1, U_2\}$, where π is the hyperelliptic map. We quote the following theorem.

Proposition 1.6. *[26, Theorem 2.2] Let $p > 2$. Let X be a hyperelliptic curve with equation $y^2 = f(x) \in k[x]$. Then $H^1_{dR}(X)$ has a basis with respect to $\{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(X)$:*

$$\gamma_i = [(\frac{y}{x^i}, \frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x, -\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)],\ i = 1, \ldots, g, \tag{1.3}$$

$$\lambda_j = [(0, \frac{x^j}{y}\mathrm{d}x, \frac{x^j}{y}\mathrm{d}x)],\ j = 0, \ldots, g-1, \tag{1.4}$$

where $\psi_i(x) = s_i^{\leq i}(x)$ and $\phi_i(x) = s_i^{>i}(x)$. Also we have $\langle \gamma_i, \lambda_j \rangle \neq 0$ if $j = i - 1$ and $\langle \gamma_i, \lambda_j \rangle = 0$ otherwise.

The authors in [26] also gave a basis of de Rham cohomology for hyperelliptic curves in characteristic 2. Later we will derive similar results for the trigonal curves and Artin-Schreier curves.

1.3 The Ekedahl-Oort stratification on the moduli space of principally polarized abelian varieties

Let $\mathcal{A}_g$ be the moduli space (stack) of principally polarized abelian varieties of dimension g defined over k. Its coarse moduli space is quasi-projective variety of dimension $g(g+1)/2$. Due to the fact that abelian varieties can degenerate, the space $\mathcal{A}_g$ is not complete. There are several compactifications of $\mathcal{A}_g$. We are interested in the smooth second Voronoi toroidal compactification $\tilde{\mathcal{A}}_g$ and the Satake compactification $\mathcal{A}_g^*$, see [17] for the precise constructions. The boundary $\mathcal{A}_g^* - \mathcal{A}_g$ of $\mathcal{A}_g^*$ is of codimension one and the points in the boundary correspond to semi-abelian varieties, see [17, Definition 2.3]. Moreover in this book, the authors also define a morphism $\rho : \tilde{\mathcal{A}}_g \mapsto \mathcal{A}_g^*$.

We now motivate and recall the Ekedahl-Oort stratification on $\mathcal{A}_g$.

For a principally polarized abelian variety (A, λ) of dimension $g > 0$ defined over k, we consider the multiplication by p morphism. It is of degree p^{2g}. The p-torsion group scheme $A[p]$ of A is defined to be the kernel of this morphism. By analysing

the action of the relative Frobenius morphism and Verschiebung morphism on $A[p]$, Ekedahl and Oort introduced a stratification on $\mathcal{A}_g$, cf. [37]. In [13, 49], Ekedahl and van der Geer gave another interpretation of the stratification using the de Rham cohomology. In this thesis, we use the de Rham cohomology to define the Ekedahl-Oort stratification.

To motivate the Ekedahl-Oort stratification, we consider principally polarized abelian varieties of dimension one, i.e. elliptic curves. Consider the multiplication by p morphism $[p]_E : E \mapsto E$, it is of degree p^2. It follows that for an elliptic curve E in characterstic 0, we have $\#E[p](\mathbb{C}) = p^2$. For an elliptic curve E defined over k of characteristic $p > 0$, by Remark 1.3 we know that $[p]_E$ is inseparable. In fact $F_{E/k}$ is inseparable but $V_{E/k}$ can be separable or not. Then $\#E[p](k) = p$ if $V_{E/k}$ is separable and $\#E[p](k) = 1$ if $V_{E/k}$ is inseparable. This shows that elliptic curves in characteristic $p > 0$ come into two types: ordinary or supersingular. This can be seen on the cohomology too. Indeed, suppose E has a p-torsion point, i.e. there is a non-zero divisor D such that pD is the divisor of a function $f \in K(E)$. This implies that for any point Q in E and t a local parameter at Q, locally f can be written as

$$f = a_n t^n + a_{n+1} t^{n+1} + \dots$$

with $n \equiv 0 \,(\mathrm{mod}\, p)$ and $a_i \in k$ and $a_n \neq 0$. Then $\mathrm{ord}_Q(\mathrm{d}f/f) \geq 0$ and $\mathrm{d}f/f \in H^0(E, \Omega^1_E)$. Applying the Cartier operator, we have $\mathcal{C}(\mathrm{d}f/f) = \mathrm{d}f/f \neq 0$. Conversely, if the Cartier operator $\mathcal{C}$ is not zero on $H^0(E, \Omega^1_E)$, then $\mathcal{C}(\omega) = c\,\omega$ for ω some holomorphic differential form and $c \in k^*$. Hence ω is a logarithm form, i.e. $\omega = c_0\, \mathrm{d}f/f$ for some $f \in K(E)$ and $c_0 \in k^*$. Now we show that the divisor of f is pD with D a non-zero divisor on E. Indeed, for a point Q in E such that $\mathrm{ord}_Q(f) = n$, if $p \nmid n$, then locally f can be written as

$$f = a_n t^n + a_{n+1} t^{n+1} + \dots$$

where t is a local parameter at Q and $a_i \in k, a_n \neq 0$. This implies $\mathrm{ord}_Q(\mathrm{d}f/f) = -1 < 0$, a contradiction. So the conclusion is that E has p-rank 0 if and only if the action of $\mathcal{C}$ on $H^0(E, \Omega^1_E)$ is zero.

For a principally polarized abelian variety (A, λ) of dimension g, since $[p]_A = V_{A/k} \circ F_{A/k}$ is of degree p^{2g} and $F_{A/k}$ is purely inseparable of degree p^g, we see that $\#A[p](k) = p^l$ with $0 \leq l \leq g$. The elliptic curve example makes it reasonable to look at the action of V and F on H^1_{dR}. More precisely, the Verschiebung homomorphism $V_{A/k} : A^{(p)} \mapsto A$ induces a map of k-vector spaces

$$V^*_{A/k} : H^1_{dR}(A) \mapsto H^1_{dR}(A^{(p)})\,.$$

The spaces $H^1_{dR}(A)$ and $H^1_{dR}(A^{(p)})$ are the "same" but with different k-action, i.e.

$$H^1_{dR}(A^{(p)}) \cong (k,\sigma) \otimes_k H^1_{dR}(A)\,. \tag{1.5}$$

Hence the linear map $V^*_{A/k} : H^1_{dR}(A) \mapsto H^1_{dR}(A^{(p)})$ is equivalent to a σ^{-1}-linear map $V : H^1_{dR}(A) \mapsto H^1_{dR}(A)$. We call V the Verschiebung operator. Similarly we can define the σ-linear Frobenius operator $F : H^1_{dR}(A) \mapsto H^1_{dR}(A)$ induced by the relative Frobenius of A. We will see that for a curve X, the Verschiebung operator coincides with the Cartier operator on $H^0(X, \Omega^1_X)$. It is well known that the structure of $A[p]$ is reflected in $H^1_{dR}(A)$ with its action of F and V, e.g. [13, 35]. Then we only need to focus on the cohomology part of A.

To introduce the stratification, we first introduce several notations. Write $G = H^1_{dR}(A)$ and $G^{(p)} = H^1_{dR}(A^{(p)})$. There is a symplectic form $\langle\,,\,\rangle_G$ on G [37, Section 12] (hence a symplectic form $\langle\,,\,\rangle_{G^{(p)}}$ on $G^{(p)}$) and the morphisms $F^*_{A/k}$ and $V^*_{A/k}$ are adjoints, i.e.

$$\langle V^*_{A/k}(\alpha), \beta\rangle_{G^{(p)}} = \langle \alpha, F^*_{A/k}(\beta)\rangle_G$$

for $\alpha \in H^1_{dR}(A)$ and $\beta \in H^1_{dR}(A^{(p)})$. Then by (1.5), we have

$$\langle V(a), b\rangle^{\sigma}_G = \langle a, F(b)\rangle_G$$

with $a, b \in H^1_{dR}(A)$. From now on, we write simply $\langle\,,\,\rangle$ for $\langle\,,\,\rangle_G$. Hence we have $\langle V(a), b\rangle = \langle a, F(b)\rangle$ by ignoring the $\sigma^{\pm}$-linearity. Denote by $H^{\perp}$ the orthogonal complement of a subspace H of G with respect to $\langle\,,\,\rangle$. Since $F_{A/k} \circ V_{A/k} = [p]_{A^{(p)}}$ and $V_{A/k} \circ F_{A/k} = [p]_A$, we have $VF = FV = 0$ and $(V(H))^{\perp} = F^{-1}(H^{\perp})$ for any subspace H of G. Moreover let $H = G$, we have $V(G) = H^0(A, \Omega^1_A) = F^{-1}(0)$ and $F(G) = V^{-1}(0)$ are maximal isotropic subspaces of dimension g with respect to $\langle\,,\,\rangle$.

For an elliptic curve E, we have seen that if E is supersingular, then $\mathcal{C}$ is zero on $H^0(E, \Omega^1_E) = \ker F$ and hence $\ker F = \ker V$. The intersection of $\ker F$ and $\ker V$ indicates interesting properties of principally polarized abelian varieties. Since $H^0(A, \Omega^1_A) \subset H^1_{dR}(A)$ as a subspace, we have the repeated images of V, F and their inverses on $H^0(A, \Omega^1_A)$. This gives us two filtrations. The idea of Ekedahl-Oort stratification is to measure the relative position of these two filtrations.

The Ekedahl-Oort stratification is defined by a final filtration on the first de Rham cohomology of abelian varieties, which we define as follows.

Definition 1.7. Let (A, λ) be a principally polarized abelian variety with $G = H^1_{dR}(A)$. A *final filtration* is a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_g = V(G) \subset \cdots \subset G_{2g} = G$$

which is stable under V and F^{-1} with the following properties being satisfied:

$$\dim(G_i) = i,\ G_i^{\perp} = G_{2g-i}, i = 1, \ldots, 2g.$$

The associated *final type* v is the increasing and surjective map

$$v : \{0, 1, 2, \ldots, 2g\} \mapsto \{0, 1, \ldots, g\}$$

such that $V(G_i) = G_{v(i)}$.

By the properties of V and F, we have $v(2g - i) = v(i) - i + g$ for $0 \leq i \leq g$.

To find a final filtration of (A, λ), we first construct a weak version of it, namely the *canonical filtration*, and refine it to a final one.

Now to construct the canonical filtration of A, we start with $0 \subset G$ and add the images of V and the orthogonal complements of the images. Repeating this process for a finite number of times, we obtain a filtration

$$0 = C_0 \subset G_1 \subset \cdots \subset G_r = V(G) \subset \cdots \subset G_{2r} = G\,,$$

which is stable under V and $^{\perp}$. Then it is also stable under F^{-1} and it is unique. This filtration is called canonical filtration and can be refined to a final one by choosing a filtration of length $2g$ which is stable under V and $^{\perp}$. In general a final filtration is not unique but the final type is unique, cf. [37].

Given a final type v, we associate to it a Young diagram, or equivalently a n-tuple $\mu = [\mu_1, \ldots, \mu_n]$ with $0 \leq n \leq g$ and $\mu_1 > \mu_2 > \cdots > \mu_n > 0$ such that

$$\mu_j = \#\{i :\ 1 \leq i \leq j, v(i) + j \leq i\}\,.$$

Denote by $\mathcal{Z}_\mu$ the set of geometric points of $\mathcal{A}_g$ with given final type v with Young diagram μ.

Theorem 1.8 (Ekedahl-Oort)**.** *The sets $\mathcal{Z}_\mu$ are locally closed and define a stratification. It is called the Ekedahl-Oort stratification and consists of 2^g strata. Moreover for $\mu = [\mu_1, \ldots, \mu_n]$ associated to a given final type v, the stratum $\mathcal{Z}_\mu$ is of codimension $\sum_{i=1}^n \mu_i$ in $\mathcal{A}_g$.*

Proof. See [37, Corollary 5.4] or [13, 49].

□

Remark 1.9. The Ekedahl-Oort stratification can be extended to $\mathcal{A}_g^*$, cf. [37, Section 6].

There is a partial order $\preceq$ on the set of n-tuples given by

$$\mu = [\mu_1, \ldots, \mu_n] \preceq \nu = [\nu_1, \ldots, \nu_m]$$

if $n \leq m$ and $\mu_i \leq \nu_i$ for $i = 1, \ldots, n$. Denote by $\overline{\mathcal{Z}_\mu}$ the Zariski closure of $\mathcal{Z}_\mu$ in $\mathcal{A}_g$. Then we have the relation for the closure $\overline{\mathcal{Z}_\mu} \supset \overline{\mathcal{Z}_\nu}$ in $\mathcal{A}_g$ if $\mu \preceq \nu$. Note that in general $\overline{\mathcal{Z}_\mu} \supsetneq \cup_{\mu \preceq \nu} \overline{\mathcal{Z}_\nu}$. But the closure of a stratum $\overline{\mathcal{Z}_\mu}$ can be larger than this, as already observed in [37] and [13]. A complete description can be found in [38, Theorem 1.4]. The largest stratum, which corresponds to the empty n-tuple $\mu = \emptyset$, is of maximal dimension $g(g+1)/2$ in $\mathcal{A}_g$ and is called the ordinary stratum. The smallest stratum, which is of dimension zero, is called superspecial stratum.

For a curve X, we can associate a principally polarized abelian variety to it, namely the Jacobian of X. It is defined as $(\mathrm{Pic}^0(X), \lambda)$, where the Picard group $\mathrm{Pic}^0(X)$ of X is the group of isomorphism classes of line bundles on X and λ is the natural principal polarization induced by the theta divisor, cf. [11]. We write $\mathrm{Jac}(X)$ for the Jacobian of X.

1.4 The p-rank and the a-number

In this section, we recall two invariants of an abelian variety, namely the p-rank and the a-number, and introduce connections of the Ekedahl-Oort types and classical invariants of abelian varieties. Let A be an abelian variety of dimension g defined over k. Denote by μ_p (resp. α_p) the group scheme which is the kernel of Frobenius endomorphism on $\mathbb{G}_m$ (resp. $\mathbb{G}_a$), cf. [11, Example 3.8, page 36].

Definition 1.10. We define the p-rank of A by

$$\#A[p](k) = p^{f_A}\,.$$

The p-rank of a curve X is defined to be the p-rank of its Jacobian $\mathrm{Jac}(X)$.

The p-rank of A can be also defined as $f_A = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A)$. We have seen that $0 \leq f_A \leq g$. Moreover, for a semi-abelian variety A' over k, the p-rank $f_{A'}$ of A' is defined by

$$\#A'[p](k) = p^{f_{A'}}\,.$$

Note that we have $\mathrm{End}(\alpha_p) \cong k$ with $\mathrm{End}(\alpha_p)$ the endomorphism ring of α_p. Then $\mathrm{Hom}(\alpha_p, A)$ is a k-vector space and we have the following definition.

Definition 1.11. The a-number of A is the number

$$a(A) = \dim(\mathrm{Hom}(\alpha_p, A)) .$$

The a-number of a curve X is the a-number of its Jacobian $\mathrm{Jac}(X)$.

There are many other ways to define the a-number of an abelian variety, e.g. in [50], we have the alternative formula for the a-number of A:

$$a(A) = \dim_k(\ker(V)),\ V:\ H^0(A, \Omega^1_A) \mapsto H^0(A, \Omega^1_A) , \tag{1.6}$$

where V is the Verschiebung operator V restricted on $H^0(X, \Omega^1_A)$.

Proposition 1.12. *If X is a curve, then V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$. Hence $a(X) = g - r$ with g the genus of X and r the rank of the Cartier operator.*

Proof. See [35, Section 5]. □

Definition 1.13. A principally polarized abelian variety is superspecial (resp. ordinary) if it has maximal a-number (resp. p-rank). A curve is superspecial (resp. ordinary) if its Jacobian is superspecial (resp. ordinary).

By the definition of Ekedahl-Oort strata, we have that a principally polarized abelian variety is superspecial (resp. ordinary) if the corresponding point in $\mathcal{A}_g$ lies in the superspecial (resp. ordinary) stratum. We have the following examples of ordinary and superspecial abelian varieties.

Example 1.14. Let g be a positive integer. The product of g ordinary (resp. supersingular) elliptic curves is an ordinary (resp. superspecial) abelian variety of dimension g.

The simplest examples of Ekedahl-Oort strata are the strata $\overline{\mathcal{Z}_\mu}$ of p-rank $\leq f$ with $\mu = [g - f]$ and the strata $\overline{\mathcal{Z}_\mu}$ of a-number $\geq a$ with $\mu = [a, a - 1, \ldots, 1]$. Moreover for the connection among the Ekedahl-Oort strata, p-rank and a-number of A, we have the following.

Proposition 1.15. *Let A be an abelian variety defined over k. If A has Ekedahl-Oort type $\mu = [\mu_1, \ldots, \mu_n]$ for $0 \leq n \leq g$, then $a(A) = n$ and $f_A = g - \mu_1$.*

Proof. The a-number part follows from the alternative formula (1.6) above and the facts that $V(H^1_{dR}(A)) = H^0(A, \Omega^1_A)$. The p-rank part follows from definition of the p-rank and the construction of the Ekedahl-Oort stratification, see [37].

□

As a corollary of Proposition 1.15, we have the following.

Corollary 1.16. *An abelian variety A is superspecial (resp. ordinary) if and only if $a(A) = g$ (resp. $f_A = g$). Moreover, we always have $a(A) + f_A \leq g$.*

Proof. Note that for a Young diagram associated to the n-tuple $\mu = [\mu_1, \ldots, \mu_n]$, the height n of the diagram is always not larger than the number μ_1 of blocks in the first row of the diagram. Then for an abelian variety with Ekedahl-Oort type μ, we have $a(A) + f_A = g - \mu_1 + n \leq g$.

□

Example 1.17. Let E be an elliptic curve defined over k. Then $a(E) + f_E = 1$, namely E either has p-rank 1 and a-number 0 or has p-rank 0 and a-number 1.

In the following, we give an example of an abelian variety A of genus g with $a(A) + f_A < g$.

Example 1.18. Consider the Jacobian of the curve X with equation $y^2 + y = x^5$ in characteristic 2. We have $H^0(X, \Omega^1_X) = \langle \mathrm{d}x, x\,\mathrm{d}x \rangle$. Applying the Cartier operator on $H^0(X, \Omega^1_X)$, we have the curve X has Ekedahl-Oort type $\mu = [2]$. Then the curve has p-rank 0 and a-number 1 and $a(X) + f_X = 1 < 2$.

1.5 The induced Ekedahl-Oort stratification on the moduli space of curves of genus g

Let $\mathcal{M}_g$ be the moduli space (stack) of (smooth projective) curves of genus $g \geq 2$ defined over k. Its coarse moduli space is a quasi-projective variety of dimension $3g - 3$. Since smooth curves can degenerate to singular ones, the space $\mathcal{M}_g$ is not compact. Deligne and Mumford [9] gave a good compactification by adding curves with certain singular points, namely the stable curves. They showed that this compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ is an irreducible and projective variety. The space $\overline{\mathcal{M}}_g$ is constructed by adding divisors of singular stable curves of genus g. Note that these divisors may intersect since different singular curves can degenerate to the same one. More precisely, we have $\overline{\mathcal{M}}_g = \mathcal{M}_g \cup \cup_{i=0}^{\lfloor g/2 \rfloor} \Delta_i$, where Δ_0 denotes the irreducible divisor whose generic point corresponds to an irreducible nodal curve and Δ_i denotes the irreducible divisor whose generic point corresponds to a reducible curve with components of genus i and $g - i$ for $1 \leq i \leq \lfloor g/2 \rfloor$, cf. [9].

By associating to a curve its Jacobian, one gets a morphism

$$\tau : \mathcal{M}_g \mapsto \mathcal{A}_g$$

called the *Torelli morphism*. The image of the Torelli morphism is called (open) *Torelli locus*. This morphism is injective on the geometric points. Moreover, the Torelli morphism can be extended to a regular map $\tau_{tor} : \overline{\mathcal{M}}_g \mapsto \tilde{\mathcal{A}}_g$, see [2, Theorem 4.1]. If we compose τ_{tor} with $\rho : \tilde{\mathcal{A}}_g \mapsto \mathcal{A}_g^*$, we have the *Satake Torelli morphism* $\bar{\tau} : \overline{\mathcal{M}}_g \mapsto \mathcal{A}_g^*$.

Via the (Satake) Torelli morphism we can pull back the (Satake) Ekedahl-Oort stratification to $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$) and study what it provides. Since the Torelli locus $\tau(\mathcal{M}_g)$ is open in $\mathcal{A}_g$, the Ekedahl-Oort indeed induces a stratification on $\mathcal{M}_g$.

Definition 1.19. For a Young diagram $\mu = [\mu_1, \ldots, \mu_n]$, we denote by $Z_\mu \subset \mathcal{M}_g$ the pull back of stratum $\mathcal{Z}_\mu$ on $\mathcal{A}_g$. We say a curve has Ekedahl-Oort type μ if the corresponding point in $\mathcal{M}_g$ lies in Z_μ.

As we indicated above, the induced strata in $\mathcal{M}_g$ did not attract as much attention as the Ekedahl-Oort strata in $\mathcal{A}_g$ and only few results are known. For the smallest stratum, namely the stratum with superspecial curve, Ekedahl [12] showed that the genus of a superspecial curve in characteristic $p > 0$ is bounded by $p(p-1)/2$. This bound is sharp and one example is a special Artin-Schreier curve.

Example 1.20. Let $p = 3$. The Artin-Schreier curve X with equation $y^3 - y = x^4$ is superspecial of genus 3.

Proof. A basis for $H^0(X, \Omega^1_X)$ is given by forms $1\,\mathrm{d}x, x\,\mathrm{d}x, y\,\mathrm{d}x$. Then one can check that

$$\begin{aligned} \mathcal{C}(\mathrm{d}x) &= \mathcal{C}(x\,\mathrm{d}x) = 0, \\ \mathcal{C}(y\,\mathrm{d}x) &= \mathcal{C}((y^3 - x^4)\,\mathrm{d}x) = \mathcal{C}(y^3\,\mathrm{d}x) - \mathcal{C}(x^4\,\mathrm{d}x) = 0\,. \end{aligned}$$

Hence the rank of the Cartier operator is zero and X is superspecial.

□

Recall that $\dim(\mathcal{M}_g) = 3g - 3$ and $\dim(\mathcal{A}_g) = g(g+1)/2$. For $g = 2, 3$, the Torelli locus is open and dense in $\mathcal{A}_g$. For large genus g, the Torelli locus has large codimension in $\mathcal{A}_g$. Then for a Young diagram μ, it is unlikely that the induced Ekedahl-Oort stratum Z_μ has the same codimension in $\mathcal{M}_g$ as the Ekedahl-Oort stratum $\mathcal{Z}_\mu$ does in $\mathcal{A}_g$. Surprisingly, for the locus of curves with given p-rank, this is the case.

Denote by $V_l(\overline{\mathcal{M}}_g)$ the locus of stable curves with p-rank $\leq l$ in $\overline{\mathcal{M}}_g$. We have the following theorem by van der Geer and Faber [16, Theorem 2.3].

Theorem 1.21. *Let $g \geq 2$. Then $V_l(\overline{\mathcal{M}}_g)$ is pure of codimension $g-l$ in $\overline{\mathcal{M}}_g$.*

Denote by $V_l(\mathcal{M}_g)$ the locus of curves with p-rank l in $\mathcal{M}_g$. Then we have the following corollary.

Corollary 1.22. *$V_l(\mathcal{M}_g)$ is pure of codimension $g-l$ in $\mathcal{M}_g$.*

Proof. We first show that $\dim(V_l(\overline{\mathcal{M}}_g) \cap \Delta_i) < \dim(V_l(\overline{\mathcal{M}}_g))$ with $0 \leq l \leq g$ and $0 \leq i \leq \lfloor g/2 \rfloor$. Then by Theorem 1.21, we have the desired conclusion. Indeed for $i = 0$, a generic point of $V_l(\overline{\mathcal{M}}_g) \cap \Delta_0$ corresponds to the isomorphism class of an irreducible singular curve X_{g-1} of genus $g-1$ with a node. Since the Jacobian of the curve X_{g-1} has a $\mathbb{G}_m$-part, the normalization $\tilde{X}_{g-1}$ of X_{g-1} is a smooth curve of genus $g-1$ with two marked points and p-rank $l-1$. It is obvious that $V_0(\overline{\mathcal{M}}_g) \cap \Delta_0$ is empty. Then the class $[\tilde{X}_{g-1}]$ lies in $V_{l-1}(\overline{\mathcal{M}}_{g-1})$. By Theorem 1.21 above, we have $\dim(V_{l-1}(\overline{\mathcal{M}}_{g-1})) = 3(g-1) - 3 - (g-1-(l-1)) = 2g+l-6$. Together with two marked points in $\tilde{X}_{g-1}$, we have

$$\dim(V_l(\overline{\mathcal{M}}_g) \cap \Delta_0) \leq 2g - 6 + l + 2 \leq 3g - 3 - g + l = \dim(V_l(\overline{\mathcal{M}}_g)) .$$

For $1 \leq i \leq \lfloor g/2 \rfloor$, a generic point of $V_l(\overline{\mathcal{M}}_g) \cap \Delta_i$ is formed from two smooth curves X_i and X_{g-i} of genus i and $g-i$ respectively each with one marked point. By the proof of Theorem 1.21 in [16, Theorem 2.3], the a-number of a generic point of $V_l(\overline{\mathcal{M}}_g) \cap \Delta_i$ is 1. Then $a(X_i) + a(X_{g-i}) = 1$ and at least one of these two curves is ordinary. If X_i is ordinary then $[X_i] \in V_i(\overline{\mathcal{M}}_i)$, $a(X_{g-i}) = 1$ and the p-rank of X_{g-i} is equal to $l-i$. We have $\dim(V_i(\overline{\mathcal{M}}_i)) = 3i-3$ and $\dim(V_{l-i}(\overline{\mathcal{M}_{g-i}})) = 3(g-i) - 3 - (g-i-(l-i))$. By Theorem 1.21, together with the fact that X_i and X_{g-i} each has a marked point, we have

$$\dim(V_l(\overline{\mathcal{M}}_g) \cap \Delta_i) \leq 3i-3+1+3(g-i)-3+1-g+l < 3g-3-g+l = \dim(V_l(\overline{\mathcal{M}}_g)) .$$

Similarly one can prove the case where X_{g-i} is ordinary by interchanging indices i and $g-i$ and the result follows by symmetry.

By Theorem 1.21, the locus $V_l(\overline{\mathcal{M}}_g)$ is pure of codimension $g-l$ in $\overline{\mathcal{M}}_g$, together with $\dim(V_l(\overline{\mathcal{M}}_g) \cap \Delta_i) < \dim(V_l(\overline{\mathcal{M}}_g))$, we have $V_l(\overline{\mathcal{M}}_g) \cap \mathcal{M}_g$ is pure of codimension $g-l$ in $\mathcal{M}_g$, i.e. the locus of curves with p-rank l is pure of codimension $g-l$ in $\mathcal{M}_g$.

□

For Z_μ with $\mu = [\mu_1]$, the corollary is equivalent to saying that the sublocus Z_μ is of codimension μ_1 in $\mathcal{M}_g$.

1.6 Newton polygons and supersingular curves

There is another way to distinguish ordinary elliptic curves from supersingular ones, namely by looking at their Newton polygon. For example, if we have an elliptic curve E over a finite field $\mathbb{F}_q$, then the Newton polygon associated to $1-aT+qT^2$ with $a = q+1-\#E(\mathbb{F}_q)$ is either a straight line segment of slope $1/2$ or two line segments of slopes 0 and 1. The first occurs if E is supersingular because $\#E(\mathbb{F}_q) \equiv 1 \pmod{p}$ and the other case is the ordinary case. This can be generalized to principally polarized abelian varieties. More precisely, to a principally polarized abelian variety (A, λ) of dimension g defined over k, we can associate a p-divisible group. To this p-divisible group, one can associate a symmetric Newton polygon, cf. [8]. It consists of finitely many line segments starting at $(0, 0)$ and ending at $(2g, g)$ with slopes such that if $b \in [0, 1]$ appears as the slope of a line segments in the Newton polygon, then so does $1 - b$. More precisely, the Newton polygon lies inside the triangle bounded by the points $(0, 0), (g, 0), (2g, g)$.

Using this one can define a stratification on $\mathcal{A}_g$, cf. [24]. The Newton polygon strata are associated to the principally polarized abelian varieties with a given Newton polygon. The p-rank strata are special examples both of Newton polygon strata and Ekedahl-Oort strata.

The largest stratum, which is of dimension $g(g+1)/2$, corresponds to the ordinary abelian varieties. The smallest stratum, which corresponds to supersingular abelian varieties with Newton polygon the line segment of slope $1/2$, is called supersingular stratum. Li and Oort prove the following result for the dimension of the supersingular Newton stratum.

Theorem 1.23. *[29, Theorem, Section 4, Page 25] The supersingular stratum is of dimension $\lfloor g^2/4 \rfloor$.*

Oort gave a criterion for supersingularity of abelian varieties as follows.

Proposition 1.24. *An abelian variety A defined over k is supersingular if and only if A is isogenous to a product of supersingular elliptic curves.*

Proof. See [36, Theorem 4.2]. □

Note that a superspecial abelian variety of dimension $g \geq 1$ is isomorphic to a product of g supersingular elliptic curves as a principally polarized abelian variety. The following proposition describes the relation of Ekedahl-Oort strata and the supersingular locus.

Proposition 1.25. *Let v be the final type associated to the Ekedahl-Oort type μ. Then the Ekedahl-Oort stratum $\mathcal{Z}_\mu$ lies in the supersingular locus if $v(\lfloor (g+1)/2 \rfloor) = 0$.*

Proof. See [8, Theorem 4.8, Step 2]. □

For a moment we denote the locus of abelian varieties with p-rank 0 by $\mathcal{Z}_{p=0}$ and denote the locus of supersingular (resp. superspecial) abelian varieties by $\mathcal{Z}_{s.s}$ (resp. $\mathcal{Z}_{s.p}$). Then $\mathcal{Z}_{s.p} \subset \mathcal{Z}_{s.s}$ by Proposition 1.25 and $\mathcal{Z}_{s.s} \subset \mathcal{Z}_{p=0}$ by Proposition 1.24.

In fact, for $g = 1$, these three loci are all the same. For $g = 2$, note that an abelian variety with Ekedahl-Oort type $\mu = [2, 1]$ is superspecial and an abelian variety with Ekedahl-Oort type $\mu = [2]$ or $\mu = [2, 1]$ is supersingular by Proposition 1.25. We have $\mathcal{Z}_{s.p} \subset \mathcal{Z}_{s.s} = \mathcal{Z}_{p=0}$. For $g \geq 3$, these three loci are distinct. Indeed, since $\dim(\mathcal{Z}_{s.s}) = \lfloor g^2/4 \rfloor$ and $\dim(\mathcal{Z}_{p=0}) = g(g-1)/2$, we have $\dim Z_{s.s} < \dim \mathcal{Z}_{p=0}$ for $g \geq 3$. Since $\dim(\mathcal{Z}_{s.p}) = 0$, we have $\mathcal{Z}_{s.p} \subsetneq \mathcal{Z}_{s.s}$. For an example of an abelian variety which is supersingular but not superspecial, we consider the Jacobian of a curve X with equation $y^2 + y = x^9$ in characteristic 2. The curve X is supersingular of genus 4 by [51]. On the other hand, we have $x\,\mathrm{d}x \in H^0(X, \Omega^1_X)$ and $\mathcal{C}(x\,\mathrm{d}x) = \mathrm{d}x \neq 0$. Hence $a(X) < 4$ and it is not superspecial.

Again the Newton polygon stratification can be pulled back to $\mathcal{M}_g$ via the Torelli map and one can ask the dimensions of the induced strata in $\mathcal{M}_g$. We are also interested in the intersection of the (induced) Ekedahl-Oort strata and the (induced) supersingular locus in $\mathcal{A}_g$ (resp. $\mathcal{M}_g$). We say a curve is called supersingular if its Jacobian is supersingular.

As examples of supersingular (resp. superspecial) curves we would like to mention Hermitian curves.

Example 1.26. Let $q = p^n$ and X be the Hermitian curve over k given by equation

$$y^q + y = x^{q+1}\,.$$

Then X is supersingular for $n \in \mathbb{Z}_{>0}$. Moreover, X is superspecial if $n = 1$.

Proof. The supersingularity follows from [20, Proposition 3.3]. For the superspeciality, let X the curve with equation $y^q + y = x^{q+1}$. Then X is of genus $g = p(p-1)/2$. One can check that a basis for $H^0(X, \Omega^1_X)$ is given by forms:

$$B = \{x^i y^j\,\mathrm{d}x | i, j \in \mathbb{Z}_{\geq 0}, ip + j(p-1) \leq (p-1)p - 2\}\,.$$

Then by showing that the rank of the Cartier operator is zero, one can prove that X is superspecial.

□

Another example we give is a family of supersingular curves in characteristic $p = 2$ constructed by van der Geer and van der Vlugt.

Example 1.27. [51, Theorem 13.7] Let $p = 2$. Let $R(x) = \sum_{i=0}^{h} a_i x^{2^i} \in k[x]$ be an additive polynomial of degree 2^h. Then the curve with equation $y^2 + y = xR(x)$ is supersingular of genus $p^h(p-1)/2$.

Since there are examples of supersingular curves, one may ask whether for a given prime p there exists a supersingular curve of genus g in characteristic p for every positive integer g. For $p = 2$, van der Geer and van der Vlugt proved that this is the case.

Proposition 1.28. *[52, Theorem 2.1] There exists a supersingular curve of genus g in characteristic 2 for every positive integer g.*

Let c be an integer such that 0 and 1 are the only coefficients in the base p expansion. It is known that there exists a supersingular curve of genus $g = cp(p-1)^2/2$ in characteristic $p > 0$, see [41, Corollary 2.6]. In [44], Re gave a table of supersingular curves of genus $g \leq 100$ in characteristic $0 < p \leq 23$.

1.7 Hyperelliptic curves and trigonal curves

As we mentioned earlier, we are also interested in the intersection of the induced Ekedahl-Oort strata and other special loci in $\mathcal{M}_g$. For example, the hyperelliptic locus $\mathcal{H}_g$ of dimension $2g-1$ parametrizing the hyperelliptic curves of genus g. More generally, one could ask how the induced Ekedahl-Oort strata intersect the locus parametrizing the curves of genus g which are degree d cover of $\mathbb{P}^1$ with $d \geq 3$. In particular, we are interested in the case where $d = 3$, i.e. the trigonal locus $\mathcal{T}_g$ in $\mathcal{M}_g$. It is of dimension $2g+1$ for $g \geq 5$, see [4, Theorem 5.3 and Formula (2.3)]. A curve is called trigonal curve if it is a degree 3 cover of $\mathbb{P}^1$.

It is well known that for $g \geq 2$, a hyperelliptic curve of genus g has unique linear system of dimension 1 and degree 2. However, a trigonal curve X may have different degree 3 maps to $\mathbb{P}^1$ up to isomorphism. Indeed, it is known that any non-hyperelliptic curve of genus 4 is the complete intersection of cubic surface and quadric surface in $\mathbb{P}^3$. If the quadric surface is not singular, then it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Hence this non-hyperelliptic curve of genus 4 is of type $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and in general has two distinct degree 3 maps to $\mathbb{P}^1$. Nevertheless, the following proposition indicates when a trigonal curve has an unique degree 3 map to $\mathbb{P}^1$.

Proposition 1.29. *Let X be a trigonal curve of genus g and $\pi : X \mapsto \mathbb{P}^1$ be a morphism of degree 3. Then π is unique if $g \geq 5$.*

Proof. One can prove it using the base point free pencil trick and Clifford's Theorem, see, for example, [28, Section 2.1].

□

For $g \leq 4$, it is well-known that a curve of genus g is either hyperelliptic or trigonal [21, Chapter IV, Section 5]. However, the following proposition shows that for $g \geq 3$, a curve can not have it both ways.

Proposition 1.30. *A trigonal curve of genus $g \geq 3$ can not be hyperelliptic.*

Proof. The proof is similar to the Proposition 1.29, see also [28, Section 2.1].

□

Chapter 2

A bound on the genus of a curve with Cartier operator of small rank

This Chapter is almost identical to my paper [54].

2.1 Introduction

In [12] Ekedahl gave a bound for the genus g of an irreducible smooth complete curve over an algebraically closed field of characteristic $p > 0$ with zero Cartier operator: $g \leq p(p-1)/2$. This bound is sharp and was generalized by Re to curves with Cartier operator of given rank [45]. He showed for hyperelliptic curves whose Cartier operator has rank m the bound $g < mp + (p+1)/2$, and for non-hyperelliptic curves

$$g \leq mp + (m+1)p(p-1)/2\,. \tag{2.1}$$

He also showed that if the Cartier operator $\mathcal{C}$ is nilpotent and $\mathcal{C}^r = 0$, then

$$g \leq p^r(p^r-1)/2\,.$$

In this paper we give a strengthening of the result (2.1) of Re. One can find other related results in [14] and [18].

Theorem 2.1. *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 1, we have $g \leq p + p(p-1)/2$.*

This is sharp for example for $p = 2$, see [40, Lemma 4.8]. In the case of higher rank we have the following result.

Theorem 2.2. *Let X be an irreducible smooth complete curve of genus g over an algebraically closed field of characteristic $p > 0$. If the rank of the Cartier operator of X equals 2, and if X possesses a point R such that linear system $|pR|$ is base point free, then $g \leq 2p + p(p-1)/2$, while if X does not have such a point, one has the bound $g \leq 2p + (4p^2 - 5p)/3$.*

2.2 The Cartier operator and linear systems

From now on, by a curve we mean an irreducible smooth complete curve over an algebraically closed field k of characteristic $p > 0$. For a curve X with function field $k(X)$, Cartier [7] defined an operator on rational differential forms with the following properties:

1) $\mathcal{C}(\omega_1 + \omega_2) = \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$,
2) $\mathcal{C}(f^p\omega) = f\,\mathcal{C}(\omega)$,
3) $\mathcal{C}(\mathrm{d}f) = 0$,
4) $\mathcal{C}(\mathrm{d}f/f) = \mathrm{d}f/f$,

where $f \in k(X)$ is non-zero. Moreover, recall that if x is a separating variable of $k(X)$, any $f \in k(X)$ can be written as

$$f = f_0^p + \ldots + f_{p-1}^p x^{p-1}, \text{ with } f_i \in k(X)\,. \tag{2.2}$$

For a rational differential form $\omega = f\,\mathrm{d}x$ with f as in (2.2), we have $\mathcal{C}(\omega) = f_{p-1}\,\mathrm{d}x$. In particular $\mathcal{C}^n(f^i\,\mathrm{d}f) = f^{(i+1)/p^n-1}\,\mathrm{d}f$ if $p^n | i+1$, and $\mathcal{C}^n(f^i\,\mathrm{d}f) = 0$ otherwise. Furthermore, for distinct points Q_1, Q_2 on X, if there is a rational differential form ω that $\mathrm{ord}_{Q_1}(\omega) \geq p$ and $\mathrm{ord}_{Q_2}(\omega) = p-1$, then by property 2) above we have $\mathrm{ord}_{Q_1}(\mathcal{C}(\omega)) \geq 1$ and $\mathrm{ord}_{Q_2}(\mathcal{C}(\omega)) = 0$.

This operator $\mathcal{C}$ induces a map $\mathcal{C} : H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ which is σ^{-1}-linear, that is, it satisfies properties 1) and 2) above, with σ denoting the Frobenius automorphism of k. We are interested in the relation between the rank of the Cartier operator, defined as $\dim_k \mathcal{C}(H^0(X, \Omega_X^1))$, and the genus g.

Re showed that there is a relation between the rank of Cartier operator and the geometry of linear systems on a curve. We will list some results that we will use and refer for the proof to Re's paper [45]. In the following, X denotes a non-hyperelliptic curve and for D a divisor on X, we will denote by $H^i(D)$ the vector space $H^i(X, \mathcal{O}_X(D))$.

We will say that a statement holds for a general effective divisor of degree n on X if the statement is true for divisors in a nonempty open set of effective divisors of degree n on X. By a general point we mean a general effective divisor of degree 1. We start with a few results of Re.

Proposition 2.3. *[45, Prop. 2.2.2] Let X be a non-hyperelliptic curve with* $\operatorname{rank}(\mathcal{C}) = m$. *Then for a general effective divisor $D = Q_1 + \ldots + Q_{m+1}$ on X with* $\deg D = m+1$, *one has*

$$h^0(pD) = 1 + h^0(pD - Q_{m+1})\,.$$

This implies for a general divisor D with $\deg D > \operatorname{rank}(\mathcal{C})$, that the linear system $|pD|$ is base point free. As a corollary, we have the following.

Corollary 2.4. *[45, Prop. 2.2.3] If X is a non-hyperelliptic curve with zero Cartier operator, then $h^0(p\,Q) \geq 2$ for any point Q on X.*

The following lemma gives a way of estimating dimensions of linear systems.

Lemma 2.5. *[45, Lemma 2.3.1] Assume that Q_1 and Q_2 are general points on a non-hyperelliptic curve X and that D is a divisor. Then we have*

$$h^0(pD + p\,Q_1 + p\,Q_2) - h^0(pD + p\,Q_1) \geq h^0(pD + p\,Q_1) - h^0(pD)\,.$$

We now give a generalization of a result of Re.

Proposition 2.6. *Let D,E,F be effective divisors on a non-hyperelliptic curve X such that*
(1) *$|F|$ is base point free;*
(2) *$D > 0$ and* $\operatorname{Supp}(D) \cap \operatorname{Supp}(E) = \emptyset$*;*
(3) *There are points $Q_1, \ldots, Q_{m+1} \in \operatorname{Supp}(D)$ and a divisor $F_1 \in |F|$ such that* $\operatorname{ord}_{Q_i}(F_1) = 1$ *for $1 \leq i \leq m+1$ and* $\operatorname{Supp}(D) \cap \operatorname{Supp}(F_1) = \{Q_1, \ldots, Q_{m+1}\}$*;*
(4) *For these points Q_i one has $h^0(E + \sum_{i=1}^{m+1} Q_i) = h^0(E)$;*
(5) *Q_i is not a base point of $|D + E + F|$ for $i = 1, \ldots, m+1$ and there exist $s_1, \ldots, s_{m+1} \in H^0(D + E + F)$ such that*

$$\operatorname{ord}_{Q_i}(s_i) = 0\,,\ \ \operatorname{ord}_{Q_i}(s_j) \geq p,\ \ i \neq j,\ i, j = 1, \ldots, m+1\,.$$

Then we have

$$h^0(D + E + F) - h^0(E + F) \geq h^0(D + E) - h^0(E) + m + 1\,.$$

Proof. Let $s_{F_1} \in H^0(F)$ with divisor F_1 and $s_D \in H^0(D)$ with divisor D. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0(E) & \xrightarrow{\cdot s_D} & H^0(D+E) & \longrightarrow & H^0(\mathcal{O}_D) \\
 & & \downarrow{\scriptstyle \cdot s_{F_1}} & & \downarrow{\scriptstyle \cdot s_{F_1}} & & \downarrow{\scriptstyle \cdot s_{F_1}|_D} \\
0 & \longrightarrow & H^0(E+F) & \xrightarrow{\cdot s_D} & H^0(D+E+F) & \longrightarrow & H^0(\mathcal{O}_D)\,.
\end{array}$$

Claim 2.7. *Multiplication by s_{F_1} induces an injective map*

$$H^0(D+E)/s_D \cdot H^0(E) \xrightarrow{s_{F_1}} H^0(D+E+F)/s_D \cdot H^0(E+F)\,.$$

This Claim follows if

$$s_{F_1} \cdot H^0(D+E) \cap s_D \cdot H^0(E+F) = s_{F_1} \cdot s_D \cdot H^0(E)\,.$$

Because of assumptions (2) and (3), the left hand side of this equation is equal to $s_D \cdot s'_{F_1} \cdot H^0(E + \sum_{i=1}^{m+1} Q_i)$, where $s'_{F_1} = s_{F_1}/s_0$ for a section $s_0 \in H^0(\sum_{i=1}^{m+1} Q_i)$ with $\mathrm{div}(s_0) = \sum_{i=1}^{m+1} Q_i$. Then (4) implies $H^0(E + \sum_{i=1}^{m+1} Q_i) = s_0 \cdot H^0(E)$. The Claim follows.

By (5), we have $s_1, \ldots, s_{m+1}$ such that for all i, j with $i \neq j$ we have $\mathrm{ord}_{Q_i}(s_i) = 0$ and $\mathrm{ord}_{Q_i}(s_j) \geq p$. Now we will show that $s_1, \ldots, s_{m+1}$ generate an $m+1$-dimensional subspace of $H^0(D+E+F)/s_D \cdot H^0(E+F)$ with zero intersection with $\mathrm{Im}(s_{F_1})$. First we will prove the zero intersection part. Assume there exist $c_1, \ldots, c_{m+1} \in k$ such that $\xi = \sum_{i=1}^{m+1} c_i s_i$ lies in $\mathrm{Im}(s_{F_1})$. That means $\xi = s_{F_1} \cdot r + s_D \cdot t$ with some $r \in H^0(D+E)$ and $t \in H^0(E+F)$. If $c_1 \neq 0$ then we obtain $\mathrm{ord}_{Q_1}(\xi) = 0$. However, because $\mathrm{ord}_{Q_1}(F_1) = \mathrm{ord}_{Q_1}(s_{F_1}) = 1$ and $\mathrm{ord}_{Q_1}(s_D) = \mathrm{ord}_{Q_1}(D) \geq 1$, we have $0 = \mathrm{ord}_{Q_1}(\xi) = \mathrm{ord}_{Q_1}(s_{F_1} \cdot r + s_D \cdot t) \geq 1$, a contradiction if $c_1 \neq 0$. Similarly, we can show $c_2 = \cdots = c_{m+1} = 0$. Then for any non-zero element ξ in $< s_1, \ldots, s_{m+1} >$ one has $\xi \notin \mathrm{Im}(s_{F_1})$.

Now for the linear independence of $s_1, \ldots, s_{m+1}$, if $\xi = \sum_{i=1}^{m+1} c_i s_i$ lies in $s_D \cdot H^0(E+F)$, then $\xi = s_D \cdot t$ with $t \in H^0(E+F)$ and we apply the same argument on the orders at Q_i as above with $r = 0$. Then we find $c_i = 0$ for $i = 1, \ldots, m+1$. So $s_1, \ldots, s_{m+1}$ are linearly independent in $H^0(D+E+F)/s_D \cdot H^0(E+F)$.

By the injectivity Claim 2.7 above we then have

$$h^0(D+E+F) - h^0(D+E) \geq h^0(E+F) - h^0(E) + m + 1\,.$$

□

2.3 Proofs of the Theorems 2.1 and 2.2

Before giving the proofs of theorems, we need several lemmas on the relation between the rank of the Cartier operator and geometrical properties of linear systems on a curve.

Lemma 2.8. *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = m \geq 1$. *Then there exists points $Q_1, \ldots, Q_m$ on X such that with $D = \sum_{i=1}^m Q_i$ we have*

$$h^0(pD) = 1 + h^0(pD - Q_m)\,.$$

Proof. Suppose that $\omega_1, \ldots, \omega_m$ are differentials that generate $\mathrm{Im}(\mathcal{C})$. Assume the lemma is not true, that is, for any m-tuple $\alpha = (Q_1, \ldots, Q_m)$, we have with $D = \sum_{i=1}^m Q_i$ that $h^0(p\,D) = h^0(p\,D - Q_m)$. Then by Serre duality and Riemann-Roch, there exists a $\omega_D \in H^0(X, \Omega^1_X)$ that

$$\mathrm{ord}_{Q_i}(\omega_D) \geq p,\ 1 \leq i \leq m-1,\ \ \mathrm{ord}_{Q_m}(\omega_D) = p-1\,. \tag{2.3}$$

Let $\eta := \mathcal{C}(\omega_D) = \sum_{i=1}^m \lambda_i\,\omega_i$ with $\lambda_i \in k$. Then one has

$$\mathrm{ord}_{Q_i}(\eta) \geq 1,\ 1 \leq i \leq m-1\,,\ \ \mathrm{ord}_{Q_m}(\eta) = 0\,. \tag{2.4}$$

Suppose now that $\omega_1, \ldots, \omega_m$ have a common base point R. Then define $Q_m = R$ and choose general points $Q_1, \ldots, Q_{m-1}$ such that $Q_1, \ldots, Q_{m-1}, R$ form m distinct points. Then with $D = \sum_{i=1}^{m-1} Q_i + R$ we have $h^0(pD) = h^0(pD - R)$, hence there exists a ω_D satisfying (2.3). Then $\eta = \mathcal{C}(\omega_D)$ satisfies (2.4) and we have $0 = \mathrm{ord}_{Q_m}(\eta) = \mathrm{ord}_{Q_m}(\sum_{i=1}^m \lambda_i\omega_i) \geq 1$, a contradiction.

So we may assume that $\omega_1, \ldots, \omega_m$ have no common base point. Choose a point Q_1 such that ω_1 does not vanish at Q_1, but $\omega_2, \ldots, \omega_m$ vanish at Q_1. More generally, assume furthermore that we have $Q_1, \ldots, Q_n$ such that $\mathrm{ord}_{Q_i}(\omega_i) = 0$ and $\mathrm{ord}_{Q_i}(\omega_j) > 0$ for $i = 1, \ldots, n$ and $i < j \leq m$.

If $\omega_{n+1}, \ldots, \omega_m$ have a base point R different from Q_i for $i = 1, \ldots, n$, then we choose $Q_{n+1}, \ldots, Q_{m-1}$ general distinct points, $Q_m = R$ and let $\alpha = (Q_1, \ldots, Q_m)$. By assumption $h^0(p\,D) = h^0(p\,D - Q_m)$ for $D = \sum_{i=1}^m Q_i$, and we find a differential form ω_D satisfying (2.3) and therefore a form $\eta = \mathcal{C}(\omega_D)$ satisfying (2.4), again a contradiction.

So we may assume that $\omega_{n+1}, \ldots, \omega_m$ do not have common base points except $Q_1, \ldots, Q_n$. Choose now a point Q_{n+1} different from $Q_1, \ldots, Q_n$ such that ω_{n+1} does not vanish at Q_{n+1}, but $\omega_{n+2}, \ldots, \omega_m$ all vanish at Q_{n+1}. By induction on n, we find points $Q_1, \ldots, Q_{m-1}$ with $\mathrm{ord}_{Q_i}(\omega_i) = 0$ and $\mathrm{ord}_{Q_i}(\omega_j) \geq 1$ for $j > i$ and $j = 2, \ldots, m$.

Now if ω_m has a zero distinct from Q_i for $i = 1, \ldots, m-1$, say Q_m, we let $\alpha = (Q_1, \ldots, Q_m)$ and $D = \sum_{i=1}^m Q_i$. The assumption $h^0(pD) = h^0(pD - Q_m)$ gives us a differential form ω_D and $\eta = \mathcal{C}(\omega_D) = \sum_{i=1}^m \lambda_i\,\omega_i$. By (2.4) we have $0 = \mathrm{ord}_{Q_m}(\eta) = \mathrm{ord}_Q(\lambda_m\omega_m) \geq 1$, a contradiction. So ω_m has no zeros outside $Q_1, \ldots, Q_{m-1}$.

Now $\deg(\omega_m) = 2g-2 \geq m$ for $g \geq 2$, so ω_m vanishes at one Q_i with multiplicity larger than one, say Q_{m-1}. Then with $D = \sum_{i=1}^{m-2} Q_i + 2Q_{m-1}$ we have $h^0(pD) = h^0(pD - Q_{m-1})$, giving us a differential form ω_D, and $\eta = \mathcal{C}(\omega_D) = \sum_{i=1}^{m} \lambda_i \omega_i$. Then we have $\mathrm{ord}_{Q_i}(\eta) \geq 1$ for $i = 1, \ldots, m-2$ and $\mathrm{ord}_{Q_{m-1}}(\eta) = 1$. However, by the induction assumption

$$\mathrm{ord}_{Q_i}(\omega_i) = 0\,, \mathrm{ord}_{Q_i}(\omega_j) \geq 1,\ 1 \leq i < j \leq m-1\,,$$
$$\mathrm{ord}_{Q_l}(\omega_m) \geq 1\,, \mathrm{ord}_{Q_{m-1}}(\omega_m) \geq 2,\ l = 1, 2, \ldots, m-2\,.$$

So we must have $\lambda_i = 0$ for $i = 1, \ldots, m-1$ and $\mathrm{ord}_{Q_{m-1}}(\eta) \geq 2$, and we therefore find $h^0(pD) = 1 + h^0(pD - Q_m)$. □

By putting $m = 1$ in the Lemma 2.8 above, we have the following.

Corollary 2.9. *Let X be a non-hyperelliptic curve. If the Cartier operator has $\mathrm{rank}(\mathcal{C}) = 1$, there exists a point R of X such that $h^0(pR) = 1 + h^0((p-1)R)$.*

Combining Lemma 2.8 above and Proposition 2.6, we have the following result. We denote the canonical divisor (class) by K_X.

Corollary 2.10. *Let X be a non-hyperelliptic curve with $\mathrm{rank}(\mathcal{C}) = 1$ and let T_n be a general effective divisor of degree n. Put $E = pT_n$ and let R be a point of X with $h^0(pR) = 1 + h^0((p-1)R)$. Then the following holds.*
i) If $h^0(K_X - E) \leq 1$, one has for general points Q_1, Q_2

$$h^0(E + pR + \sum_{i=1}^{2} pQ_i) - h^0(E + \sum_{i=1}^{2} pQ_i) = p\,.$$

ii) If $h^0(K_X - E) \geq 2$, one has for general points Q_1, Q_2

$$h^0(E + pR + \sum_{i=1}^{2} pQ_i) - h^0(E + \sum_{i=1}^{2} pQ_i) \geq 2 + h^0(E + pR) - h^0(E)\,.$$

Proof. Note that the existence of R is provided by Corollary 2.9 above.
i) If $h^0(K_X - E) = 0$, i.e. E is non-special, Riemann-Roch implies statement *i)*. If $h^0(K_X - E) = 1$, we choose Q_1 a non-base point of $|K_X - E|$, then $h^0(K_X - E - Q_1) = 0$, hence $h^0(K_X - E - pQ_1) = 0$. Therefore $h^0(K_X - E - \sum_{i=1}^{2} pQ_i) = h^0(K_X - E - \sum_{i=1}^{2} pQ_i - pR) = 0$ and by Riemann-Roch we have $h^0(E + \sum_{i=1}^{2} pQ_i + pR) - h^0(E + \sum_{i=1}^{2} pQ_i) = p$.
ii) If $h^0(K_X - E) \geq 2$, we write $D = pQ_1 + pQ_2$, $E = pT_n$ and $F = pR$ and we proceed to verify the conditions $(1) - (5)$ of Proposition 2.6 in this case. Conditions

(1) and (2) are easy consequences of the generality assumptions of Q_1, Q_2 and R. For condition (3), if the linear system $|pR|$ induces a separable map to projective space, then we can choose Q_1 and Q_2 to be points where the map is smooth and find an effective divisor F_1 such that $\mathrm{ord}_{Q_1}(F_1) = \mathrm{ord}_{Q_2}(F_1) = 1$. If, on the contrary, the map induced by $|pR|$ is inseparable, then $\dim |R| \geq 1$, which is not true for curves of genus larger than zero.

Condition (4) is satisfied once we choose Q_1 to be a non-base point of $|K_X - E|$ and Q_2 a non-base point of $|K_X - E - Q_1|$, since $h^0(K_X - E) \geq 2$. Then we have $h^0(K_X - E - \sum_{i=1}^{2} Q_i) = h^0(K_X - E) - 2$.

Condition (5) holds as $|E + pR + pQ_1 + pQ_2|$ is base point free by Proposition 2.3 if Q_1 and Q_2 are general. Furthermore by Proposition 2.3, we have $h^0(E + pQ_i) = 1 + h^0(E + (p-1)Q_i)$ for $i = 1, 2$. Then we obtain s_1 and s_2 in $H^0(E + pR + pQ_1 + pQ_2) = H^0(D + E + F)$ such that for all i, j we have $\mathrm{ord}_{Q_i}(s_i) = 0$ and $\mathrm{ord}_{Q_i}(s_j) \geq p$ for $j \neq i$.

Then we conclude by Proposition 2.6 above.

□

Now we can state some numerical consequences of Corollary 2.10.

Corollary 2.11. *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 1$. *Denote by D_n a general divisor of degree n. Then for any integer $n \geq 1$, one has*

i) $p \geq h^0(pD_{2n}) - h^0(pD_{2n-1}) \geq \min(2n-1, p)$.

ii) *For $1 \leq n \leq \lceil (p+1)/2 \rceil$, one has $p \geq h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 2n-2$.*

iii) *pD_p is non-special, i.e. $h^0(K_X - pD_p) = 0$.*

iv) *For $1 \leq n \leq \lfloor (p+1)/2 \rfloor$, one has $h^0(pD_{2n}) - h^0(pD_{2n-2}) \geq 4n-3$.*

v) *For $1 \leq n \leq \lfloor (p+1)/2 \rfloor$, one has*

$$h^0(K_X - pD_{2n-2}) - h^0(K_X - pD_{2n}) \leq 2p - 4n + 3\,.$$

vi) $h^0(K_X - pD_{p-1}) \leq 1$ *for $p \geq 3$.*

Proof. *i)* For $n \in \mathbb{Z}_{>0}$, one can always has $p \geq h^0(pD_{2n}) - h^0(pD_{2n-1})$. We will prove the second inequality in *i)* by induction on n.

In the case $n = 1$, by Proposition 2.3, for general points Q_1, Q_2 one has

$$h^0(pQ_1 + pQ_2) = 1 + h^0(pQ_1 + (p-1)Q_2)\,,$$

and thus with $D_2 = Q_1 + Q_2$ and $D_1 = Q_1$, we see $h^0(pD_2) \geq 1 + h^0(pD_1)$. Now we do induction and assume $h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n-3$. We apply Corollary

2.10 with $E = pD_{2n-3}$ for $n \geq 2$. If $h^0(K_X - E) \leq 2$, then we have $h^0(pD_{2n}) - h^0(pD_{2n-1}) = p$. Otherwise, Corollary 2.10 implies

$$h^0(pD_{2n}) - h^0(pD_{2n-1}) \geq 2 + h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n - 1\,.$$

and we are done.
ii) The case $n = 1$ is trivial. Assuming the assertion for $n - 1$, we will prove

$$h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 1 + h^0(pD_{2n-2}) - h^0(pD_{2n-3})\,, \tag{2.5}$$

and by *i*) the right hand side is at least $2n - 2$, which suffices for *ii*). To prove the inequality (2.5), take $D = p\,Q_1$, $E = p\,D_{2n-3}$ and $F = p\,R$ with the point R satisfying $h^0(pR) = 1 + h^0((p-1)R)$ (see Corollary 2.9) and Q_1 a general point. We are going to verify the conditions $(1) - (5)$ of Proposition 2.6 in the case of $m = 0$. Conditions (1) and (2) are obvious by the property of R and generality of Q_1. For condition (3), the map induced by $|pR|$ is separable for curves of genus $g > 0$. We can choose Q_1 to be a point where the map is smooth.

For condition (4) we can choose Q_1 to be a non-base point of $|K_X - E|$ as it is non-empty. For condition (5), as $E = pD_{2n-3}$ with $n \geq 2$, we have for any point Q in $\mathrm{Supp}(E)$, $|pQ + pQ_1|$ is base point free due to Proposition 2.3. Then $|D + E + F|$ is base point free and by Proposition 2.6 we have

$$h^0(pD_{2n-1}) - h^0(pD_{2n-2}) \geq 1 + h^0(pD_{2n-2}) - h^0(pD_{2n-3}) \geq 2n - 2\,.$$

iii) For p odd, we let $n = (p+1)/2$ and apply *i*) get $h^0(pD_{p+1}) - h^0(pD_p) \geq p$. For $p = 2$, we let $n = 2$ and apply *ii*) we also get $h^0(pD_{p+1}) - h^0(pD_p) \geq p$. So we have $h^0(K_X - pD_p) = h^0(K_X - pD_{p+1})$. In other words, for a general point Q, we see that $p\,Q$ lies in the base locus of $|K_X - pD_p|$. This can only happen when $h^0(K_X - pD_p) = 0$. Property *iv*) follows by combining *i*) and *ii*). Property *v*) follows by *iv*) and Riemann-Roch. For property *vi*), by *ii*) and *iii*), it is known that

$$h^0(pD_p) - h^0(pD_{p-1}) \geq p - 1\,,\ h^0(K_X - pD_p) = 0\,.$$

We have

$$\begin{aligned} h^0(K_X - pD_{p-1}) &= h^0(pD_{p-1}) - 1 + g - p(p-1) \\ &\leq h^0(pD_p) - 1 + g - p(p-1) - (p-1) \\ &= h^0(K_X - pD_p) + 1 = 1\,. \end{aligned}$$

□

Using the inequalities in Corollary 2.11, we can easily prove Theorem 2.1.

Proof of Theorem 2.1. We estimate $g = h^0(K_X)$ by

$$\begin{aligned} h^0(K_X) &= \sum_{n=1}^{\lceil (p-1)/2 \rceil} (h^0(K_X - pD_{2n-2}) - h^0(K_X - pD_{2n})) + h^0(K_X - pD_{2\lceil (p-1)/2 \rceil}) \\ &\leq \sum_{n=1}^{\lceil (p-1)/2 \rceil} (2p - 4n + 3) + h^0(K_X - pD_{2\lceil (p-1)/2 \rceil}) = p + p(p-1)/2\,. \end{aligned}$$

□

Our approach to the case $\operatorname{rank}(\mathcal{C}) = 2$ is similar, but there are differences due to the existence of special linear systems. We now give the analogue of Corollary 2.10.

Corollary 2.12. *Let X be a non-hyperelliptic curve with $\operatorname{rank}(\mathcal{C}) = 2$, and let T_n be a general effective divisor of degree n and put $E = p\,T_n$. Let Q_1, Q_2, Q_3 be general points of X and put $D = E + \sum_{i=1}^{3} p\,Q_i$.*
1) *Assume there exists R of X such that $h^0(p\,R) = 1 + h^0((p-1)\,R)$.*
   a) If $h^0(K_X - E) \leq 2$, then one has $h^0(D + pR) - h^0(D) = p$.
   b) If $h^0(K_X - E) \geq 3$, then one has

$$h^0(D + pR) - h^0(D) \geq h^0(E + pR) - h^0(E) + 3\,.$$

2) *If there does not exist such a point R, we choose points R_1,R_2 satisfying $h^0(\sum_{i=1}^{2} p\,R_i) = 1 + h^0(\sum_{i=1}^{2} p\,R_i - R_2)$ and let $\deg E \geq 2p$.*
   a) If $h^0(K_X - E) \leq 2$, then one has $h^0(D + \sum_{j=1}^{2} pR_j) - h^0(D) = 2p$.
   b) If $h^0(K_X - E) \geq 3$, then one has

$$h^0(D + \sum_{i=1}^{2} pR_j) - h^0(D) \geq h^0(E + \sum_{j=1}^{2} pR_j) - h^0(E) + 3\,.$$

Note that in 2) we can choose such R_1 and R_2 by Lemma 2.8. The proof is similar to the proof of Corollary 2.10. But we point out that in the proof of part 2), instead of using the separable map induced by $|p\,R|$ in part *ii)* of the proof of Corollary 2.10, we consider the map induced by $|p\,R_1 + p\,R_2|$ with points R_1 and R_2. This map is separable, otherwise $\dim |R_1 + R_2| \geq 1$, contradicting that X is non-hyperelliptic.

The following two corollaries are the analogues of Corollary 2.11.

Corollary 2.13. *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 2$. *Denote by D_n a general divisor of degree n. If there exists a point R of X that $|p\,R|$ is base point free, then for any integer $n \geq 1$, one has*

i) $p \geq h^0(pD_{3n}) - h^0(pD_{3n-1}) \geq \min(3\,n-2, p)$.

ii) *For $1 \leq n \leq \lceil (p+2)/3 \rceil$, one has*

$$2\,p \geq h^0(pD_{3n-1}) - h^0(pD_{3n-3}) \geq \max(6\,n-7, 0)\,.$$

iii) pD_{p+1} *is non-special, i.e.* $h^0(K_X - p\,D_{p+1}) = 0$.

iv) *For $1 \leq n \leq [(p+2)/3]$, one has* $h^0(pD_{3n}) - h^0(pD_{3n-3}) \geq 9\,n-9$.

v) *For $1 \leq n \leq [(p+2)/3]$, one has*

$$h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n}) \leq 3\,p - 9\,n + 9\,.$$

vi) $h^0(K_X - pD_{3\lceil (p-1)/3 \rceil}) \leq 3$.

Corollary 2.14. *Let X be a non-hyperelliptic curve with* $\mathrm{rank}(\mathcal{C}) = 2$. *Denote by D_n a general divisor of degree n. If X does not possess a point R such that $|p\,R|$ is base point free, then for any integer $n \geq 1$, one has*

i) $2\,p \geq h^0(pD_{3n}) - h^0(pD_{3n-2}) \geq \min(3\,n-2, 2p)$.

ii) *For $2 \leq n \leq \lceil (2p+2)/3 \rceil$, one has*

$$2\,p \geq h^0(pD_{3n-2}) - h^0(pD_{3n-3}) \geq 1\,.$$

iii) pD_{2p} *is non-special, i.e.* $h^0(K_X - pD_{2p}) = 0$.

iv) *For $2 \leq n \leq [(2p+2)/3]$, one has* $h^0(pD_{3n}) - h^0(pD_{3n-3}) \geq 3\,n-1$.
For $n = 1$, one has $h^0(pD_3) - h^0(pD_0) \geq 1$.

v) *For $2 \leq n \leq [(2p+2)/3]$, one has*

$$h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n}) \leq 3\,p - 3\,n + 1\,.$$

For $n = 1$, one has $h^0(K_X) - h^0(K_X - pD_3) \leq 3\,p - 1$.

vi) $h^0(K_X - pD_{3\lceil (2p-1)/3 \rceil}) \leq p-1$.

The proofs of two corollaries above are similar to the proof of Corollary 2.11 and therefore we omit these. The corollaries above now readily imply the proof of theorem in the case of $\mathrm{rank}(\mathcal{C}) = 2$.

Proof of Theorem 2.2. (1) If $|p\,R|$ is base point free, then by Corollary 2.13 we have

$$\begin{aligned} h^0(K_X) &\leq \sum_{n=1}^{\lceil (p-1)/3 \rceil} (h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n})) + h^0(K_X - pD_{3\lceil (p-1)/3 \rceil}) \\ &\leq \sum_{n=2}^{\lceil (p-1)/3 \rceil} (3\,p - 9\,n + 9) + 1 + 3 = p(p-1)/2 + 2\,p\,. \end{aligned}$$

(2) Otherwise, by Corollary 2.14 we have

$$\begin{aligned} h^0(K_X) &\le \sum_{n=1}^{\lceil (2p-1)/3 \rceil} (h^0(K_X - pD_{3n-3}) - h^0(K_X - pD_{3n})) + h^0(K_X - pD_{3\lceil (2p-1)/3 \rceil}) \\ &\le \sum_{n=2}^{\lceil (2p-1)/3 \rceil} (3\,p - 3\,n + 1) + 3\,p - 1 + p - 1 = 2\,p + (4\,p^2 - 5\,p)/3\,. \end{aligned}$$

□

Chapter 3

Ekedahl-Oort strata on moduli of hyperelliptic curves of genus 4 in characteristic 3

In this chapter, we study the intersection of the Ekedahl-Oort strata and the hyperelliptic locus in $\mathcal{M}_g$ in characteristic 3. We describe the Ekedahl-Oort stratification on the moduli of hyperelliptic curves of genus 4 in characteristic 3.

3.1 Main result

Let k be an algebraically closed field of characteristic $p = 3$. Recall that we denote by $\mathcal{M}_g$ the moduli space (stack) of (smooth projective) curves of genus $g \geq 2$ defined over k and by Z_μ (resp. $\mathcal{Z}_\mu$) the induced Ekedahl-Oort stratum (resp. Ekedahl-Oort stratum) in $\mathcal{M}_g$ (resp. $\mathcal{A}_g$) associated to the type $\mu = [\mu_1, \ldots, \mu_n]$. There is a partial order $\preceq$ on the set of Ekedahl-Oort strata by

$$\mu = [\mu_1, \ldots, \mu_n] \preceq \nu = [\nu_1, \ldots, \nu_m]$$

if $n \leq m$ and $\mu_i \leq \nu_i$ for $i = 1, \ldots, n$. Denote by $\overline{\mathcal{Z}_\mu}$ the Zariski closure of $\mathcal{Z}_\mu$ in $\mathcal{A}_g$. Then we have the relation for the closure $\overline{\mathcal{Z}_\mu} \supset \overline{\mathcal{Z}_\nu}$ in $\mathcal{A}_g$ if $\mu \preceq \nu$.

The following result describes the Ekedahl-Oort stratification on $\mathcal{H}_4$ in characteristic 3.

Theorem 3.1. *Let k be an algebraically closed field with* $\mathrm{char}(k) = 3$. *A smooth hyperelliptic curve of genus 4 over k has a-number ≤ 2. In particular, $Z_\mu \cap \mathcal{H}_4$ is empty if $\mu \succeq [3, 2, 1]$. If $\mu \not\succeq [3, 2, 1]$, the codimension of $Z_\mu \cap \mathcal{H}_4$ in $\mathcal{H}_4$ equals the*

expected codimension $\sum_{i=1}^{n} \mu_i$. *Moreover, in the cases* $\mu = [4,1], [3,1], [3,2], [2,1]$ *and* $[1]$ *the intersection* $Z_\mu \cap \mathcal{H}_4$ *is irreducible.*

Part of Theorem 3.1 was known. Frei [18] proved that hyperelliptic curves in odd characteristic can not have a-number $g-1$. Glass and Pries ([19, Theorem 1]) showed that the intersection of $\mathcal{H}_g$ with the locus V_l of p-rank $\leq l$ has codimension $g-l$ in characteristic $p>0$. Pries ([40, Theorem 4.2]) showed that $Z_{[2]} \cap \mathcal{H}_4$ has dimension 5 for $p \geq 3$.

3.2 The a-number and a basis of $H^1_{dR}(X)$

Now let X be a hyperelliptic curve of genus 4 defined over k. Before giving the proof of Theorem 3.1, we prove several propositions needed for Theorem 3.1 and give a basis of the de Rham cohomology of a hyperelliptic curve of genus 4 defined over k.

We first show that any smooth hyperelliptic curve of genus 4 has a-number at most 2.

Proposition 3.2. *A hyperelliptic curve of genus* 4 *in characteristic* 3 *has a-number at most* 2.

Proof. Any smooth hyperelliptic curve X can be written as $y^2 = f(x)$ with $f(x) = \sum_{i=0}^{9} a_i x^i \in k[x]$ and $\mathrm{disc}(f) \neq 0$. By putting a branch point at 0 and by scaling we may assume that $a_1 = a_9 = 1$ and

$$f(x) = x^9 + a_8 x^8 + \cdots + a_2 x^2 + x \tag{3.1}$$

with $a_i \in k$ for $i = 2, \ldots, 8$.

As a basis of $H^0(X, \Omega^1_X)$ we choose $\omega_i = x^i/y\,\mathrm{d}x$ for $i = 0, \ldots, 3$. Then the Cartier-Manin matrix H of curve X is

$$H = \begin{pmatrix} a_2 & 1 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_8 & a_7 & a_6 & a_5 \\ 0 & 0 & 1 & a_8 \end{pmatrix}^{1/3}, \tag{3.2}$$

where $H^{1/3} = (h_{ij}^{1/3})$ if $H = (h_{ij})$. Since $\mathrm{rank}(H) \geq 2$, we have for the a-number $a = 4 - \mathrm{rank}(H) \leq 2$.

□

Remark 3.3. Note that the map from the parameter space of the a_i $(i = 2, \ldots, 8)$ to the hyperelliptic locus has finite fibres. Indeed, if ϕ is an isomorphism between two smooth hyperelliptic curves given by $f_1(x) = \sum_{i=1}^{9} a_i x^i$ and $f_2(x) = \sum_{i=1}^{9} b_i x^i$ as in (3.1) that induces an isomorphism of $\mathbb{P}^1$ fixing 0 and ∞, then ϕ is given by scaling $x \mapsto \alpha x$ and $y \mapsto \beta y$. We obtain $\alpha^9/\beta^2 = \alpha/\beta^2 = 1$ and hence $\alpha^8 = 1, \beta^2 = \alpha$.

We let Y be the open subset of affine space with coordinates $(a_2, \ldots, a_8)$ such that $\mathrm{disc}(f) \neq 0$. Denote by T_a the locus of curves of genus g with a-number $\geq a$ in $\mathcal{M}_4$ and by X_f the smooth projective hyperelliptic curve defined by the equation $y^2 = f(x)$ as in (3.1). Let H_f be the Cartier-Manin matrix of the curve X_f. In the following we simply write X (resp. H) for X_f (resp. H_f). Now we give a result about the intersection $\mathcal{H}_4 \cap T_a$ with $a \leq 2$.

Proposition 3.4. *The locus of $\mathcal{H}_4 \cap T_a$ with $a \leq 2$ is irreducible of codimension $a(a+1)/2$.*

Proof. For $a = 0$, we consider the curve with equation $y^2 = f(x) = x^9 + tx^5 + x$ defined over k where t is a primitive element in $\mathbb{F}_9$. Then $\mathrm{disc}(f) = 2 \neq 0$ and by (3.2) we have $\mathrm{rank}(H) = 4$. Hence there is a curve with $a = 0$ and note that $\mathcal{H}_4$ is irreducible of dimension 7. Then by semicontinuity the generic hyperelliptic curve is ordinary and $T_0 \cap \mathcal{H}_4$ is irreducible of dimension 7.

The condition $a = 1$ means $\mathrm{rank}(H) = 3$. We show that the locus in Y with $\mathrm{rank}(H) = 3$ is given by

$$(a_8 a_6 - a_5)(a_2 a_4 - a_5) + (a_2 - a_3 a_8)(a_2 a_7 - a_8) = 0\,.$$

Indeed if $a_2 = a_8 = 0$ and $\mathrm{disc}(f) \neq 0$, then the rank of H is equal to the rank of

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{1/3} .$$

Since we want $\mathrm{rank}(H) > 2$ we must have $a_5 \neq 0$. Then this implies $\mathrm{rank}(H) = 4$ and the curve is ordinary.

Suppose one of a_2, a_8 is not zero; by symmetry we can assume $a_2 \neq 0$ and the rank of H is equal to the rank of

$$\begin{pmatrix} a_2 & 1 & 0 & 0 \\ 0 & a_4 - a_5/a_2 & a_3 & a_2 \\ 0 & a_7 - a_8/a_2 & a_6 & a_5 \\ 0 & 0 & 1 & a_8 \end{pmatrix}^{1/3} .$$

We have $\det(H) = 0$ as $\mathrm{rank}(H) = 3$ and hence

$$(a_8a_6 - a_5)(a_2a_4 - a_5) + (a_2 - a_3a_8)(a_2a_7 - a_8) = 0\,. \tag{3.3}$$

Note that equation (3.3) can be rewritten as

$$a_7a_2^2 + 2(a_3a_7a_8 + a_4a_5 + 2a_2a_4a_6a_8 + a_2a_8)a_2 + a_3a_8^2 + a_5^2 + 2a_5a_6a_8 = 0\,.$$

This is a 6-dimensional subspace of Y, which is irreducible. Also if we take $a_2 = a_7 = a_8 = 1$ and $a_i = 0$ for $i \neq 2, 7, 8$, then $\mathrm{disc}(f) = 2 \neq 0$ and $\mathrm{rank}(H) = 3$. Hence there is a curve with $a = 1$ and by semicontinuity $T_1 \cap \mathcal{H}_4$ is irreducible of codimension 1.

For $a = 2$, we want to show that the locus in Y with $a = 2$ is given by $a_2 = a_5 = a_8 = 0$. Since we want $\mathrm{rank}(H) = 2$ and the first and fourth row of H are linearly independent, we have several situations to deal with: *i)* $a_2 = 0$, *ii)* $a_8 = 0$ and *iii)* $a_2a_8 \neq 0$.

For the first two cases, if the $\mathrm{rank}(H) = 2$, then the second and third rows of H are linear combinations of the first and fourth rows. Therefore we have $a_2 = a_5 = a_8 = 0$. For the third case, if $a_2a_8 \neq 0$, let e_i to be the i-th row of H, then with some $b, c, s, t \in k$ we have

$$be_1 + ce_4 = e_2,\ \ se_1 + te_4 = e_3\,.$$

This implies $a_3 = a_2/a_8, a_4 = a_5/a_2, a_6 = a_5/a_8, a_7 = a_8/a_2$ and hence

$$f(x) = (x^2 + (a_5/a_8)^{1/3}x + (a_2/a_8)^{1/3})^3(x^3 + a_8x^2 + (a_8/a_2)x)\,,$$

which does not have distinct roots, a contradiction. Then we have $a_2 = a_5 = a_8 = 0$, which defines an irreducible sublocus in Y. Indeed, if we take $a_3 = 1, a_7 = 2$ and $a_i = 0$ for $i \neq 3, 7$, then $\mathrm{disc}(f) = 1 \neq 0$ and $\mathrm{rank}(H) = 2$. So we find a curve with $\mathrm{rank}(H) = 2$. Hence by semicontinuity $T_2 \cap \mathcal{H}_4$ is irreducible of codimension 3.

□

We have seen that any hyperelliptic curve over k with a-number 2 is given by an equation $y^2 = f(x)$ as in (3.1) with $(a_2, \ldots, a_8) \in Y$ and $a_2 = a_5 = a_8 = 0$. Now we consider the action of Verschiebung V on the de Rham cohomology of a curve X given by equation (3.1) with a-number 2. First we give a basis of the de Rham cohomology of a hyperelliptic curve with a-number 2. Let X be a smooth irreducible complete curve over k with equation

$$y^2 = f(x) = x^9 + a_7x^7 + a_6x^6 + a_4x^4 + a_3x^3 + x,\ \ a_i \in k\,, \tag{3.4}$$

where the discriminant of $f(x)$ is non-zero. Let $\pi : X \to \mathbb{P}^1$ be the hyperelliptic map. Take an open affine cover $\mathcal{U} = \{U_1, U_2\}$ with $U_1 = \pi^{-1}(\mathbb{P}^1 - \{0\})$ and $U_2 =$

$\pi^{-1}(\mathbb{P}^1 - \{\infty\})$. By Section 1.2, the de Rham cohomology $H^1_{dR}(X)$ can be described as

$$H^1_{dR}(X) = Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U})$$

with

$$Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2) |\ t \in \mathcal{O}_X(U_1 \cap U_2), \omega_i \in \Omega^1_X(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$$

and $B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2) |\ t_i \in \mathcal{O}_X(U_i)\}$. Then $V(H^1_{dR}(X)) = H^0(X, \Omega^1_X)$ and V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$.

For $1 \leq i \leq 4$, put $s_i(x) = xf'(x) - 2if(x)$ with $f'(x)$ the formal derivative of $f(x)$ and write $s_i(x) = s_i^{\leq i}(x) + s_i^{>i}(x)$ with $s_i^{\leq i}(x)$ the sum of monomials of degree $\leq i$. By Proposition 1.6, $H^1_{dR}(X)$ has a basis with respect to $\{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(X)$:

$$\gamma_i = [(\frac{y}{x^i}, \frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x, -\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)],\ i = 1, \ldots, 4, \tag{3.5}$$

$$\lambda_j = [(0, \frac{x^j}{y}\mathrm{d}x, \frac{x^j}{y}\mathrm{d}x)],\ j = 0, \ldots, 3, \tag{3.6}$$

where $\psi_i(x) = s_i^{\leq i}(x)$ and $\phi_i(x) = s_i^{>i}(x)$. Also we have $\langle \gamma_i, \lambda_j \rangle \neq 0$ if $j = i - 1$ and $\langle \gamma_i, \lambda_j \rangle = 0$ otherwise. Now we give the action of Verschiebung.

Lemma 3.5. *Let X be a smooth hyperelliptic curve over k with equation* (3.4)*. Let $\{U_1, U_2\}$ be a covering of X as above. Then for the basis of $H^1_{dR}(X)$ given by* (3.5) *and* (3.6) *, we have*

$$\begin{aligned}
&V(\lambda_0) = V(\lambda_3) = 0\,, V(\lambda_1) = a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0\,,\\
&V(\lambda_2) = \lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1\,,\\
&V(\gamma_1) = \lambda_2 + a_6^{1/3}\lambda_1 + a_3^{1/3}\lambda_0\,, V(\gamma_2) = V(\gamma_3) = 0\,,\\
&V(\gamma_4) = a_4^{1/3}\lambda_2 + (1 - (a_3a_7)^{1/3} + (a_4a_6)^{1/3})\lambda_1 + a_6^{1/3}\lambda_0\,.
\end{aligned}$$

Proof. Since V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$, we have $V(h\mathrm{d}x) = (-\mathrm{d}^2h/\mathrm{d}x^2)^{1/3}\mathrm{d}x$ with $h \in k(x)$. We will compute $V(\gamma_1)$ and the rest of the lemma will follow easily by using a similar argument. Note that we always have for $1 \leq i \leq 4$

$$V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x) = V(-\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)$$

as $0 = V(\mathrm{d}(y/x^i)) = V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x) - V(-\frac{\phi_i(x)}{2x^{i+1}y}\mathrm{d}x)$. So it suffices to compute $V(\frac{\psi_i(x)}{2x^{i+1}y}\mathrm{d}x)$ instead of computing $V(\gamma_i)$. For $i = 1$, we have

$$V(\frac{\psi_1(x)}{2x^{1+1}y}\mathrm{d}x) = V(\frac{1}{xy}\mathrm{d}x) = (-\frac{\mathrm{d}(\mathrm{d}(\frac{1}{xy})/\mathrm{d}x)}{dx})^{1/3}\,\mathrm{d}x\,.$$

Note that $\mathrm{d}f'(x)/\mathrm{d}x = 0$ and by a calculation we have

$$-\frac{\mathrm{d}(\mathrm{d}(\frac{1}{xy})/\mathrm{d}x)}{\mathrm{d}x} = \frac{x^9 + a_6x^6 + a_3x^3}{x^3y^3}\,.$$

Hence $V(\psi_1(x)/(2x^{1+1}y)\mathrm{d}x) = (x^2/y + a_6^{1/3}x/y + a_3^{1/3}/y)\,\mathrm{d}x$ and we have $V(\gamma_1) = \lambda_2 + a_6^{1/3}\lambda_1 + a_3^{1/3}\lambda_0$.

□

3.3 The proof of Theorem 3.1

Now we give a proof of Theorem 3.1.

Proof of Theorem 3.1. The theorem holds for cases $\mu = [0]$ and $[1]$ by Proposition 3.4 where we showed that $T_a \cap \mathcal{H}_4$ is irreducible with codimension $a(a+1)/2$ for $a \leq 2$. Also $T_3 \cap \mathcal{H}_4$ is empty by Proposition 3.2, hence $Z_\mu \cap \mathcal{H}_4 = \varnothing$ for any $\mu \succeq [3,2,1]$.

We only need to prove that the theorem is true for the remaining nine Ekedahl-Oort strata, that is six strata consisting of curves with a-number 2 and three strata consisting of curves with a-number 1.

As an outline of the proof, we first show that for $\mu = [2,1], [3,1], [3,2], [4,1], [4,2]$ and $[4,3]$ the codimension of $Z_\mu \cap \mathcal{H}_4$ in $\mathcal{H}_4$ equals the expected codimension $\sum_{i=1}^n \mu_i$ with $\mu = [\mu_1, \ldots, \mu_n]$. For $\mu = [2], [3]$ and $[4]$, combined with the fact $V_l \cap \mathcal{H}_4$ is non-empty of codimension $4 - l$ in $\mathcal{H}_4$ for $l = 0, 1, 2$ by Glass and Pries [19, Theorem 1], the intersection $Z_\mu \cap \mathcal{H}_4$ also has the expected codimension. In the cases $\mu = [2,1], [3,1], [3,2], [4,1], [4,2]$ and $[4,3]$, a curve with Ekedahl-Oort type μ can be written as equation (3.4) by the proof of Proposition 3.4.

Throughout the proof, denote by X a smooth hyperelliptic curve given by equation $y^2 = f(x)$ as in (3.4) with Ekedahl-Oort type μ. Denote

$$Y_2 := V(H^0(X, \Omega^1_X)) = V(\langle \lambda_0, \ldots, \lambda_3 \rangle) \text{ and } Y_6 := Y_2^\perp$$

with respect to the paring $\langle\,,\,\rangle$ on $H^1_{dR}(X)$. Put $v : \{0, 1, \ldots, 8\} \to \{0, 1, \ldots, 4\}$ the final type of X. From Lemma 3.5 above, we know that Y_2 is a 2-dimensional subspace of $H^0(X, \Omega^1_X)$ generated by $V(\lambda_1)$ and $V(\lambda_2)$.

Let $\mu = [2,1]$. By Proposition 3.4 the intersection $T_2 \cap \mathcal{H}_4$ is irreducible of dimension 4. For any curve X corresponding to a point in $\mathcal{H}_4 \cap T_2$, we have by Lemma 3.5

$$\begin{aligned} V(Y_2) &= \langle V^2(\lambda_1), V^2(\lambda_2)\rangle \\ &= \langle V(a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0), V(\lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1)\rangle \\ &= \langle a_7^{1/9}V(\lambda_2) + a_4^{1/9}V(\lambda_1), a_6^{1/9}V(\lambda_2) + a_3^{1/9}V(\lambda_1)\rangle\,. \end{aligned} \tag{3.7}$$

We consider the curve associated to $(a_3, a_4, a_6, a_7) = (1, 0, 0, 2)$. Then $\mathrm{disc}(f) = 1 \neq 0$. Moreover, $V^n(Y_2) = Y_2$ for any $n \in \mathbb{Z}_{>0}$. Hence the semi-simple rank of V acting on $H^0(X, \Omega^1_X)$ is 2 and the Ekedahl-Oort type of this curve is $[2,1]$. Since the p-rank can only decrease under specialization, the generic point of $\mathcal{H}_4 \cap T_2$ has Ekedahl-Oort type $[2,1]$ and $Z_{[2,1]} \cap \mathcal{H}_4$ is irreducible of dimension 4.

Now we move to the case $\mu = [3,1]$. We show that a curve with Ekedahl-Oort type $[3,1]$ has equation (3.4) with $a_7a_3 = a_6a_4$ and $\mathrm{disc}(f) = a_3a_4^2 + a_6a_7 + 1 \neq 0$. Then the irreducibility and dimension will follow naturally.

Suppose X has Ekedahl-Oort type $[3,1]$, then X is given by equation (3.4) with $\dim(V(Y_2)) = 1$. Then by Lemma 3.5 and relation (3.7), we have $a_3a_7 = a_4a_6$.

Put $Y_3 := V(Y_6)$ then we have

$$\dim Y_3 = v(6) = v(2) + 4 - 2 = 3$$

by the properties of the final type v. If we take $(a_3, a_4, a_6, a_7) = (0, 1, 0, 0)$, then $\mathrm{disc}(f) = 1 \neq 0$. Note that $V(\gamma_1) = \lambda_2$ and $V(\gamma_4) = \lambda_2 + \lambda_1$, hence $Y_2 = \langle \lambda_3, \lambda_1 + \lambda_0\rangle$ by the Lemma 3.5. Furthermore, it is easy to check that $V^2(\lambda_2) = 0$ and $V^n(\lambda_1) = \lambda_1$. Then we get $v(1) = 1$. Also there exists an element $\gamma = \sum_{i=1}^3 b_i\gamma_i$ with $b_i \in k$ in Y_6 such that $b_1 \neq 0$, otherwise it contradicts that $\langle \gamma, \lambda_0 + \lambda_1\rangle = 0$. Thus $b_1^{1/3}\lambda_2 = V(\gamma) \in Y_3$ and by simple computation we have $\dim V(Y_3) = 2$. Then there is a curve with Ekedahl-Oort type $[3,1]$ and by semicontinuity we have the $Z_{[3,1]} \cap \mathcal{H}_4$ is irreducible of dimension 3.

Let $\mu = [3,2]$, we show that the smooth hyperelliptic curve X with Ekedahl-Oort type $[3,2]$ can be written as

$$y^2 = f(x) = x^9 + a_7x^7 + \alpha^3a_7x^6 + a_4x^4 + \alpha^3a_4x^3 + x \tag{3.8}$$

with $a_4, a_7, \alpha \in k^*$ satisfying $\alpha^3a_7^2 + \alpha a_7 = a_4 + \alpha a_4^2$ and $\mathrm{disc}(f) = (a_4\alpha + a_7\alpha^2 + 1)^9 \neq 0$.

Indeed, if the curve X is given by equation (3.4) with Ekedahl-Oort type $[3,2]$, then we have $v(2) = 1$ and $v(1) = 1$. By Lemma 3.5 and relation (3.7), the condition

$v(2) = 1$ implies $a_3a_7 = a_4a_6$. Also Y_6 is generated by λ_i for $i = 0, \ldots, 3$ and $\sum_{j=1}^4 b_j\gamma_j$ with $b_j \in k$ and $\langle \sum_{j=1}^4 b_j\gamma_j, Y_2\rangle = 0$.

If $a_7 = 0$, by $a_3a_7 = a_4a_6$ we have (i) $a_6 = 0$ or (ii) $a_4 = 0$.

If we suppose $a_7 = a_6 = 0$, then $Y_2 = \langle a_4^{1/3}\lambda_1 + \lambda_0, \lambda_3 + a_3^{1/3}\lambda_1\rangle$ and Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with

$$b_1 + b_2a_4^{1/3} = b_2a_3^{1/3} + b_4 = 0,\ b_1, \ldots, b_4 \in k\,. \tag{3.9}$$

Write $Y_3 = V(Y_6)$. If $a_4 = 0$, then we have $V^2(Y_2) = \langle\, 0\,\rangle$, a contradiction since X has Ekedahl-Oort type $[3, 2]$. Now suppose $a_4 \neq 0$, then for all b_1, b_4 satisfying (3.9) we have

$$V(Y_6) = Y_3 = \langle Y_2, (1 + a_4^{2/9}a_3^{1/9})\lambda_2 + (\frac{a_3}{a_4})^{1/9}\lambda_1 + a_3^{1/3}\lambda_0)\rangle\,.$$

Since $v(3) = 1$, we have $1 + a_4^{2/9}a_3^{1/9} = 0$, which implies $a_3^1a_4^2 = -1$. In this case we have $\mathrm{disc}(f) = a_3^3a_4^6 + 1 = 0$, a contradiction.

Now if $a_7 = a_4 = 0$, Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with $b_1 = b_2a_3^{1/3} + a_6^{1/3}b_3 + b_4 = 0$. By Lemma 3.5 we have $V(\gamma_4) = \lambda_1 + a_6^{1/3}\lambda_0$, hence

$$\begin{aligned} Y_3 = V(Y_6) &= \langle Y_2, V(b_2\gamma_2 + \cdots + b_4\gamma_4)\rangle = \langle Y_2, V(\gamma_4)\rangle \\ &= \langle \lambda_0, \lambda_3 + a_6^{1/3}\lambda_2 + a_3^{1/3}\lambda_1, \lambda_1 + a_6^{1/3}\lambda_0\rangle\,. \end{aligned}$$

Therefore we have $V(Y_3) = Y_2$, a contradiction with $v(3) = 1$.

Now assume $a_7 \neq 0$ and put $\alpha = (a_6/a_7)^{1/3}$. Then we have $a_3 = \alpha^3a_4$ by relation $a_7a_3 = a_6a_4$, and

$$Y_2 = \langle a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1 + \lambda_0, \lambda_3 + \alpha a_7^{1/3}\lambda_2 + \alpha a_4^{1/3}\lambda_1\rangle\,. \tag{3.10}$$

By a similar argument to the above, Y_6 is generated by λ_i and $\sum_{j=1}^4 b_j\gamma_j$ with $\langle \sum_{j=1}^4 b_j\gamma_j, Y_2\rangle = 0$, this implies

$$b_4 - \alpha b_1 = b_3a_7^{1/3} + b_2a_4^{1/3} + b_1 = 0, b_i \in k\,.$$

Then $Y_3 = V(Y_6) = \langle Y_2, V(b_1\gamma_1 + b_4\gamma_4)\rangle$ and this equals $\langle Y_2, \xi\rangle$ with

$$\xi = (1 + \alpha^{1/3}a_4^{1/3})\lambda_2 + (\alpha a_7^{1/3} + \alpha^{1/3})\lambda_1 + (\alpha^{1/3}a_4^{1/3} + \alpha^{4/3}a_7^{1/3})\lambda_0\rangle\,.$$

Since X has Ekedahl-Oort type $\mu = [3, 2]$, we have $v(3) = 1$. Then $V(\langle\, \xi\,\rangle) = V(Y_2) = V(a_7^{1/3}\lambda_2 + a_4^{1/3}\lambda_1)$ by relation (3.10) and Lemma 3.5. Thus we have

$$\alpha a_7^{2/3} + (\alpha a_7)^{1/3} = a_4^{1/3} + \alpha^{1/3}a_4^{2/3}$$

and hence

$$\alpha^3 a_7^2 + \alpha a_7 = a_4 + \alpha a_4^2 . \tag{3.11}$$

If $\alpha = 0$, by equality (3.11) we have $a_4 = 0$ and in equation (3.8) we have $f = x^9 + a_7 x^7 + x$ and one can easily show that $v(1) = 0$, a contradiction as X has Ekedahl-Oort type $\mu = [3, 2]$. By a similar argument we can prove $a_4 \neq 0$. If we take $(a_7, \alpha, a_4) = (2, 2, 1)$ in equation (3.8), we have $\mathrm{disc}(f) = 2 \neq 0$. Then there is a curve with Ekedahl-Oort type $[3, 2]$ and by semicontinuity we have $Z_{[3,2]} \cap \mathcal{H}_4$ is irreducible of dimension 2.

Let $\mu = [4, 1]$. We show that any smooth hyperelliptic curve with Ekedahl-Oort type $[4, 1]$ can be written as

$$y^2 = f(x) = x^9 + a_7 x^7 + \alpha^3 a_7 x^6 - \alpha^9 a_7 x^4 - \alpha^{12} a_7 x^3 + x \tag{3.12}$$

with $a_7, \alpha \in k^*$ and $\mathrm{disc}(f) = 2\alpha^{10} a_7 + \alpha^2 a_7 + 1 \neq 0$. Then it will follow that $Z_{[4,1]} \cap \mathcal{H}_4$ is irreducible of dimension 2. Indeed, if X is given by equation (3.4) with Ekedahl-Oort type [4,1], then $v(2) = 1, v(1) = 0$ and by Lemma 3.5 and relation (3.7) we have $a_3 a_7 = a_4 a_6$.

a) : If $a_7 = 0$, we have $a_6 = 0$ or $a_4 = 0$. Assume $a_6 = a_7 = 0$, then by relation (3.7) we have $V(Y_2) = \langle V(\lambda_1) \rangle$. By Lemma 3.5, we have $a_4 = 0$ since the p-rank of X is zero. But then X has Ekedahl-Oort type $[4, 2]$ by a similar argument with $Y_6 = Y_2^{\perp}$ and $Y_3 = V(Y_6)$ as in case $\mu = [3, 2]$. Now suppose $a_7 = a_4 = 0$. We have $a_6 = 0$ since X has p-rank 0. Then again X has Ekedahl-Oort type $[4, 2]$.

b) : Now assume $a_7 \neq 0$. Put $\alpha = (a_6/a_7)^{1/3}$ and we have $a_3 = \alpha^3 a_4$ by equation $a_7 a_3 = a_6 a_4$. Write $Y_1 = V(Y_2) = \langle a_7^{1/9} V(\lambda_2) + a_4^{1/9} V(\lambda_1) \rangle$. Suppose we have $V^m(Y_1) = 0$ and $V^{m-1}(Y_1) \neq 0$ for some $m \in \mathbb{Z}_{>0}$. For $0 \leq i \leq m$, put $V^i(Y_1) = \langle g_i(\lambda_0, \lambda_3) + c_i \lambda_2 + d_i \lambda_1 \rangle$ with $g_i(x, y) \in k[x, y]$. Then we have

$$c_i = (\alpha c_{i-1}^{1/3} + d_{i-1}^{1/3}) a_7^{1/3}, d_i = (\alpha c_{i-1}^{1/3} + d_{i-1}^{1/3}) a_4^{1/3} \tag{3.13}$$

for $1 \leq i \leq m$. Now $V(V^{m-1}(Y_1)) = 0$. Therefore by Lemma 3.5 we have

$$V^m(Y_1) = V(\langle c_{m-1}\lambda_2 + d_{m-1}\lambda_1 + g_{m-1}(\lambda_0, \lambda_3) \rangle) = 0 .$$

Hence we have $c_m = d_m = 0$ as $V(\langle \lambda_0, \lambda_3 \rangle) = 0$ by Lemma 3.5. Thus we obtain $(\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3}) a_7^{1/3} = (\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3}) a_4^{1/3} = 0$, which implies $\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3} = 0$ as $a_7 \neq 0$. Using the inductive assumption (3.13) again, we have

$$\alpha c_{m-1}^{1/3} + d_{m-1}^{1/3} = ((\alpha^3 a_7^{1/3} + a_4^{1/3})(\alpha^{1/3} c_{m-2} + d_{m-2}))^{1/3} = 0 .$$

Since $V^{m-1}(Y_1) \neq 0$, we have $\alpha^{1/3}c_{m-2} + d_{m-2} \neq 0$ and hence $(\alpha^3 a_7^{1/3} + a_4^{1/3}) = 0$. This implies $a_4 = -\alpha^9 a_7$ and $a_3 = \alpha^3 a_4 = -\alpha^{12} a_7$. Now we compute $Y_3 = V(Y_6)$ and this equals

$$\langle Y_2, (1 - \alpha^{10/3} a_7^{1/3})\lambda_2 + (\alpha a_7^{1/3} + \alpha^{1/3})\lambda_1 + g(\lambda_0, \lambda_3)\rangle$$

for some $g(x, y) \in k[x, y]$. Combined with

$$\begin{aligned} Y_2 &= \langle a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1) + \lambda_0, \lambda_3 + \alpha a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1)\rangle \\ &= \langle \lambda_3 - \alpha\lambda_0, a_7^{1/3}(\lambda_2 - \alpha^3\lambda_1) + \lambda_0\rangle\,, \end{aligned}$$

we have $v(3) = 1$ if

$$\alpha^3(-1 + \alpha^{10/3} a_7^{1/3}) = (\alpha a_7^{1/3} + \alpha^{1/3})\,,$$

this is equivalent to $\alpha^3(\alpha^{16} - 1)a_7 = \alpha^9 + \alpha$. Otherwise X has Ekedahl-Oort type $[4, 1]$ for general pair $(a_7, \alpha) \in \mathbb{A}_k^2$. Hence we have the desired conclusion for $\mu = [4, 1]$. Moreover if in equation (3.12) we take $(a_7, \alpha) = (v^{10}, v^9)$ with v a primitive element in $\mathbb{F}_{27}$, then $\mathrm{disc}(f) = v^{21} \neq 0$ and there is a curve with equation (3.12) has Ekedahl-Oort type associated to $\mu = [4, 1]$. Hence by semicontinuity we have proved the theorem is true for $\mu = [4, 1]$.

For $\mu = [4, 2]$, from the discussion in the case $[4, 1]$ above, a hyperelliptic curve X with Ekedahl-Oort type $[4, 2]$ is either given by equation (3.4) with $a_7 = a_6 = a_4 = 0$, or it can be written as

$$y^2 = f(x) = x^9 + a_7 x^7 + \alpha^3 a_7 x^6 - \alpha^9 a_7 x^4 - \alpha^{12} a_7 x^3 + x$$

with $a_7, \alpha \in k, a_7 \neq 0$ satisfying $\alpha^3(\alpha^{16} - 1)a_7 = \alpha^9 + \alpha$ and $\mathrm{disc}(f) \neq 0$. Moreover,the curve with equation $y^2 = x^9 + x^7 + x$ has $\mathrm{disc}(f) = 1 \neq 0$ and Ekedahl-Oort type $\mu = [4, 2]$. Hence $Z_{[4,2]} \cap \mathcal{H}_4$ is non-empty of dimension 1.

For $\mu = [4, 3]$, a curve X with Ekedahl-Oort type $[4, 3]$ is given by (3.4) with $V(Y_2) = \langle 0\rangle$. Then by Lemma 3.5 we have $a_7 = a_6 = a_4 = a_3 = 0$. This implies X is isomorphic to the curve with equation

$$y^2 = x^9 + x\,.$$

Now we have proved the theorem for $\mu = [2, 1], [3, 1], [3, 2], [4, 1], [4, 2]$ and $[4, 3]$. Also for $\mu = [2], [3]$ and $[4]$, by Glass and Pries [19, Theorem 1] the intersection $V_l \cap \mathcal{H}_4$ has codimension $4 - l$ in $\mathcal{H}_4$ for $l = 0, 1, 2$. Since we have showed that $Z_{[2,1]}$ (resp. $Z_{[3,1]}$ and $Z_{[4,1]}$) intersects $\mathcal{H}_4$ with codimension 3 (resp. 4 and 5), it follows that $Z_\mu \cap \mathcal{H}_4$ has the expected codimension for $\mu = [2], [3]$ and $[4]$.

□

Example 3.6. 1) A hyperelliptic curve $y^2 = x^9 + t^{10}x^7 + t^{11}x^6 + x^4 + tx^3 + x$ defined over k with t a primitive element of F_{27} has Ekedahl-Oort type $[4, 1]$.
2) A hyperelliptic curve $y^2 = x^9 + a_7x^7 + x$ defined over k with $a_7 \in k^*$ has Ekedahl-Oort type $[4, 2]$.

Chapter 4

Cyclic covers of $\mathbb{P}^1$ with given Ekedahl-Oort type

In this chapter, we prove the existence of curves with given Ekedahl-Oort type using cyclic covers of the projective line in characteristic $p > 0$. We first give a basis of the first de Rham cohomology for cyclic covers of the projective line and study the action of Verschiebung operator. In particular for $g = 4$, we show the non-emptiness of certain Ekedahl-Oort strata in $\mathcal{M}_4$. We also show that there is a superspecial curve of genus 4 for any $p \equiv \pm 1 \pmod 5$. Furthermore, we give an alternative proof of a result of Kudo [27] showing that there exists a superspecial curve of genus 4 in characteristic p for all p with $p \equiv 2 \pmod 3$.

4.1 Cyclic covers of the projective line

Let k be an algebraically closed field of characteristic $p > 0$. By a curve we mean a smooth irreducible projective curve defined over k. We fix an integer $m \geq 2$ with $p \nmid m$. Write $a = (a_1, \ldots, a_N)$ for an N-tuple of positive integers with $N \geq 3$. We say a is a monodromy vector of length N if

$$\sum_{i=1}^{N} a_i \equiv 0 \pmod{m}, \qquad \gcd(a_i, m) = 1,\ i = 1, \ldots, N. \tag{4.1}$$

There is an action of $(\mathbb{Z}/m\mathbb{Z})^* \times \mathfrak{S}_N$ on the set of monodromy vectors of length N, where the symmetric group $\mathfrak{S}_N$ acts by permutation of indices and $(\mathbb{Z}/m\mathbb{Z})^*$ acts by multiplication on vectors. Two monodromy vectors a and a' are called equivalent if they are in the same orbit.

Let $P_1, \ldots, P_N$ be the distinct points in $\mathbb{P}^1$ and x be a coordinate on $\mathbb{P}^1$. By a change of coordinates, we may assume $P_1 = 0$ and $P_N = \infty$. Denote by $x - \xi_i$ the local parameter of P_i ($\xi_1 = 0$) for $1 \leq i \leq N-1$. We consider a smooth projective curve X given by equation

$$y^m = f_a(x) = \Pi_{i=1}^{N-1}(x - \xi_i)^{a_i} \,. \tag{4.2}$$

Note that the isomorphism class of the curve depends only on the orbit of monodromy vector a. For $N = 3$, the supersingularity of cyclic covers of the projective line has been studied and examples of supersingular curves was given for $4 \leq g \leq 11$, see [31]. In [14], Elkin gave a bound for the a-number of X for $m \geq 2$ and $N \geq 3$.

A curve defined by equation (4.2) is equipped with a $\mathbb{Z}/m\mathbb{Z}$ action generated by $\epsilon : (x, y) \mapsto (x, \zeta^{-1}y)$ with $\zeta \in k$ a primitive m-th root of unity. This ϵ also induces an automorphism on $H^0(X, \Omega^1_X)$. Then we can decompose

$$H^0(X, \Omega^1_X) = \bigoplus_{n=1}^{m-1} H^0(X, \Omega^1_X)_{(n)} \,, \tag{4.3}$$

where $H^0(X, \Omega^1_X)_{(n)} := \{\omega \in H^0(X, \Omega^1_X) \mid \epsilon^*(\omega) = \zeta^n \omega\}$ is the ζ^n-eigenspace of $H^0(X, \Omega^1_X)$. Denote by $\langle z \rangle := z - \lfloor z \rfloor$ the fractional part of z for any $z \in \mathbb{R}$. Put

$$\begin{aligned} b(i, n) &:= \lfloor (na_i)/m \rfloor, \\ \omega_{n,l} &:= y^{-n} x^l \Pi_{i=1}^{N-1} (x - \xi_i)^{b(i,n)} \, \mathrm{d}x \,. \end{aligned}$$

Then (see for example [32]) the space $H^0(X, \Omega^1_X)_{(n)}$ is generated by $\omega_{n,l}$ with $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$ and

$$d_n = \dim H^0(X, \Omega^1_X)_{(n)} = -1 + \sum_{i=1}^N \langle na_i/m \rangle \,.$$

By the Hurwitz formula, the curve has genus

$$g = 1 + \frac{(N-2)m - N}{2} \,.$$

4.2 A basis of the de Rham cohomology

Now we introduce a basis of de Rham cohomology of a curve X given by equation (4.2). Fix a monodromy vector a together with an ordered N-tuple $(\xi_1, \ldots, \xi_N)$. Let $\pi : X \to \mathbb{P}^1$ be the $\mathbb{Z}/m\mathbb{Z}$-cover. Put $U_1 = \pi^{-1}(\mathbb{P}^1 - \{0\})$ and $U_2 = \pi^{-1}(\mathbb{P}^1 - \{\infty\})$.

For the open affine cover $\mathcal{U} = \{U_1, U_2\}$, we consider the de Rham cohomology $H^1_{dR}(X)$ as in Section 1.2, i.e.

$$H^1_{dR}(X) \cong Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U})$$

with

$$Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2) | t \in \mathcal{O}_X(U_1 \cap U_2), \omega_i \in \Omega^1_X(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$$

and $B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2) | t_i \in \mathcal{O}_X(U_i)\}$.

By (4.3), the vector space $H^0(X, \Omega^1_X)$ is generated by $\omega_{n,l}$ with $1 \leq n \leq m-1$ and $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$. Moreover for the basis of $H^1(X, \mathcal{O}_X)$, we have the following result.

Lemma 4.1. *Let $K(X)$ be the function field of X. For all integers n, l such that $1 \leq n \leq m-1$ and $0 \leq l \leq -2+\sum_{i=1}^N \langle na_i/m \rangle$, the elements $f_{n,l} := y^n x^{-l-1} \Pi_{i=1}^{N-1}(x - \xi_i)^{-b(i,n)} \in K(X)$ are regular on $U_1 \cap U_2$ and their residue classes $[f_{n,l}]$ form a basis of $H^1(X, \mathcal{O}_X)$ with respect to $\{U_1, U_2\}$.*

Proof. It suffices to show that the $f_{n,l}$ are regular on $U_1 \cap U_2$ for all integers n, l such that $\omega_{n,l} \in H^0(X, \Omega^1_X)$. One can check the linear independence by checking the order of $f_{n,l}$ at ∞.

Note that for P_i with $i = 2, \ldots, N-1$, we have

$$\begin{aligned} \mathrm{ord}_{P_i}\left(\frac{y^n}{x^{l+1}\Pi_{i=1}^{N-1}(x-\xi_i)^{b(i,n)}}\right) &= n\,\mathrm{ord}_{P_i}(y) - b(i,n)\,\mathrm{ord}_{P_i}(x-\xi_i) \\ &= na_i - mb(i,n) = na_i - m\lfloor \frac{na_i}{m} \rfloor \geq 0\,. \end{aligned}$$

Then $f_{n,l}$ is regular on $U_1 \cap U_2$ for $1 \leq n \leq m-1$ and $0 \leq l \leq -2 + \sum_{i=1}^N \langle na_i/m \rangle$. □

Put

$$s_a(x) := \Pi_{i=1}^{N-1}(x - \xi_i)^{b(i,n)}\,.$$

Denote by $h_a(x)$ the polynomial in $k[x]$ such that

$$\frac{nxs_a(x)f_a'(x) + ((l+1)s_a(x) + xs_a'(x))f_a(x)}{s_a^2(x)} = \Pi_{i=1}^{N-1}(x-\xi_i)^{a_i - b(i,n) - 1} h_a(x)\,,$$

where $f_a'(x)$ (resp. $s_a'(x)$) is the formal derivative of $f_a(x)$ (resp. $s_a(x)$). In the following we write simply $s(x)$ (resp. $f(x)$, $h(x)$) for $s_a(x)$ (resp. $f_a(x)$, $h_a(x)$).

Then we have the following result.

Theorem 4.2. *Let X be a smooth projective curve over k given by equation* (4.2). *Then $H^1_{dR}(X)$ has a basis with respect to $\mathcal{U} = \{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(\mathcal{U})$:*

$$\alpha_{n,l} = [(0, \omega_{n,l}, \omega_{n,l})],\ 1 \le n \le m-1,\ 0 \le l \le -2 + \sum_{i=1}^{N} \langle na_i/m \rangle\,, \tag{4.4}$$

$$\beta_{n,l} = [(f_{n,l}, \frac{\psi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x, -\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x)]\,, \tag{4.5}$$

where $t(x) = \Pi_{i=1}^{N-1}(x-\xi_i)^{a_i - b(i,n)-1}$ and $\psi(x) + \phi(x) = h(x)$ with $\psi(x)$ the sum of monomials of degree $\le l+1$.

Proof. We use the exact sequence

$$0 \to H^0(X, \Omega^1_X) \to H^1_{dR}(X) \to H^1(X, \mathcal{O}_X) \to 0\,.$$

The elements $\alpha_{n,l}$ are images of $\omega_{n,l}$ under the canonical map.

By Lemma 4.1, we have $[f_{n,l}] \in H^1(X, \mathcal{O}_X)$ for any n, l such that $\omega_{n,l} \in H^0(X, \Omega^1_X)$. To prove the theorem, we need to show that the elements $\beta_{n,l}$ are well defined and are mapped to the element $[f_{n,l}]$ in $H^1(X, \mathcal{O}_X)$. We first show that $\psi_{n,l}(x)/(x^{l+2}y^{m-n}) \in \mathcal{O}(U_1)$ and $\phi_{n,l}(x)/(x^{l+2}y^{m-n}) \in \mathcal{O}(U_2)$. Next we show

$$\mathrm{d}f_{n,l} = \frac{\psi_{n,l}(x)}{x^{l+2}y^{m-n}} - (-\frac{\phi_{n,l}(x)}{x^{l+2}y^{m-n}})$$

and then we will have the desired conclusion.

Note that for any P_i with $i = 1, \ldots, N-1$, we have

$$\begin{aligned}
\mathrm{ord}_{P_i}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) &= \mathrm{ord}_{P_i}(\frac{\Pi_{i=1}^{N-1}(x-\xi_i)^{a_i-b(i,n)-1}}{y^{m-n}}) + \mathrm{ord}_{P_i}(\mathrm{d}x) \\
&= (a_i - b(i,n) - 1)\,\mathrm{ord}_{P_i}(x-\xi_i) - (m-n)\,\mathrm{ord}_{P_i}(y) + m - 1 \\
&= m(a_i - \lfloor \frac{na_i}{m} \rfloor - 1) - a_i(m-n) + m - 1 \\
&= m(a_i - \frac{na_i}{m} - 1 + \langle \frac{na_i}{m} \rangle) - a_i(m-n) + m - 1 \\
&= m\langle \frac{na_i}{m} \rangle - 1 \ge 0\,,
\end{aligned}$$

since $\gcd(a_i, m) = 1$. Hence

$$\mathrm{ord}_{P_i}(\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\mathrm{d}x)) \ge \mathrm{ord}_{P_i}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) \ge 0$$

and $\phi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is regular at P_i for $i = 2, \ldots, N-1$. By a similar argument, $\psi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is also regular at P_i for $i = 2, \ldots, N-1$. For $P_1 = 0$, we have

$$\mathrm{ord}_{P_1}(\frac{\phi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x) \geq \mathrm{ord}_{P_1}(\frac{t(x)}{y^{m-n}}\,\mathrm{d}x) \geq 0\,,$$

since all the monomials of $\phi_{n,l}(x)$ has degree $\geq l+2$. Then the residue class of the element $\phi_{n,l}(x)t(x)/(x^{l+2}y^{m-n})\mathrm{d}x$ is regular on U_2. For $P_N = \infty$,

$$\mathrm{ord}_{P_N}(\frac{\psi_{n,l}(x)t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x) \geq \mathrm{ord}_{P_N}(\frac{x^{l+1}t(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x)\,.$$

The right hand side is equal to

$$\begin{aligned}
&\mathrm{ord}_{P_N}(1/x\,\mathrm{d}x) + \mathrm{ord}_{P_N}(\frac{t(x)}{y^{m-n}}) \\
&= -1 + \sum_{i=1}^{N-1}(a_i - b(i,n) - 1)\,\mathrm{ord}_{P_N}(x) - (m-n)\,\mathrm{ord}_{P_N}(y) \\
&= -1 + ((m-n)\sum_{i=1}^{N-1} a_i - m(N-1) - m\sum_{i=1}^{N-1}\langle\frac{na_i}{m}\rangle) + (m-n)\sum_{i=1}^{N-1} a_i \\
&= m(N-1) - 1 - m\sum_{i=1}^{N-1}\langle\frac{na_i}{m}\rangle \geq 0\,.
\end{aligned}$$

Let n, l be the integers such that $\omega_{n,l} \in H^0(X, \Omega_X^1)$. Then

$$\begin{aligned}
\mathrm{d}(f_{n,l}) = \mathrm{d}(\frac{y^n}{x^{l+1}s(x)}) &= \frac{nxs(x)f'(x) + ((l+1)s(x) + xs'(x))f(x)}{x^{l+2}s^2(x)y^{m-n}}\,\mathrm{d}x \\
&= \frac{t(x)h(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x = \frac{t(x)\psi_{n,l}(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x - \frac{-t(x)\phi_{n,l}(x)}{x^{l+2}y^{m-n}}\,\mathrm{d}x\,.
\end{aligned}$$

□

Remark 4.3. Note that the vector space $H^1_{dR}(X)$ is equipped with a perfect pairing $\langle\ ,\ \rangle$, see Section 1.3. We have $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle \neq 0$ if $(i_1, j_1) = (i_2, j_2)$ and $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle = 0$ otherwise. Indeed, for $(i_1, j_1) = (i_2, j_2)$ we have $\mathrm{ord}_\infty(1/x\,\mathrm{d}x) = -1$ and hence $\langle\alpha_{i_1,j_1}, \beta_{i_2,j_2}\rangle \neq 0$. For the other cases, the proof is similar to the proof of [48, Theorem 4.2.1].

Now for $p = 2$ and $N = 4$, we have the following

Corollary 4.4. *Let k be an algebraically closed field of characteristic p and a be a monodromy vector satisfying relation (4.1) with $a = (1, 1, 1, m-3)$. Let X be a curve of genus $m-1$ over k given by equation*

$$y^m = x(x-1)(x-\xi)\,,$$

where $\xi \neq 0, 1 \in k$. Then $H^1_{dR}(X)$ has a basis with respect to $\{U_1, U_2\}$ consisting of the following residue classes with representatives in $Z^1_{dR}(\mathcal{U})$:

$$\alpha_{i,0} = [(0, \frac{1}{y^i}\,\mathrm{d}x, \frac{1}{y^i}\,\mathrm{d}x)],\ \frac{m}{3} < i \leq m-1,\ ,$$

$$\alpha_{j,1} = [(0, \frac{x}{y^j}\,\mathrm{d}x, \frac{x}{y^j}\,\mathrm{d}x)],\ \frac{2m}{3} < j \leq m-1,\ ,$$

$$\beta_{i,0} = [(\frac{y^i}{x}, \frac{\xi}{xy^{m-i}}\mathrm{d}x, -\frac{x+(\xi+1)}{y^{m-i}}\mathrm{d}x)],\ i \text{ even and } \frac{m}{3} < i \leq m-1,$$

$$\beta_{i,0} = [(\frac{y^i}{x}, 0, -\frac{(\xi+1)}{y^{m-i}}\mathrm{d}x)],\ i \text{ odd and } \frac{m}{3} < i \leq m-1,$$

$$\beta_{j,1} = [(\frac{y^j}{x^2}, 0, 0)],\ j \text{ even and } \frac{2m}{3} < j \leq m-1,$$

$$\beta_{j,1} = [(\frac{y^j}{x^2}, \frac{\xi}{x^2y^{m-j}}\mathrm{d}x, -\frac{1}{y^{m-j}}\mathrm{d}x)],\ j \text{ odd and } \frac{2m}{3} < j \leq m-1,$$

Proof. Note that $a = (a_1, a_2, a_3, a_4) = (1, 1, 1, m-3)$. Then by definition $b(i, n) = \langle na_i/m\rangle = 0$ for any $1 \leq n \leq m-1$ and $1 \leq i \leq 3$. Moreover, the differential form $\omega_{n,l} = y^{-n}x^l\,\mathrm{d}x$ is holomorphic for $1 \leq n \leq m-1$ if and only if $0 \leq l \leq -2 + \sum_{i=1}^4 \langle na_i/m\rangle \leq -2+3 = 1$. If $0 \leq n \leq m/3$, then $\sum_{i=1}^4 \langle na_i/m\rangle = 1$ and $H^0(X, \Omega^1_X)_{(n)} = \langle 0\rangle$. The rest of the corollary follows from Theorem 4.2.

□

4.3 Existence of cyclic covers of $\mathbb{P}^1$ with given Ekedahl-Oort type

We can compute the Ekedahl-Oort type of the curve given by Corollary 4.4.

Proposition 4.5. *Let k be an algebraically closed field of characteristic $p = \pm 2 \pmod 5$. Consider the curve X of genus 4 with equation*

$$y^5 = x(x-1)(x-\xi)\,,$$

where $\xi \in k \backslash \mathbb{F}_p$. Then X has Ekedahl-Oort type $[4, 2]$.

Proof. We prove the case where $p = 2 \,(\mathrm{mod}\, 5)$. The case $p = -2 \,(\mathrm{mod}\, 5)$ is similar and hence we omit it. Write $p = 5r + 2$ with $r \in \mathbb{Z}_{\geq 0}$. Since p is a prime, either $r = 0$ or r is an odd positive integer.

Let $Y_8 := H^1_{dR}(X)$ and $Y_4 := H^0(X, \Omega^1_X)$. By Corollary 4.4, we have $V(Y_8) = Y_4$ and $Y_4 = \langle \alpha_{2,0}, \alpha_{3,0}, \alpha_{4,0}, \alpha_{4,1} \rangle$.

Note that

$$\begin{aligned}
\mathcal{C}(\frac{1}{y^2}\,\mathrm{d}x) &= \mathcal{C}(\frac{y^{5r}}{y^{5r+2}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}((x(x-1)(x-\xi))^r\,\mathrm{d}x) = 0\,,\\
\mathcal{C}(\frac{1}{y^3}\,\mathrm{d}x) &= \mathcal{C}(\frac{y^{20r+5}}{y^{20r+5+3}}\,\mathrm{d}x) = \frac{1}{y^4}\mathcal{C}((x(x-1)(x-\xi))^{4r+1}\,\mathrm{d}x)\,,\\
\mathcal{C}(\frac{1}{y^4}\,\mathrm{d}x) &= \mathcal{C}(\frac{y^{10r}}{y^{10r+4}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}((x(x-1)(x-\xi))^{2r}\,\mathrm{d}x)\,,\\
\mathcal{C}(\frac{x}{y^4}\,\mathrm{d}x) &= \mathcal{C}(\frac{xy^{10r}}{y^{10r+4}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r+1}((x-1)(x-\xi))^{2r}\,\mathrm{d}x)\,.
\end{aligned}$$

One can show that the coefficient of $x^{p-1} = x^{5r+1}$ in $x^{2r+i}((x-1)(x-\xi))^{2r}$ can not be simultaneously zero for $i = 0, 1$ and $\xi \in k \backslash \mathbb{F}_p$. Similarly, the coefficients of x^{p-1} and x^{2p-1} in $(x(x-1)(x-\xi))^{4r+1}$ are both not zero. Then $Y_2 := V(Y_4) = \langle \alpha_{2,0},\, \gamma\alpha_{4,0} + \eta\alpha_{4,1} \rangle$ with $\gamma, \eta \in k^*$. Denote by $Y_6 = Y_2^{\perp}$ the orthogonal complement with respect to the pairing on $H^1_{dR}(X)$. Hence by a calculation using Corollary 4.4, we have

$$Y_6 = \langle \alpha_{2,0}, \alpha_{3,0}, \alpha_{4,0}, \alpha_{4,1}, \beta_{3,0}, \lambda_0\beta_{4,0} + \lambda_1\beta_{4,1} \rangle\,,$$

where $\lambda_i \in k^*$. This implies $Y_3 := V(Y_6) = \langle \alpha_{2,0}, \alpha_{4,0}, \alpha_{4,1} \rangle$ and $V(Y_3) = \langle \alpha_{2,0} \rangle$. We obtain that X has Ekedahl-Oort type $[4, 2]$.

□

Corollary 4.6. *The locus $Z_{[4,2]}$ in $\mathcal{M}_4$ is non-empty of dimension at least* 1 *for any prime $p \equiv \pm 2 \,(\mathrm{mod}\, 5)$.*

If $\xi = -1$ and $p \neq 2$, the curve X in Proposition 4.5 has Ekedahl-Oort $[4, 3]$. Moreover, denote by v the final type of the curve, see Section 1.3. We have $v(2) = 0$ and the curve is supersingular by Proposition 1.25.

Take now $m = 5$ and monodromy vector $a = (1, 1, 1, 2)$. Then we have the following result.

Theorem 4.7. *Let k be an algebraically closed field of characteristic p. For any odd prime p with $p \equiv \pm 2 (\mathrm{mod}\, 5)$, we have $Z_{[4,3]}$ in $\mathcal{M}_4$ is non-empty. For any prime $p \equiv -1 (\mathrm{mod}\, 5)$, there exist superspecial curves of genus* 4 *in characteristic p.*

Proof. Consider a curve X given by equation

$$y^5 = x(x-\xi)(x+\xi)\,,$$

where $\xi \in k^*$. For $p \neq 2, 5$, the curve is of genus 4. Moreover by Lemma 4.1, the vector space $H^0(X, \Omega^1_X)$ has a basis given by forms $y^2\,\mathrm{d}x, y^3\,\mathrm{d}x, y^4\,\mathrm{d}x$ and $xy^4\,\mathrm{d}x$. Now if $p > 2$ and $p \equiv 2 (\bmod 5)$, then write $p = 5r + 2$ with r an odd positive integer and we consider the action of the Cartier operator $\mathcal{C}$. We have

$$\begin{aligned}
\mathcal{C}(\frac{\mathrm{d}x}{y^2}) &= \mathcal{C}(\frac{y^{5r}}{y^{5r+2}}\,\mathrm{d}x) = \mathcal{C}(\frac{x^r(x^2-v^2)^r\,\mathrm{d}x}{y^{5r+2}}) \\
&= \frac{1}{y}\mathcal{C}(x^r(x^2-v^2)^r\,\mathrm{d}x) = 0\,, \\
\mathcal{C}(\frac{\mathrm{d}x}{y^3}) &= \mathcal{C}(\frac{y^{5(4r+1)}}{y^{20r+5+3}}\,\mathrm{d}x) = \frac{1}{y^4}\mathcal{C}(x^{4r+1}(x^2-v^2)^{4r+1}\,\mathrm{d}x) \\
&= \binom{4r+1}{r}(-v)^{2r/p}\frac{x}{y^4}\,\mathrm{d}x\,, \\
\mathcal{C}(\frac{x\,\mathrm{d}x}{y^4}) &= \mathcal{C}(\frac{xy^{10r}}{y^{10r+4}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r+1}(x^2-v^2)^{2r}\,\mathrm{d}x) = 0\,, \\
\mathcal{C}(\frac{\mathrm{d}x}{y^4}) &= \mathcal{C}(\frac{y^{10r}}{y^{10r+4}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r}(x^2-v^2)^{2r}\,\mathrm{d}x) \\
&= \binom{2r}{(3r+1)/2}(-1)^{(r-1)/2}v^{(r-1)/p}\frac{1}{y^2}\,\mathrm{d}x\,.
\end{aligned}$$

The third equation follows from the fact that r is odd and the coefficient of $x^{p-1} = x^{5r+1}$ in the polynomial $x^{2r+1}(x^2-v^2)^{2r}$ is zero. Then X has Ekedahl-Oort type $[4,3]$ and $v(2) = 0$ with v the final type. This implies X is supersingular by Proposition 1.25. By a similar argument, one can show that for $p \equiv 3 (\bmod 5)$, the curve has Ekedahl-Oort type $[4,3]$ and hence is supersingular.

Now for $p \equiv -1 (\bmod 5)$, write $p = 5r + 4$ with r some positive integer. Then

$$\mathcal{C}(\frac{\mathrm{d}x}{y^2}) = \mathcal{C}(\frac{y^{15r+10}}{y^{15r+10+2}}\,\mathrm{d}x) = \frac{1}{y^3}\mathcal{C}(x^{3r+2}(x^2-v^2)^{3r+2}\,\mathrm{d}x) = 0$$

since the coefficient of $x^{p-1} = x^{5r+3}$ in polynomial $x^{3r+2}(x^2-v^2)^{3r+2}$ is zero. In a

similar fashion, we have

$$\begin{aligned}
\mathcal{C}(\frac{\mathrm{d}x}{y^3}) = \mathcal{C}(\frac{y^{10r+5}}{y^{10r+5+3}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r+1}(x^2-v^2)^{2r+1}\,\mathrm{d}x) = 0\,,\\
\mathcal{C}(\frac{x\,\mathrm{d}x}{y^4}) = \mathcal{C}(\frac{xy^{5r}}{y^{5r+4}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}(x^{r+1}(x^2-v^2)^{r}\,\mathrm{d}x) = 0\,,\\
\mathcal{C}(\frac{\mathrm{d}x}{y^4}) = \mathcal{C}(\frac{y^{5r}}{y^{5r+4}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}(x^{r}(x^2-v^2)^{r}\,\mathrm{d}x) = 0\,.
\end{aligned}$$

Then $\mathcal{C}(H^0(X,\Omega^1_X)) = 0$ and X is superspecial.

□

In [27], Kudo showed that there is a superspecial non-hyperelliptic curve of genus 4 over k for any odd prime $p \equiv 2 \pmod 3$ by using the description as an intersection of a quadric and a cubic in $\mathbb{P}^3$. Using our approach and considering a curve X of degree 3 together with monodromy vector $a = (1,1,1,1,1,1)$, we can give an alternative proof of Kudo's result.

Proposition 4.8. *[27, Theorem 3.1] There exists a superspecial curve of genus 4 in characteristic $p \equiv 2 \pmod 3$ and hence the superspecial stratum $Z_{[4,3,2,1]}$ in $\mathcal{M}_4$ is non-empty.*

Proof. Consider the monodromy vector $a = (1,1,1,1,1,1)$. Let X be the smooth projective curve of degree 3 with equation

$$y^3 = x(x-\xi)(x-\xi^3)(x-\xi^5)(x-\xi^7) = x^5 + x\,,$$

where $\xi \in k$ is a primitive 8-th root of unity.

By Lemma 4.1, the vector space $H^0(X,\Omega^1_X)$ has a basis consisting of forms $1/y\,\mathrm{d}x$, $1/y^2\,\mathrm{d}x, x/y^2\,\mathrm{d}x$ and $x^2/y^2\,\mathrm{d}x$. Write $p = 3r+2$ with r an odd positive integer. Then

$$\mathcal{C}(\frac{1}{y}\,\mathrm{d}x) = \mathcal{C}(\frac{y^{6r+3}}{y^{6r+3+1}}\,\mathrm{d}x) = \frac{1}{y^2}\mathcal{C}(x^{2r+1}(x^4+1)^{2r+1}\,\mathrm{d}x)\,.$$

Since r is an odd integer, the coefficient of x^{np-1} in $x^{2r+1}(x^4+1)^{2r+1}$ for any $n \in \mathbb{Z}_{>0}$ is zero. Similarly, we have

$$\begin{aligned}
\mathcal{C}(\frac{1}{y^2}\,\mathrm{d}x) = \mathcal{C}(\frac{y^{3r}}{y^{3r+2}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}(x^{r}(x^4+1)^{r}\,\mathrm{d}x) = 0\,,\\
\mathcal{C}(\frac{x}{y^2}\,\mathrm{d}x) = \mathcal{C}(\frac{xy^{3r}}{y^{3r+2}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}(x^{r+1}(x^4+1)^{r}\,\mathrm{d}x) = 0\,,\\
\mathcal{C}(\frac{x^2}{y^2}\,\mathrm{d}x) = \mathcal{C}(\frac{x^2y^{3r}}{y^{3r+2}}\,\mathrm{d}x) = \frac{1}{y}\mathcal{C}(x^{r+2}(x^4+1)^{r}\,\mathrm{d}x) = 0\,.
\end{aligned}$$

Since the coefficients of x^{p-1} in $x^{r+i}(x^4+1)^r$ are zero for $i = 0, 1, 2$.

□

Remark 4.9. Note that Li, Mantovan, Pries and Tang have some results about the Newton polygons of cyclic covers of the projective line and the existence of curves with given Newton polygon, see [30, 31].

Chapter 5

Ekedahl-Oort strata in $\mathcal{M}_4$ in characteristic 3

In this chapter, we study the induced Ekedahl-Oort stratification on $\mathcal{M}_4$ in characteristic $p = 3$. We show that for certain induced Ekedahl-Oort strata in $\mathcal{M}_4$, they have the same codimension in $\mathcal{M}_4$ as the corresponding Ekedahl-Oort strata in $\mathcal{A}_4$. Our strategies to prove this are to construct families of curves with given Ekedahl-Oort type and compute the dimension of the intersection of the induced Ekedahl-Oort strata and the boundary divisor classes.

5.1 Non-emptiness of Ekedahl-Oort strata in $\mathcal{M}_4$ in characteristic 3

Let k be an algebraically closed field of characteristic 3. Recall that we denote by $\mathcal{M}_g$ the moduli space (stack) of (smooth projective) curves of genus g defined over k and by $\overline{\mathcal{M}}_g$ its Deligne-Mumford compactification. We also recall that $\mathcal{Z}_\mu$ (resp. Z_μ) is the Ekedahl-Oort stratum in $\mathcal{A}_g$ (resp. $\mathcal{M}_g$) associated to Ekedahl-Oort type $\mu = [\mu_1, \ldots, \mu_n]$. Write $\overline{\mathcal{Z}_\mu}$ for the Zariski closure of $\mathcal{Z}_\mu$ in $\mathcal{A}_g$. Put $\widetilde{\mathcal{M}}_g := \overline{\mathcal{M}}_g - \Delta_0$. It is an open substack of $\overline{\mathcal{M}}_g$ parametrising stable curves of compact type of genus g. Note that a stable curve is said to be of compact type if the dual graph of the curve is a tree, we refer the definition of the dual graph of a curve to [3]. For $g = 1$, a stable curve of genus 1 of compact type is equivalent to an irreducible smooth elliptic curve. For $g \geq 2$, the Jacobian of a stable curve of genus $g \geq 2$ with a tree as its dual graph is an abelian variety.

The Torelli morphism $\tau : \mathcal{M}_g \mapsto \mathcal{A}_g$ can be extended to a regular map:

$$\tilde{\tau} : \widetilde{\mathcal{M}}_g \mapsto \mathcal{A}_g \,.$$

Since curve with different marked point can have same Jacobian, the map $\tilde{\tau}$ has positive dimensional fibres. It is known that this morphism is proper and its image $\tilde{\tau}(\widetilde{\mathcal{M}}_g)$ is a reduced closed subscheme of $\mathcal{A}_g$, cf. [33]. For $g = 4$, we have that $\dim(\mathcal{M}_4) = 9$ and $\dim(\mathcal{A}_4) = 10$. Moreover, by a work of Igusa [22], the image $\tilde{\tau}(\widetilde{\mathcal{M}}_4)$ is an ample divisor on $\mathcal{A}_4$. Hence $\tilde{\tau}(\widetilde{\mathcal{M}}_4) \cap \mathcal{Z}_\mu$ is of codimension at most $\sum_{i=1}^n \mu_i$ with $\mu = [\mu_1, \ldots, \mu_n]$. Recall that there is a partial order $\preceq$ on the set of Ekedahl-Oort strata. We have $\overline{Z_\mu} \supset \overline{Z_\upsilon}$ if $\mu \preceq \upsilon$. For the Ekedahl-Oort strata in $\mathcal{M}_4$, we have the following result.

Theorem 5.1. *Let $p = 3$. The locus Z_μ is non-empty of codimension $\sum_{i=1}^n \mu_i$ in $\mathcal{M}_4$ if $\mu \preceq [4, 1]$. Moreover, the locus Z_μ is non-empty of codimension at most $\sum_{i=1}^n \mu_i$ for $\mu = [3, 2], [3, 2, 1]$ and $Z_{[4,3]}$ is non-empty of dimension 3.*

Proof. The p-rank part were known by Theorem 1.21 and Corollary 1.22. For $\mu = [2, 1]$, we compute the dimension of $\tilde{\tau}(\Delta_i) \cap \mathcal{Z}_{[2,1]}$ in $\mathcal{A}_4$ for $i = 1, 2$. A curve corresponding to a point in the pull back of a generic point in $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[2,1]}$ is formed from two smooth pointed curves X_1 and X_3 of genus 1 and 3 respectively. Then X_1 is ordinary and X_3 has Ekedahl-Oort type $\mu = [2, 1]$ or X_1 is supersingular and X_3 has Ekedahl-Oort type $\mu = [1]$. In the first case, we consider the action of Verschiebung operator V on the de Rham cohomology. Let $A_2 = H^1_{dR}(X_1), B_6 = H^1_{dR}(X_3)$ and $C_8 = H^1_{dR}(X_1 \times X_3)$. We now compute the final filtration on $H^1_{dR}(X_1 \times X_3)$ as explained in Section 1.3. Recall that we have a pairing $\langle\,,\,\rangle$ on $H^1_{dR}(X)$ for any smooth irreducible projective curve X. We have

$$V(A_2) = A_1,\ V(A_1) = A_1,\ A_1^\perp = A_1\,,$$

where $A_1^\perp$ is the orthogonal complement of A_1 in A with respect to the pairing $\langle\,,\,\rangle$. Similarly, $V(B_6) = B_3, V(B_3) = B_1, V(B_1) = B_1,\ B_1^\perp = B_5$. For a final filtration $0 \subset G_1 \subset G_2 \subset \cdots \subset G_g \subset \cdots \subset G_{2g}$, we have

$$\dim V(G_{2g-n}) = \dim V(G_n) + g - n \tag{5.1}$$

with $0 \leq n \leq g$. Then $\dim V(B_5) = \dim V(B_1) + 3 - 1 = 3$. It follows that $V(B_5) = B_3$ and

$$\begin{aligned} V(C_8) &= V(A_2 + B_6) = A_1 + B_3 = C_4, \\ V(C_4) &= V(A_1 + B_3) = A_1 + B_1 = C_2, \\ V(C_2) &= V(A_1 + B_1) = A_1 + B_1\,. \end{aligned}$$

Furthermore,

$$C_2^{\perp} = A_1 + B_5 = C_6, V(C_6) = V(A_1 + B_5) = A_1 + B_3 = C_4\,.$$

This gives the canonical filtration $0 \subset C_2 \subset C_4 \subset C_6 \subset C_8$ for $C_8 = H^1_{dR}(X_1 \times X_3)$ and the action of V. So the Ekedahl-Oort type is $[2,1]$. Note that the moduli space of pointed elliptic curves is irreducible of dimension 1 and $Z_{[2,1]} \subset \mathcal{M}_3$ is irreducible of dimension 3 by [13, Theorem 11.3]. Hence in this case the component in $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[2,1]}$ is of dimension 4. By a similar argument, in the other case, a generic point also has Ekedahl-Oort type $[2,1]$ and the component of $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[2,1]}$ is non-empty of dimension 5. Hence $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[2,1]}$ is of dimension at most 5.

For $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[2,1]}$, a generic point in $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[2,1]}$ is the union of two pointed curves X_2 and $\check{X}_2$, both of genus 2, where X_2 and $\check{X}_2$ both have 3-rank 1 or X_2 is ordinary and $\check{X}_2$ is superspecial. In the first case, let $A_4 = H^1_{dR}(X_2), B_4 = H^1_{dR}(\check{X}_2)$ and $C_8 = H^1_{dR}(X_2 \times \check{X}_2)$, then we have

$$V(A_4) = A_2,\, V(A_2) = A_1,\, A_1^{\perp} = A_3\,.$$

By equation (5.1), we have $V(A_3) = A_2$. We also have V acts similarly on B_4. Furthermore,

$$\begin{aligned} V(C_8) = V(A_4 + B_4) &= A_2 + B_2 = C_4, \\ V(C_4) = V(A_2 + B_2) &= A_1 + B_1 = C_2, \\ V(C_2) = V(A_1 + B_1) &= A_1 + B_1\,. \end{aligned}$$

Combined with $C_2^{\perp} = A_3 + B_3 = C_6$ and $V(C_6) = A_2 + B_2 = C_4$, we have the canonical filtration $0 \subset C_2 \subset C_4 \subset C_6 \subset C_8$ and the action of V. Hence the generic point has Ekedahl-Oort type $[2,1]$ and the component is of dimension 4. In a similar fashion we can prove in the other case that the component is of dimension 3.

On the other hand, $\tilde{\tau}(\widetilde{\mathcal{M}}_4)$ is ample in $\mathcal{A}_4$ and hence $\tilde{\tau}(\widetilde{\mathcal{M}}_4) \cap \mathcal{Z}_{[2,1]}$ is of dimension at least 6. Furthermore, $\mathcal{Z}_{[2,1]} \subset \mathcal{A}_4$ is irreducible of dimension 7 by [13, Theorem 11.5]. Suppose $Z_{[2,1]} \subset \mathcal{M}_4$ is of dimension 7. By taking the closure in $\mathcal{A}_4$ and by the irreducibility of $\mathcal{Z}_{[2,1]}$, we have $\overline{\mathcal{Z}_{[2,1]}} = \overline{\tilde{\tau}(Z_{[2,1]})}$. Since $\tilde{\tau}(\widetilde{\mathcal{M}}_4)$ is closed in $\mathcal{A}_4$, we have $\overline{\mathcal{Z}_{[2,1]}} \subset \tilde{\tau}(\widetilde{\mathcal{M}}_4)$. This implies $\overline{\mathcal{Z}_{[4,1]}} \subset \overline{\mathcal{Z}_{[2,1]}} \subset \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ with $\overline{\mathcal{Z}_{[4,1]}}$ irreducible of dimension 5 by [13, Theorem 11.3]. Note that we also have $\overline{\mathcal{Z}_{[4,1]}} \subset \overline{\mathcal{Z}_{[4]}}$. Then $\overline{\mathcal{Z}_{[4,1]}} \subset \overline{\mathcal{Z}_{[4]}} \cap \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ with $\dim(\overline{\mathcal{Z}_{[4,1]}}) = \dim(\overline{\mathcal{Z}_{[4]}} \cap \tilde{\tau}(\widetilde{\mathcal{M}}_4)) = 5$ by Theorem 1.21. Then $\overline{\mathcal{Z}_{[4,1]}}$ is an irreducible component of $\overline{\mathcal{Z}_{[4]}} \cap \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ and every abelian variety corresponds to a point of this component of $\overline{\mathcal{Z}_{[4]}} \cap \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ has a-number ≥ 2. Write

$V_0(\widetilde{\mathcal{M}}_4)$ for the locus of curves with p-rank 0 in $\widetilde{\mathcal{M}}_4$. By the proof of Theorem 1.21 (see [16, Theorem 2.3]), a generic point of any irreducible component of the locus $\tilde{\tau}(V_0(\widetilde{\mathcal{M}}_4))$ has a-number 1. This contradiction implies $\overline{\mathcal{Z}_{[2,1]}} \not\subset \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ and $Z_{[2,1]}$ is non-empty of dimension 6.

For $\mu = [3,1]$, similarly we can compute that $\tilde{\tau}(\Delta_i) \cap \mathcal{Z}_{[3,1]}$ has dimension at most 3 for $i = 1, 2$. On the other hand, $\tilde{\tau}(\widetilde{\mathcal{M}}_4) \cap \mathcal{Z}_{[3,1]}$ is of dimension at least 5 as $\tilde{\tau}(\widetilde{\mathcal{M}}_4)$ in $\mathcal{A}_4$ is ample. Moreover, $\mathcal{Z}_{[3,1]} \subset \mathcal{A}_4$ is irreducible of dimension 7 by [13, Theorem 11.5]. Suppose $Z_{[3,1]}$ is of dimension 6. By taking the closure in $\mathcal{A}_4$ and by the irreducibility of $\mathcal{Z}_{[3,1]}$, we have $\overline{\mathcal{Z}_{[3,1]}} = \overline{\tilde{\tau}(Z_{[3,1]})}$. Note that $\overline{\mathcal{Z}_{[4,1]}} \subset \overline{\mathcal{Z}_{[3,1]}}$. By the similar argument in the case $Z_{[2,1]}$, we have $\overline{\mathcal{Z}_{[3,1]}} \not\subset \tilde{\tau}(\widetilde{\mathcal{M}}_4)$ and the locus $Z_{[3,1]}$ is non-empty of dimension 5.

For $\mu = [3,2]$, we show that a smooth curve with affine equation

$$y^3 + y^2 + by = x^5 + a_3x^3 + a_2x^2 + a_0\,, \tag{5.2}$$

where $a_i, b \in k$ and $a_2 \neq 0$ has Ekedahl-Oort type $[3,2]$.

We first show that the map from the parameter space with coordinates (a_3, a_2, a_0, b) to $\mathcal{M}_4$ has finite fibres. Denote by σ an isomorphism between two smooth Artin-Schreier curves given by $y^3 + y^2 + by = x^5 + a_3x^3 + a_2x^2 + a_0$ and $y^3 + y^2 + b_1y = x^5 + c_3x^3 + c_2x^2 + c_0$ as in (5.3). After possibly composing with an inversion $x \mapsto 1/x$, we may assume $\sigma(x) = \alpha x + \beta$. Also since σ is invertible, we have $\sigma(y) = zy + \delta$ with z a unit in k and $\delta \in k(x)$. Hence

$$z^3y^3 + \delta^3 + (zy+\delta)^2 + b(zy+\delta) = (\alpha x+\beta)^5 + a_3(\alpha x+\beta)^3 + a_2(\alpha x+\beta)^2 + a_0\,.$$

Then we have $\delta = \beta = 0$ and $y^3 + y^2/z + by/z^2 = 1/z^3(\alpha^5x^5 + a_3\alpha^3x^3 + a_2\alpha^2x^2 + a_0)$. This implies $\beta = \delta = 0, \alpha^5 = z = 1$.

Now we show that a curve X with equation (5.2) has Ekedahl-Oort type $[3,2]$. A basis of $H^0(X, \Omega^1_X)$ is given by $\omega_1 = 1/(y-b)\,\mathrm{d}x, \omega_2 = x/(y-b)\,\mathrm{d}x, \omega_3 = x^2/(y-b)\,\mathrm{d}x, \omega_4 = 1\,\mathrm{d}x$. Then the Cartier-Manin matrix is given by

$$\begin{pmatrix} a_2 & 0 & a_0 - b^3 & 0 \\ 1 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{1/3}$$

and it has rank 2 and semisimple rank 1. One can compute that curve X has final filtration

$$0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = H^0(X, \Omega^1_X) \subset \cdots \subset G_8 = H^1_{dR}(X)\,,$$

where $G_1 = \langle a_2^{1/3}\omega_1 + \omega_2\rangle, G_2 = \langle a_2\omega_1 + \omega_2, (a_0 - b^3)^{1/3}\omega_1 + a_3^{1/3}\omega_2 + \omega_4\rangle$ and $G_3 = \langle \omega_1, \omega_2, \omega_4\rangle$. Hence the curve has Ekedahl-Oort type $[3, 2]$ and $Z_{[3,2]}$ is non-empty of dimension at least 4.

For $\mu = [3, 2, 1]$, we show that the family of curves given by equation

$$y^3 - bx^3(y^2 + y) = x^5 + cx^3 + dx^2 + 1,\ b, c, d \in k,\ bd \neq 0 \tag{5.3}$$

has Ekedahl-Oort type $[3, 2, 1]$.

In a similar fashion, we first show that the map from the parameter space with coordinates (b, c, d) to $\mathcal{M}_4$ has finite fibres. Denote by σ an isomorphism between two smooth Artin-Schreier curves given by $y^3 - bx^3(y^2 + y) = x^5 + cx^3 + dx^2 + 1$ and $y^3 - b_1x^3(y^2 + y) = x^5 + c_1x^3 + d_1x^2 + 1$ as in (5.3). After possibly composing with an inversion $x \mapsto 1/x$, we may assume $\sigma(x) = \alpha x + \beta$. Also since σ is invertible, we have $\sigma(y) = zy + \delta$ with z a unit in k and $\delta \in k(x)$. Hence

$$z^3y^3 + \delta^3 + b(\alpha x + \beta)^3(z^2y^2 + (2\delta + z)y + \delta^2 + \delta) = \sigma(x)^5 + c\sigma(x)^3 + d\sigma(x)^2 + 1\,.$$

By comparing the coefficients of x^i for $i = 0, 1, 2, 3, 4, 5$, we have $\beta = \delta = 0$ and $z = 1$. Furthermore, we have $y^3 - b\alpha^3x^3(y^2 + y) = \alpha^5x^5 + \alpha^2x^3 + dx^2 + 1$ and hence $\alpha^5 = 1$. This implies $\beta = \delta = 0, \alpha^5 = z = 1$.

Now we compute the Ekedahl-Oort type of a curve X given by equation (5.3). For a basis of $H^0(X, \Omega^1_X)$ we choose

$$1/s(x, y)\,\mathrm{d}x, x/s(x, y)\,\mathrm{d}x, x^2/s(x, y)\,\mathrm{d}x, y/s(x, y)\,\mathrm{d}x$$

with $s(x, y) = x^3(y - 1)$. Then the Cartier-Manin matrix has rank 1 and semisimple rank 1. Hence $Z_{[3,2,1]}$ is non-empty with dimension at least 3.

For $\mu = [4, 1]$, we have $\mathcal{Z}_{[4,1]}$ is irreducible by [13, Theorem 11.5]. For $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[4,1]}$, a curve corresponding to a point in the pull back of a generic point in $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[4,1]}$ is formed from two pointed curves X_1 and X_3 of genus 1 and 3 respectively. Moreover, X_1 is superspecial and X_3 has Ekedahl-Oort type $\mu = [3]$. One can easily compute that in this case the curve formed from pointed curves X_1 and X_3 has Ekedahl-Oort type $[4, 2]$ and $\tilde{\tau}(\Delta_1) \cap \mathcal{Z}_{[4,1]}$ is of dimension 3. For $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[4,1]}$, a point in the pull back of a generic point in $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[4,1]}$ is formed from two pointed curves with Ekedahl-Oort type $\mu = [2]$. Then one can compute that a generic point of $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[4,1]}$ has Ekedahl-Oort type $[4, 3]$ and $\tilde{\tau}(\Delta_2) \cap \mathcal{Z}_{[4,1]}$ is of dimension 2.

On the other hand, $\tilde{\tau}(\widetilde{\mathcal{M}}_4) \cap \mathcal{Z}_{[4,1]}$ is of dimension at least 4. Hence $Z_{[4,1]}$ is of dimension at least 4. Note that if $Z_{[4,1]}$ is of dimension 5. Then by taking the closure in $\mathcal{A}_4$ and by the irreducibility of $\mathcal{Z}_{[4,1]}$, we have $\overline{\mathcal{Z}_{[4,1]}} \subset \tilde{\tau}(\widetilde{\mathcal{M}}_4)$, which is a

contradiction by the proof of the case $Z_{[2,1]}$. Hence $Z_{[4,1]}$ is non-empty of dimension 4.

For $\mu = [4,2]$ and $[4,3]$, we have $\mathcal{H}_4 \cap Z_\mu$ non-empty hence Z_μ is non-empty by Theorem 3.1.

Denote $\theta(x) = x^3 + x^2 + b_1 x + b_2$ and $h(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x$. We consider the family of curves parametrized by equation

$$y^3 + y^2 + \theta(x) y = h(x)\,, \tag{5.4}$$

where $a_i, b_j \in k$ such that $b_2 = b_1^2 + a_2,\ a_4 = b_1 - 1$ and $a_3 = b_1^3 + b_1^2 + b_1 + a_2$.

Note that the map from the parameter space to the $\mathcal{M}_4$ is finite and hence it gives a 3-dimensional sublocus in $\mathcal{M}_4$. Indeed, denote by σ an isomorphism between two smooth Artin-Schreier curves given by $\theta_1(x), h_1(x)$ and $\theta_2(x), h_2(x)$ as in (5.4). After possibly composing with an inversion $x \mapsto 1/x$, we may assume $\sigma(x) = \alpha x + \beta$. Also since σ is invertible, we have $\sigma(y) = zy + \delta$ with z a unit in k and $\delta \in k(x)$. Then one can easily show that $z = \alpha = 1,\ \beta^3 + \beta^2 = 0$ and $-(\delta^3 + \delta^2 + b_2\delta) + h_1(\beta) = 0$ with $\theta_1(x) = x^3 + x^2 + b_1 x + b_2$ and $h(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x$.

Indeed, denote by σ an isomorphism between two smooth Artin-Schreier curves given by $y^3 - y = f_1(x) = \sum_{i=0} a_i x^i$ and $y^3 - y = f_2(x) = \sum_{i=0} b_i x^i$ as in (5.5). By [39, Lemma 2.1.5], two Artin-Schreier curves are isomorphic if and only if $f_2(x) = z f_1(x) + \delta^3 - \delta$ with $z \in \mathbb{F}_3^*$ and $\delta \in k(x)$. Moreover, the σ is defined by $\sigma(x) = \alpha x + \beta$ and $\sigma(y) = zy + \delta$. Hence we have $\beta = 0, \alpha^5 = z \in \mathbb{F}_3^*$ and $\delta \in \mathbb{F}_3$.

Now we show that any smooth curve X given by equation (5.4) above has Ekedahl-Oort type $[4,3]$. For a basis of $H^0(X, \Omega^1_X)$, we choose

$$1/s(x,y)\,\mathrm{d}x, x/s(x,y)\,\mathrm{d}x, x^2/s(x,y)\,\mathrm{d}x, y/s(x,y)\,\mathrm{d}x$$

with $s(x,y) = y - (x^3 + x^2 + b_1 x + b_2)$. For the Cartier operator $\mathcal{C}$ we have

$$\begin{aligned}\mathcal{C}\Big(\frac{1}{s(x,y)}\,\mathrm{d}x\Big) &= \mathcal{C}\Big(\frac{1}{y - \theta(x)}\Big) = \mathcal{C}\Big(\frac{(y-\theta(x))^2}{(y-\theta(x))^3}\Big) \\ &= \frac{1}{y - \theta(x)} \mathcal{C}(y^2 + \theta^2(x) + \theta(x) y)\,.\end{aligned}$$

Note that $y^2 + \theta(x)y = -y^3 + h(x)$ and $\theta^2(x) = x^6 + 2x^5 + (2b_1 + 1)x^4 + (2b_1 + 2b_2)x^3 + (b_1^2 + 2b_2)x^2 + 2b_1 b_2 x + b_2^2$. Hence

$$\begin{aligned}y^2 + \theta^2(x) + \theta(x)y = -y^3 + x^6 + (2b_1 + 1 + a_4)x^5 + (2b_1 + 2b_2 + a_3)x^3 \\ + (b_1^2 + 2b_2 + a_2)x^2 + (2b_1 b_2 + a_1)x + b_2^2\,,\end{aligned}$$

and we have

$$\begin{aligned}\mathcal{C}(\frac{1}{s(x,y)}\,\mathrm{d}x) &= \frac{1}{y-\theta(x)}\mathcal{C}(y^2+\theta^2(x)+\theta(x)y)\\ &= \frac{(2b_2+b_1^2+a_2)^{1/3}}{y-\theta(x)} = (2b_2+b_1^2+a_2)^{1/3}\frac{1}{s(x,y)}\,\mathrm{d}x\,.\end{aligned}$$

Similarly one can compute $\mathcal{C}(x/s(x,y)), \mathcal{C}(x^2/s(x,y))$ and $\mathcal{C}(y/s(x,y))$. Then we have the Cartier-Manin matrix which equals to

$$\begin{pmatrix} 2b_2+b_1^2+a_2 & 2b_1b_2+a_1 & b_2^2 & b_2^2+b_1(2b_1b_2+a_1)\\ 0 & 2b_1+1+a_4 & 2b_1+2b_2+a_3 & 2b_1+2b_2+a_3\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & -1 \end{pmatrix}^{1/3}$$

$$= \begin{pmatrix} 0 & 2b_1b_2+a_1 & b_2^2 & b_2^2+b_1(2b_1b_2+a_1)\\ 0 & 0 & b_1^3 & b_1^3\\ 0 & 0 & 1 & 1\\ 0 & 0 & -1 & -1 \end{pmatrix}^{1/3}.$$

We obtain that $\mathrm{rank}(\mathcal{C}) = 2$ and $\mathcal{C}^2 = 0$ on $H^0(X, \Omega_X^1)$. Hence X with equation (5.4) has Ekedahl-Oort type $[4,3]$ and $Z_{[4,3]}$ is of dimension 3.

For $\mu = [4,3,2,1]$, a curve corresponding to a point in $\mathcal{Z}_{[4,3,2,1]}$ is superspecial. Then $\mathcal{Z}_{[4,3,2,1]}$ is empty as Ekedahl [12] showed that any superspecial curve of genus g satisfies :

$$g \le 3(3-1)/2 = 3\,.$$

□

Remark 5.2. As an example, a family of curves given by the equation

$$y^3 + bx^2y = x^5 + ax^4 + x,\ a, b \in k, b \neq 0$$

has Ekedahl-Oort type $[2,1]$. Moreover, the family of curves parametrized by equation

$$y^3 - y = x^5 + a_2x^2 + a_1x,\ a_1, a_2 \in k \tag{5.5}$$

gives a 2-dimensional locus in $Z_{[4,3]}$. On the other hand, the family of curves parametrized by equation

$$y^3 + y^2 + (x^3+x^2)y = x^5 + 2x^4 + a_1x + a_2\,, \tag{5.6}$$

where $a_1, a_2 \in k$ and $a_1 \neq 0$, gives another 2-dimensional locus in $Z_{[4,3]}$. Consider a curve with equation (5.6) above, with respect to the basis given by

$$1/s(x,y)\,\mathrm{d}x, x/s(x,y)\,\mathrm{d}x, x^2/s(x,y)\,\mathrm{d}x, y/s(x,y)\,\mathrm{d}x$$

with $s(x,y) = y + 2x^3 + 2x^2$, one can easily compute the Cartier-Manin matrix which equals to

$$\begin{pmatrix} 0 & a_1 & a_2 & a_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}^{1/3}.$$

Then the rank and semisimple rank of the Cartier-Manin matrix is 2 and 0 respectively and the curve with equation (5.6) has Ekedahl-Oort type $[4, 3]$.

Remark 5.3. Note that Pries [40, Theorem 4.3] showed that there is a family (with dimension 6) of smooth curves of genus 4 with p-rank 1 and a-number 1. Moreover, if $p \geq 5$, then by [40, Corollary 4.5] there is a family (with dimension 6) of smooth curves of genus 4 with p-rank 2 and a-number 2. Note also Achter and Pries [1] have some result about generic Newton polygon of curves of given genus and p-rank.

Chapter 6

On the existence of Artin-Schreier curves with prescribed a-number

In this chapter we study the existence of Artin-Schreier curves in characteristic $p > 0$ with prescribed a-number. More precisely, we describe Artin Schreier curves genus $g \geq 2$ with a-number equals to $g, g-1$ and $g-2$. Our main tool is the Cartier operator. Moreover, we give a basis of de Rham cohomology for Artin-Schreier curves and use it to study the supersingularity of Artin-Schreier curves.

6.1 Superspecial Artin-Schreier curves

Let k be an algebraically closed field of characteristic $p > 0$. By an Artin-Schreier curve we mean a smooth irreducible projective curve defined over k which is a $\mathbb{Z}/p\mathbb{Z}$-Galois cover of $\mathbb{P}^1$. Such an Artin-Schreier curve can be written as $y^p - y = f(x)$, where $f(x) \in k(x) \backslash k$ is a rational function on $\mathbb{P}^1$.

Now let $\mathrm{char}(k) = p \geq 3$. For superspecial Artin-Schreier curves we have the following result due to Irokawa and Sasaki [23].

Theorem 6.1. *Let k be an algebraically closed field of* $\mathrm{char}(k) = p \geq 3$. *Let X be an superspecial Artin-Schreier curve with equation $y^p - y = f(x)$, where $f(x) \in k[x]$ and* $\deg f(x) = d \geq 2$ *with* $\gcd(p, d) = 1$. *Then X is isomorphic to a curve given by $y^p - y = x^d$ with $d | p + 1$.*

Proof. Since X is superspecial, the Cartier operator $\mathcal{C}$ on $H^0(X, \Omega^1_X)$ is zero. Moreover, the curve is of genus $g = (p-1)(d-1)/2$ with $d \leq p+1$ by Ekedahl [12]. There will be two parts:

i): We show that if $d < p+1$, then X can be written as

$$y^p - y = a_d x^d, \text{ where } d|p+1,\ a_d \neq 0\,.$$

ii): We show that if $d = p+1$, then X is isomorphic to curve with equation

$$y^p - y = a_d x^d, \text{ where } a_d \neq 0\,.$$

Then by scaling we can get $a_d = 1$. By [47, Lemma 1], a basis for $H^0(X, \Omega^1_X)$ is given by forms:

$$B = \{x^i y^j \,\mathrm{d}x | i, j \in \mathbb{Z}_{\geq 0}, ip + jd \leq (p-1)(d-1) - 2\}$$

with the genus $g = (p-1)(d-1)/2$. Since $y = y^p - f(x)$, we have

$$x^i y^j \,\mathrm{d}x = x^i (y^p - f(x))^j \,\mathrm{d}x = \sum_{t=0}^{j} (-1)^t \binom{j}{t} x^i f^t(x) (y^{j-t})^p \,\mathrm{d}x.$$

By assumption, we have $\mathcal{C}(\omega) = 0$ for any $\omega \in H^0(X, \Omega^1_X)$. This implies $\mathcal{C}(x^i y^j \,\mathrm{d}x) = 0$ for any $i, j \in \mathbb{Z}_{\geq 0}$ such that $ip + jd \leq (p-1)(d-1) - 2$. Note that

$$\begin{aligned}\mathcal{C}(x^i y^j \,\mathrm{d}x) &= \mathcal{C}(\sum_{t=0}^{j} \binom{j}{t} (-1)^t x^i f^t(x) (y^{j-t})^p \,\mathrm{d}x) \\ &= \sum_{t=0}^{j} \binom{j}{t} (-1)^t (y^{j-t}) \mathcal{C}(x^i f^t(x) \,\mathrm{d}x)\,.\end{aligned}$$

It follows that $\mathcal{C}(x^i f^t(x) \,\mathrm{d}x) = 0$ for any $0 \leq t \leq j$ such that $p \nmid \binom{j}{t}$.

Now we prove case *i*). Since $\gcd(d, p) = 1$, we obtain $d \leq p-1$. If $d = 2$, by a change of coordinates we may assume that $f(x) = a_2 x^2 + a_0$ since $p \neq 2$. If $2 < d \leq p-1$, there always exist $l \in \mathbb{Z}_{>0}$ such that $ld < p < (l+1)d$. Then we have $ld \leq p-1 < (l+1)d$. Now let $a = p-1-ld$, then $0 \leq a \leq d-2$. Suppose $0 \leq a \leq d-3$, we show that $x^{i_0} y^{j_0} \,\mathrm{d}x \in B$ with $i_0 = a$ and $j_0 = l$. Indeed, since $ld \leq p-1, d < p$ and hence $(l+1)d + 1 - 2p < 0$, we have

$$\begin{aligned}ap + ld &\leq (d-3)p + ld \\ &= dp - p - d - 1 + (l+1)d - 2p + 1 \\ &\leq dp - p - d - 1 = (d-1)(p-1) - 2\,.\end{aligned}$$

Then we have $\mathcal{C}(x^a y^l \,\mathrm{d}x) = \sum_{t=0}^{l} (-1)^t \binom{l}{t} (y^{l-t}) \mathcal{C}(x^a f^t(x) \,\mathrm{d}x) = 0$. By an appropriate change of coordinates, we may assume

$$f(x) = a_d x^d + a_{d-2} x^{d-2} + \cdots + a_1 x\,.$$

On the other hand, by taking $t = l$, we have $x^a f^l(x)\,\mathrm{d}x = (a_d^l x^{ld+a} + \cdots)\,\mathrm{d}x = (a_d^l x^{p-1} + \cdots)\,\mathrm{d}x$, which implies $\mathcal{C}(x^a f^l(x)\,\mathrm{d}x) \neq 0$, a contradiction. Thus we have $a = d-2$ and $p+1 = ld+a+2 = (l+1)d$, which indicates $d \mid p+1$.

Now let $1 \leq n \leq d-2$ and we show that $a_n = 0$ for every $1 \leq n \leq d-2$ in $f(x)$. Note that we have $2 < d \leq p-1$ and $p+1 = (l+1)d$. This implies $np-d-2 \geq p-d-2 \geq ld-3 \geq d-3 \geq 0$. We show that $x^{d-2-n}y^{l+1}\,\mathrm{d}x \in B$. Note that we have $0 \leq d-2-n \leq d-3$. Then we have

$$\begin{aligned}(d-2-n)p + (l+1)d &= dp - p - d - 1 + (l+2)d - (n+1)p + 1 \\ &= (d-1)(p-1) - 2 + d + 2 - np \\ &\leq (d-1)(p-1) - 2\,.\end{aligned}$$

Hence $x^{d-2-n}y^{l+1}\,\mathrm{d}x \in B$.

Since $\mathcal{C} = 0$, we have $\mathcal{C}(x^{d-2-n}y^{l+1}\,\mathrm{d}x) = 0$ for any $1 \leq n \leq d-2$. Note that this implies $\mathcal{C}(x^{d-2-n}f^{l+1}(x)\,\mathrm{d}x) = 0$. Additionally, $x^{d-2-n}f^{l+1}(x) = x^{d-2-n}(a_d x^d + \cdots + a_n x^n + \cdots + a_0)^{l+1}$. Put $n = d-2$ and we have the coefficient of $x^{p-1} = x^{ld+d-2}$ in $x^{d-2-n}f^{l+1}(x)$ is $(l+1)a_d^l a_{d-2}$. Since $p+1 = (l+1)d$ we have $p \nmid (l+1)$. Then $\mathcal{C} = 0$ implies $a_{d-2} = 0$. We now do induction on n. Assume we have $a_{n+1} = a_{n+2} = \cdots = a_{d-2} = 0$ in $f(x)$ for some $n \geq 1$. Note that the coefficient of $x^{p-1} = x^{ld+d-2}$ in $x^{d-2-n}f^{l+1}(x)$ is $(l+1)a_d^l a_n$ and hence we have $a_n = 0$. Then we have $a_t = 0$ for any $1 \leq t \leq d-2$ and $f(x) = a_d x^d$ with d a divisor of $p+1$.

Now we prove case *ii*). If $d = p+1$, for $2 \leq n \leq d-2$, we check that $x^{d-2-n}y\,\mathrm{d}x \in B$. Indeed, one has $0 \leq d-2-n \leq d-4$ and

$$\begin{aligned}(d-2-n)p + d &= (p+1-2-n)p + p + 1 \\ &= p^2 - p - 2 + (1-n)p + 3 \\ &\leq p^2 - p - 2 = (d-1)(p-1) - 2\end{aligned}$$

as $(1-n)p+3 \leq 3-p \leq 0$. Hence $\mathcal{C}(x^{d-2-n}y\,\mathrm{d}x) = 0$. This implies $\mathcal{C}(x^{d-2-n}f(x)\,\mathrm{d}x) = 0$. By a similar argument as in part (*i*), we have $\mathcal{C}(x^{d-2-n}a_n x^n\,\mathrm{d}x) = 0$ and $a_n = 0$ for any $2 \leq n \leq d-2$. Then we obtain $f(x) = a_{p+1}x^{p+1} + a_1 x + a_0 \in k[x]$. By a change of coordinates $y \mapsto y + b_0 + b_1 x, x \mapsto x + \alpha$, where $b_0, b_1, \alpha \in k$ such that $b_0^p - b_0 = a_{p+1}\alpha^{p+1} + a_1\alpha + a_0$, $b_1^p = a_{p+1}\alpha$ and $b_1 = a_{p+1}\alpha^p + a_1$, we may assume $a_0 = a_1 = 0$, which ends the proof.

□

6.2 Artin-Schreier curves with a-number $g-1$ and de Rham cohomology

We now treat the case that the a-number of a curve X is equal to $g-1$ with g the genus of X.

Theorem 6.2. *Let k be an algebraically closed field with $\mathrm{char}(k) = p > 0$. Let X be an Artin-Schreier curve of genus $g \geq 1$ with equation $y^p - y = f(x)$, where $f(x) \in k(x)$. Assume that the a-number of X is $g-1$. Then*
(1) : if $p = 2$, we have $g \leq 3$. Moreover, the curve X can be either written as

$$y^2 + y = f(x)\,,$$

where $f(x) \in k[x]$ and $\deg f(x) = 5$ or 7, or written as $y^2 + y = f_0(x) + 1/x$ with $\deg f_0(x) = 1$ or 3 and $f_0(x) \in xk[x]$;
(2) : if $p \geq 3$, then $g \leq (p-1)p/2$. More precisely, X is isomorphic to a curve with equation

$$y^p - y = x^d + a_{d-2}x^{d-2} + \cdots + a_1 x,$$

where $d|p+1$. Moreover if $d = p+1$, then at least one of a_i with $2 \leq i \leq d-2$ is non-zero. If $d < p+1, d \mid p+1$, then at least one of a_i with $1 \leq i \leq d-2$ is non-zero.

Proof. Suppose that $f(x)$ has poles at $\infty, Q_1, \ldots, Q_m$ for some $m \in \mathbb{Z}_{\geq 0}$. Let $x - \xi_i$ be a local parameter at Q_i. Write $x_i = 1/(x-\xi_i)$ for $i = 1, \ldots, m$ and $x_0 = x$. Then $f(x)$ can be written as

$$f(x) = f_0(x) + \sum_{i=1}^{m} f_i(1/(x-\xi_i)) = \sum_{i=0}^{m} f_i(x_i)\,, \tag{6.1}$$

where $\deg f_i(x) = d_i$. By [47, Lemma 1], a basis of $H^0(X, \Omega^1_X)$ is given by $B = \cup_{s=0}^{m} B_s$ where

$$B_0 = \{x^i y^j \,\mathrm{d}x | i, j \in \mathbb{Z}_{\geq 0},\ ip + jd \leq (p-1)(d_0 - 1) - 2\},$$
$$B_s = \{x_s^i y^j \,\mathrm{d}x |\ i \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\geq 0},\ ip + jd \leq (p-1)(d_s+1)\}, s = 1, \ldots, m\,.$$

The condition $a(X) = g-1$ is equivalent to the rank of the Cartier operator $\mathrm{rank}(\mathcal{C})$ is equal to 1. Note that if $f(x) = \sum_{i=0}^{m} f_i(x_i)$ as in (6.1), we always have $x_s \,\mathrm{d}x \in B$ for $1 \leq s \leq m$. Note that $\mathcal{C}(x_s \,\mathrm{d}x) \neq 0$ and we get $\mathrm{rank}(\mathcal{C}) \geq m$.

(1): For $p = 2$, we consider distinct cases for $f(x)$. If $f(x) \in k[x]$ with $\deg f(x) = d$, then a basis of $H^0(X, \Omega^1_X)$ is

$$\begin{aligned} B &= \{x^i y^j \,\mathrm{d}x |\ i, j \in \mathbb{Z}_{\geq 0}, 2i + jd \leq d - 3\} \\ &= \{x^i \,\mathrm{d}x |\ i \in \mathbb{Z}_{\geq 0}, 2i \leq d - 3,\ d \geq 3\}. \end{aligned}$$

Since $\mathrm{rank}(\mathcal{C}) = 1$, we must have $x \,\mathrm{d}x \in B$ and hence $d \geq 5$.

Suppose $d \geq 9$, we have $x \,\mathrm{d}x$, $x^3 \,\mathrm{d}x \in B$ and hence $\mathcal{C}(x^i \,\mathrm{d}x) = x^{(i-1)/2} \,\mathrm{d}x$ for $i = 1, 3$, a contradiction since $\mathrm{rank}(\mathcal{C}) = 1$. Now if $d = 5$ (resp. 7), then $B = \{\mathrm{d}x, x \,\mathrm{d}x\}$ (resp. $B = \{\mathrm{d}x, x \,\mathrm{d}x, x^2 \,\mathrm{d}x\}$) with $\mathcal{C}(\mathrm{d}x) = \mathcal{C}(x^2 \,\mathrm{d}x) = 0$ and $\mathcal{C}(x \,\mathrm{d}x) = \mathrm{d}x \neq 0$. Hence we have $\mathrm{rank}(\mathcal{C}) = 1$ for both cases.

Now if $f(x) = f_0(x_0) + f_1(x_1)$ with $m = 1$ as in (6.1), we have $\deg f_0(x_0) = d_0 \leq 3$ and $\deg f_1(x_1) = d_1 = 1$. By putting a pole at 0 and by scaling, we arrive at $f_1(x_1) = x_1$. Then the curve X can be written as $y^2 + y = f_0(x) + 1/x$ with $\deg f_0(x) \leq 3$. Again by a change of coordinates $y \mapsto y + b_0$ with $b_0 \in k$ and $b_0^2 + b_0 = f_0(0)$ we get $f_0(x) \in xk[x]$.
(2): If $p \geq 3$, we show the following:
(a): For all $p \geq 3$, then $d \leq p + 1$;
(b): For $p = 5$ and $d = 4$, then $\mathrm{rank}(\mathcal{C}) \geq 2$;
(c): For $p \geq 7$ and $d \geq 3$ with $d \nmid p + 1$, then $\mathrm{rank}(\mathcal{C}) \geq 2$.
Then after excluding the cases where $\mathrm{rank}(\mathcal{C}) \geq 2$ and the cases where $\mathrm{rank}(\mathcal{C}) = 0$, what is left are curves with a-number $g - 1$. Note that if $d = 2$ and $f(x) \in k[x]$, the curve with equation $y^p - y = f(x)$ is superspecial.

By a change of coordinates, we may assume

$$f(x) = x^d + a_{d-2}x^{d-2} + \cdots + a_1 x + a_0,\ d \geq 3\,.$$

(a): If $d \geq p + 2$, then by definition we have $x^{p-1} \,\mathrm{d}x \in B$. There exists $l, b \in \mathbb{Z}_{\geq 0}$ such that $d = lp + b$ with $l = 1$ and $2 \leq b \leq p - 1$ or $l \geq 2$ and $1 \leq b \leq p - 1$. One can show $x^{p-1-b} y \,\mathrm{d}x \in B$ by checking $(p - 1 - b)p + d \leq (p - 1)(d - 1) - 2$. Then

$$\begin{aligned} \mathcal{C}(x^{p-1-b} y \,\mathrm{d}x) &= \mathcal{C}(x^{p-1-b}(y^p - f(x)) \,\mathrm{d}x) \\ &= \mathcal{C}(x^{p-1-b} y^p \,\mathrm{d}x - x^{p-1-b} f(x) \,\mathrm{d}x) \\ &= y\mathcal{C}(x^{p-1-b} \,\mathrm{d}x) - \mathcal{C}(x^{p-1-b} f(x) \,\mathrm{d}x) \neq 0 \end{aligned}$$

as the leading term of $x^{p-1-b} f(x)$ is x^{lp+p-1}. This contradiction shows that $d \leq p+1$.
(b): For $p = 5$, by (a) we have $d \leq p + 1$. Suppose $d \nmid p + 1$, we have $d = 4$ and $y^1 \,\mathrm{d}x, y^2 \,\mathrm{d}x \in B$. Additionally, we have $\mathcal{C}(y^i \,\mathrm{d}x) = y^{i-1} \,\mathrm{d}x$ for $i = 1, 2$ and hence $\mathrm{rank}(\mathcal{C}) \geq 2$, a contradiction. We therefore have $d | p + 1$.

(c): For $p \geq 7$ and $d \leq p+1$, assume we have $d \nmid p+1$. Then there exists $l \in \mathbb{Z}_{>0}$ such that $ld \leq p \leq (l+1)d$. Furthermore, we have $ld \leq p-1$ and $(l+1)d \geq p+2$ as $\gcd(d,p) = 1$ and $d \nmid p+1$. Then there exists b' satisfying $ld + b' = p-1$ for $0 \leq b' \leq d-3$.

If $d = p-1$, then $l = 1$, $b' = 0$, we get $y\,\mathrm{d}x,\ y^2\,\mathrm{d}x \in B$ and $\mathcal{C}(y^i\,\mathrm{d}x) = y^{i-1}\,\mathrm{d}x$ for $i = 1, 2$. This implies $\mathrm{rank}(\mathcal{C}) \geq 2$, a contradiction. If $d = p-2$, $l = 1$ and $b' = 1$, then we have $i(p-1) \leq (p-1)(p-2)-2$, which implies $xy\,\mathrm{d}x,\ xy^2\,\mathrm{d}x \in B$. Then $\mathcal{C}(xy\,\mathrm{d}x)$ and $\mathcal{C}(xy^2\,\mathrm{d}x)$ are linearly independent and hence $\mathrm{rank}(\mathcal{C}) \geq 2$. Now if $d \leq p-3$, we show that $x^{b'}y^l\,\mathrm{d}x \in B$. This is equivalent to showing $ld + b'p \leq (p-1)(d-1) - 2$. By substituting b' with $b' = p-1-ld$ in the inequality, we only need to show $d(l+1)(p-1) \geq p^2+1$, which is clear since $(l+1)d \geq p+2$.

Now we show that $x^{b'}y^l\,\mathrm{d}x, x^{b'}y^{l+1}\,\mathrm{d}x \in B$. It suffices to show $ld + d + b'p \leq (p-1)(d-1) - 2$. We have $ld + d + b'p \leq (p-1) - b' + d + b'p$ as $b' = p-1-ld$. Hence we only need to show

$$d \leq (d - b' - 2)(p-1) - 2\,. \tag{6.2}$$

Note that $b' \leq d-3$, we have $(d-b'-2)(p-1) - 2 \geq p-3 \geq d$. Then $x^{b'}y^l\,\mathrm{d}x, x^{b'}y^{l+1}\,\mathrm{d}x \in B$ and

$$\mathcal{C}(x^{b'}y^j\,\mathrm{d}x) = \sum_{t=0}^{j}(-1)^t\binom{j}{t}(y^{l-t})\mathcal{C}(x^{b'}f^t(x)\,\mathrm{d}x) = 0,\ j = l, l+1\,.$$

Put $t = l$, then $\mathcal{C}(x^{b'}f^l(x)) = \mathcal{C}(x^{b'+ld} + \cdots)\,\mathrm{d}x) \neq 0$, which implies $\mathrm{rank}(\mathcal{C}) \geq 2$. Therefore we have $d|p+1$.

□

For Artin-Schreier curves for $p \geq 3$ with a-number $g-1$ we have the following result.

Theorem 6.3. *Let k be an algebraically closed field with $\mathrm{char}(k) = p \geq 3$. Let X be an Artin-Schreier curve of genus $g > 0$ with equation $y^p - y = f(x)$, where $f(x) \in k[x]$ and $\deg f(x) = d$. If $d|p+1$ and $a(X) = g-1$, then either $p = 5, d = 3$ and X is isomorphic to a supersingular curve of genus 4 with equation:*

$$y^5 - y = x^3 + a_1 x,\ a_1 \in k^*\,, \tag{6.3}$$

or $p = 3, d = 4$ and X is isomorphic to a supersingular curve of genus 3 with equation:

$$y^3 - y = x^4 + a_2 x^2, a_2 \in k^*\,. \tag{6.4}$$

Moreover, the family of curves given by equation (6.3) *(resp.* (6.4)*) defines an irreducible component of* $Z_{[4,3,2]}$ *(resp.* $Z_{[3,2]}$*) in* $\mathcal{M}_4$ *(resp.* $\mathcal{M}_3$*) in characteristic* 5 *(resp.* 3*).*

Before giving the proof of theorem, we introduce a basis of de Rham cohomology of an Artin-Schreier curve. Let X be an Artin-Schreier curve over k of genus g with equation

$$y^p - y = h(x)\,, \tag{6.5}$$

where $h(x) \in k[x]\backslash k$ is non-zero of degree d. Let $\pi : X \to \mathbb{P}^1$ be the $\mathbb{Z}/p$-cover. Put $U_1 = \pi^{-1}(\mathbb{P}^1 - \{0\})$ and $U_2 = \pi^{-1}(\mathbb{P}^1 - \{\infty\})$. For the open affine cover $\mathcal{U} = \{U_1, U_2\}$, we consider the de Rham cohomology $H^1_{dR}(X)$ as in Section 1.2, i.e.

$$H^1_{dR}(X) = Z^1_{dR}(\mathcal{U})/B^1_{dR}(\mathcal{U})$$

with

$$Z^1_{dR}(\mathcal{U}) = \{(t, \omega_1, \omega_2) | t \in \mathcal{O}_X(U_1 \cap U_2), \omega_i \in \Omega^1_X(U_i), \mathrm{d}t = \omega_1 - \omega_2\}$$

and $B^1_{dR}(\mathcal{U}) = \{(t_1 - t_2, \mathrm{d}t_1, \mathrm{d}t_2) | t_i \in \mathcal{O}_X(U_i)\}$.

Under the action of Verschiebung operator V on $H^1_{dR}(X)$, one has $V(H^1_{dR}(X)) = H^0(X, \Omega^1_X)$ and V coincides with the Cartier operator on $H^0(X, \Omega^1_X)$.

For $1 \leq i \leq g$, put $s(x) = xh'(x)$ with $h'(x)$ the formal derivative of $h(x)$ and write $s(x) = s^{\leq i}(x) + s^{>i}(x)$ with $s^{\leq i}(x)$ the sum of monomials of degree $\leq i$. Then we have the following proposition.

Proposition 6.4. *Let* X *be an Artin-Schreier curve over* k *with equation* $y^p - y = h(x)$*, where* $h(x) \in k[x]$ *and* $\deg h(x) = d$*. Then* $H^1_{dR}(X)$ *has a basis with respect to* $\mathcal{U} = \{U_1, U_2\}$ *consisting of the following residue classes with representatives in* $Z^1_{dR}(\mathcal{U})$*:*

$$\alpha_{i,j} = [(0, x^i y^j \,\mathrm{d}x, x^i y^j \,\mathrm{d}x)], \tag{6.6}$$

$$\beta_{i,j} = [(\frac{y^{p-1-j}}{x^{i+1}}, -\frac{\phi_{i,j}(x,y)}{x^{i+2}}\mathrm{d}x, \frac{(p-1-j)s^{>i+2}(x)y^{p-2-j}}{x^{i+2}}\mathrm{d}x)], \tag{6.7}$$

where $i, j \in \mathbb{Z}_{\geq 0}, pi + jd \leq (p-1)(d-1) - 2$ *and* $\phi_{i,j}(x,y) = (p-1-j)s^{\leq i+2}(x)y^{p-2-j} + (i+1)y^{p-1-j}$*.*

Proof. Clearly, $\omega_{i,j} = x^i y^j \,\mathrm{d}x$ form a basis of $H^0(X, \Omega^1_X)$ for $i, j \in \mathbb{Z}_{\geq 0}$ with $pi + dj \leq (p-1)(d-1) - 2$. On the other hand, we may identify $\mathcal{O}_X(U_2)$ with the k-algebra

$k[x,y]$ defined by (6.5). Moreover, x^iy^j with $i \geq 0, 0 \leq j \leq p-1$ form a basis of the image of $\mathcal{O}_X(U_2)$ in $\mathcal{O}_X(U_1 \cap U_2)$. Additionally, we have $x^iy^j \in \mathcal{O}_X(U_1)$ for $0 \leq j \leq p-1$ and $-pi \geq dj$. Then the residue classes $[x^iy^j]$ form a basis of $H^1(X, \mathcal{O}_X)$ for $i < 0, 0 \leq j \leq p-1$ and $-pi-dj < 0$. By substituting $i = -(i'+1), j = p-1-j'$, the residue classes $[x^{i+1}y^{p-1-j}]$ form a basis with $i \geq 0, 0 \leq j \leq p-1$ and $pi + jd \leq (d-1)(p-1) - 2$.

Now we check the equality that $\mathrm{d}f_{i,j} = \omega_{i,j,1} - \omega_{i,j,2}$ for residue classes $\beta_{i,j} = [(f_{i,j}, \omega_{i,j,1}, \omega_{i,j,2})]$. Note that

$$\begin{aligned}
\mathrm{d}f_{i,j} &= \mathrm{d}\frac{y^{p-1-j}}{x^{i+1}} = \frac{(p-1-j)x^{i+1}y^{p-2-j}\,\mathrm{d}y}{x^{2i+2}} - \frac{(i+1)x^iy^{p-1-j}\,\mathrm{d}x}{x^{2i+2}} \\
&= \frac{-(p-1-j)x^{i+1}y^{p-2-j}h'(x)\,\mathrm{d}x}{x^{2i+2}} - \frac{(i+1)x^iy^{p-1-j}\,\mathrm{d}x}{x^{2i+2}} \\
&= \frac{-(p-1-j)y^{p-2-j}xh'(x)\,\mathrm{d}x}{x^{i+2}} - \frac{(i+1)y^{p-1-j}\,\mathrm{d}x}{x^{i+2}} \\
&= -\frac{\psi_{i,j}(x,y)\,\mathrm{d}x}{x^{i+2}} - \frac{(p-1-j)y^{p-2-j}s^{>i+2}(x)\,\mathrm{d}x}{x^{i+2}} \\
&= \omega_{i,j,1} - \omega_{i,j,2}\,,
\end{aligned}$$

which ends the proof.

□

Remark 6.5. Note that the vector space $H^1_{dR}(X)$ is provided with a perfect pairing $\langle\ ,\ \rangle$, see Section 1.3. Moreover, we have $\langle \alpha_{i_1,j_1}, \beta_{i_2,j_2} \rangle \neq 0$ if $(i_1,j_1) = (i_2,j_2)$ and $\langle \alpha_{i_1,j_1}, \beta_{i_2,j_2} \rangle = 0$ otherwise. Indeed, for $(i_1,j_1) = (i_2,j_2)$ we have $\mathrm{ord}_\infty(y^{p-1}/x\,\mathrm{d}x) = -1$ and hence $\langle \alpha_{i_1,j_1}, \beta_{i_2,j_2} \rangle \neq 0$. For other cases, the proof is similar to the proof of [48, Theorem 4.2.1].

Proof of Theorem 6.3. We show that $\mathrm{rank}(\mathcal{C}) = 1$. For $d \leq 2$, the situation is trivial and $\mathrm{rank}(\mathcal{C}) = 0$ for all $p > 0$. We may assume that the polynomial $f(x)$ has the form:

$$f(x) = x^d + a_{d-2}x^{d-2} + \cdots + a_1x\,.$$

Also a basis of $H^0(X, \Omega^1_X)$ is given by forms below:

$$B = \{x^iy^j\,\mathrm{d}x | ip + jd \leq (p-1)(d-1) - 2\}\,.$$

(1): For $p \geq 7$, if $a_i = 0$ for any integer $i \in \{1, 2, \ldots, d-2, d\}$, then according to Theorem 6.1 we have the $\mathrm{rank}(\mathcal{C}) = 0$. Otherwise there exists at least one n such that

$a_n \neq 0$ with $n \in \{1, 2, \ldots, d-2\}$. Denote by i_0 the largest integer in $\{1, 2, \ldots, d-2\}$ such that $a_{i_0} \neq 0$. There are non-negative integers l, m, b satisfying $ld = p+1$ and $d-2 = mi_0 + b$ with $b \leq i_0 - 1$.

Suppose $2 \leq i_0 \leq d-2$, we show that $x^b y^{l-1+m}\,\mathrm{d}x \in B$. This is equivalent to showing

$$bp + (l+m-1)d \leq (d-1)(p-1) - 2,$$

for $m \geq 1$, $i_0 \geq 2$. By substituting $b = d-2-mi_0$, one can show this is equivalent to $m(pi_0 - d) \geq 2$, which is trivial as $d|p+1$ and $m(pi_0 - d) \geq 2p - d \geq 2$.

Now if $d = p+1$, then we have $l = 1$ and $x^b y^m\,\mathrm{d}x \in B$ as showed above. If $b = 0$, we have $d - 2 = p - 1 = mi_0$. By $i_0 \geq 2$, we have $m \leq (p-1)/2$. We show that $y^{m+1}\,\mathrm{d}x \in B$ if $p \geq 5$. It is sufficient to show that $(m+1) \leq (p-1)(d-1) - 2 = p(p-1) - 2$. This is true for $p \geq 5$. Then $\mathcal{C}(y^{m+1}\,\mathrm{d}x) \neq 0$ and $\mathcal{C}(y^m\,\mathrm{d}x) \neq 0$ are linearly independent. This implies $\mathrm{rank}(\mathcal{C}) \geq 2$. Now suppose $b \geq 1$, we show that $x^{b-1}y^{m+1} \in B$. Note that $d - 2 = p - 1 = mi_0 + b$. By a similar fashion, we only need to show $m(i_0 - 1)(p-1) \geq 4$, which is true if $p \geq 5$. Then we have

$$\begin{aligned}
\omega_{b,m} :&= \mathcal{C}(x^b y^m\,\mathrm{d}x) = \mathcal{C}(x^b(y^p - f(x))^m\,\mathrm{d}x) \\
&= \mathcal{C}((-1)^m x^b a_{i_0}^m (x^{i_0})^m\,\mathrm{d}x) + \cdots \\
&= \mathcal{C}((-1)^m a_{i_0}^m x^{(b+mi_0)}\,\mathrm{d}x) + \cdots \\
&= \mathcal{C}((-1)^m a_{i_0}^m x^{p-1}\,\mathrm{d}x) + \cdots \neq 0\,.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\omega_{b-1,m+1} :&= \mathcal{C}(x^{b-1}y^{m+1}\,\mathrm{d}x) = \mathcal{C}(x^{b-1}(y^p - f(x))^{m+1}\,\mathrm{d}x) \\
&= \mathcal{C}(\binom{m+1}{1}(-1)^{m+1}x^{b-1}a_{i_0}^m(x^{i_0})^m a_{p+1}x^{p+1}\,\mathrm{d}x) + \cdots \\
&= \mathcal{C}((m+1)(-1)^{m+1}a_{p+1}a_{i_0}^m x^{2p-1}\,\mathrm{d}x) + \cdots \\
&\neq 0\,.
\end{aligned}$$

Since $\omega_{b,m}$ and $\omega_{b-1,m+1}$ are k-linearly independent, we have $\mathrm{rank}(\mathcal{C}) \geq 2$.

Now if $d|p+1$ and $d \leq (p+1)/2$, we show that $x^b y^{l+m}\,\mathrm{d}x \in B$, which is equivalent to $bp + (l+m)d \leq (d-1)(p-1) - 2$. Since $m(pi_0 - d) \geq 2p - d$, we only need to show $m(pi_0 - d) - d \leq 2$, which is true for $p \geq 7$. Hence $\mathcal{C}(x^b y^{l+m}\,\mathrm{d}x) \neq 0$ and $\mathcal{C}(x^b y^{l+m-1}\,\mathrm{d}x) \neq 0$ by the same method above.

Now assume $i_0 = 1$ and $a_i = 0$ for any $i \in 2, 3, \ldots, d-2$, if $d = p+1$, by a simple change of coordinates the curve is superspecial and $\mathrm{rank}(\mathcal{C}) = 0$. Otherwise we have

$d < p+1$, in this case we have $d-2 = m+b$. We show that $y^{l+m+b-1}\,\mathrm{d}x, y^{l+m+b}\,\mathrm{d}x \in B$, which is equivalent to showing

$$\begin{aligned}(l+m+b-1)d &\leq (d-1)(p-1)-2 \text{ and}\\ (l+m+b)d &\leq (d-1)(p-1)-2\,,\end{aligned}$$

respectively. These can be simplified to

$$\begin{aligned}d^2-(p+2)d+2p+2 &\leq 0,\\ d^2-(p+1)d+2p+2 &\leq 0\,.\end{aligned}$$

These two inequalities hold for $p \geq 11$. For $p = 7$, we have $d|p+1 = 8$ and hence $d \geq 4$. Then those two inequalities also hold.

Moreover, we have

$$\begin{aligned}\mathcal{C}(y^{l+m+b-1}\,\mathrm{d}x) &= \mathcal{C}((y^p-f(x))^{l+m+b-1}\,\mathrm{d}x)\\ &= \mathcal{C}((-1)^{l+m+b-1}(x^d)^{l-1}(a_1x)^{m+b}\,\mathrm{d}x)+\cdots\\ &= \mathcal{C}((-1)^{l+m+b-1}a_1^{m+b}x^{p-1}\,\mathrm{d}x)+\cdots \neq 0\end{aligned}$$

and $\mathcal{C}(y^{l+m+b}\,\mathrm{d}x) = \mathcal{C}((-1)^{l+m+b-1}a_1^{m+b}x^{p-1}y^p\,\mathrm{d}x)+\cdots \neq 0$. Then we have $\operatorname{rank}(\mathcal{C}) \geq 2$.

(2): For $p = 5$ and $d = p+1 = 6$, to get $\operatorname{rank}(\mathcal{C}) = 1$ we must have $i_0 \geq 2$, otherwise X is superspecial by Theorem 6.1. Then $x^{b-1}y^{m+1}\,\mathrm{d}x, x^by^m\,\mathrm{d}x \in B$ for $b \geq 1$ and $y^{m+1}\,\mathrm{d}x, y^m\,\mathrm{d}x \in B$ for $b = 0$ (similar to the case $p = 7$). This implies $\operatorname{rank}(\mathcal{C}) \geq 2$. As for $d = 3$, if $a_{d-2} = a_1 = 0$, then it is superspecial by Theorem 6.1. If $a_1 \neq 0$, then $y^2\,\mathrm{d}x \in B$ and $\operatorname{rank}(\mathcal{C}) = 1$. For the supersingularity, let X be a curve given by equation $y^5-y = x^3+a_1x$ with $a_1 \in k^*$. Then we have $H^0(X,\Omega^1_X) = \langle \mathrm{d}x, x\,\mathrm{d}x, y\,\mathrm{d}x, y^2\,\mathrm{d}x\rangle$ and $\mathcal{C}(H^0(X,\Omega^1_X)) = \langle\, \mathrm{d}x\,\rangle$. Moreover by a calculation using Proposition 6.4, one can compute that X has Ekedahl-Oort type $[4,3,2]$. Hence for the final type v, we have $v(\lfloor (g+1)/2 \rfloor) = v(2) = 0$ and the curve X is supersingular by Proposition 1.25. One can easily check that the map from the parameter space with coordinate (a_1) to $\mathcal{M}_4$ has finite fibre. Note that $Z_{[4,3,2]}$ is of dimension at most 1 in $\mathcal{M}_4$. Then the family of curves given by equation (6.3) is a component of $Z_{[4,3,2]}$.

(3): For $p = 3$, if $d = 2$ the curve is superspecial. If $d = 4$, then we may assume that $a_2 \neq 0$ in $f(x)$, otherwise by a simple change of coordinates we may assume the curve is given by equation $y^3-y = a_4x^4$, which is superspecial by the Theorem 6.1.

If $a_2 \neq 0$, then by a change of coordinate we get $f(x) = x^4+a_2x^2$. Then a basis of $H^0(X,\Omega^1_X)$ is $B = \{\mathrm{d}x, x\,\mathrm{d}x, y\,\mathrm{d}x\}$ with $\mathcal{C}(\mathrm{d}x) = \mathcal{C}(x\,\mathrm{d}x) = 0$ and $\mathcal{C}(y\,\mathrm{d}x) = \mathcal{C}(-a_2x^2\,\mathrm{d}x) = -a_2^{1/3}\,\mathrm{d}x \neq 0$. This implies $\operatorname{rank}(\mathcal{C}) = 1$. Moreover by a calculation

using Proposition 6.4, a curve given by equation $y^3 - y = x^4 + a_2x^2$ with $a_2 \in k^*$ has Ekedahl-Oort type $[3, 2]$ and hence is supersingular by Proposition 1.25. One can easily check that the map from the parameter space with coordinate (a_2) to $\mathcal{M}_3$ has finite fibre. Note that $Z_{[3,2]}$ is of dimension 1 in $\mathcal{M}_3$. Then the family of curves given by equation (6.4) is a component of $Z_{[3,2]}$.

□

6.3 Artin-Schreier curves with a-number $g - 2$

For an Artin-Schreier curve X given by equation $y^p - y = f(x)$ with $\deg f(x) = d | p+1$ and $\operatorname{rank}(\mathcal{C}) = 2$, we have the following.

Corollary 6.6. *Let X be an Artin-Schreier curve of genus $g > 0$ given by an equation $y^p - y = f(x)$, where $f(x) \in k[x]$ and $\deg f(x) = d$. If $d | p+1$ and the a-number of X is equal to $g-2$, then we have $p = 7, d = 4$ and X is isomorphic to the supersingular curve of genus 9 with equation*

$$y^7 - y = x^4 + a_1x,\ a_1 \in k^*\,.$$

Proof. We show that $\operatorname{rank}(\mathcal{C}) = 2$. We may assume that the polynomial $f(x)$ has the form:

$$f(x) = x^d + a_{d-2}x^{d-2} + \cdots + a_1x\,.$$

By the proof of Theorem 6.3, there is an integer $n \in \{1, 2, \ldots, d-2\}$ such that $a_n \neq 0$. Again denote by i_0 the largest integer in $\{1, 2, \ldots, d-2\}$ such that $a_{i_0} \neq 0$ and let l, m, b be the same as in the proof of Theorem 6.3.

For $p \geq 7$, if $d = p+1$, we show that $\operatorname{rank}(\mathcal{C}) \geq 3$. Indeed by Theorem 6.1, we have $i_0 \geq 2$ and $d-2 = p-1 = mi_0+b$. If $b = 0$, then $d-2 = p-1 = mi_0$ and $m \leq (p-1)/2$. Moreover from the proof of Theorem 6.3, part (1), we have $y^m\,\mathrm{d}x, y^{m+1}\,\mathrm{d}x \in B$. We show that $y^{m+2}\,\mathrm{d}x \in B$. It suffices to show that $(m+2)d \leq (p-1)(d-1)-2$, which is equivalent to showing $(m+2)(p+1) \leq p^2-p-2$ for any $1 \leq m \leq (p-1)/2$. This is true for $p \geq 7$. On the other hand, note that $\mathcal{C}(y^m\,\mathrm{d}x), \mathcal{C}(y^{m+1}\,\mathrm{d}x)$ and $\mathcal{C}(y^{m+2}\,\mathrm{d}x)$ are linearly independent. Then $\operatorname{rank}(\mathcal{C}) \geq 3$ in this case. Now if $b \geq 1$, we showed that $x^by^m\,\mathrm{d}x, x^{b-1}y^{m+1}\,\mathrm{d}x \in B$. By a similar argument as in the case $b = 0$ above, one can show that $x^by^{m+1}\,\mathrm{d}x \in B$. Additionally, $\mathcal{C}(x^by^m\,\mathrm{d}x), \mathcal{C}(x^{b-1}y^{m+1}\,\mathrm{d}x)$ and $\mathcal{C}(x^by^{m+1}\,\mathrm{d}x)$ are linearly independent. Then we have $\operatorname{rank}(\mathcal{C}) \geq 3$ for $p \geq 7$ and $d = p+1$.

Now if $d | p+1$ and $d < p+1$, then $l = (p+1)/d \geq 2$. If $i_0 \geq 2$, we show that $\operatorname{rank}(\mathcal{C}) \geq 3$ for $p \geq 7$. Note that we have $x^by^{l+m-1}\,\mathrm{d}x, x^by^{l+m}\,\mathrm{d}x \in B$ by the part

(1) of the proof of Theorem 6.3. We now claim that $x^b y^{l+m+1}\,\mathrm{d}x \in B$. By definition of B, it suffices to show

$$(l+m+1)d + bp \leq (p-1)(d-1) - 2\,.$$

By substituting $b = d - 2 - mi_0$ and $p = ld - 1$, the inequality can be simplified to $(i_0 l - 1)m \geq 3$. This is true as $i_0 \geq 2, l \geq 2$ and $m \geq 1$. For $i_0 = 1$, we show that $\mathrm{rank}(\mathcal{C}) \geq 3$ for $p \geq 11$. Note that in this case we have $d - 2 = m$. One can easily show that $y^{l+m}\,\mathrm{d}x, y^{l+m-1}\,\mathrm{d}x \in B$ by the definition of B. Additionally, we show that $y^{l+m+1}\,\mathrm{d}x \in B$ for $p \geq 11$. Indeed, it suffices to show $(l+m+1)d \leq (p-1)(d-1)-2$, which can be simplified to $2l + d \leq p$. Note that $ld = p + 1$, we only need to show $2(p+1)/d + d \leq p$ which can be rewritten as $d^2 - dp + 2(p+1) \leq 0$. This is true for $3 \leq d \leq (p+1)/2$.

For $p = 7$ and $i_0 = 1$, we have $d = 4$ and the curve is given by equation $y^7 - y = x^4 + a_1 x$ with $a_1 \in k^*$. Then

$$B = \{x^i y^j\,\mathrm{d}x, |i, j \in \mathbb{Z}_{\geq 0}, 7i + 4j \leq 16\}$$

and $\mathcal{C}(x^i y^j\,\mathrm{d}x) = 0$ for all i, j except $(i, j) = (0, 4), (0, 3), (1, 2)$. Moreover, $\mathcal{C}(y^4\,\mathrm{d}x)$ and $\mathcal{C}(y^3\,\mathrm{d}x)$ are linearly independent and $\mathcal{C}(y^3\,\mathrm{d}x) = \xi\mathcal{C}(xy^2\,\mathrm{d}x)$ for some $\xi \in k^*$. Then $\mathrm{rank}(\mathcal{C}) = 2$. Moreover by a calculation using Proposition 6.4, we have $v(n) = 0$ for $n \leq 6$ with v the final type of the curve. This implies the curve is supersingular.

Now let $p = 5$. If $d = 3$, then by Theorem 6.1 and Theorem 6.3, we have $a(X) = g$ or $g - 1$. For $d = 6$, we get $d - 2 = 4 = mi_0 + b$. Additionally for $i_0 = 2, 3, 4$, one can easily show that $y^2\,\mathrm{d}x, y^3\,\mathrm{d}x, xy^3\,\mathrm{d}x \in B$ and $\mathcal{C}(y^2\,\mathrm{d}x), \mathcal{C}(y^3\,\mathrm{d}x)$ and $\mathcal{C}(xy^2\,\mathrm{d}x)$ are linearly independent. Hence $\mathrm{rank}(\mathcal{C}) \geq 3$ and $a(X) \leq g - 3$ with $g = 10$.

For $p = 3$ and $d \,|\, p + 1 = 4$, by Theorem 6.1 and Theorem 6.3, we have $a(X) \geq g - 1$.

□

Chapter 7

Existence of trigonal curves and hyperelliptic curves with prescribed a-number

In this chapter, we study the existence of trigonal and hyperelliptic curves with prescribed a-number. We give bounds on the a-number of trigonal curves of genus 5 in characteristic 2 and 3. We also bound the a-number of hyperelliptic curves of genus $g > p \geq 3$ in characteristic p.

7.1 Main results

Let k be an algebraically closed field of characteristic $p > 0$. By a curve we mean a smooth irreducible projective curve over k and by a trigonal curve we mean a curve that is a degree 3 cover of $\mathbb{P}^1$.

It is well known that a trigonal curve X of genus 5 is a normalization of a quintic curve C in $\mathbb{P}^2$ with an unique singular point [5, Exercise I-6, page 279], see also [28, Lemma 2.2.1]. Recall that a result of Re [45] states that if X is a non-hyperelliptic curve of genus g, then

$$a(X) \leq \frac{p-1}{p+1}\left(\frac{2g}{p} + g + 1\right).$$

We improve Re's bound for the trigonal curve of genus 5 in characteristic 2 and 3. Recall that a trigonal curve of genus 5 is not hyperelliptic by Proposition 1.30.

Theorem 7.1. *Let k be an algebraically closed field of characteristic 2. If X is a trigonal curve of genus 5 defined over k, then the a-number of X satisfies $a(X) \leq 2$.*

Theorem 7.2. *Let k be an algebraically closed field of characteristic 3. If X is a trigonal curve of genus 5 defined over k, then the a-number of X satisfies $a(X) \leq 3$.*

For $g = 5$ and $p = 2$, Re's bound says that $a(X) \leq 11/3$ while our result implies that $a(X) \leq 2$. Also for $g = 5$ and $p = 3$, Re's bound says that $a(X) \leq 14/3$ whiles our result implies $a(X) \leq 3$.

In [45] Re also gave a bound for the a-number of a hyperelliptic curve of genus g defined over k:

$$a < \frac{(p-1)g}{p} + \frac{p+1}{2p}\,.$$

In the following theorem we give a strengthening of this result.

Theorem 7.3. *A smooth projective hyperelliptic curve over k in characteristic $p \geq 3$ of genus $g \geq p$ has a-number*

$$a \leq \lceil (p-1)g/p - (3p-1)/2p \rceil\,.$$

A related result was given by Elkin [14, Corollary 1.2]; he showed that the a-number is bounded by $a \leq g(p-2)/p + 1$. In general we have

$$\lceil (p-1)g/p - (3p-1)/2p \rceil < (p-1)g/p + (p+1)/2p - 1\,.$$

For example in the case $p = g = 3$, Re's bound shows that $a < 2 + 2/3$ and Elkin's bound shows that $a \leq 2$, while our bound shows that $a \leq 1$, which is sharp. Note also that for $p = g = 3$, Frei [18] ruled out the case $a = 2$.

By Proposition 1.12, the a-number $a(X)$ of a curve X of genus g defined over k is equal to $g - r$ where r is the rank of the Cartier operator acting on $H^0(X, \Omega^1_X)$. We have r is also equal to the rank of the Hasse-Witt matrix. To prove Theorems 7.1, 7.2 and 7.3, we compute the rank of the Hasse-Witt matrix with respect to a basis of $H^1(X, \mathcal{O}_X)$.

7.2 Set up

For a trigonal curve X of genus 5 defined over k, let $\phi : X \to \mathbb{P}^1$ be a morphism of degree 3. By Propositions 1.29 and 1.30, ϕ is unique (up to isomorphism of $\mathbb{P}^1$) and X is not hyperelliptic.

Lemma 7.4. *Let p be either 2 or 3. If X is a trigonal curve of genus 5 over k, then (1) : X is a normalization of a quintic C in $\mathbb{P}^2$, where C has an unique singular point with multiplicity 2. Moreover,*

(*i*) : *If C has a node, then C is given by a homogeneous polynomial $F \in k[x, y, z]$ of degree* 5 *with*

$$F = xyz^3 + f\,,$$

where f is the sum of monomials which not divisible by z^3.

(*ii*) : *If C has a cusp, then C is given by a homogeneous polynomial $F \in k[x, y, z]$ of degree* 5 *with*

$$F = x^2z^3 + f\,,$$

where f is the sum of monomials not divisible by z^3 and the coefficient of y^3z^2 in f is non-zero.

(2) : *Conversely, for C in* (*i*) *and* (*ii*) *with an unique singular point, its normalization is a trigonal curve of genus* 5.

Proof. Kudo and Harashita proved the lemma for $p \neq 2$ in [28, Lemma 2.2.1]. For $p = 2$, we show that the part (1) is true and the rest will be similar to the case $p \geq 3$ and hence we omit it.

The curve C has an unique singular point and we may assume it is $(0, 0, 1)$. Then C is given by equation

$$F = Qz^3 + f\,,$$

where Q is a quadratic form in $k[x, y]$ and f is the sum of monomials, which have degree > 2 in x, y.

Now if Q is non-degenerate, then C has a node. Moreover, any non-degenerate quadratic form Q can be written as xy. Then we can write $F = xyz^3 + f$ with f the sum of monomials which has degree > 2 in x, y.

If Q is degenerate, then C has a cusp. By an appropriate change of coordinates, Q can be written as x^2 and hence $F = x^2z^3 + f$ with f the sum of monomials which has degree > 2 in x, y.

□

The following proposition gives a way of computing the Hasse-Witt matrix of a trigonal curve of genus 5, which is a matrix representation of the Frobenius operator on $H^1(X, \mathcal{O}_X)$. For a basis of $H^1(X, \mathcal{O}_X)$, we refer to [28].

Proposition 7.5. *[28, Proposition* 2.3.1*] Let X be a trigonal curve of genus* 5 *defined over k. Let C be an associated quintic curve in $\mathbb{P}^2$ given by Lemma 7.4. Let $h_{l,m}$ $(1 \leq$*

$l, m \leq 5$) *be the coefficient of the monomial* $x^{pi_l - i_m} y^{pj_l - j_m} z^{pk_l - k_m}$ *in* F^{p-1}, *where*

l	1	2	3	4	5
i_l	3	1	2	2	1
j_l	1	3	2	1	2
k_l	1	1	1	2	2

.

Then the Hasse-Witt matrix H *of* X *is given by* $H = (h_{l,m})$.

Remark 7.6. The trigonal curve X of genus 5 is superspecial if and only if the Hasse-Witt matrix is zero. It is known that there is no superspecial curve of genus 5 in characteristic 2 or 3 by Ekedahl [12]. One can also prove it by showing that if the curve has zero Hasse-Witt matrix, then the curve has to have at least two singular points or the curve has to be reducible.

7.3 The proof of Theorem 7.1

Let $p = 2$ and X be a trigonal curve of genus 5 defined over k. In order to compute the Hasse-Witt matrix of X using Proposition 7.5, we need to simplify the defining equation of the singular model $C \subset \mathbb{P}^2$ of X.

Lemma 7.7. *Let* k *be an algebraically closed field with* $\mathrm{char}(k) = 2$. *In the notation of Lemma 7.4 case* (i), *we can choose the equation as*

$$\begin{aligned} f =&(x^3 + b_1 y^3)z^2 + (a_1 x^4 + a_2 x^3 y + a_3 x^2 y^2 + a_4 x y^3 + a_5 y^4)z \\ &+ a_6 x^5 + a_7 x^4 y + a_8 x^3 y^2 + a_9 x^2 y^3 + a_{10} x y^4 + a_{11} y^5 , \end{aligned} \tag{7.1}$$

or

$$\begin{aligned} f =&(a_1 x^4 + a_2 x^3 y + a_3 x^2 y^2 + a_4 x y^3 + a_5 y^4)z \\ &+ a_6 x^5 + a_7 x^4 y + a_8 x^3 y^2 + a_9 x^2 y^3 + a_{10} x y^4 + a_{11} y^5 . \end{aligned} \tag{7.2}$$

For case (ii), *the equation can be written as*

$$\begin{aligned} f =&y^3 z^2 + (a_1 x^4 + a_2 x^3 y + a_3 x^2 y^2 + a_4 x y^3 + a_5 y^4)z \\ &+ a_6 x^5 + a_7 x^4 y + a_8 x^3 y^2 + a_9 x^2 y^3 + a_{10} x y^4 + a_{11} y^5 . \end{aligned} \tag{7.3}$$

Proof. For the case (i) of Lemma 7.4, the curve C is given by $F = xyz^3 + f$, where f is the sum of monomials, which have degree > 2 in x, y. By a linear transformation $z \mapsto z + \alpha x + \beta y$, we may assume the coefficients of $x^2 y z^2$ and $x y^2 z^2$ are zero. Then

$$\begin{aligned} f =&(b_0 x^3 + b_1 y^3)z^2 + (a_1 x^4 + a_2 x^3 y + a_3 x^2 y^2 + a_4 x y^3 + a_5 y^4)z \\ &+ a_6 x^5 + a_7 x^4 y + a_8 x^3 y^2 + a_9 x^2 y^3 + a_{10} x y^4 + a_{11} y^5 , \end{aligned}$$

where $b_0, b_1, a_1, \ldots, a_{11} \in k$. Note that if $(b_0, b_1) \neq (0,0)$, by symmetry we may assume $b_0 \neq 0$. By scaling $x \mapsto \alpha x, y \mapsto \beta y$ with $\alpha\beta = 1$ and $\alpha^3 = 1$. Then we may assume $b_0 = 1$. On the other hand, if $b_0 = b_1 = 0$ in f, then we have

$$\begin{aligned} f =&(a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z \\ &+ a_6x^5 + a_7x^4y + a_8x^3y^2 + a_9x^2y^3 + a_{10}xy^4 + a_{11}y^5\,. \end{aligned}$$

For the case (ii) of Lemma 7.4, the curve C is given by $F = x^2z^3 + f$, where f is the sum of monomials, which have degree > 2 in x, y and the coefficient of y^3z^2 is non-zero. Consider $y \mapsto y + \gamma x$ and then consider $z \mapsto z + \alpha x + \beta y$, we may assume the coefficients of x^3z^2, x^2yz^2 and xy^2z^2 are zero. Moreover, by scaling $y \mapsto \delta y$ with $\delta^3 = 1$, we may assume the coefficient of y^3z^2 is equal to 1. Then we have

$$\begin{aligned} f =&y^3z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z \\ &+ a_6x^5 + a_7x^4y + a_8x^3y^2 + a_9x^2y^3 + a_{10}xy^4 + a_{11}y^5\,, \end{aligned}$$

where $b_1, b_2, a_1, \ldots, a_{11} \in k$.

□

Now we can give a proof of Theorem 7.1.

Proof of Theorem 7.1. Let C be a singular model of X given by Lemma 7.4. Moreover, if the curve C has a node, then by Lemma 7.7, f is either given by equation (7.1) or equation (7.2). If f is given by equation (7.1), then by Proposition 7.5, the Hasse-Witt matrix H of X is equal to

$$\begin{pmatrix} a_2 & 0 & a_1 & a_7 & a_6 \\ 0 & a_4 & a_5 & a_{11} & a_{10} \\ a_4 & a_2 & a_3 & a_9 & a_8 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & b_1 & 0 \end{pmatrix}. \tag{7.4}$$

Let $e_i = (e_{i,1}, \ldots, e_{i,5})$ be the i-th row of H. Then $\mathrm{rank}(H) \geq 2$ since e_4 and e_5 are linearly independent.

Now we show $\mathrm{rank}(H) \geq 3$ in this case. Indeed, if $\mathrm{rank}(H) = 2$, then e_i for $i = 1, 2, 3$ is a linear combination of e_4 and e_5. By the shape of H, we have the following:

$$\begin{aligned} &a_1 = a_3 = a_5 = a_7 = a_{10} = 0, a_4 = a_8, \\ &a_2 = a_6,\, b_1a_4 = a_{11}, b_1a_2 = a_9\,. \end{aligned}$$

Hence C is given by

$$\begin{aligned} F &= xyz^3 + (x^3 + b_1y^3)z^2 + (a_2x^3y + a_4xy^3) + a_2x^5 + a_4x^3y^2 + b_1a_2x^2y^3 + b_1a_4y^5 \\ &= b_1y^3(z^2 + a_2x^2 + a_4y^2) + xyz(z^2 + a_2x^2 + a_4y^2) + x^3(z^2 + a_2x^2 + a_4y^2) \\ &= (z + a_2^{1/2}x + a_4^{1/2}y)^2(b_1y^3 + x^3 + xyz)\,. \end{aligned}$$

We have C is reducible. This contradiction shows that $\mathrm{rank}(H) \geq 3$.

Now if f is given by equation (7.2), then again by Proposition 7.5 the Hasse-Witt matrix H of X is equal to

$$\begin{pmatrix} a_2 & 0 & a_1 & a_7 & a_6 \\ 0 & a_4 & a_5 & a_{11} & a_{10} \\ a_4 & a_2 & a_3 & a_9 & a_8 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7.5}$$

Then we have $\mathrm{rank}(H) \geq 2$. Moreover, if $\mathrm{rank}(H) = 2$, then we have $a_i = 0$ for all $i \in \{1, \ldots, 11\}$ except $i = 2, 4$. This implies

$$F = xyz^3 + a_2x^3yz + a_4xy^3z = xy(z^3 + a_2x^2z + a_4y^2)\,,$$

a contradiction. Hence we have $\mathrm{rank}(H) \geq 3$ if C has a node.

If the curve C has a cusp, then by Lemma 7.7, f is given by equation (7.3). Hence by Proposition 7.5, the Hasse-Witt matrix of X is equal to

$$\begin{pmatrix} a_2 & 0 & a_1 & a_7 & a_6 \\ 0 & a_4 & a_5 & a_{11} & a_{10} \\ a_4 & a_2 & a_3 & a_9 & a_8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{7.6}$$

Then we still have $\mathrm{rank}(H) \geq 2$. We show $\mathrm{rank}(H) \geq 3$ by showing that C has at least two singular points if $\mathrm{rank}(H) = 2$. Indeed, suppose $\mathrm{rank}(H) = 2$. By (7.6), we obtain $a_2 = a_4 = a_6 = a_8 = a_{10} = 0$. This implies

$$F = x^2z^3 + y^3z^2 + (a_1x^4 + a_3x^2y^2 + a_5y^4)z + a_7x^4y + a_9x^2y^3 + a_{11}y^5\,.$$

Denote by F_x (resp. F_y, F_z) the formal partial derivative with respect to the variable x (resp. y, z). Note that we have

$$\begin{aligned} &F_x = 0, F_y = y^2z^2 + a_7x^4 + a_9x^2y^2 + a_{11}y^4, \\ &F_z = x^2z^2 + a_1x^4 + a_3x^2y^2 + a_5y^4\,. \end{aligned}$$

By letting $x = 1$ in F_x, F_y and F_z, one can easily show that there exists a point $(1, a, b)$ which is different from (0,0,1) such that

$$F(1, a, b) = F_x(1, a, b) = F_y(1, a, b) = F_z(1, a, b) = 0\,.$$

Then there are at least two singular points on C. By the genus formula for plane curves, the genus of X is less than 5, a contradiction.

Now we have $\mathrm{rank}(H) \geq 3$ for any trigonal curve X of genus 5 over k. Then $a(X) \leq 2$.

□

7.4 The proof of Theorem 7.2

Let $p = 3$ and X be a trigonal curve of genus 5 defined over k. We now give the reductions of the defining equations of the singular model $C \subset \mathbb{P}^2$ of X given by Lemma 7.4.

Lemma 7.8. *Let k be an algebraically closed field with* $\mathrm{char}(k) = 3$. *In the notation of Lemma 7.4 case (i), we can choose the equation as*

$$\begin{aligned}&(b_0x^3 + b_1y^3 + b_2x^2y + b_3xy^2)z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z\\ &+ a_6x^5 + a_8x^3y^2 + a_9x^2y^3 + a_{11}y^5\,.\end{aligned} \tag{7.7}$$

For the case (ii), the equation can be written as

$$\begin{aligned}f =&(y^3 + b_2x^2y + b_3xy^2)z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z\\ &+ a_7x^4y + a_8x^3y^2 + a_{10}xy^4 + a_{11}y^5\,.\end{aligned} \tag{7.8}$$

Proof. If C has a node, then by Lemma 7.4, $F = xyz^3 + f$ with f the sum of monomials, which have degree > 2 in x, y. By a linear transform $z \mapsto z + \alpha x + \beta y$ we may assume the coefficient of x^4y and xy^4 is zero. Then f is equal to

$$\begin{aligned}&(b_0x^3 + b_1y^3 + b_2x^2y + b_3xy^2)z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z\\ &+ a_6x^5 + a_8x^3y^2 + a_9x^2y^3 + a_{11}y^5\,.\end{aligned}$$

By Lemma 7.4, if C has a cusp, then $F = xyz^3 + f$ with f the sum of monomials, which have degree > 2 in x, y and the coefficient of y^3z^2 is non-zero. By a linear transform $z \mapsto z + \alpha x + \beta y$, we may assume the coefficient of x^5 and x^2y^3 is zero.

Moreover, by scaling $y \mapsto \delta y$, we may assume the coefficient of y^3z^2 is 1. Then we have

$$\begin{aligned} f =&(y^3 + b_2x^2y + b_3xy^2)z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4)z \\ &+ a_7x^4y + a_8x^3y^2 + a_{10}xy^4 + a_{11}y^5 \,. \end{aligned}$$

□

Now we give the proof of Theorem 7.2.

Proof of Theorem 7.2. Let C be the singular model given by Lemma 7.4. Denote by H the Hasse-Witt matrix of X given by Proposition 7.5 and by $e_i = (e_{i,1}, \ldots, e_{i,5})$ the i-th row of H. Then we have $\operatorname{rank}(H) \geq 1$ because of the Ekedahl's genus bound for superspecial curve [12]. Suppose $\operatorname{rank}(H) = 1$. We consider different cases for the singular point of C.

If the curve C has a node, by Lemma 7.8, f is given by equation (7.7). If at least one of b_0, b_1 is non-zero, by symmetry we may assume $b_0 \neq 0$. By scaling we may assume $b_0 = 1$. Moreover, by Proposition 7.5, we have $e_4 = (2b_2, 0, 2, b_2^2 + 2b_3 + 2a_2, b_2 + 2a_1)$ which is non-zero. Then $e_i = \lambda_i e_4$ with $\lambda_i \in k$ for $i = 1, 2, 3, 5$. In particular, we have

$$e_5 = (0, 2b_3, 2b_1, 2b_1b_3 + 2a_5, 2b_1b_2 + b_3^2 + 2a_4) = \lambda_5 e_4 \,.$$

This implies that $b_3 = 0$ and $b_1b_2 = 0$.

If $b_1 = 0$, then $e_5 = (0, 0, 0, 2a_5, 2a_4)$ is the zero vector. Hence $a_4 = a_5 = 0$. Note that in this case we have $e_{3,1} = 2a_{11}$ and $e_{3,1} = \lambda_3 e_{4,1} = 0$. Then $a_{11} = 0$ and

$$\begin{aligned} F &= xyz^3 + (x^3 + b_2x^2y)z^2 + (a_1x^4 + a_2x^3y + a_3x^2y^2)z \\ &\quad + a_6x^5 + a_8x^3y^2 + a_9x^2y^3 \,, \\ F_x &= z^3y + 2b_2xyz^2 + (a_1x^3 + 2a_3xy^2)z + 2a_6x^4 + 2a_9xy^3 \,, \\ F_y &= xz^3 + b_2x^2z^2 + (a_2x^3 + 2a_3x^2y)z + 2a_8x^3y \,, \\ F_z &= 2(x^3 + b_2x^2y)z + a_1x^4 + a_2x^3y + a_3x^2y^2 \end{aligned}$$

with F_x (resp. F_y, F_z) the formal derivative with respect to x (resp. y, z). One can easily check that $(0, 0, 1)$ and $(0, 1, 0)$ are common zeros of $F = F_x = F_y = F_z = 0$. Then C has at least two singular points, a contradiction.

Now if $b_1 \neq 0$, then the equation $b_1b_2 = 0$ implies $b_2 = 0$. Consider a change of coordinate $x \mapsto \alpha x, y \mapsto \beta y$ and multiply F by $1/(\alpha\beta)$. The coefficients of x^3 and

y^3 in F become α^2/β and $\beta^2 b_1/\alpha$. By taking $\beta = \alpha^2$ and $\alpha = b_1^{-1/3}$, we may assume $b_0 = b_1 = 1$. Then $e_5 = \lambda_5 e_4$ implies $a_2 = a_5$ and $a_1 = a_4$. Additionally, we have

$$\begin{aligned} e_1 &= (2a_1a_3 + a_2^2 + 2a_8, a_1^2 + 2a_6, 2a_1a_2, 2a_1a_8 + 2a_3a_6, 2a_2a_6), \\ e_2 &= (a_2^2 + 2a_{11}, a_1^2 + 2a_2a_3 + 2a_9, 2a_1a_2, 2a_1a_{11}, 2a_2a_9 + 2a_3a_{11})\,. \end{aligned}$$

Then by $e_1 = \lambda_1 e_4$ and $e_2 = \lambda_2 e_4$, we have

$$a_6 = a_1^2, a_8 = a_2^2 - a_1a_3, a_9 = a_1^2 - a_2a_3, a_{11} = a_2^2\,.$$

Similarly we have

$$\begin{aligned} F_x &= yz^3 + (a_1x^3 + 2a_3xy^2 + a_1y^3)z + 2a_1^2x^4 + 2(a_1^2 - a_2a_3)xy^3\,, \\ F_y &= xz^3 + (a_2x^3 + 2a_3x^2y + a_2y^3)z + 2a_2^2y^4 + 2(a_2^2 - a_1a_3)x^3y\,, \\ F_z &= 2(x^3 + y^3)z + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_1xy^3 + a_2y^4\,. \end{aligned}$$

If $a_3 = 0$, then one can easily show that $(-1, 1, 0)$ and $(0, 0, 1)$ are singular points of C, a contradiction.

Suppose $a_3 \neq 0$. By substituting

$$z = (a_1x^4 + a_2x^3y + a_3x^2y^2 + a_1xy^3 + a_2y^4)/(x + y)^3$$

in F_x and F_y and by letting $y = 1$, we have $(x(x+y)^9F_x)|_{y=1} = ((x+y)^9F_y)|_{y=1}$ and

$$\begin{aligned} ((x + y)^9F_x)|_{y=1} = (a_1^3 + a_1a_2)x^{12} + (a_1a_2 + a_2^3 + 2a_3^2)x^9 + (a_3^3 + a_3^2)x^6 \\ + (a_1^3 + a_1a_2 + 2a_3^2)x^3 + a_1a_2 + a_2^3\,. \end{aligned}$$

Since $a_3 \neq 0$, one can check that it has solutions with $x \neq -1, 0$. Then there exists another singular point on C which is distinct from $(0, 0, 1)$, a contradiction.

Now if $b_0 = b_1 = 0$, we have $e_4 = (2b_2, 0, 0, b_2^2 + 2a_2, 2a_1)$ and $e_5 = (0, 2b_3, 0, 2a_5, b_3^2 + 2a_4)$. Since $\mathrm{rank}(H) = 1$, we have at least one of b_2, b_3 is zero. By symmetry we may assume $b_3 = 0$. If $b_2 \neq 0$, by scaling we may assume $b_2 = 1$. Note that we have $e_i = \lambda_i e_4$ with $\lambda_i \in k$ for $i = 1, 2, 3, 5$. In particular for $i = 5$, it is straightforward to see that $\lambda_5 = 0$ and $a_4 = a_5 = 0$. Moreover, $e_{2,2} = 2a_{11} = \lambda_2 e_{4,2} = 0$. Then by a similar fashion, one can show that $(0, 0, 1)$ and $(0, 1, 0)$ are singular points of C, a contradiction. If $b_3 = b_2 = 0$, then

$$\begin{aligned} e_1 &= (2a_1a_3 + a_2^2, a_1^2, 2a_1a_2, 2a_1a_8 + 2a_3a_6, 2a_2a_6), \\ e_4 &= (0, 0, 0, 2a_2, 2a_1)\,. \end{aligned}$$

This implies $a_1 = a_2 = 0$ otherwise e_1 and e_4 are linearly independent. Similarly one has $a_4 = a_5 = 0$ by checking the linearly independence of e_2 and e_5. Now we have $e_4 = e_5 = 0$ and

$$e_1 = (0, 0, 0, 2a_3a_6, 0),\ e_2 = (0, 0, 0, 0, 2a_3a_{11}),$$
$$e_3 = (0, 0, a_3, 2a_3a_9, 2a_3a_8)\,.$$

Since $\mathrm{rank}(H) = 1$, we obtain that $a_6 = a_{11} = 0$ and $F = y(xz^3 + a_3x^2yz + a_8x^3y + a_9x^2y^2)$, a contradiction.

Now if C has a cusp, then by Lemma 7.8, f is given by equation (7.8). Moreover, by Proposition 7.5, we have

$$e_4 = (2b_3, 0, 2b_2, b_2^2 + 2a_3, 2a_2),\ e_5 = (0, 2, 0, 2b_3, 2b_2 + b_3^2 + 2a_5)\,.$$

If $\mathrm{rank}(H) = 1$, then $e_i = \lambda_i e_5$ with $\lambda_i \in k$ for $i = 1, 2, 3, 4$. In particular, $e_4 = \lambda_4 e_5$ implies $\lambda_4 = b_2 = b_3 = a_2 = a_3 = 0$. Then we obtain

$$e_1 = (0, a_1^2, 0, 2a_1a_8, 2a_1a_7),$$
$$e_2 = (a_5^2 + 2a_{11}, a_4^2, 2a_4a_5 + 2a_{10}, 2a_4a_{11} + 2a_5a_{10}, 2a_4a_{10}),$$
$$e_3 = (2a_8, 2a_1a_4, 2a_1a_5 + 2a_7, 2a_1a_{11} + 2a_4a_8 + 2a_5a_7, 2a_1a_{10} + 2a_4a_7)\,.$$

Hence by $e_i = \lambda_i e_5$ for $i = 1, 2, 3$, we get

$$a_7 = 2a_1a_5, a_8 = 0, a_{10} = 2a_4a_5, a_{11} = a_5^2\,.$$

Additionally, one can easily check that

$$F_x = 2xz^3 + (a_1x^3 + a_4y^3)z + 2a_1a_5x^3y + 2a_4a_5y^4,$$
$$F_y = a_5y^3z + 2a_4a_5xy^3 + 2a_5^2y^4,$$
$$F_z = 2y^3z + a_1x^4 + a_4xy^3 + a_5y^4$$

have common zeros $(0, 0, 1), (0, 1, 0)$ if $a_5 = 0$ and $(0, 0, 1), (0, 1/(2a_5), 1)$ if $a_5 \neq 0$, a contradiction. Then $\mathrm{rank}(H) \geq 2$ if C has a cusp.

In any case, we have $\mathrm{rank}(H) \geq 2$ and hence $a(X) \leq 3$.

□

7.5 The proof of Theorem 7.3

Now we treat the hyperelliptic case. Let $p \geq 3$. Write $g = pl + d \geq p$ for some $l, d \in \mathbb{Z}_{\geq 0}$ with $0 \leq d < p$. We have the following lemma.

Lemma 7.9. *Let X be a hyperelliptic curve of genus $g \geq p$ over k. The Cartier-Manin matrix H of X has*

$$\begin{aligned} &\operatorname{rank}(H) \geq l+1 \text{ if } 1 \leq d \leq n, \\ &\operatorname{rank}(H) \geq l+2 \text{ if } n+1 \leq d < p\,. \end{aligned}$$

Proof. For $p > 2$ we know that any hyperelliptic curve of genus g can be written as $y^2 = f$ of $\deg f = 2g+1$ with non-zero discriminant. By putting a branch point at 0 and by scaling we may assume that

$$f = x^{2g+1} + a_{2g}x^{2g} + \cdots + a_2x^2 + x\,, \tag{7.9}$$

where $a_i \in k$ for $i = 2, \ldots, 2g$. For $i \in \mathbb{Z}$, we denote a_i the coefficient of x^i in f. We also fix $n := (p-1)/2$ and put c_i the coefficient of x^i in f^n. Note that $c_i = 0$ for $i > g(p-1)+n$ and $i < n$ and $c_n = c_{(p-1)g+n} = 1$.

As a basis of $H^0(X, \Omega^1_X)$ we choose $\omega_i = x^i\,\mathrm{d}x/y$ for $i = 0, \ldots, g-1$. Recall that the Cartier-Manin matrix is the matrix representation of the Cartier operator on $H^0(X, \Omega^1_X)$. By [53], the Cartier-Manin matrix H of X has entries $H_{i,j} = c_{pi-j}^{1/p}$ with $1 \leq i, j \leq g$. Also we put e_i to be i-th row of H for $1 \leq i \leq g$. Then

$$e_i = (e_{i,1}, \ldots, e_{i,g}) = (c_{pi-1}, \ldots, c_{pi-g})^{1/p}\,.$$

Firstly, suppose $g = pl + d$ with $1 \leq d \leq n$. Recall that $c_i = 0$ for any $i > g(p-1)+n$ and $i < n$ and $c_n = c_{(p-1)g+n} = 1$. Then $e_{i,(i-1)p+n+1} = 1$ and $e_{i,j} = 0$ for $j > (i-1)p+n+1$. By the shape of row e_i for $i = 1, \ldots, l$, we obtain that $e_1, \ldots, e_l$ are linearly independent. Additionally, we have $e_{g,g-n} = c_{pg-g+n} = c_{(p-1)g+n} = 1 = c_n = e_{l,g-d-n}$ and $e_{i,g-d-n+1} = \cdots = e_{i,g} = 0$ for $1 \leq i \leq l$. Then $e_1, \ldots, e_l, e_g$ are linearly independent and hence $\operatorname{rank}(H) \geq l+1$.

Secondly, suppose $g = pl+d$ with $n+1 \leq d < p$. Similarly $e_1, \ldots, e_{l+1}$ are linearly independent with $e_{l+1.g-d+n+1} = c_n = 1$. If $\operatorname{rank}(H) = 1+l$, then $e_i \in \langle e_1, \ldots, e_{l+1}\rangle$ for $l+2 \leq i \leq g$. In particular, we have

$$\begin{aligned} e_g = (0, \ldots, 0, e_{g,g-n}, \ldots, e_{g,g}) = (0, \ldots, 0, 1, c_{(p-1)g+n-1}, \ldots, c_{(p-1)g})\,, \\ e_{j,pl+1} = \cdots = e_{j,g} = 0 \text{ for } j = 1, \ldots, l. \end{aligned} \tag{7.10}$$

Then the condition $e_g \in \langle e_1, \ldots, e_{l+1}\rangle$ implies

$$e_{l+1,pl+1} = \cdots = e_{l+1,g-n-1} = 0\,. \tag{7.11}$$

Combined with the relations (7.10) and (7.11), we get

$$e_{i,g-d+n+2} = \cdots = e_{i,g} = 0,\ e_{i,j} = \lambda_{i-l-1}e_{l+1,j}$$

for $l+1 \leq i \leq g$ and $g-n \leq j \leq g-d+n+1$ with $\lambda_{i-l-1} \in k$. In particular, λ_{g-l-1} and c_{p+n-d} are both non-zero as $e_{g,g-n} = c_{(p-1)g+n} = 1 = \lambda_{g-l-1}e_{l+1,g-n} = \lambda_{g-l-1}c_{p+n-d}$. Then $c_m = \lambda_{i-l-1}c_{pl+p-j}$ if $m = pi - j$ for some i, j such that $l+1 \leq i \leq g$ and $g - n \leq j \leq g - d + n + 1$ and $c_m = 0$ otherwise.

Note that

$$\begin{aligned} f^n &= \sum_{i=n}^{(p-1)g+n} c_i x^i = (\sum_{i=0}^{p-d} c_{n+i}x^{n+i})(\sum_{j=0}^{g-l-1} \lambda_j x^{jp}) \\ &= (\sum_{i=0}^{p-d} c_{n+i}x^{n+i})(\sum_{j=0}^{g-l-1} \lambda_j^{1/p} x^j)^p \,. \end{aligned}$$

This implies $f^{(p-1/2)}$ has a root with multiplicity $\geq p$. Then f also has multiple root, a contradiction. Hence $\operatorname{rank}(H) \geq l+2$. □

Now we can easily prove Theorem 7.3.

Proof of Theorem 7.3. Since the a-number $a = g - \operatorname{rank}(H)$, by the Lemma 7.9, we have

$$a \leq \lceil (p-1)g/p - (3p-1)/2p \rceil$$

as desired.

□

So we know the a-number of a hyperelliptic curve in characteristic $p \geq 3$ of genus $g \geq p$ satisfies certain condition. But for the other direction, we have the following result.

Proposition 7.10. *Suppose $g \geq p$ and $p \geq 3$. Then there exists hyperelliptic curve X with $a(X) \geq \lfloor g/p \rfloor (p-1)/2$.*

Proof. If $g \not\equiv 0 \pmod{p}$, then write $g = pl + d$ for some $l, d \in \mathbb{Z}_{>0}$ such that $1 \leq d \leq p-1$. We consider the curve X given by equation $y^2 = f$ with

$$f = x^{2g+1} + x \,.$$

One can easily show that $\operatorname{disc}(f) = 0$ if and only if $g \equiv 0 \pmod{p}$. Again we let $n := (p-1)/2$ and denote by c_i the coefficient of x^i in f^n. Then we have

$$c_j = \begin{cases} 1 & \text{if } j = 2ig + n \text{ for } 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{7.12}$$

As a basis of $H^0(X, \Omega^1_X)$, we choose

$$\omega_i = \frac{x^i \, \mathrm{d}x}{y}, \ i = 0, \ldots, g-1 \, . \tag{7.13}$$

Then the Cartier-Manin matrix H has entries $H_{i,j} = c_{pi-j}^{1/p}$ with $1 \leq i, j \leq g$. Also we put $e_i = (e_{i,1}, \ldots, e_{i,g})$ to be i-th row of H for $1 \leq i \leq g$.

We show there are at least nl zero rows in H, then in this case the a-number of the curve $a = g - \mathrm{rank}(H) \geq nl = (p-1)/2\lfloor g/p \rfloor$.

For $0 \leq i \leq n-1$, suppose $(m-1)p \leq 2id + n \leq mp - 1$ for some $m \in \mathbb{Z}_{>0}$ and write $mp - 1 = 2id + n + s$ with $s \in \mathbb{Z}_{>0}$. Now if $d \geq s+1$, by (7.12) we have $e_{(2i+1)l+m+1} = e_{(2i+1)l+m+2} = \cdots = e_{(2i+2)l+m} = (0, \ldots, 0)$. Otherwise for $1 \leq d \leq s$, we have

$$\begin{aligned} e_{(2i+1)l+m,g} &= c_{(2i+1)pl+mp-g} = 0, \\ (2i+1)pl + mp - g &= 2ipl + mp - d > 2ipl + mp - 1 - s = 2ig + n \, . \end{aligned}$$

Then we have $e_{(2i+1)l+m} = e_{(2i+1)l+m+1} = \cdots = e_{(2i+2)l+m-1} = (0, \ldots, 0)$. In any case, we always have l zero rows for any i with $0 \leq i \leq n-1$. Hence there are at least nl zero rows in H and we have the desired conclusion.

Now if $g = pl$ for some $l \in \mathbb{Z}_{>0}$, then we consider the curve X given by equation $f = x^{2g+1} + a_2 x^2 + x$ with $a_2 \in k^*$. One can easily show that $\mathrm{disc}(f) \neq 0$. Let n, c_i be the same as above. With respect to the basis as in (7.13), we show that the Cartier-Manin matrix has nl zero rows.

First we claim that $c_j = 0$ for $j \notin [2ipl + n, \, 2ipl + p - 1 - i]$ and $i = 0, 1 \ldots, n$. Indeed, put $a_{2g+1} = a_1 = 1$ we always have

$$c_j = \sum_{j_1 + j_2 + j_3 = n} b_{j_i, j_2, j_3} a_{2g+1}^{j_1} a_2^{j_2} a_1^{j_3}$$

with $b_{j_i, j_2, j_3} \in k$, $j_1, j_2, j_3 \in \mathbb{Z}_{\geq 0}$ and $j = (2g+1)j_1 + 2j_2 + j_3$. For an integer j, write $j = mpl + d$ for some $m, d \in \mathbb{Z}$ such that $0 \leq d \leq pl - 1$. Suppose there are $j_1, j_2, j_3 \in \mathbb{Z}_{\geq 0}$ satisfying equations $j_1 + j_2 + j_3 = n$ and $(2g+1)j_1 + 2j_2 + j_3 = j$. This implies $2j_1 pl + j_2 = j - n = mpl + d - n$ and hence $m = 2j_1$ and $j_2 = d - n$. Moreover, the inequality $0 \leq j_2 \leq n - j_1$ implies $n \leq d \leq p - 1 - j_1$ and the claim has been proven.

By the claim we know that $e_{(2i+1)l+1} = e_{(2i+1)l+2} = \cdots = e_{(2i+2)l} = (0, \ldots, 0)$ for $i = 0, \ldots, n$. Additionally, we have

$$\begin{aligned} e_{(2il+m)} &= (0, \ldots, 0, c_{2ipl+p-1}, \ldots, c_{2ipl+p-1-i}, \ldots, c_{2ipl+n}, 0, \ldots, 0) \\ &= (0 \ldots, 0, c_{2ipl+p-1-i}, \ldots, c_{2ipl+n}, 0, \ldots, 0) \end{aligned}$$

for any $0 \leq i \leq n$ and $1 \leq m \leq l$. Hence rows e_{2il+m} with $0 \leq i \leq n$ and $1 \leq m \leq l$ are linearly independent.

We see that $\operatorname{rank}(H) = (n+1)l$ and hence $a(X) = g - \operatorname{rank}(H) = nl$.

□

Remark 7.11. Take $p = g = 5$. We consider a family of curves given by the equation $y^2 = x^{11} + a_2 x^2 + x$ with $a_2 \in k^*$. Then computing the rank and semi-simple rank of the Cartier-Manin matrix, we have that this family parametrises a 1-dimensional locus in $\mathcal{H}_5$ of curves with constant 5-rank 2 and a-number 2.

7.6 Hyperelliptic curves in characteristic 5

In Theorem 7.3, we studied the a-number of hyperelliptic curve of genus $g \geq p$. Now if $p = 5$ and $g = 4$, then we have the following result.

Proposition 7.12. *Let k be an algebraically closed field with characteristic 5. Then for any smooth hyperelliptic curve X of genus $g = 4$ over k, we have $a(X) \leq 2$.*

Proof. Any smooth hyperelliptic curve X of genus g can be written as $y^2 = f$ with $f \in k[x]$ of $\deg f = 2g + 1$ and non-zero discriminant. By putting a branch point at 0 and by scaling we may assume that

$$f = x^{2g+1} + a_{2g}x^{2g} + \cdots + a_2 x^2 + x \tag{7.14}$$

with $a_i \in k$ for $i = 2, \ldots, 2g$. For $i \in \mathbb{Z}$, we denote by c_i the coefficient of x^i in $f^2 \in k[x]$. As a basis of $H^0(X, \Omega^1_X)$ we choose $\omega_i = x^i \, \mathrm{d}x/y$ for $i = 0, \ldots, g-1$. Then the Cartier-Manin matrix H has entries $H_{i,j} = h_{ij}^{1/5} = c_{pi-j}^{1/5}$ with $1 \leq i, j \leq g$. Also we put e_i to be i-th row of H for $1 \leq i \leq g$.

For $g = 4$, we find the Cartier-Manin matrix $H = (h_{ij}^{1/5})$ with

$$\begin{aligned}
&h_{11} = a_2^2 + 2a_3\,; h_{12} = 2a_2\,; h_{13} = 1\,; h_{14} = 0\,;\\
&h_{21} = 2a_2a_7 + 2a_3a_6 + 2a_4a_5 + 2a_8\,; h_{22} = 2a_2a_6 + 2a_3a_5 + a_4^2 + 2a_7\,;\\
&h_{23} = 2a_2a_5 + 2a_3a_4 + 2a_6\,; h_{24} = 2a_2a_4 + a_3^2 + 2a_5\,;\\
&h_{31} = 2a_5 + 2a_6a_8 + a_7^2\,; h_{32} = 2a_4 + 2a_5a_8 + 2a_6a_7\,;\\
&h_{33} = 2a_3 + 2a_4a_8 + 2a_5a_7 + a_6^2\,; h_{34} = 2a_2 + 2a_3a_8 + 2a_4a_7 + 2a_5a_6\,;\\
&h_{41} = 0\,; h_{42} = 1\,; h_{43} = 2a_8\,; h_{44} = 2a_7 + a_8^2\,.
\end{aligned}$$

Hence $\operatorname{rank}(H) \geq 1$. If $\operatorname{rank}(H) = 1$, then e_i is multiply of e_1 for $2 \leq i \leq 4$. This gives $a_2^2 + 2a_3 = 0, 2a_7 + a_8^2 = 0, a_2a_8 = -1$. Hence we have $e_1 = (0, 2a_2, 1, 0)$ and

$h_{i1} = h_{i4} = 0$ for all i. Then we obtain a system of equations

$$1 + 2a_2^3 a_6 + a_2 a_4 a_5 = 0\,, \tag{7.15}$$
$$a_2^4 a_5 - a_2^3 a_6 + 2 = 0\,, \tag{7.16}$$
$$a_2 a_4 + a_5 + 2a_2^4 = 0\,, \tag{7.17}$$
$$-a_2^3 + 2a_4 + a_2^3 a_5 a_6 = 0\,. \tag{7.18}$$

Eliminate a_6 by equations (7.15)+2 $\times$ (7.16) and we get $a_2 a_4 a_5 + 2a_2^4 a_5 = 0$. Then we have $a_4 = -2a_2^3$ or $a_5 = 0$.

If $a_5 = 0$, then by solving these equations above with $a_5 = 0$ we have $a_3 = 2a_2^2, a_4 = -2a_2^3, a_5 = 0, a_6 = 2/a_2^3, a_7 = 2/a_2^2$ and $a_8 = -1/a_2$. This implies $f = (x^4 - 1/a_2 x^3 + 2/a_2^2 x^2 + 2/a_2^3 x)(x + 3a_2^{3/5})^5$, a contradiction.

If $a_5 \neq 0$ and $a_4 = -2a_2^3$, then by equation (7.18) we have $a_2^3 a_5 a_6 = 0$ and hence $a_6 = 0$. Since the second row is a multiple of first row, we have $h_{22} = 2a_2 h_{23}$, which can be simplified to $a_2^6 + 1/a_2^2 = a_2^6$, a contradiction since $a_2 \neq 0$.

From the discussion above, we always have $\operatorname{rank}(H) \geq 2$ for $g = 4$ and 5. Then $a(X) = g - \operatorname{rank}(H) \leq g - 2 = 2$.

□

Remark 7.13. The bound we find is sharp because if $a_2 = a_3 = \cdots = a_8 = 0$ in (7.14), then the curve X given by equation $y^2 = x^9 + x$ has $\operatorname{rank}(H) = 2$ and $a(X) = 2$.

Remark 7.14. Note that Frei [18] gave a condition when a hyperelliptic curve in characteristic $p > 3$ of genus $g = p - 1$ has a-number $g - 1$.

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Samenvatting

Dit proefschrift is gewijd aan krommen in positieve karakteristiek. Het is welbekend dat krommen in karakteristiek $p > 0$ zich anders gedragen dan krommen in karakteristiek 0. Een eerste voorbeeld is dat van elliptische krommen, waar de kern van vermenigvuldiging met p twee vormen kan aannemen, en dit geeft twee gevallen: de elliptische kromme heet gewoon of supersingulier. Dit leidt tot een stratificatie van de moduliruimte (of parameterruimte) van elliptische krommen in positieve karakteristiek.

Dit idee is uitgebreid naar het geval van abelse variëteiten met een hoofdpolarisatie waarvoor Ekedahl en Oort een stratificatie op de moduliruimte hebben geïntroduceerd die de stratificatie op de moduliruimte van elliptische krommen generaliseert. Deze stratificatie heeft veel aandacht aangetrokken in recente jaren, en er is veel bekend over de strata, zoals dimensies, reduceerbaarheid, en hun cohomologieklassen. Sommige van de strata zijn gedefinieerd door een enkele invariant, zoals de p-rang van en abelse variëteit, of zijn a-getal. In het algemeen zijn de strata gedefinieerd door de werking van Frobenius en Verschiebung op de eerste de-Rham-cohomologie.

Door aan een kromme van geslacht g zijn Jacobiaan (een abelse variëteit met hoofdpolarisatie) te koppelen, krijgen we een afbeelding van de moduliruimte van krommen van geslacht g naar de moduliruimte van abelse variëteiten met hoofdpolarisatie van dimensie g, de Torelli afbeelding. Langs deze Torelli afbeelding kunnen we de stratificatie op de moduliruimte van abelse variëteiten met hoofdpolarisatie terugtrekken naar de moduliruimte van krommen. We kunnen bijvoorbeeld naar de p-rang of het a-getal van de Jacobiaan kijken; het a-getal is in dit geval gelijk aan het maximale aantal van lineair onafhankelijke exacte differentiaalvormen op de kromme.

In deze situatie is het natuurlijk om vragen te stellen over eigenschappen van de geïnduceerde strata op de moduliruimte van krommen. Maar er is veel minder bekend voor het geval van krommen dan in het geval van abelse variëteiten. Bijvoorbeeld, de dimensies van de strata zijn welbekend voor abelse variëteiten, maar er is veel minder bekend over de dimensies van de strata op de moduliruimte van krommen

van gegeven geslacht. Er zijn resultaten over de dimensies van de loci met gegeven p-rang (door Faber en van der Geer) en er is een resultaat van Ekedahl dat zegt dat krommen in karakteristiek p met maximaal a-getal (dat wil zeggen, gelijk aan het geslacht g) aan $g \leq p(p-1)/2$ voldoen, maar afgezien daarvan zijn slechts enkele resultaten bekend.

We behandelen in dit proefschrift de a-getallen van krommen in karakteristiek p, en de Ekedahl-Oort strata op de moduliruimte van krommen, voornamelijk in laag geslacht en lage karakteristiek. Naast de bovenstaande karakterisering is het a-getal van een kromme ook gelijk aan de dimensie van de kern van de Cartier operator werkend op de ruimte van reguliere differentiaalvormen op de kromme.

Een van onze resultaten (in Hoofdstuk 2) is een versterking van een resultaat van R. Re. We geven een bovengrens voor het geslacht van een kromme waarvoor de Cartier operator rang 1 heeft. Re behandelt het geval van (nilpotente) rang 0. Wij doen dit door lineaire systemen op de kromme en de gerelateerde werking van de Cartier operator te bestuderen.

Andere resultaten gaan over de moduliruimte van hyperelliptische krommen van geslacht 4 in karakteristiek 3. Om de geïnduceerde stratificatie te bestuderen geven een expliciete beschrijving van de de-Rham-cohomologie en hoe Verschiebung erop werkt. Dit wordt gedaan in hoofdstuk 3. In het hierop volgende hoofdstuk behandelen we overdekkingen van de projectieve lijn. In geslacht 4 laten we zien dat bepaalde strata in de locus van p-rang 0 alleen voorkomen als aan bepaalde congruenties op de karakteristiek is voldaan. We bestuderen de dimensies van alle strata op de moduliruimte van geslacht 4 krommen in karakteristiek 3. In de laatste twee hoofdstukken kijken we naar krommen van verscheidene speciale types, zoals Artin-Schreier krommen, hyperelliptische krommen en trigonale krommen. Door de werking van de Cartier operator op differentiaalvormen expliciet te bestuderen kunnen Artin-Schreier krommen met hoog a-getal construeren. Voor hyperelliptische krommen gebruiken we de Hasse-Witt matrix om de a-getallen te begrenzen in het geval waar het geslacht lager is dan de karakteristiek. Voor trigonale krommen in lage karakteristiek begrenzen we ook de a-getallen.

Summary

This thesis deals with curves in positive characteristic. It is well known that curves in positive characteristic p display a behaviour different from curves in characteristic zero. A first example is the case of elliptic curves, where the kernel of multiplication by p can assume two types: the elliptic curve is called ordinary or supersingular. This leads to a stratification of the moduli space (parameter space) of elliptic curves in positive characteristic.

This idea is extended to the case of (principally polarized) abelian varieties where Ekedahl and Oort introduced a stratification on the moduli space generalizing the stratification on the moduli of elliptic curves. This stratification has attracted a lot of attention in recent years and many results are known about the strata, like dimensions, reducibility and their cohomology classes. Some of the strata are defined by a single invariant, like the p-rank of an abelian variety, or its a-number. In general it is defined by the action of Frobenius and Verschiebung on the first de Rham cohomology.

By associating to a curve of genus g its Jacobian variety, a principally polarized abelian variety of dimension g, we get a map from the moduli of curves of genus g to the moduli of principally polarized abelian varieties of dimension g, the Torelli map. Using this Torelli map one can pull back the stratification of the moduli of principally polarized abelian varieties of dimension g to the moduli space of curves of genus g. For example, one can look at the p-rank of the Jacobian or its a-number; the a-number has an interpretation as the maximal number of linearly independent global exact differentials on the curve.

Given this situation it is natural to ask for the properties of the induced strata on the moduli space of curves. But much less is known for the case of curves than for the case of abelian varieties. For example the dimensions of the strata are well-known for the case of abelian varieties, but about the dimensions of the strata on the moduli of curves of given genus not much is known. There are results on the dimensions of the loci of fixed p-rank (by Faber-van der Geer) and there is a result of Ekedahl that says that curves in characteristic p with maximal a-number (that

is, equal to the genus g) satisfy $g \leq p(p-1)/2$, but apart from that only scattered results are available.

In this thesis we deal with the a-numbers of curves in characteristic p and with the Ekedahl-Oort strata on the moduli spaces of curves, mostly in low genus or for small characteristic. Besides the characterization given above, the a-number of a curve is also equal to the dimension of the kernel of the Cartier operator acting on the space of regular differential forms on the curve.

One of our results (in Chapter 2) is an improvement of a result of R. Re. We give an upper bound on the genus of a curve for which the Cartier operator has rank 1. Re dealt with the case of (nilpotent) rank 0. We do this by studying linear systems on the curve and the related action of the Cartier operator.

Other results deal with the moduli of hyperelliptic curves of genus 4 and characteristic 3. To study the induced stratification we give an explicit description of the de Rham cohomology and how Verschiebung acts on it. This covers Chapter 3. In the next chapter we deal with covers of the projective line. In genus 4 we show that certain strata inside the p-rank 0 locus only do occur if certain congruence conditions on the characteristics are fulfilled. We study the dimensions of all strata for the moduli of genus 4 in characteristic 3 . In the last two chapters we look at curves of various special types, like Artin-Schreier curves, hyperelliptic curves and trigonal curves. By studying the action of the Cartier operator on differential forms explicitly we are able to construct Artin-Schreier curves with large a-numbers. For hyperelliptic curves we use the Hasse-Witt matrix to give bounds on the a-numbers for the case where the genus is larger that the characteristic. Similarly for trigonal curves in small characteristics, we bound their a-numbers.