Estimating the Intensity of a Cyclic Poisson Process
Wayan Mangku, I. 

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 2

Estimation of the global intensity

2.1 Introduction

In this chapter we focus on estimation of the global intensity $\theta$, using only a single realization $X(\omega)$ of the cyclic Poisson process $X$ observed in $W_n$. This chapter is a revised version of section 2 of Helmers and Mangku (2000). An estimator for this parameter of interest is given by

$$\hat{\theta}_n = \frac{X(W_n)}{|W_n|}. \quad (2.1)$$

One way to obtain the estimator $\hat{\theta}_n$ in (2.1) is as follows. If the Poisson process $X$ is homogeneous, $\mu(B) = \lambda_0 \nu(B) = \lambda_0 |B|$, for some constant $\lambda_0 > 0$ and all Borel sets $B$, the local intensity is constant, i.e. $\lambda(s) = \lambda_0$ for all $s \in \mathbb{R}$. The global intensity $\theta$ is precisely equal to $\lambda_0$ in this very special case, and the maximum likelihood method can be applied to estimate $\theta$. Let $s_i, i = 1, \ldots, X(W_n)$ denote the locations of the points in the realization $X(\omega)$ of the Poisson process, observed in $W_n$. Then, the likelihood of $(s_1, \ldots, s_{X(W_n)})$ is equal to

$$L_n = e^{-\int_{W_n} \lambda(s) ds} \prod_{i=1}^{X(W_n)} \lambda(s_i) = e^{-\lambda_0 |W_n|} \lambda_0^{X(W_n)},$$

where $X(W_n)$ denotes the observed number of points in $W_n$ (cf. Cressie, 1993, p. 655). Maximizing $\ln L_n$ gives us:

$$\frac{d \ln L_n}{d \lambda_0} = \frac{d}{d \lambda_0} \left( -\lambda_0 |W_n| + X(W_n) \ln \lambda_0 \right) = -|W_n| + \frac{X(W_n)}{\lambda_0} = 0,$$

which directly yields the MLE $\hat{\theta}_n = \frac{X(W_n)}{|W_n|}$ of $\lambda_0$ and hence of $\theta$ as well.
In Lemma 2.1 we prove that for the cyclic Poisson process \( \theta \) is well-defined by (1.3) and can now also be written as

\[
\theta = \frac{1}{\tau} \int_{U_\tau} \lambda(s)ds,
\]

where \( U_\tau \) denote any interval of length \( \tau \). In Lemma 2.2 we will show that \( \hat{\theta}_n \) is a consistent estimator of the global intensity \( \theta \) of \( X \). Complete convergence (implying strong consistency) of \( \hat{\theta}_n \) is established in Lemma 2.3, while the asymptotic normality \( \hat{\theta}_n - \theta \), properly normalized, is derived in Theorem 2.4. A bootstrap CLT for \( \hat{\theta}_n - \theta \) is established in Theorem 2.5.

### 2.2 Consistency

**Lemma 2.1** If \( \lambda \) is periodic (with period \( \tau \)) and locally integrable, then

\[
\theta_n = \frac{\mathbb{E}X(W_n)}{|W_n|} = \frac{1}{|W_n|} \int_{W_n} \lambda(s)ds \to \theta,
\]

as \( n \to \infty \), with \( \theta \) as in (2.2). Hence \( \hat{\theta}_n \) is asymptotically unbiased in estimating \( \theta \).

**Proof:** Let \( N_{n\tau} = \lfloor \frac{|W_n|}{\tau} \rfloor \). Let \( W_{N_{n\tau}} \) denote an interval of length \( \tau N_{n\tau} \) contained in \( W_n \), and \( R_{n\tau} = W_n \setminus W_{N_{n\tau}} \). Then we can write

\[
\theta_n = \frac{|W_{N_{n\tau}}|}{|W_n|} \cdot \frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s)ds + \frac{1}{|W_n|} \int_{R_{n\tau}} \lambda(s)ds.
\]

First note that

\[
\frac{1}{|W_{N_{n\tau}}|} \int_{W_{N_{n\tau}}} \lambda(s)ds = \theta
\]

because \( \lambda \) is periodic with period \( \tau \). Since \( |R_{n\tau}| < \tau \) for all \( n \), we have that

\[
\frac{|W_{N_{n\tau}}|}{|W_n|} = \frac{|W_n| - |R_{n\tau}|}{|W_n|} \to 1,
\]

as \( n \to \infty \). Because \( \lambda \) is locally integrable and \( |R_{n\tau}| = \mathcal{O}(1) \), as \( n \to \infty \), we also know that

\[
\int_{R_{n\tau}} \lambda(s)ds = \mathcal{O}(1), \text{ as } n \to \infty.
\]

Hence, the first term on the r.h.s. of (2.4) converges to \( \theta \), while its second term (by (1.2)) converges to zero, as \( n \to \infty \). This completes the proof of Lemma 2.1. \( \square \)
Lemma 2.2 If $\lambda$ is periodic (with period $\tau$) and locally integrable, then
\[ \hat{\theta}_n \xrightarrow{p} \theta, \]
(2.7)
as $n \rightarrow \infty$.

Proof: To prove (2.7) we must show, for each $\epsilon > 0$,
\[ P(|\hat{\theta}_n - \theta| \geq \epsilon) \rightarrow 0, \]
(2.8)
as $n \rightarrow \infty$. Since $X(W_n)$ has Poisson distribution with parameter $\mu(W_n) = \int_{W_n} \lambda(s)ds$, we know that
\[ E(X(W_n)) = Var(X(W_n)) = \int_{W_n} \lambda(s)ds. \]
Then we have
\[ E(\hat{\theta}_n) = \frac{1}{|W_n|} \int_{W_n} \lambda(s)ds, \quad \text{and} \quad Var(\hat{\theta}_n) = \frac{1}{|W_n|^2} \int_{W_n} \lambda(s)ds. \]
Now we write
\[ P \left( |\hat{\theta}_n - \theta| \geq \epsilon \right) \leq P \left( |\hat{\theta}_n - E\hat{\theta}_n| + |E\hat{\theta}_n - \theta| \geq \epsilon \right). \]
By Lemma 2.1, for sufficiently large $n$, we have $|E\hat{\theta}_n - \theta| \leq \epsilon/2$. Then, for sufficiently large $n$, we have
\[ P \left( |\hat{\theta}_n - \theta| \geq \epsilon \right) \leq P \left( |\hat{\theta}_n - E\hat{\theta}_n| \geq \frac{\epsilon}{2} \right). \]
By Chebyshev's inequality and Lemma 2.1, the r.h.s. of (2.9) does not exceed
\[ \frac{4Var(\hat{\theta}_n)}{\epsilon^2} = \frac{4}{\epsilon^2 |W_n|^2} \int_{W_n} \lambda(s)ds = \frac{4}{\epsilon^2 |W_n|}(\theta + o(1)), \]
(2.10)
as $n \rightarrow \infty$. By (1.2), the r.h.s. of (2.10) is $o(1)$, as $n \rightarrow \infty$. This completes the proof of Lemma 2.2. □

Throughout this thesis, for any random variables $Y_n$ and $Y$ on a probability space $(\Omega, \mathcal{A}, P)$, we write $Y_n \xrightarrow{c} Y$ to denote that $Y_n$ converges completely to $Y$, as $n \rightarrow \infty$. We say that $Y_n$ converges completely to $Y$ if $\sum_{n=1}^{\infty} P(|Y_n - Y| > \epsilon) < \infty$, for every $\epsilon > 0$. 
Lemma 2.3 Suppose that \( \lambda \) is periodic (with period \( \tau \)) and locally integrable. If, in addition, for each \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} \exp\{-\epsilon|W_n|\} < \infty, \tag{2.11}
\]

then, as \( n \to \infty \),

\[
\hat{\theta}_n \overset{\mathcal{L}}{\to} \theta. \tag{2.12}
\]

Proof: To establish (2.12) we must show

\[
\sum_{n=1}^{\infty} \mathbb{P}\left(|\hat{\theta}_n - \theta| > \epsilon \right) < \infty, \tag{2.13}
\]

for each \( \epsilon > 0 \). Now, recall from the proof of Lemma 2.2 that, for sufficiently large \( n \), the probability on the l.h.s. of (2.13) does not exceed that on the r.h.s. of (2.9). Then, to prove (2.13), it suffices to check that the probability on the r.h.s. of (2.9) is summable. By an application of Lemma A.1 (see Appendix), and with \( \theta_n \) as in (2.3), the probability on the r.h.s. of (2.9) can be bounded above as follows.

\[
P\left(|\hat{\theta}_n - \mathbb{E}\hat{\theta}_n| \geq \frac{\epsilon}{2}\right) = \mathbb{P}\left(|W_n|^{-1}|X(W_n) - \mathbb{E}X(W_n)| \geq \frac{\epsilon}{2}\right)
\]

\[
= \mathbb{P}\left((\mathbb{E}X(W_n))^{-1/2}|X(W_n) - \mathbb{E}X(W_n)| \geq \frac{\epsilon|W_n|}{2(\mathbb{E}X(W_n))^{1/2}}\right)
\]

\[
\leq 2 \exp\left\{-\frac{\epsilon^2 d-1|W_n|^2(\mathbb{E}X(W_n))^{-1}}{2 + \epsilon^2 |W_n|(\mathbb{E}X(W_n))^{-1}}\right\} = 2 \exp\left\{-\frac{\epsilon^2 |W_n|}{8\theta_n + 2\epsilon}\right\}. \tag{2.14}
\]

For sufficiently large \( n \), since by Lemma 2.1 we have \( \theta_n = \theta + o(1) \), as \( n \to \infty \), the r.h.s. of (2.14) does not exceed \( 2 \exp\{-\epsilon^2 |W_n|(16\theta + 2\epsilon)^{-1}\} \). By assumption (2.11), we can conclude that the quantity on the r.h.s. of (2.14) is summable. This completes the proof of Lemma 2.3. \( \square \)

In view of Lemma 2.3, we may replace definition (1.3) of the global intensity \( \theta \) by the following one

\[
\theta = \lim_{n \to \infty} \frac{X(W_n)}{|W_n|} \quad \text{a.s.}[\mathcal{P}] \tag{2.15}
\]

provided (2.11) holds. Note that (2.15) is similar to the notion of global intensity described in (8.3.22) of Cressie (1993, p. 629).
2.3 Asymptotic normality

Asymptotic normality of $\hat{\theta}_n$, properly normalized, is established in the following theorem.

**Theorem 2.4** If $\lambda$ is periodic (with period $\tau$) and locally integrable, then

$$|W_n|^{1/2} \left( \hat{\theta}_n - \theta \right) \xrightarrow{d} N(0, \theta), \quad (2.16)$$

as $n \to \infty$.

**Proof:** First we write

$$|W_n|^{1/2} \left( \hat{\theta}_n - \theta \right) = |W_n|^{1/2} \left( \hat{\theta}_n - \theta_n \right) + |W_n|^{1/2} (\theta_n - \theta), \quad (2.17)$$

where $\theta_n$ is given by the l.h.s. of (2.3). Then, to prove (2.16), it suffices to check

$$|W_n|^{1/2} (\hat{\theta}_n - \theta_n) \xrightarrow{d} N(0, \theta), \quad (2.18)$$

and

$$|W_n|^{1/2} (\theta_n - \theta) \to 0, \quad (2.19)$$

as $n \to \infty$.

First we prove (2.18). The l.h.s. of (2.18) can be written as

$$|W_n|^{1/2} \left( \frac{X(W_n)}{|W_n|} - \frac{\int_{W_n} \lambda(s)ds}{|W_n|} \right)$$

$$= \left( \frac{\int_{W_n} \lambda(s)ds}{|W_n|^{1/2}} \right)^{1/2} \left( \frac{X(W_n) - \int_{W_n} \lambda(s)ds}{(\int_{W_n} \lambda(s)ds)^{1/2}} \right). \quad (2.20)$$

By Lemma 2.1, (1.2), and the normal approximation to the Poisson distribution, the r.h.s. of (2.20) can be written as $(\theta^{1/2} + o(1))(N(0, 1) + o_p(1))$, which converges in distribution to $N(0, \theta)$ as $n \to \infty$.

Next we prove (2.19). Substituting (2.5) into the r.h.s. of (2.4), and by writing $|W_{n\tau}|$ as $(|W_n| - |R_{n\tau}|)$, we can simplify the r.h.s. of (2.4) to get

$$\theta_n = \theta - \frac{\theta |R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|} \int_{R_{n\tau}} \lambda(s)ds. \quad (2.21)$$

The l.h.s. of (2.19) now reduces to

$$|W_n|^{1/2} \left( \frac{\theta |R_{n\tau}|}{|W_n|} + \frac{1}{|W_n|} \int_{R_{n\tau}} \lambda(s)ds \right)$$

$$= \left( \frac{\theta |R_{n\tau}|}{|W_n|^{1/2}} + \frac{\int_{R_{n\tau}} \lambda(s)ds}{|W_n|^{1/2}} \right). \quad (2.22)$$
Since $|R_{n\tau}| < \tau$ for all $n$ and $\int_{R_{n\tau}} \lambda(s)ds = \mathcal{O}(1)$, as $n \to \infty$, then by (1.2), the r.h.s. of (2.22) is $o(1)$, as $n \to \infty$. This completes the proof Theorem 2.4. \(\square\)

To conclude this section we derive a bootstrap CLT, parallel to Theorem 2.4. Conditionally given $X(W_n)$, let $X^*(W_n)$ denote a realization from a Poisson distribution with parameter $X(W_n)$. If $X(W_n)$ happens to be equal to zero, we set $X^*(W_n) = 0$. Define

$$\hat{\theta}_n^* = \frac{X^*(W_n)}{|W_n|^i}, \quad (2.23)$$

To obtain a bootstrap counterpart of (2.16), we replace $\hat{\theta}_n - \theta$ by $\hat{\theta}_n^* - \hat{\theta}_n$, with $\hat{\theta}_n$ as in (2.1), and establish bootstrap consistency, i.e. we shall prove that $|W_n|^{i/2}(\hat{\theta}_n^* - \hat{\theta}_n)$ has - in P-probability - the same limit distribution as $|W_n|^{i/2}(\hat{\theta}_n - \theta)$, that is a normal $(0, \theta)$ distribution. Note that we have employed a 'parametric bootstrap' here. There is no use for Efron's bootstrap, instead our bootstrap is based on a parametric model, namely a Poisson distribution with estimated parameter.

**Theorem 2.5** If $\lambda$ is periodic (with period $\tau$) and locally integrable, then

$$|W_n|^{1/2} (\hat{\theta}_n^* - \hat{\theta}_n) \overset{d}{\to} N(0, \theta), \quad (2.24)$$

as $n \to \infty$, in P-probability. Hence our parametric bootstrap works.

**Proof:** Since $X^*(W_n)$ has Poisson distribution with parameter $X(W_n)$, it suffices to write the l.h.s. of (2.24) as

$$|W_n|^{1/2} \left( \frac{X^*(W_n)}{|W_n|} - \frac{X(W_n)}{|W_n|} \right) = \left( \frac{X(W_n)}{|W_n|} \right)^{1/2} \left( \frac{X^*(W_n) - X(W_n)}{(X(W_n))^{1/2}} \right). \quad (2.25)$$

By Lemma 2.2, (1.2), and the normal approximation to the Poisson distribution, the r.h.s. of (2.25) can be written as

$$\left( \theta^{1/2} + o_p(1) \right) (N(0, 1) + o_p(1)), \quad (2.26)$$

since $X(W_n) \to \infty$, in P-probability, as $\int_{W_n} \lambda(s)ds \to \infty$, which is implied by $|W_n| \to \infty$ (cf.(1.2)), because $\theta > 0$. Hence, by Slutsky (cf. Serfling (1980), p. 19), the quantity in (2.26) converges in distribution to $N(0, \theta)$, as $n \to \infty$, in P-probability. This completes the proof Theorem 2.5. \(\square\)