Estimating the Intensity of a Cyclic Poisson Process
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Chapter 4

Nearest neighbor estimation of the local intensity

4.1 Introduction

In this chapter we consider nearest neighbor estimation of the intensity function \( \lambda \) at a given point \( s \in W_n \), using only a single realization \( X(\omega) \) of the cyclic Poisson process \( X \) observed in \( W_n \). The requirement \( s \in W_n \) can be dropped if we know the period \( \tau \). The first part of this chapter is a revised version of Mangku (1999).

As in chapter 3, let \( \hat{\tau} \) be any consistent estimator of the period \( \tau \), e.g. the one proposed and studied in chapter 5 or perhaps the estimator investigated by Vere-Jones (1982).

Let \( s_i, i = 1, \ldots, X(W_n, \omega) \), denote the locations of the points in the realization \( X(\omega) \) of the Poisson process \( X \), observed in window \( W_n \). Here \( X(W_n, \omega) \) is nothing but the cardinality of the data set \( \{s_i\} \).

It is well-known (see, e.g. Cressie (1993), p. 651) that, for any positive integer \( m \), conditionally given \( X(W_n) = m, (s_1, \ldots, s_m) \) can be viewed as a random sample of size \( m \) from a distribution with density \( f \), which is given by

\[
f(u) = \frac{\lambda(u)}{\int_{W_n} \lambda(v)dv} I(u \in W_n),
\]

while the simultaneous density \( f(s_1, \ldots, s_m) \), of \( (s_1, \ldots, s_m) \) is given by

\[
f(s_1, \ldots, s_m) = \frac{\prod_{i=1}^{m} \lambda(s_i)}{\left( \int_{W_n} \lambda(v)dv \right)^m} I((s_1, \ldots, s_m) \in W_n^m).
\]

Let \( \hat{s}_i, i = 1, \ldots, m \), denote the location of the point \( s_i \) \( (i = 1, \ldots, m) \), after translation by a multiple of \( \hat{\tau}_n \) such that \( \hat{s}_i \in \tilde{B}_{\hat{\tau}_n}(s) \), for all \( i = 1, \ldots, m \).
1, \ldots, m$, where $B_{\tau_n}(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. The translation can be described more precisely as follows. We cover the window $W_n$ by $N_n\tau_n$ adjacent disjoint intervals $B_{\tau_n}(s + j\tau_n)$, for some integer $j$, and let $N_n\tau_n$ denote the number of such intervals, with $B_{\tau_n}(s + j\tau_n) \cap W_n \neq \emptyset$. Then, for each $j$, we shift the interval $B_{\tau_n}(s + j\tau_n)$ (together with the data points of $X(\omega)$ contained in this interval) by the amount $j\tau_n$ such that after translation the interval coincide with $B_{\tau_n}(s)$.

Let $k = k_n$ be a sequence of positive integers such that

\[ k_n \to \infty, \quad (4.3) \]

and

\[ \frac{k_n}{|W_n|} \downarrow 0, \quad (4.4) \]

as $n \to \infty$.

Let now $|\hat{s}_{(k_n)} - s|$ denote the $k_n$-th order statistics of $|\hat{s}_1 - s|, \ldots, |\hat{s}_m - s|$, given $X(W_n) = m$. A nearest neighbor estimator for $\lambda$ at the point $s$, is given by

\[ \hat{\lambda}_n(s) = \frac{\hat{\tau}_n k_n}{2|W_n||\hat{s}_{(k_n)} - s|}, \quad (4.5) \]

if $X(W_n) \geq k_n$, and $\hat{\lambda}_n(s) = 0$ otherwise.

Let us briefly indicate the relation between the kernel type estimator investigated in chapter 3 and the idea behind the construction of our nearest neighbor estimator. Let, for each $\omega$, $\hat{X}_n(\omega)$ denote the set $\{\hat{s}_i\}$, where for any data point $s_i \in X(\omega)$, $\hat{s}_i$ is obtained from $s_i$ by shifting over a random multiple of $\tau_n$ such that $\hat{s}_i \in B_{\tau_n}(s)$. Here and elsewhere in this chapter let, for any set $A$, $\hat{X}_n(A)$ denote the number of points $\hat{s}_i$ in $A$. Then, the 'uniform' kernel estimator in (3.5) can also be written as

\[ \hat{\lambda}_{n, K}(s) = \frac{\hat{\tau}_n \hat{X}_n(B_{h_n}(s))}{|W_n| 2h_n}. \quad (4.6) \]

To obtain our nearest neighbor estimator (4.5), we replace the (random) number $\hat{X}_n(B_{h_n}(s))$ in (4.6) by a (non-random) positive integer $k_n$, i.e. $\hat{X}_n(B_{h_n}(s)) = k_n$, which directly yields that we may take $h_n = |\hat{s}_{(k_n)} - s|$, and (4.6) reduces to (4.5). A detailed comparison of (4.5) and (4.6) is given in section 4.4.

We remark that nearest neighbor estimators for estimating an unknown density function have been studied by Loftsgaarden and Quesenberry (1965), Wagner (1973), Moore and Yackel (1977), Ralescu (1995), among others. The condition (4.9) also appears in Wagner (1973).
the construction of our nearest neighbor estimator (4.5) we employ the periodicity of \( \lambda \) (cf. (1.10)) to combine different pieces from our data set, in order to mimic the 'infill asymptotic' framework.

### 4.2 Consistency

#### 4.2.1 Results

**Theorem 4.1** Suppose that \( \lambda \) is periodic and locally integrable. If, in addition (4.3) and (4.4) hold true, and

\[
\frac{|W_n|^2}{k_n} |\hat{\tau}_n - \tau| \overset{p}{\to} 0, \\
(4.7)
\]

as \( n \to \infty \), then

\[
\hat{\lambda}_n(s) \overset{p}{\to} \lambda(s), \\
(4.8)
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Theorem 4.2** Suppose that \( \lambda \) is periodic and locally integrable. If, in addition

\[
\sum_{n=1}^{\infty} \exp(-\epsilon k_n) < \infty, \\
(4.9)
\]

for each \( \epsilon > 0 \), (4.4) holds, and

\[
\frac{|W_n|^2}{k_n} |\hat{\tau}_n - \tau| \overset{p}{\to} 0, \\
(4.10)
\]

then

\[
\hat{\lambda}_n(s) \overset{p}{\to} \lambda(s), \\
(4.11)
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Remark 4.1** Since

\[
P(k_n \leq X(W_n)) = P(k_n/|W_n| \leq X(W_n)/|W_n|) \to 1, \\
(4.12)
\]

as \( n \to \infty \), (because of (4.4) and by Lemma 2.2 we have \( X(W_n)/|W_n| \overset{p}{\to} \theta \), with \( \theta > 0 \)), we can conclude that no matter how we define \( \hat{\lambda}_n(s) \) in case \( k_n > X(W_n) \), Theorem 4.1 remains valid. To check that the above conclusion also holds for Theorem 4.2, we need to show that

\[
\sum_{n=1}^{\infty} P(k_n > X(W_n)) < \infty.
\]

But, by (4.4), Lemma A.1 (see Appendix), and (4.9), it is easy to show that \( P(k_n > X(W_n)) \) is summable. \( \Box \)
4.2.2 Proofs: the case $\tau$ is known

We first consider the situation where we know the period $\tau$. Let $\tilde{s}_i$, $i = 1, \ldots, X(W_n, \omega)$, denote the location of the points $s_i$ ($i = 1, \ldots, X(W_n, \omega)$), after translation by a multiple of $\tau$ such that $\tilde{s}_i \in \tilde{B}_\tau(s)$, for all $i = 1, \ldots, X(W_n, \omega)$, where $\tilde{B}_\tau(s) = [s - \frac{\tau}{2}, s + \frac{\tau}{2})$. By periodicity of $\lambda$, we have that $\lambda(\tilde{s}_i) = \lambda(s_i)$, for each $i = 1, \ldots, X(W_n, \omega)$. For any $A \subset \tilde{B}_\tau(s)$, let $X_n(A)$ denote the number of points $\tilde{s}_i$ in $A$. Then, of course, $X_n(\tilde{B}_\tau(s)) = X(W_n)$, where $X_n$ is a Poisson process with intensity function

$$\lambda_n(u) = \lambda(u) \sum_{j=-\infty}^{\infty} I(u + j\tau \in W_n)$$

(cf. Kingman (1993), Superposition Theorem and Restriction Theorem, p. 16-17). As a result, (cf. (4.1) and (4.2)), conditionally given $X_n(\tilde{B}_\tau(s)) = m$, $(\tilde{s}_1, \ldots, \tilde{s}_m)$ can be viewed as a random sample of size $m$ from a distribution with density $\tilde{f}$, which is given by

$$\tilde{f}(u) = \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} I(u \in \tilde{B}_\tau(s)) = \frac{\lambda_n(u)}{\int_{\tilde{B}_\tau(s)} \lambda_n(v)dv} I(u \in \tilde{B}_\tau(s)),$$

while the simultaneous density $\tilde{f}(\tilde{s}_1, \ldots, \tilde{s}_m)$, of $(\tilde{s}_1, \ldots, \tilde{s}_m)$ is given by

$$\tilde{f}(\tilde{s}_1, \ldots, \tilde{s}_m) = \frac{\prod_{i=1}^{m} \lambda_n(\tilde{s}_i)}{\left(\int_{W_n} \lambda(v)dv\right)^m} I((\tilde{s}_1, \ldots, \tilde{s}_m) \in \tilde{B}_\tau(s)^m).$$

For any real number $x \geq 0$, define

$$H_n(x) = P(|\tilde{s}_i - s| \leq x \mid X(W_n) = m) = P(s - x \leq s_i \leq s + x \mid X(W_n) = m) = \int_{s-x}^{s+x} \frac{\lambda_n(u)}{\int_{W_n} \lambda(v)dv} I(u \in \tilde{B}_\tau(s))du. \quad (4.13)$$

Now we consider the order statistics of the random sample

$$|\tilde{s}_1 - s|, \ldots, |\tilde{s}_m - s|$$

of size $m$ from $H_n$. Let $|\tilde{s}_{(k)} - s|$ denote the $k$-th order statistic of the sample $|\tilde{s}_1 - s|, \ldots, |\tilde{s}_m - s|$. Define

$$\tilde{\lambda}_n(s) = \frac{\tau k_n}{2|W_n||\tilde{s}_{(k_n)} - s|}. \quad (4.14)$$

Note that, if we replace $\tau$ and $\tilde{s}_{(k_n)}$ in $\tilde{\lambda}_n(s)$ by $\tau_n$ and $\hat{s}_{(k_n)}$ respectively, then $\tilde{\lambda}_n(s)$ reduces to the estimator $\hat{\lambda}_n(s)$ given in (4.5). We will now
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first prove that our Theorems are true, when \( \hat{\lambda}_n(s) \) is replaced by \( \tilde{\lambda}_n(s) \).
In section 3 we will show that our Theorems are valid for \( \tilde{\lambda}_n(s) \) as well.

**Lemma 4.3** Suppose that \( \lambda \) is periodic (with period \( \tau \)), and locally integrable. If, in addition (4.3) and (4.4) hold, then

\[
\hat{\lambda}_n(s) \xrightarrow{p} \lambda(s),
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Lemma 4.4** Suppose that \( \lambda \) is periodic (with period \( \tau \)), and locally integrable. If, in addition (4.9) and (4.4) hold, then

\[
\tilde{\lambda}_n(s) \xrightarrow{p} \lambda(s),
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Proof of Lemma 4.3** In view of Remark 4.1, we may assume, without loss of generality, that \( k_n \leq X(W_n) \).

To prove (4.15), we must show that,

\[
P\left( \left| \frac{\tau k_n}{2|W_n||\tilde{s}_n| - s} - \lambda(s) \right| \geq \varepsilon \right) \to 0
\]

as \( n \to \infty \), for each sufficiently small \( \varepsilon > 0 \). Choose \( \varepsilon < \lambda(s) \). Then, a simple calculation shows that, the probability on the l.h.s. of (4.17) is equal to

\[
P\left( \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| - \varepsilon)} \leq |\tilde{s}_n| - s \right) \text{ or } P\left( \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| + \varepsilon)} \geq |\tilde{s}_n| - s \right)
\]

\[
\leq P\left( |\tilde{s}_n| - s \geq \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| - \varepsilon)} \right) + P\left( |\tilde{s}_n| - s \leq \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| + \varepsilon)} \right).
\]

Then, to prove (4.17), it suffices to check that

\[
P\left( |\tilde{s}_n| - s \geq \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| - \varepsilon)} \right) \to 0
\]

and

\[
P\left( |\tilde{s}_n| - s \leq \frac{\tau k_n}{2|W_n|(|\tilde{s}_n| + \varepsilon)} \right) \to 0
\]

as \( n \to \infty \), for each \( \varepsilon > 0 \). Here we only give proof of (4.19), because the proof of (4.20) is similar.
Recall that $X(W_n)$ is a Poisson random variable with

$$\mathbb{E}X(W_n) = \text{Var}(X(W_n)) = \int_{W_n} \lambda(s)ds.$$ 

Since $\lambda$ is cyclic (with period $\tau$), by Lemma 2.1 we have that

$$\int_{W_n} \lambda(s)ds = \theta|W_n| + O(1),$$

as $n \to \infty$. Let

$$C_{1,n} = [\theta|W_n| - (\theta|W_n|)^{1/2}a_n], \quad (4.21)$$
$$C_{2,n} = [\theta|W_n| + (\theta|W_n|)^{1/2}a_n], \quad (4.22)$$

where $a_n$ is an arbitrary sequence such that $a_n \to \infty$ and $a_n = o(|W_n|^{1/2})$, as $n \to \infty$. Then, we can write the probability on the l.h.s. of (4.19) as

$$\sum_{m=k_n}^{\infty} \mathbb{P}\left(|\bar{s}(k_n) - s| \geq \frac{\tau k_n}{2|W_n|(|\lambda(s)| - \epsilon)} |X(W_n) = m\right) \mathbb{P}(X(W_n) = m)$$

$$\lesssim \sum_{m=k_n}^{\infty} \mathbb{P}(X(W_n) = m) + \sum_{m=C_{1,n}+1}^{\infty} \mathbb{P}(X(W_n) = m)$$

$$+ \max_{m=C_{1,n}} \mathbb{P}(X(W_n) = m) \cdot \sum_{m=C_{2,n}}^{\infty} \mathbb{P}\left(|\bar{s}(k_n) - s| \geq \frac{\tau k_n}{2|W_n|(|\lambda(s)| - \epsilon)} |X(W_n) = m\right). \quad (4.23)$$

It suffices now to show that each term on the r.h.s. of (4.23) converges to zero, as $n \to \infty$.

First we show that the first term on the r.h.s. of (4.23) is $o(1)$, as $n \to \infty$. Since $|\mathbb{E}X(W_n) - \theta|W_n|| = O(1)$, as $n \to \infty$, this quantity is equal to

$$\mathbb{P}(X(W_n) \leq C_{1,n} - 1) \leq \mathbb{P}\left(X(W_n) \leq \theta|W_n| - (\theta|W_n|)^{1/2}a_n\right)$$

$$\leq \mathbb{P}\left(|X(W_n) - \mathbb{E}X(W_n)| \geq (\theta|W_n|)^{1/2}a_n - |\mathbb{E}X(W_n) - \theta|W_n||\right)$$

$$= \mathbb{P}\left((\mathbb{E}X(W_n))^{-1/2}|X(W_n) - \mathbb{E}X(W_n)| \geq O(1)a_n\right)$$

$$\leq O(1) \exp\left(-\frac{a_n^2}{2 + o(1)}\right), \quad (4.24)$$

which is $o(1)$, since $a_n \to \infty$, as $n \to \infty$. Here we have used Lemma A.1 of the Appendix. A similar argument also shows that the second term on the r.h.s. of (4.23) is $o(1)$, as $n \to \infty$. 


Next we prove that the third term on the r.h.s. of (4.23) is \(o(1)\), as \(n \to \infty\). Let \(m = m_n\) be a positive integer, such that \(C_{1,n} \leq m_n \leq C_{2,n}\). Then \(m_n \sim \theta |W_n|\), which implies that \(k_n/m_n = o(1)\), as \(n \to \infty\) (by (4.4)). Recall that \(X(W_n)\) has a Poisson distribution with parameter \(\mu(W_n) = \int_{W_n} \lambda(s)ds\). A simple calculation, using Stirling's formula, shows that

\[
\max_{m_n, C_{1,n} \leq m_n \leq C_{2,n}} P \left( X(W_n) = m_n \right) = O(|W_n|^{-1/2}),
\]
as \(n \to \infty\). It is well-known (see, e.g. Reiss (1989), p. 15) that, conditionally given \(X_n(\tilde{B}_r(s)) = X(W_n) = m_n\), \(|\tilde{s}(k_n) - s|\) has exactly the same distribution as \(H_n^{-1}(Z_{k_n:m_n})\), where \(Z_{k_n:m_n}\) is the \(k_n\)-th order statistic of a sample \(Z_1, \ldots, Z_{m_n}\) of size \(m_n\) from the uniform \((0,1)\) distribution. (We remark in passing that \(k_n \leq m_n\) for all \(n\) sufficiently large). Note that a similar device was employed by Ralescu (1995) in his analysis of multivariate nearest neighbor density estimators. As a result, the third term on the r.h.s. of (4.23) is equal to

\[
O(|W_n|^{-1/2}) \sum_{m_n = C_{1,n}}^{C_{2,n}} P \left( H_n^{-1}(Z_{k_n:m_n}) \geq \frac{\tau k_n}{2|W_n|(|\lambda(s) - \epsilon|)} \right). \tag{4.25}
\]

First note that, by choosing \(\epsilon < \lambda(s)\), we have

\[
\frac{\tau k_n}{2|W_n|(|\lambda(s) - \epsilon|)} = \frac{\tau k_n}{2\lambda(s)|W_n| \left(1 - \frac{\epsilon}{\lambda(s)}\right)} \geq \frac{\tau k_n}{2\lambda(s)|W_n|} \left(1 + \frac{\epsilon}{\lambda(s)}\right)
\]

\[
= \frac{\tau k_n}{2\lambda(s)|W_n|} + \frac{\tau \epsilon k_n}{2\lambda^2(s)|W_n|}. \tag{4.26}
\]

We know that, for each \(m_n\),

\[
E Z_{k_n:m_n} = k_n/(m_n + 1)
\]

and

\[
V a r(Z_{k_n:m_n}) = O(k_n/(m_n^2)).
\]

We now need a stochastic expansion for \(H_n^{-1}(Z_{k_n:m_n})\). First we simplify the r.h.s. of (4.13) to get for any \(x \geq 0\)

\[
H_n(x) = \frac{(|W_n|/\tau + O(1))}{(\theta |W_n| + O(1))} \int_{s-z}^{s+z} \lambda(u)I(u \in \tilde{B}_r(s))du = \left(1 + O(||W_n||^{-1})\right) \int_{s-z}^{s+z} \lambda(u)I(u \in \tilde{B}_r(s))du = \frac{1}{\theta \tau} \int_{s-z}^{s+z} \lambda(u)I(u \in \tilde{B}_r(s))du + O(||W_n||^{-1}), \tag{4.27}
\]
as \(n \to \infty\), uniformly in \(x\). This because \(\int_{s-z}^{s+z} \lambda(u)I(u \in \tilde{B}_r(s))du \leq \theta \tau\). Define function \(H(x)\), which is equal to the first term on the r.h.s. of
for $x \geq 0$, and zero otherwise. The density $h$ of $H$ is given by
\[
h(x) = \frac{\lambda(s + x)I(s + x \in B_r(s))}{\theta r} + \frac{\lambda(s - x)I(s - x \in B_r(s))}{\theta r},
\]
(4.28)
for any $x > 0$, while $h(0)$ denote the right hand derivative of $H$ at zero. Next note that
\[
H^{-1}(Z_{k_n:m_n}) = \inf\{x : H_n(x) > Z_{k_n:m_n}\}
= \inf\{x : H(x) > Z_{k_n:m_n} + O(|W_n|^{-1})\}
= H^{-1}(Z_{k_n:m_n} + O(|W_n|^{-1})),
\]
(4.29)
as $n \to \infty$. Here and elsewhere in this chapter we define $H^{-1}(t) = \inf\{x : H(x) > t\}$, $0 \leq t < 1$. Now we compute $H^{-1}(0)$. Since $\lambda(s) > 0$ and $\lambda$ is continuous at $s$, we see from the first term on the r.h.s. of (4.27) that $H(x) > 0$, while $x > 0$. In other words, the first term on the r.h.s. of (4.27) is equal to zero, if and only if, $x = 0$. Hence $H^{-1}(0) = 0$. Since $h$ is right continuous at 0, the first (right hand) derivative of $H^{-1}$ at 0 can be computed as
\[
H^{-1'}(0) = \frac{1}{h(H^{-1}(0))} = \frac{1}{h(0)} = \frac{\theta r}{2\lambda(s)},
\]
(4.30)
Since $H^{-1'}(0)$ is finite, by Young’s form for Taylor’s theorem (Serfling (1980), p. 45), we can write
\[
H^{-1}\left(\frac{k_n}{m_n + 1} + O(|W_n|^{-1})\right)
= H^{-1}(0) + \left(\frac{k_n}{m_n + 1} + O(|W_n|^{-1})\right) H^{-1'}(0) (1 + o(1))
= \frac{\theta r k_n}{2\lambda(s)(m_n + 1)} + o\left(\frac{k_n}{|W_n|}\right),
\]
(4.31)
as $n \to \infty$. Because $\lambda$ is continuous at $s$, we have
\[
H^{-1'}\left(\frac{k_n}{m_n + 1} + O(|W_n|^{-1})\right) = \frac{1}{h(H^{-1}\left(\frac{k_n}{m_n + 1} + O(|W_n|^{-1})\right))}
= \frac{1}{h(|o(1)|)} = \frac{\theta r}{2\lambda(s + |o(1)|)} = \frac{\theta r}{2\lambda(s)} + o(1),
\]
(4.32)
as $n \to \infty$. Let $\tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - \mathbb{E}Z_{k_n:m_n} = Z_{k_n:m_n} - k_n/(m_n + 1)$. Let us write
\[
H^{-1}(Z_{k_n:m_n}) = H^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n)
+ H^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| > \epsilon_n),
\]
where \( \epsilon_n \) is a sequence of positive real numbers such that \( \epsilon_n \downarrow 0 \) as \( n \to \infty \). Because
\[
H^{-1} \left( \frac{k_n}{m_n + 1} + O(|W_n|^{-1}) \right) = O(1),
\]
as \( n \to \infty \), by Young's form for Taylor's theorem, we can write
\[
H_n^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) \quad \text{as (cf. (4.29))}
\]
\[
= H_n^{-1}(Z_{k_n:m_n} + O(|W_n|^{-1})) I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n)
\]
\[
= \left\{ H^{-1} \left( \frac{k_n}{m_n + 1} + O(|W_n|^{-1}) \right) \right. \\
\left. + \left( Z_{k_n:m_n} - \frac{k_n}{m_n + 1} \right) H^{-1} \left( \frac{k_n}{m_n + 1} + O(|W_n|^{-1}) \right) \right. \\
\left. + o \left( Z_{k_n:m_n} - \frac{k_n}{m_n + 1} \right) \right\} I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n),
\]
(4.33)
as \( n \to \infty \). Substituting (4.31) and (4.32) into the r.h.s. of (4.33), we then have
\[
H_n^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) = \left\{ \frac{\theta \tau k_n}{2\lambda(s)(m_n + 1)} + o \left( \frac{k_n}{|W_n|} \right) \right. \\
\left. + \left( \frac{\theta \tau}{2\lambda(s)} \right) Z_{k_n:m_n} + o \left( \tilde{Z}_{k_n:m_n} \right) \right\} I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n),
\]
(4.34)
as \( n \to \infty \). Since \( m_n \geq C_{1,n} \), the first term on the r.h.s. of (4.34) does not exceed
\[
\frac{\theta \tau k_n}{2\lambda(s) \left( |\theta| |W_n| - (\theta |W_n|)^{1/2} a_n \right) + 1} \leq \frac{\theta \tau k_n}{2\lambda(s) \left( |\theta| |W_n| - (\theta |W_n|)^{1/2} a_n \right)}
\]
\[
= \frac{\theta \tau k_n}{2\lambda(s) |W_n| \left( (1 - (\theta |W_n|)^{-1/2} a_n \right)} = \frac{\tau k_n}{2\lambda(s) |W_n|} + o \left( \frac{k_n}{|W_n|} \right),
\]
(4.35)
as \( n \to \infty \). Combining (4.34), (4.35), and (4.26), and by noting also that the first term on the r.h.s. of (4.35) cancels with the first term on the r.h.s. of (4.26), we find that, for sufficiently large \( n \), the probability appearing in (4.25) does not exceed
\[
P \left( \frac{\theta \tau}{\lambda(s)} |Z_{k_n:m_n}|I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) + o \left( \tilde{Z}_{k_n:m_n} \right) I(|\tilde{Z}_{k_n:m_n}| \leq \epsilon_n) \right)
\]
\[
+ H_n^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau \epsilon k_n}{4\lambda^2(s)|W_n|}
\]
\[
\leq P \left( |\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta \lambda(s)|W_n|} \right) + P \left( |o(\tilde{Z}_{k_n:m_n})| > \frac{\tau \epsilon k_n}{12\lambda^2(s)|W_n|} \right)
\]
\[
+ P \left( H_n^{-1}(Z_{k_n:m_n})I(|\tilde{Z}_{k_n:m_n}| > \epsilon_n) > \frac{\tau \epsilon k_n}{12\lambda^2(s)|W_n|} \right).
\]
(4.36)
First note that, for sufficiently large $n$, the second term on the r.h.s. of (4.36) does not exceed its first term. Now we notice that $H_n^{-1}(Z_{k_n:m_n}) \leq H_n^{-1}(1) = \frac{\tau}{2}$. Then we find that the third probability on the r.h.s. of (4.36) does not exceed $P(|\tilde{Z}_{k_n:m_n}| > \epsilon_n)$. For convenience we take $\epsilon_n = (\epsilon k_n)/(12\theta \lambda(s)|W_n|)$. Then, the r.h.s. of (4.36) does not exceed

$$3P\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta \lambda(s)|W_n|}\right).$$

Therefore, for sufficiently large $n$, the quantity in (4.25) does not exceed

$$O(|W_n|^{-1/2}) (C_{2,n} - C_{1,n} + 1) P\left(|\tilde{Z}_{k_n:m_n}| > \frac{\epsilon k_n}{12\theta \lambda(s)|W_n|}\right)$$

$$\leq O(1) a_n P\left(|Z_{k_n:m_n} - \frac{k_n}{m_n + 1}| > \frac{\epsilon k_n}{12\theta \lambda(s)|W_n|}\right),$$

(4.37)

as $n \to \infty$. By Chebyshev's inequality, we find that the probability on the r.h.s. of (4.37) is of order $O(k_n^{-1})$, as $n \to \infty$. By (4.3) and choosing now $a_n = o(k_n^{-1})$, as $n \to \infty$, we have that the r.h.s. of (4.37) is $o(1)$ as $n \to \infty$. Hence (4.19) is proved. This completes the proof of Lemma 4.3. \Box

Proof of Lemma 4.4 To establish (4.16), we must show that

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\tau k_n}{|W_n|(|\tilde{s}(k_n)| - s)} - \lambda(s)\right| > \epsilon\right) < \infty,$$

(4.38)

for each $\epsilon > 0$. By (4.18), to prove (4.38) it suffices to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|\tilde{s}(k_n) - s| \geq \frac{\tau k_n}{2|W_n|(\lambda(s) - \epsilon)}\right) < \infty.$$

(4.39)

and

$$\sum_{n=1}^{\infty} P\left(|\tilde{s}(k_n) - s| \leq \frac{\tau k_n}{2|W_n|(\lambda(s) + \epsilon)}\right) < \infty.$$

(4.40)

Here we only give the proof of (4.39), because the proof of (4.40) is similar. To prove (4.39), it suffices clearly to show that, each of the terms on the r.h.s. of (4.23) converges completely to zero, as $n \to \infty$.

Let $C_{1,n}$ and $C_{2,n}$ be as given in (4.21) and (4.22). In order to deal with the first and second term of (4.23), the sequence $a_n$ will now have to
satisfy, in addition to the assumption \( a_n = o(|W_n|^{1/2}) \) which was already needed in the proof of Lemma 4.3, the additional requirement

\[
\sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty.
\]

The argument given in (4.24) will then imply that these terms converge completely to zero, as \( n \to \infty \).

It remains to show that the third term on the r.h.s. of (4.23) also converges completely to zero, as \( n \to \infty \). To do this, it is clear from the proof of Lemma 4.3, that it suffices now to check that the r.h.s. of (4.37) is summable, for each \( \epsilon > 0 \).

Let us now consider the probability appearing on the r.h.s. of (4.37). For sufficiently large \( n \), by Lemma A.4 (see Appendix), there exists a positive constant \( C_0 \) such that the probability on the r.h.s. of (4.37) does not exceed

\[
2 \exp \left\{ -C_0 t_n^2 \right\},
\]

where

\[
t_n = \left( \frac{m_n}{k_n/(m_n + 1)(1 - k_n/(m_n + 1))} \right)^{1/2} \frac{k_n \epsilon}{120 \lambda(s)|W_n|}
\]

which (for sufficiently large \( n \)) can be replaced with impunity by

\[
\epsilon k_n^{\frac{1}{2}}/(24 \lambda(s)).
\]

Hence, for sufficiently large \( n \), the r.h.s. of (4.37) does not exceed

\[
O(1) \exp \left\{ -\frac{C_0 \epsilon^2}{576(\lambda(s))^2} k_n \right\} = O(1) \exp \left\{ \log a_n - \frac{C_0 \epsilon^2}{1152(\lambda(s))^2} k_n \right\} \exp \left\{ -\frac{C_0 \epsilon^2}{1152(\lambda(s))^2} k_n \right\}
\]

provided we require \( a_n \) to satisfy \( \log a_n = o(k_n) \), as \( n \to \infty \). Note that, e.g. the choice \( a_n = (k_n)^{1/2} \) satisfies each of the three conditions imposed on \( a_n \), namely \( a_n = o(|W_n|^{1/2}) \), \( \sum_{n=1}^{\infty} \exp(-a_n^2/3) < \infty \), and \( \log a_n = o(k_n) \), provided (4.4) and (4.9). By assumption (4.9), we have that the r.h.s. of (4.41) is summable. Hence (4.39) is proved. This completes the proof of Lemma 4.4. \( \Box \)
4.2.3 Proofs

Proof of Theorem 4.1 To prove (4.8), it suffices to check that
\[
\frac{\tau n}{2|W_n||\hat{s}(k_n) - s|} \xrightarrow{p} \lambda(s),
\]
and
\[
\left| \frac{\hat{n}n}{2|W_n||\hat{s}(k_n) - s|} - \frac{\tau n}{2|W_n||\hat{s}(k_n) - s|} \right| \xrightarrow{p} 0,
\]
as \(n \to \infty\), for each \(s\) at which \(\lambda\) is continuous and positive.

First, we prove (4.42). To do this, we must show that
\[
P \left( \left| \frac{\tau n}{2|W_n||\hat{s}(k_n) - s|} - \lambda(s) \right| \geq \epsilon \right) \to 0
\]
as \(n \to \infty\), for each sufficiently small \(\epsilon > 0\). Choose \(\epsilon < \lambda(s)\). Then, a simple calculation like the one leading from (4.17) to (4.19) and (4.20), shows that it suffices to check
\[
P \left( |\hat{s}(k_n) - s| \geq \frac{\tau n}{2|W_n|}\lambda(s) - \epsilon \right) \to 0
\]
and
\[
P \left( |\hat{s}(k_n) - s| \leq \frac{\tau n}{2|W_n|}\lambda(s) + \epsilon \right) \to 0
\]
as \(n \to \infty\), for each \(\epsilon > 0\). We only prove (4.45), because the proof of (4.46) is similar.

Recall that \(s_i\), \((i = 1, \ldots, m)\) denotes the location of the points in the realization \(X(\omega)\) of the Poisson process \(X\). Let \(\hat{j}_i\) denote the random integer, depending on \(\hat{t}_n\) and \(s_i\), such that \(\hat{s}_i = s_i + \hat{j}_i \hat{t}_n\). Similarly, let \(\hat{j}_i\) denote an integer, depending on \(\tau\) and \(s_i\), such that \(\hat{s}_i = s_i + \hat{j}_i \tau\). If \(s(k_n)\) denotes the point corresponding to \(\hat{s}(k_n)\) before translation, then obviously \(\hat{s}(k_n) = s(k_n) + \hat{j}_n \hat{t}_n\). Furthermore we have that
\[
|\hat{s}(k_n) - s| = |s(k_n) + \hat{j}_n \hat{t}_n - s|
\leq |s(k_n) + \hat{j}_n \tau - s| + |\hat{j}_n \hat{t}_n - \hat{j}_n \tau|
\leq |s(k_n) - s| + |\hat{j}_n ||\hat{t}_n - \tau| + \tau |\hat{j}_n - \hat{j}_n|
\]
To prove (4.45), it suffices now to check, for each \(\epsilon > 0\),
\[
P \left( |\hat{s}(k_n) - s| \geq \frac{\tau n}{6|W_n|}\lambda(s) - \epsilon \right) \to 0,
\]
\[
P \left( |\hat{j}_n ||\hat{t}_n - \tau| \geq \frac{\tau n}{6|W_n|}\lambda(s) - \epsilon \right) \to 0,
\]
and
\[ P \left( |\hat{j}_{kn} - \bar{j}_{kn}| \geq \frac{k_n}{6|W_n| (\lambda(s) - \epsilon)} \right) \to 0, \tag{4.50} \]
as \( n \to \infty \). First note that, the proof of (4.19) also yields (4.48). Since
\[ |\hat{j}_{kn}| = O_p(|W_n|), \]
as \( n \to \infty \), assumption (4.7) yields that
\[ |\hat{j}_{kn}||\hat{r}_n - \tau| = o_p(k_n/|W_n|), \]
as \( n \to \infty \), which directly implies (4.49). Hence, it remains to check (4.50).

Here we only give the proof of (4.50) for the case \( \hat{r}_n \geq \tau \) and \( \hat{j}_{kn}, \bar{j}_{kn} \) are both positive; because the proofs of the other seven cases are similar and therefore omitted. Since \( \hat{r}_n \geq \tau \), we also know that \( \hat{j}_{kn} \leq \bar{j}_{kn} \). Hence we have that \( \hat{r}_n = \tau + |\hat{r}_n - \tau| \) and \( \hat{j}_{kn} = \bar{j}_{kn} - (|\hat{j}_{kn} - \bar{j}_{kn}|) \). Then, we can write
\[
\hat{s}(k_n) = s_{kn} + \hat{j}_{kn} \hat{r}_n \\
= s_{kn} + (\hat{j}_{kn} - |\hat{j}_{kn} - \bar{j}_{kn}|) (\tau + |\hat{r}_n - \tau|) \\
= \hat{s}_{kn} + \hat{j}_{kn} |\hat{r}_n - \tau| - \tau |\hat{j}_{kn} - \bar{j}_{kn}| - |\hat{j}_{kn} - \bar{j}_{kn}| |\hat{r}_n - \tau|. \tag{4.51}
\]
Since \( \hat{s}(k_n) \in [s - \frac{\hat{r}_n}{2}, s + \frac{\hat{r}_n}{2}] \), it follows now from (4.51) that
\[
s - \frac{\tau}{2} - \frac{|\hat{r}_n - \tau|}{2} \leq \hat{s}_{kn} + \hat{j}_{kn} |\hat{r}_n - \tau| - \tau |\hat{j}_{kn} - \bar{j}_{kn}| - |\hat{j}_{kn} - \bar{j}_{kn}| |\hat{r}_n - \tau| \\
< s + \frac{\tau}{2} + \frac{|\hat{r}_n - \tau|}{2}. \tag{4.52}
\]
Since we also know that (4.52) holds true for any value \( \tilde{s}_{kn} \in [s - \frac{\hat{r}_n}{2}, s + \frac{\hat{r}_n}{2}] \), (4.52) directly yields that
\[
-\frac{|\hat{r}_n - \tau|}{2} \leq \tilde{j}_{kn} |\hat{r}_n - \tau| - \tau |\hat{j}_{kn} - \bar{j}_{kn}| - |\hat{j}_{kn} - \bar{j}_{kn}| |\hat{r}_n - \tau| \leq \frac{|\hat{r}_n - \tau|}{2},
\]
which is equivalent to
\[
\left( \frac{1}{2} \right) |\hat{r}_n - \tau| < (\tau + o_p(1)) |\hat{j}_{kn} - \bar{j}_{kn}| \leq \left( \frac{1}{2} \right) |\hat{r}_n - \tau|. \tag{4.53}
\]
Since \( \tilde{j}_{kn} = O(|W_n|) \), as \( n \to \infty \), together with assumption (4.7), we find that
\[ |\hat{j}_{kn} - \bar{j}_{kn}| = o_p(k_n |W_n|^{-1}), \]
as \( n \to \infty \), which implies (4.50). Hence (4.42) is proved.
Next we prove (4.43). The l.h.s. of (4.43) can be written as
\[
\frac{\tau k_n}{2|W_n||\hat{s}(k_n) - s|} \frac{1}{\tau} |\hat{\tau}_n - \tau| = o_p(1) \cdot o_p(k_n|W_n|^{-2}) = o_p(1),
\]
as \(n \to \infty\). Here we have used (4.42) and assumption (4.7). Hence (4.43) is proved. This completes the proof of Theorem 4.1.

**Proof of Theorem 4.2** To establish (4.11), it suffices to check that (4.42) and (4.43) remain valid, when \(\Rightarrow\) is replaced by \(\Rightarrow\), as \(n \to \infty\), for each \(s\) at which \(\lambda\) is continuous and positive.

First, we prove that the l.h.s. of (4.42) converges completely to \(\lambda(s)\), as \(n \to \infty\). Following the structure of the proof of Theorem 1.1, it suffices to check that the probabilities appearing on the l.h.s. of (4.45) and (4.46) are summable, for each \(\epsilon > 0\). We shall prove that the probability appearing on the l.h.s. of (4.45) is summable; the proof of the other case is similar.

In view of (4.47), it suffices now to show that the probabilities appearing on the l.h.s. of (4.48), (4.49), and (4.50), are summable, for each \(\epsilon > 0\). The proof of the probability on the l.h.s. of (4.48) is summable is exactly the same as the proof of (4.39). Since, by assumption (4.10), we have
\[
|\bar{j}_{k_n}| \leq \frac{|W_n|}{\tau} (1 + o_c(1)),
\]
as \(n \to \infty\), (for any r.v. \(Y_n\) we write \(Y_n = o_c(1)\) to denote that \(Y_n\) converges completely to zero, as \(n \to \infty\)), then by assumption (4.10) once more, we have that the probability on the l.h.s. of (4.49) is summable, for each \(\epsilon > 0\). It remains to prove that the probability on the l.h.s. of (4.50) is summable. We only consider the case that \(\hat{\tau}_n \geq \tau\) and \(\bar{j}_{k_n}, \bar{j}_{k_n}\) are both positive; the proofs for the other seven cases are similar. An application of inequality (4.53), by using now assumption (4.10), yields that
\[
|\bar{j}_{k_n} - \tilde{j}_{k_n}| \leq \left(\bar{j}_{k_n} + \frac{1}{2}\right) (\tau + o_c(1))^{-1} |\hat{\tau}_n - \tau|,
\]
as \(n \to \infty\). Since \(\bar{j}_{k_n} = O(|W_n|)\), as \(n \to \infty\), by assumption (4.10) once more, we have that the probability on the l.h.s. of (4.50) is summable. Hence we have proved (4.42) with \(\Rightarrow\) replaced by \(\Rightarrow\).

Next we prove (4.43) with \(\Rightarrow\) replaced by \(\Rightarrow\). First note that, the l.h.s. of (4.43) is the same as the l.h.s. of (4.54). Because we have that the l.h.s. of (4.42) converges completely to \(\lambda(s)\), as \(n \to \infty\), by assumption (4.10)
and Lemma A.5 (see Appendix), we also have that the l.h.s. of (4.54) converges completely to zero, which of course implies that the l.h.s. of (4.43) converges completely to zero, as \( n \to \infty \). This completes the proof of Theorem 4.2. \( \square \)

### 4.3 Statistical properties

In this section we focus on statistical properties of our estimator, i.e. we compute the bias, variance, and mean squared error (MSE) of \( \hat{\lambda}_n \). We refer to section 3.3 for a more precise description of the type of assumption we will need for the estimator \( \hat{\tau}_n \) of \( \tau \).

#### 4.3.1 Results

**Theorem 4.5** Suppose that \( \lambda \) is periodic and locally integrable. If, in addition, (4.3) and (4.4) hold true, and

\[
|W_n| |\hat{\tau}_n - \tau| = O \left( \frac{\delta_n k_n}{|W_n|} \right)
\]

with probability 1 as \( n \to \infty \), for some fixed sequence \( \delta_n \downarrow 0 \) as \( n \to \infty \), then

\[
E \hat{\lambda}_n(s) \to \lambda(s)
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Theorem 4.6** Suppose that \( \lambda \) is periodic and locally integrable. If, in addition, (4.3), (4.4) and (4.55) hold, then

\[
\text{Var} \left( \hat{\lambda}_n(s) \right) \to 0
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

**Theorem 4.7** Suppose that \( \lambda \) is periodic and locally integrable. If, in addition, (4.3) and (4.4) hold true, and

\[
|W_n| |\hat{\tau}_n - \tau| = O \left( \frac{\delta_n k_n^{1/2}}{|W_n|} \right)
\]

with probability 1 as \( n \to \infty \), for some fixed sequence \( \delta_n \downarrow 0 \) as \( n \to \infty \), then we have

\[
\text{Var} \left( \hat{\lambda}_n(s) \right) = \frac{\lambda^2(s)}{k_n} + o \left( \frac{1}{k_n} \right)
\]

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.
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Theorem 4.8 Suppose that \( \lambda \) is periodic and locally integrable, (4.3) and (4.4) hold true, and

\[
|W_n| |\hat{\tau}_n - \tau| = O \left( \delta_n \frac{k_n^3}{|W_n|^3} \right)
\]

(4.60)

with probability 1 as \( n \to \infty \), for some fixed sequence \( \delta_n \downarrow 0 \) as \( n \to \infty \). If, in addition, \( \lambda \) has finite second derivative \( \lambda'' \) at \( s \), then

\[
\mathbb{E} \hat{\lambda}_n(s) = \lambda(s) + \frac{\tau^2 \lambda''(s) k_n^2}{24 \lambda^2(s) |W_n|^2} + o \left( \frac{k_n^2}{|W_n|^2} \right) + O \left( \frac{1}{|W_n|^{1/2-\epsilon_0} + \frac{1}{k_n}} \right),
\]

(4.61)

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive, where \( \epsilon_0 \) is an arbitrary small positive real number.

Note that, the r.h.s. of (4.61) yields an asymptotic approximation for the bias of \( \hat{\lambda}_n(s) \) provided \( k_n \to \infty \) faster than \( |W_n|^{3/4+\epsilon_0} \) for some arbitrary small \( \epsilon_0 > 0 \); otherwise the \( O(|W_n|^{\sigma_0-1/2} + k_n^{-1}) \) remainder term will dominate. However, the optimal choice of \( k_n \) (cf. (4.64) ) satisfies this restriction.

Corollary 4.9 Suppose that \( \lambda \) is periodic, locally integrable, (4.3) and (4.4) hold.

(i) If, in addition, (4.55) holds true, then

\[
MSE \left( \hat{\lambda}_n(s) \right) = Var \left( \hat{\lambda}_n(s) \right) + Bias^2 \left( \hat{\lambda}_n(s) \right) \to 0
\]

(4.62)

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive.

(ii) If (4.60) hold true, and \( \lambda \) has finite second derivative \( \lambda'' \) at \( s \), then

\[
MSE \left( \hat{\lambda}_n(s) \right) = \frac{\lambda^2(s)}{k_n} + \frac{\tau^4 \lambda''(s)^2 k_n^4}{576 \lambda^4(s) |W_n|^4} + o \left( \frac{1}{k_n} \right) + O \left( \frac{k_n^4}{|W_n|^4} \right) + O \left( |W_n|^{\epsilon_0-1} \right)
\]

(4.63)

as \( n \to \infty \), for each \( s \) at which \( \lambda \) is continuous and positive, where \( \epsilon_0 \) is an arbitrary small positive real number.
4.3 Statistical properties

The first statement of Corollary 4.9 is implied by Theorems 4.5 and 4.6, while its second statement is due to Theorems 4.7 and 4.8.

Now, we consider the r.h.s. of (4.63). By minimizing the sum of the first and second term of (4.63) (the leading terms for the variance and the squared bias), we obtain the optimal choice of $k_n$, which is given by

$$k_n = \left[ \frac{144\lambda^6(s)}{\tau^4 (\lambda''(s))^2} \right]^{1/5} |W_n|^{4/5}. \quad (4.64)$$

With this choice of $k_n$, the optimal rate of decrease of $MSE(\hat{\lambda}_n(s))$ is of order $O(|W_n|^{-4/5})$ as $n \to \infty$; and also in this important special case both (4.58) and (4.60) reduce to the same condition

$$|W_n| |\hat{\tau}_n - \tau| = O \left( \delta_n |W_n|^{-3/5} \right) \quad (4.65)$$

with probability 1 as $n \to \infty$, for some fixed sequence $\delta_n \downarrow 0$ as $n \to \infty$.

**Remark 4.2** The formulas (4.59), (4.61), (4.63), and (4.64) resemble closely corresponding ones in the 'classical' nearest neighbor density estimation for one dimensional case. To see this, let us consider for moment estimation of a density $f$, proportional to the intensity function $\lambda$ and having support in $[0, \tau]$. For simplicity, we consider here only the (unrealistic) case where we know $\theta \tau$, where $\theta \tau = \int_0^\tau \lambda(s)ds$ (we assume here that $\theta > 0$). Then we have that $f(s) = \lambda(s)(\theta \tau)^{-1}$, for all $s \in [0, \tau]$. Consequently, the quantity $\hat{f}_n(s) = \hat{\lambda}_n(s)(\theta \tau)^{-1}$ can be viewed as an estimate of $f$ at a given point $s$. Since $\lambda(s) = f(s)\theta \tau$, we also have that $\lambda''(s) = f''(s)\theta \tau$, for all $s \in (0, \tau)$. From (4.59), we can compute $Var(\hat{f}_n(s))$ as follows

$$Var(\hat{f}_n(s)) = Var \left( \frac{\hat{\lambda}_n(s)}{\theta \tau} \right) = \frac{1}{(\theta \tau)^2} \frac{f(s)\theta \tau^2}{k_n} + o \left( \frac{1}{k_n} \right)$$

as $n \to \infty$. Note that the r.h.s. of (4.66) is the same as the well known asymptotic approximation to the variance in nearest neighbor density estimator for one dimensional case. From (4.61), we have that

$$E\hat{f}_n(s) = E \frac{\hat{\lambda}_n(s)}{\theta \tau}$$

$$= \frac{\lambda(s)}{\theta \tau} + \frac{\tau^2 f''(s)\theta \tau k_n^2}{24 \theta \tau (f(s)\theta \tau)^2 |W_n|^2} + o \left( \frac{k_n^2}{|W_n|^2} \right)$$
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\[ + \mathcal{O}\left( \frac{1}{|W_n|^{1/2-\epsilon_0}} + \frac{1}{k_n} \right) \]

\[ = f(s) + \frac{f''(s)k_n^2}{24f^2(s)|W_n|^{2}} + o\left( \frac{k_n^2}{|W_n|^2} \right) \]

\[ + \mathcal{O}\left( \frac{1}{|W_n|^{1/2-\epsilon_0}} + \frac{1}{k_n} \right), \quad (4.67) \]

as \( n \to \infty \). Note that, due to our 'increasing domain asymptotic framework', the number of observations \( X(W_n) \) in a given window \( W_n \) is random. However, it is easy to check that \( \mathbb{E}X(W_n) \sim \theta|W_n| \). Hence, it seems appropriate to compare \( \theta|W_n| \) with the 'sample size \( n \)' in the 'classical' density estimation case. If we replace \( \theta|W_n| \) on the r.h.s. of (4.67) by \( n \), the r.h.s. of (4.67) indeed reduces to the well-known expression for the asymptotic approximation to the bias in nearest neighbor density estimation. From (4.66) and (4.67), we also can find formulas for \( MSE(\hat{f}_n(s)) \) and optimal choice of \( k_n \), when estimating \( f \). These expressions also reduce to the corresponding ones in nearest neighbor density estimation, if we replace \( \theta|W_n| \) by \( n \). For example, the formula for \( MSE(\hat{f}_n(s)) \) reduces to 'one dimensional case' of formula (26) in Fukunaga and Hostetler (1973) (cf. also Mack and Rosenblatt (1979) and Prakasa Rao (1983)).

**Remark 4.3** Since \( \hat{\lambda}_n(s) = 0 \) if \( X(W_n) < k_n \), we have that

\[ \mathbb{E}\hat{\lambda}_n(s)I(X(W_n) < k_n) = \text{Var}(\hat{\lambda}_n(s)I(X(W_n) < k_n)) = 0. \]

This implies

\[ \mathbb{E}\hat{\lambda}_n(s) = \mathbb{E}\hat{\lambda}_n(s)I(X(W_n) \geq k_n), \]

and

\[ \text{Var}(\hat{\lambda}_n(s)) = \text{Var}(\hat{\lambda}_n(s))I(X(W_n) \geq k_n). \]

Hence, in all of our proofs in this subsection, we only need to consider the case \( X(W_n) \geq k_n \) (cf (4.12)).

**4.3.2 Proofs**

We begin with a simple lemma, which we will need in our proofs.

**Lemma 4.10** If (4.4) and (4.55) hold true, then we have with probability 1 that

\[ |\hat{s}(k_n) - s| = |\hat{s}(k_n) - s| + \mathcal{O}\left( \frac{k_n}{|W_n|} \right) \quad (4.68) \]

as \( n \to \infty \), provided \( X(W_n) \geq k_n \).
4.3 Statistical properties

Proof: Similar to (4.47), we can write

\[ (\tilde{s}(k_n) - s) = (s(k_n) + j_{k_n} \hat{\tau}_n - s) \]
\[ = (s(k_n) + \bar{j}_{k_n} \tau - s) + (j_{k_n} \hat{\tau}_n - j_{k_n} \tau) \]
\[ = (\hat{s}(k_n) - s) + j_{k_n} (\hat{\tau}_n - \tau) + \tau (j_{k_n} - j_{k_n}) . \quad (4.69) \]

First we will show that the second term on the r.h.s. of (4.69) is of order \( \mathcal{O}(\delta_n |W_n|^{-1}) \) with probability 1, as \( n \to \infty \). To do this, we argue as follows. By (4.55), there exists a positive constant \( C \) such that we have with probability 1

\[ |\hat{\tau}_n - \tau| \leq C \delta_n |W_n|^{-2} . \quad (4.70) \]

Since \( s \in W_n \), by (4.4) and (4.70), we have with probability 1 that \( |j_{k_n}| = \mathcal{O}(|W_n|) \) as \( n \to \infty \). Combining this order bound and (4.70), we then have with probability 1 that the second term on the r.h.s. of (4.69) is of order \( \mathcal{O}(\delta_n |W_n|^{-1}) \) as \( n \to \infty \).

Next we will show that the third term on the r.h.s. of (4.69) is of order \( \mathcal{O}(\delta_n |W_n|^{-1}) \) with probability 1, as \( n \to \infty \). Here we only give the proof for the case \( \hat{\tau}_n \geq \tau \) and \( j_{k_n} \), \( \bar{j}_{k_n} \) are both positive; because the proofs of the other seven cases are similar. Recall (4.53). Since \( s \in W_n \), we have that \( \bar{j}_{k_n} = \mathcal{O}(W_n) \) as \( n \to \infty \). Then, by (4.70) and (4.53), we have with probability 1 that the third term on the r.h.s. of (4.69) is of order \( \mathcal{O}(\delta_n |W_n|^{-1}) \) as \( n \to \infty \). Therefore we have that

\[ (\tilde{s}(k_n) - s) = (\hat{s}(k_n) - s) + \mathcal{O}(\delta_n |W_n|) \] as \( n \to \infty \). By the triangle inequality, we have

\[ |\hat{s}(k_n) - s| - |\mathcal{O}(\delta_n |W_n|^{-1})| \leq |\tilde{s}(k_n) - s| \]
\[ \leq |\hat{s}(k_n) - s| + |\mathcal{O}(\delta_n |W_n|^{-1})| \]

which implies this lemma. This completes the proof of Lemma 4.10. □

Proof of Theorem 4.5

By Remark 4.3, the l.h.s. of (4.56) is equal to

\[ \frac{k_n}{2|W_n|} \mathbb{E} \frac{\hat{\tau}_n}{|\tilde{s}(k_n) - s|} I(X(W_n) \geq k_n) \]
\[ = \frac{\tau k_n}{2|W_n|} \mathbb{E} \frac{1}{|\tilde{s}(k_n) - s|} I(X(W_n) \geq k_n) \]
\[ + \frac{k_n}{2|W_n|} \mathbb{E} \frac{(\hat{\tau}_n - \tau)}{|\tilde{s}(k_n) - s|} I(X(W_n) \geq k_n) . \quad (4.71) \]
We will prove (4.56) by showing that the first term on the r.h.s. of (4.71) is equal to \( \lambda(s) + o(1) \) as \( n \to \infty \), while its second term is of order \( o(1) \) as \( n \to \infty \).

First we consider the first term on the r.h.s. of (4.71). For each \( n \), let \( A_n \) denote the set of all integers \( m_n \), where \( C_{1,n} \leq m_n \leq C_{2,n} \), with \( C_{1,n} \) and \( C_{2,n} \) are given respectively by (4.21) and (4.22). Let \( A_n^c = [k_n, \infty) \setminus A_n \). Then, the expectation in the first term on the r.h.s. of (4.71) can be computed as follows

\[
\begin{align*}
&= \mathbb{E} \left( \mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \mathbb{I}(X(W_n) \geq k_n) \bigg| X(W_n) = m \right) \right) \\
&= \sum_{m_n \in A_n} \left( \mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \bigg| X(W_n) = m_n \right) \right) \mathbb{P}(X(W_n) = m_n) \\
&\quad + \sum_{m = k_n}^{C_{1,n} - 1} \left( \mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \bigg| X(W_n) = m \right) \right) \mathbb{P}(X(W_n) = m) \\
&\quad + \sum_{m = C_{2,n} + 1}^{\infty} \left( \mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \bigg| X(W_n) = m \right) \right) \mathbb{P}(X(W_n) = m). \tag{4.72}
\end{align*}
\]

First we consider the first term on the r.h.s. of (4.72). To begin with, we first consider this term with \( |\hat{s}(k_n) - s| \) replaced by \( |\tilde{s}(k_n) - s| \), where \( |\tilde{s}(k_n) - s| \) is defined as in the paragraph preceding (4.14). Recall that, conditionally given \( X(W_n) = m_n \in A_n \), \( |\tilde{s}(k_n) - s| \) has the same distribution as \( H_n^{-1}(Z_{k_n:m_n}) \), where \( Z_{k_n:m_n} \) denotes the \( k_n \)-th order statistics of a sample \( Z_1, \ldots, Z_{m_n} \) of size \( m_n \) from the uniform \( (0,1) \) distribution. First we write the expectation appearing in the first term on the r.h.s. of (4.72) as

\[
\mathbb{E} \left( \frac{1}{|\tilde{s}(k_n) - s|} \mathbb{I} \left( \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right) + \mathbb{E} \left( \frac{1}{|\tilde{s}(k_n) - s|} \mathbb{I} \left( \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right), \tag{4.73}
\]

for some sequence of positive real numbers \( \epsilon_n \downarrow 0 \) as \( n \to \infty \), and \( \tilde{Z}_{k_n:m_n} = Z_{k_n:m_n} - \mathbb{E} Z_{k_n:m_n} = Z_{k_n,m_n} - k_n/(m_n + 1) \). By a similar argument as in (4.34) (cf. also (4.35)), conditionally given \( X(W_n) = m_n \), we have

\[
|\tilde{s}(k_n) - s| \mathbb{I} \left( \frac{k_n}{m_n} \right) \leq \epsilon_n \frac{k_n}{m_n}
\]

\[
\frac{\theta \tau k_n}{2\lambda(s)(m_n + 1)} + o \left( \frac{k_n}{W_n} \right) + \left( \frac{\theta \tau}{2\lambda(s)} \right) \tilde{Z}_{k_n:m_n} + o \left( \tilde{Z}_{k_n:m_n} \right)
\]

\[
\cdot \mathbb{I} \left( \frac{k_n}{m_n} \right)
\]
as \( n \to \infty \). By Lemma A.4 (see Appendix), there exists a positive constant \( C_0 \) such that

\[
\mathbb{P} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) \leq 2 \exp \left\{ -C_0 \epsilon_n^2 k_n \right\} \leq 2 \exp \left\{ -C_0 k_n^{1/2} \right\}, \tag{4.76}
\]

as \( n \to \infty \), provided \( \epsilon_n^{-1} = o(k_n^{1/4}) \) as \( n \to \infty \) (cf. also the r.h.s. of (4.41) with \( \epsilon \) replaced by \( \epsilon_n \)). Throughout this proof, we take \( \epsilon_n^{-1} = o(k_n^{1/4}) \) as \( n \to \infty \). From (4.76), since \( k_n \to \infty \) which implies the r.h.s. of (4.76) is \( o(1) \) as \( n \to \infty \), we obtain

\[
\mathbb{P} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) = 1 - o(1), \tag{4.77}
\]

as \( n \to \infty \). By (4.75) and (4.77), we can compute the following conditional expectation

\[
\mathbb{E} \left( \frac{1}{\hat{s}(k_n) - s} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \middle| X(W_n) = m_n \right) = \mathbb{E} \left( \frac{1}{(\hat{s}(k_n)(2\lambda(s)|W_n|))^{-1} (1 + o(1))} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \right) = \mathbb{E} \left( \frac{2\lambda(s)|W_n|}{\tau k_n} (1 + o(1)) \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \right) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o \left( \frac{|W_n|}{k_n} \right), \tag{4.78}
\]

as \( n \to \infty \).

Next we consider the second term of (4.73). First note that

\[
\mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) = \mathbb{I} \left( Z_{k_n:m_n} > \frac{k_n}{m_n + 1} + \epsilon_n \frac{k_n}{m_n} \right) + \mathbb{I} \left( Z_{k_n:m_n} < \frac{k_n}{m_n + 1} - \epsilon_n \frac{k_n}{m_n} \right).
\]
For the case $Z_{kn,mn} > \frac{k_n}{m_n + 1} + \epsilon_n \frac{k_n}{m_n}$, by Lemma 4.10 and (4.31), conditionally given $X(W_n) = m_n$, we have

$$|\hat{s}(k_n) - s| = |\hat{s}(k_n) - s| + o\left( \frac{k_n}{|W_n|} \right) = H_n^{-1}(Z_{kn,mn}) + o\left( \frac{k_n}{|W_n|} \right)$$

$$\geq H_n^{-1}\left( \frac{k_n}{m_n + 1} \right) + o\left( \frac{k_n}{|W_n|} \right) = H^{-1}\left( \frac{k_n}{m_n + 1} + o(|W_n|^{-1}) \right) + o\left( \frac{k_n}{|W_n|} \right)$$

$$= \frac{\theta\tau k_n}{2\lambda(s)(m_n + 1)} + o\left( \frac{k_n}{|W_n|} \right) \geq \frac{\tau k_n}{4\lambda(s)|W_n|},$$

for sufficiently large $n$. Hence, for sufficiently large $n$, conditionally given $X(W_n) = m_n$, we have

$$\frac{1}{|\hat{s}(k_n) - s|} I\left( Z_{kn,mn} > \frac{k_n}{m_n + 1} + \epsilon_n \frac{k_n}{m_n} \right) X(W_n) = m_n = o\left( \frac{|W_n|}{k_n} \right)$$

which in combination with (4.76), implies

$$\mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} I\left( Z_{kn,mn} > \frac{k_n}{m_n + 1} + \epsilon_n \frac{k_n}{m_n} \right) X(W_n) = m_n \right) = o\left( \frac{|W_n|}{k_n} \right)$$

as $n \to \infty$. Next we will show

$$\mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} I\left( Z_{kn,mn} < \frac{k_n}{m_n + 1} - \epsilon_n \frac{k_n}{m_n} \right) X(W_n) = m_n \right) = o\left( \frac{|W_n|}{k_n} \right)$$

as $n \to \infty$. By Lemma 4.10, the fact that $|\hat{s}(k_n) - s| = H_n^{-1}(Z_{kn,mn})$, and an application of mean value theorem, together with a little calculation showing that $H_n^{-1'}(\xi_n) = (\theta\tau)(2\lambda(s))^{-1} + o(1)$ as $n \to \infty$, for any (random) point $\xi_n \in (Z_{kn,mn}, k_n(m_n + 1) - 1)$, whenever $I(Z_{kn,mn} < k_n(m_n + 1) - 1 - \epsilon_n k_n m_n^{-1}) = 1$, shows that $|\hat{s}(k_n) - s| = ((\theta\tau)(2\lambda(s))^{-1} + o(1))Z_{kn,mn} + o(k_n|W_n|^{-1})$, as $n \to \infty$. Since $\mathbb{E}Z_{kn,mn}^2 = O(m_n^2 k_n^{-2})$ as $n \to \infty$, by an application of Cauchy-Schwarz inequality and (4.76), we can easily completes the proof of (4.81). Combining (4.78), (4.80) and (4.81), we have

$$\mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} X(W_n) = m_n \right) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left( \frac{|W_n|}{k_n} \right)$$

as $n \to \infty$. By an exponential bound for the Poisson probabilities (Lemma A.1), we know that (cf. also (4.24))

$$P(X(W_n) \in A_n^c) \leq O(1) \exp \left( -\frac{\alpha_n^2}{2 + o(1)} \right),$$

(4.83)
which is $o(1)$ as $n \to \infty$, since $a_n \to \infty$ as $n \to \infty$. This implies

$$
P(X(W_n) \in A_n) = (1 - o(1)),
$$

(4.84)
as $n \to \infty$. By (4.82) and (4.84), the first term on the r.h.s. of (4.72) is equal to

$$
\left( \frac{2\lambda(s)|W_n|}{\tau k_n} + o \left( \frac{|W_n|}{k_n} \right) \right) P(X(W_n) \in A_n)
= \frac{2\lambda(s)|W_n|}{\tau k_n} + o \left( \frac{|W_n|}{k_n} \right),
$$

(4.85)
as $n \to \infty$.

Next we consider the second and third term on the r.h.s. of (4.72). First, for any integer $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, we write the expectation appearing in this term as (4.73) with $m_n$ replaced by $m$. For any integer $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, similar to that in (4.74) with $m_n$ replaced by $m$, we have a stochastic expansion for $|\hat{s}(k_n) - s|I(|\tilde{Z}_{k_n:m}| \leq \epsilon_n k_n m^{-1})$, conditionally given $X(W_n) = m$, as follows

$$
|\hat{s}(k_n) - s|I \left( |\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m} \right) \overset{d}{=} \left\{ \frac{\theta \tau k_n}{2\lambda(s)m} + o \left( \frac{k_n}{m} \right) + O \left( \frac{1}{|W_n|} \right) + O(\tilde{Z}_{k_n:m}) \right\} I \left( |\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m} \right),
$$

(4.86)
as $n \to \infty$. Combining (4.86) and Lemma 4.10, conditionally given $X(W_n) = m$, we have

$$
|\hat{s}(k_n) - s|I \left( |\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m} \right) \overset{d}{=} \left\{ \frac{\theta \tau k_n}{2\lambda(s)m} + o \left( \frac{k_n}{m} \right) + o \left( \frac{k_n}{|W_n|} \right) \right\} I \left( |\tilde{Z}_{k_n:m}| \leq \epsilon_n \frac{k_n}{m} \right),
$$

(4.87)
as $n \to \infty$. Note that (4.76) and (4.77) remain hold true when $m_n \in A_n$ is now replaced by $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$. By a similar argument as the one used to prove (4.80) and (4.81), but with $m_n \in A_n$ replaced by $m \in \{[k_n, C_{1,n}) \cup (C_{2,n}, \infty)\}$, conditionally given $X(W_n) = m$, we have

$$
E \left( \frac{1}{|\hat{s}(k_n) - s|} I \left( |\tilde{Z}_{k_n:m}| > \epsilon_n \frac{k_n}{m} \right) \right) = O \left( \frac{m}{k_n} + \frac{|W_n|}{k_n} \right)
$$

(4.88)
as $n \to \infty$. Then, by (4.77) with $m_n$ replaced by $m \in [k_n, C_{1,n})$ and (4.87), in combination with (4.88), we have

$$
E \left( \frac{1}{|\hat{s}(k_n) - s|} |X(W_n) = m| \right) = O \left( \frac{|W_n|}{k_n} \right),
$$

(4.89)
as \( n \to \infty \), uniformly for all \( m \in [k_n, C_1,n] \). By (4.89), the second term on the r.h.s. of (4.72) is equal to \( \mathcal{O}(|W_n|^{-1})P(X(W_n) \in [k_n, C_1,n]) \). Since by (4.83) we have \( P(X(W_n) \in [k_n, C_1,n]) \leq P(X(W_n) \in A^c_n) = o(1) \), as \( n \to \infty \), this term is of order \( o(|W_n|^{-1}) \) as \( n \to \infty \).

For any \( m \in (C_2,n, \infty) \), since \( m > (\theta|W_n|) + (\theta|W_n|)^{1/2}a_n \) (for some sequence \( a_n \to \infty \) and \( a_n = o(|W_n|^{1/2}) \), we may have the absolute value of the third term on the r.h.s. of (4.87) is bigger than its first term. If the first term on the r.h.s. of (4.87) is the leading term, a similar argument as the one used to prove (4.82) shows that

\[
E(|\hat{s}(k_n) - s|^{-1}I(|\hat{Z}_{k_n,m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(mk_n^{-1}),
\]

as \( n \to \infty \). If the third term on the r.h.s. of (4.87) is the leading term, then there exists a sequence \( c_n \to 0 \) as \( n \to \infty \), such that this term can be written as \( c_n k_n|W_n|^{-1} \) with \( |c_n| > (\theta \tau |W_n|)/(2\lambda(s)m) \). For this case, a similar argument as the one used to prove (4.82) shows that \( E(|\hat{s}(k_n) - s|^{-1}I(|\hat{Z}_{k_n,m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(|W_n|^{-1}c_n^{-1}) \) as \( n \to \infty \). Since \( |c_n| > (\theta \tau |W_n|)/(2\lambda(s)m) \) which implies \( |c_n| < (2\lambda(s)m)/(\theta \tau |W_n|) \), we also have

\[
E(|\hat{s}(k_n) - s|^{-1}I(|\hat{Z}_{k_n,m}| \leq \epsilon_n k_n m^{-1})|X(W_n) = m) = \mathcal{O}(mk_n^{-1}),
\]

as \( n \to \infty \). A similar argument also holds true when the first and third terms on the r.h.s. of (4.87) are of the same order. Combining this result with (4.88), uniformly in \( m \in (C_2,n, \infty) \), we have

\[
E \left( \frac{1}{|\hat{s}(k_n) - s|} \right| X(W_n) = m) = \mathcal{O} \left( \frac{m}{k_n} \right),
\]

(4.90)

as \( n \to \infty \). By (4.90), the third term on the r.h.s. of (4.72) can be computed as follows

\[
\mathcal{O} \left( \frac{1}{k_n} \right) \sum_{m=C_2,n+1}^{\infty} m P(X(W_n) = m)
\]

\[
= \mathcal{O} \left( \frac{1}{k_n} \right) \mathbb{E}X(W_n)I(X(W_n) > C_2,n)
\]

\[
\leq \mathcal{O} \left( \frac{1}{k_n} \right) \left( \mathbb{E}X^2(W_n) \right)^{1/2} P^{1/2}(X(W_n) > C_2,n) = o \left( \frac{|W_n|}{k_n} \right),
\]

(4.91)

as \( n \to \infty \), because by periodicity of \( \lambda \) we have \( \mathbb{E}X^2(W_n) \) as \( n \to \infty \), and by (4.83) we have \( P^{1/2}(X(W_n) > C_2,n) \leq P^{1/2}(X(W_n) \in A^c_n) = o(1) \), as \( n \to \infty \).
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Since the first term on the r.h.s. of (4.72) is equal to the r.h.s. of (4.85), while the other terms are of order \( o(|W_n|k_n^{-1}) \) as \( n \to \infty \), we then have

\[
\mathbb{E} \frac{1}{|\hat{s}(k_n) - s|} I(X(W_n) \geq k_n) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o\left(\frac{|W_n|}{k_n}\right),
\]

(4.92)
as \( n \to \infty \), which implies the first term on the r.h.s. of (4.71) is equal to \( \lambda(s) + o(1) \) as \( n \to \infty \).

Next we show that the second term on the r.h.s. of (4.71) is of order \( o(1) \) as \( n \to \infty \). By (4.70) and (4.92), the absolute value of this term does not exceed

\[
\frac{C\delta_n k_n^2}{2|W_n|^3} \mathbb{E} \frac{1}{|\hat{s}(k_n) - s|} I(X(W_n) \geq k_n) = \frac{C\delta_n k_n^2}{2|W_n|^3} \mathcal{O}\left(\frac{|W_n|}{k_n}\right)
\]

(4.93)
as \( n \to \infty \). This completes the proof of Theorem 4.5. □

**Proof of Theorem 4.6**

By Remark 4.3, we can write

\[
\text{Var} \left( \hat{\lambda}_n(s) \right) = \text{Var} \left( \hat{\lambda}_n(s) I(X(W_n) \geq k_n) \right)
\]

\[
= \mathbb{E} \left( \hat{\lambda}_n(s) I(X(W_n) \geq k_n) \right)^2 - \left( \mathbb{E} \hat{\lambda}_n(s) I(X(W_n) \geq k_n) \right)^2.
\]

(4.93)

By Remark 4.3 and Theorem 4.5, we have \( \mathbb{E} \hat{\lambda}_n(s) I(X(W_n) \geq k_n) = \mathbb{E} \hat{\lambda}_n(s) = \lambda(s) + o(1) \) as \( n \to \infty \). This implies the second term on the r.h.s. of (4.93) is equal to \(-\lambda^2(s) + o(1) \) as \( n \to \infty \). Then, to prove this theorem, it suffices to show that the first term on the r.h.s. of (4.93) is equal to \( \lambda^2(s) + o(1) \) as \( n \to \infty \). To do this we argue as follows. The first term on the r.h.s. of (4.93) is equal to

\[
\mathbb{E} \left( \frac{\tilde{\tau}_n k_n}{2|W_n||\hat{s}(k_n) - s|} I(X(W_n) \geq k_n) \right)^2
\]

\[
= \mathbb{E} \left( \frac{\tau k_n}{2|W_n||\hat{s}(k_n) - s|} I(X(W_n) \geq k_n) \right)^2
\]

\[
+ \mathbb{E} \left( \frac{(\tilde{\tau}_n - \tau) k_n}{2|W_n||\hat{s}(k_n) - s|} I(X(W_n) \geq k_n) \right)^2
\]

\[
+ 2\mathbb{E} \left( \frac{\tau k_n}{2|W_n||\hat{s}(k_n) - s|} \right) \left( \frac{(\tilde{\tau}_n - \tau) k_n}{2|W_n||\hat{s}(k_n) - s|} \right) I(X(W_n) \geq k_n).
\]

(4.94)
We will show that the first term on the r.h.s. of (4.94) is equal to \( \lambda^2(s) + o(1) \) as \( n \to \infty \), while its second and third terms are of order \( o(1) \) as \( n \to \infty \).

First we consider the first term on the r.h.s. of (4.94). This term can be written as

\[
\frac{\tau^2 k_n^2}{4|W_n|^2} \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} I(X(W_n) \geq k_n) \right)^2 \right) = \frac{\tau^2 k_n^2}{4|W_n|^2} \mathbb{E} \left\{ \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} I(X(W_n) \geq k_n) \right)^2 \bigg| X(W_n) = m \right) \right\} .
\]

(4.95)

Expectation of the quantity within curly brackets on the r.h.s. of (4.95) can be computed as follows

\[
\sum_{m_n \in A_n} \left\{ \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 \bigg| X(W_n) = m_n \right) \right\} \mathbb{P} (X(W_n) = m_n) + \sum_{m = k_n}^{C_{1,n}-1} \left\{ \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 \bigg| X(W_n) = m \right) \right\} \mathbb{P} (X(W_n) = m) + \sum_{C_{2,n+1}}^{\infty} \left\{ \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 \bigg| X(W_n) = m \right) \right\} \mathbb{P} (X(W_n) = m) .
\]

(4.96)

First we consider the first term of (4.96). The expectation appearing in this term can be written as

\[
\mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 I \left( |\tilde{X}_{k_n,m_n} - s| \leq \epsilon_n \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right) + \mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 I \left( |\tilde{X}_{k_n,m_n} - s| > \epsilon_n \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right) ,
\]

where \( \epsilon_n \) a sequence of positive real numbers converging to zero and \( \epsilon_n^{-1} = o(k_n^{1/4}) \), as \( n \to \infty \). By (4.75) and (4.77), we can compute the following conditional expectation

\[
\mathbb{E} \left( \left( \frac{1}{|\delta(k_n) - s|} \right)^2 I \left( |\tilde{X}_{k_n,m_n} - s| \leq \epsilon_n \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right) = \frac{1}{(\tau^2 k_n^2)(4\lambda^2(s)|W_n|^2)^{-1}(1 + o(1))^2} \mathbb{I} \left( |\tilde{X}_{k_n,m_n} - s| \leq \epsilon_n \frac{k_n}{m_n} \right)
\]

\[
= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} \left( 1 + o(1) \right) \mathbb{P} \left( |\tilde{X}_{k_n,m_n} - s| \leq \epsilon_n \frac{k_n}{m_n} \right)
\]

\[
= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} \left( 1 + o(1) \right) \mathbb{P} \left( \tilde{X}_{k_n,m_n} \leq k_n \right)
\]

\[
= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} \left( 1 + o(1) \right) \mathbb{P} \left( X(W_n) \leq k_n \right)
\]

(4.97)
as \( n \to \infty \). By (4.79) and (4.76), together with a similar argument as the one used to prove (4.81), we have

\[
\mathbb{E} \left( \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \right)^2 I \left( |\hat{Z}_{k_n} - n| \geq \epsilon_n \frac{k_n}{m_n} \right) \mid X(W_n) = m_n \right)
\]

\[
= o \left( \frac{|W_n|^2}{k_n^2} \right),
\]

(4.98)
as \( n \to \infty \). By (4.97) and (4.98), we have

\[
\mathbb{E} \left( \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \right)^2 \mid X(W_n) = m_n \right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o \left( \frac{|W_n|^2}{k_n^2} \right),
\]

(4.99)as \( n \to \infty \). By (4.99), the first term of (4.96) is equal to

\[
\left( \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o \left( \frac{|W_n|^2}{k_n^2} \right) \right) P(X(W_n) \in A_n)
\]

\[
= \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o \left( \frac{|W_n|^2}{k_n^2} \right),
\]

(4.100)as \( n \to \infty \), because by (4.84) we have \( P(X(W_n) \in A_n) = 1 - o(1) \) as \( n \to \infty \).

Next we consider the second and third term of (4.96). By a similar argument as the one used to compute the expectation in (4.99), but with \( m_n \) replaced by \( m \), we have that

\[
\mathbb{E} \left( \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \right)^2 \mid X(W_n) = m \right) = \mathcal{O} \left( \frac{|W_n|^2}{k_n^2} \right),
\]

(4.101)as \( n \to \infty \), uniformly for all \( m \in [k_n, C_1, n) \), and

\[
\mathbb{E} \left( \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \right)^2 \mid X(W_n) = m \right) = \mathcal{O} \left( \frac{m^2}{k_n^2} \right),
\]

(4.102)as \( n \to \infty \), for each \( m \in (C_2, n, \infty) \) (cf. also the argument used to prove (4.90) to handle possibility that the first term on the r.h.s. of (4.87) is of smaller order than its third term when \( m \in (C_2, \infty) \)). By (4.101), the second term of (4.96) is equal to \( \mathcal{O}(|W_n|^2k_n^{-2})P(X(W_n) \in [k_n, C_1, n)) \). Since by (4.83) we have \( P(X(W_n) \in [k_n, C_1, n)) \leq P(X(W_n) \in A_n) = o(1) \), as \( n \to \infty \), this term is of order \( o(|W_n|^2k_n^{-2}) \) as \( n \to \infty \). By (4.102),
the third term of (4.96) can be computed as follows

\[
\mathcal{O}\left(\frac{1}{k_n^2}\right) \sum_{m=C_2,n+1}^{\infty} m^2 P(X(W_n) = m)
\]

\[
= \mathcal{O}\left(\frac{1}{k_n^2}\right) \mathbb{E}X^2(W_n)I(X(W_n) > C_{2,n})
\]

\[
\leq \mathcal{O}\left(\frac{1}{k_n^2}\right) \left(\mathbb{E}X^4(W_n)\right)^{1/2} P^{1/2}(X(W_n) > C_{2,n}) = o\left(|W_n|^2 k_n^{-2}\right),
\]

as \(n \to \infty\), because by periodicity of \(\lambda\) we have \((\mathbb{E}X^4(W_n))^{1/2} = \mathcal{O}(|W_n|^2)\) as \(n \to \infty\), and by (4.83) we have \(P^{1/2}(X(W_n) > C_{2,n}) \leq P^{1/2}(X(W_n) \in A_{C_n}^c) = o(1)\), as \(n \to \infty\).

Since the first term of (4.96) is equal to the r.h.s. of (4.100), while its second and third terms are of order \(o(|W_n|^2 k_n^{-2})\) as \(n \to \infty\), we then have

\[
\mathbb{E}\left(\frac{1}{\delta(k_n) - \delta} I(X(W_n) \geq k_n)\right)^2 = \frac{4\lambda^2(s)|W_n|^2}{\tau^2 k_n^2} + o\left(\frac{|W_n|^2}{k_n^2}\right),
\]

(4.103)

as \(n \to \infty\), which implies the quantity in (4.95) is equal to \(\lambda^2(s) + o(1)\) as \(n \to \infty\). Hence, the first term on the r.h.s. of (4.94) is equal to \(\lambda^2(s) + o(1)\) as \(n \to \infty\).

It remains to show that the second and third term on the r.h.s. of (4.94) are of order \(o(1)\) as \(n \to \infty\). By (4.70) and (4.103), sum of the second term and the absolute value of the third term on the r.h.s. of (4.94) does not exceed

\[
\left(\frac{C^2\delta_n^2 k_n^4}{4|W_n|^6} + \frac{C\tau\delta_n k_n^3}{2|W_n|^4}\right) \mathbb{E}\left(\frac{1}{|\delta(k_n) - \delta|} I(X(W_n) \geq k_n)\right)^2
\]

\[
= \left(\frac{C^2\delta_n^2 k_n^4}{4|W_n|^6} + \frac{C\tau\delta_n k_n^3}{2|W_n|^4}\right) \mathcal{O}\left(\frac{|W_n|^2}{k_n^2}\right) = \mathcal{O}\left(\frac{\delta_n^2 k_n^2}{|W_n|^4} + \frac{\delta_n k_n}{|W_n|^2}\right)
\]

\[= o(1),\]

as \(n \to \infty\). This completes the proof of Theorem 4.6. □

**Proof of Theorem 4.7**

Since we want to prove (4.59) instead of (4.57), it is not enough now to use the result from Theorem 4.5 to simplify the expression for \(Var(\hat{\lambda}_n(s))\). Hence, instead of writing \(Var(\hat{\lambda}_n(s))\) as in (4.93), here we have to directly compute \(Var(\hat{\lambda}_n(s))\) as below. By Remark 4.3 we have

\[
Var(\hat{\lambda}_n(s)) = Var(\hat{\lambda}_n(s)I(X(W_n) \geq k_n)).
\]
Then, the l.h.s. of (4.59) can be written as

\[ \text{Var} \left( \frac{\tau k_n}{2|W_n||{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \right) \]

\[ = \text{Var} \left( \frac{\tau k_n}{2|W_n||{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \right) \]

\[ + \text{Var} \left( \frac{(\hat{\mu}_n - \tau)k_n}{2|W_n||{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \right) \]

\[ + 2 \text{Cov} \left( \frac{\tau k_n}{2|W_n||{\hat{s}}(k_n) - s|}, \frac{(\hat{\mu}_n - \tau)k_n}{2|W_n||{\hat{s}}(k_n) - s|} \right) I(X(W_n) \geq k_n). \]

(4.104)

The first term on the r.h.s. of (4.104) can be written as

\[ \frac{\tau^2 k_n^2}{4|W_n|^2} \text{Var} \left( \frac{1}{|{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \right) \]

\[ + \frac{\tau^2 k_n^2}{4|W_n|^2} \text{Var} \left( \left( \frac{1}{|{\hat{s}}(k_n) - s|} - \frac{1}{|{\hat{s}}(k_n) - s|} \right) I(X(W_n) \geq k_n) \right) \]

\[ + \frac{\tau^2 k_n^2}{2|W_n|^2} \text{Cov} \left( \frac{1}{|{\hat{s}}(k_n) - s|}, \frac{1}{|{\hat{s}}(k_n) - s|} - \frac{1}{|{\hat{s}}(k_n) - s|} \right) I(X(W_n) \geq k_n). \]

(4.105)

We will prove this theorem by showing that the first term of (4.105) is equal to \( \lambda^2(s)k_n^{-1} + o(k_n^{-1}) \) as \( n \to \infty \), while the second and third terms of (4.105), as well as the second and third terms on the r.h.s. of (4.104) are of order \( o(k_n^{-1}) \) as \( n \to \infty \).

First we show that the first term of (4.105) is equal to \( \lambda^2(s)k_n^{-1} + o(k_n^{-1}) \) as \( n \to \infty \). The variance appearing in this term can be computed as follows

\[ \mathbb{E} \left( \text{Var} \left( \frac{1}{|{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \bigg| X(W_n) = m \right) \right) \]

\[ + \text{Var} \left( \mathbb{E} \left( \frac{1}{|{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) \bigg| X(W_n) = m \right) \right). \]

(4.106)

Similar to (4.92) (note that the r.h.s. of (4.74) is equal to the r.h.s. of (4.75)), we also have

\[ \mathbb{E} \frac{1}{|{\hat{s}}(k_n) - s|} I(X(W_n) \geq k_n) = \frac{2\lambda(s)|W_n|}{\tau k_n} + o \left( \frac{|W_n|}{k_n} \right), \]

(4.107)

as \( n \to \infty \), which is deterministic. Hence, the second term of (4.106) is equal to zero. The first term of (4.106) is equal to

\[ \sum_{m_n \in A_n} \left\{ \text{Var} \left( \frac{1}{|{\hat{s}}(k_n) - s|} \bigg| X(W_n) = m_n \right) \right\} P(X(W_n) = m_n) \]
First we consider the first term of (4.108). The variance appearing in this term is equal to

\[
\begin{align*}
  V\text{ar} \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) X(W_n) = m_n \\
  \quad + V\text{ar} \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \left| \mathcal{Z}_{k_n:m_n} > \epsilon_n \frac{k_n}{m_n} \right| \right) X(W_n) = m_n \\
  \quad + 2 \text{Cov} \left\{ \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) \right\} X(W_n) = m_n \quad \text{(4.109)}
\end{align*}
\]

To compute the first term of (4.109), we argue as follows. From (4.74), conditionally given \( X(W_n) = m_n \), we have the following stochastic expansion

\[
|\hat{\sigma}(k_n) - s| \mathbf{I} \left( \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) = \frac{\tau k_n}{2\lambda(s)|W_n|} \left( 1 + o(1) + \left( \theta |W_n| k_n + o \left( \frac{|W_n|}{k_n} \right) \right) \mathbf{I} \left( \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) \right) \quad \text{(4.110)}
\]

By (4.110), and by noting that \( |W_n|k_n^{-1} \mathcal{Z}_{k_n:m_n} | \mathcal{Z}_{k_n:m_n} \leq \epsilon_n k_n m_n^{-1} | = o(1) \) as \( n \to \infty \), we can obtain the following stochastic expansion

\[
\begin{align*}
  \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) X(W_n) = m_n \\
  \quad = \frac{\tau k_n}{2\lambda(s)|W_n|} \left\{ \frac{1}{\{ 1 + o(1) + (\theta |W_n| k_n^{-1} + o(|W_n|k_n^{-1})) \mathbf{I} \left( \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) \} \right\} \\
  \quad \cdot \mathbf{I} \left( \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) \\
  \quad = \frac{2\lambda(s)|W_n|}{\tau k_n} \left\{ 1 + o(1) - \left( \frac{\theta |W_n|}{k_n} + o \left( \frac{|W_n|}{k_n} \right) \right) \mathbf{I} \left( \left| \mathcal{Z}_{k_n:m_n} \leq \epsilon_n \frac{k_n}{m_n} \right| \right) \right\}
\end{align*}
\]
4.3 Statistical properties

\[ + O \left( \frac{|W_n|^2}{k_n^2} \right) \tilde{Z}_{k_n:m_n}^2 \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \]

\[ = \left\{ \frac{2\lambda(s)|W_n|}{\tau k_n} + o \left( \frac{|W_n|^2}{k_n^2} \right) \right\} \tilde{Z}_{k_n:m_n} + O \left( \frac{|W_n|^3}{k_n^3} \right) \tilde{Z}_{k_n:m_n}^2 \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right). \quad (4.111) \]

By (4.111), we can compute the following conditional variance

\[ \text{Var} \left( \frac{1}{|\tilde{s}(k_n) - s|} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) | X(W_n) = m_n \right) \]

\[ = \left( \frac{2\lambda(s)\theta|W_n|^2}{\tau k_n^2} + o \left( \frac{|W_n|^2}{k_n^2} \right) \right)^2 \text{Var} \left( \tilde{Z}_{k_n:m_n} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \right) \]

\[ + O \left( \frac{|W_n|^6}{k_n^6} \right) \text{Var} \left( \tilde{Z}_{k_n:m_n}^2 \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \right) \]

\[ + O \left( \frac{|W_n|^6}{k_n^5} \right) \text{Cov} \left\{ \left( \tilde{Z}_{k_n:m_n}, \tilde{Z}_{k_n:m_n}^2 \right) \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right) \right\}. \quad (4.112) \]

The variance appearing in the first term on the r.h.s. of (4.112) can be computed as follows

\[ = \text{Var} \left( \tilde{Z}_{k_n:m_n} - \tilde{Z}_{k_n:m_n} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) \right) \]

\[ = \text{Var} \left( \tilde{Z}_{k_n:m_n} \right) + \text{Var} \left( \tilde{Z}_{k_n:m_n} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) \right) \]

\[ - 2 \text{Cov} \left( \tilde{Z}_{k_n:m_n}, \tilde{Z}_{k_n:m_n} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) \right). \quad (4.113) \]

The first term on the r.h.s. of (4.113) is equal to (cf. Reiss (1989), p. 45)

\[ \frac{k_n(m_n - k_n + 1)}{(m_n + 1)^2(m_n + 2)} = \frac{k_n}{\theta^2|W_n|^2} + o \left( \frac{k_n}{|W_n|^2} \right), \]

as \( n \to \infty \). A simple calculation (using formula (1.7.4) of Reiss (1989, p. 45) shows that

\[ \mathbb{E} \tilde{Z}_{k_n:m_n}^4 = O \left( k_n^2|W_n|^{-4} \right) \quad (4.114) \]

as \( n \to \infty \). By (4.76), (4.114) and an application of Cauchy-Schwarz inequality, we have that the second term on the r.h.s. of (4.113) does not exceed

\[ \mathbb{E} \left( \tilde{Z}_{k_n:m_n} \mathbb{I} \left( \left| \tilde{Z}_{k_n:m_n} \right| > \epsilon_n \frac{k_n}{m_n} \right) \right)^2 = o \left( \frac{k_n}{|W_n|^2} \right), \]
as \( n \to \infty \). Another application of Cauchy-Schwarz inequality shows that the third term on the r.h.s. of (4.113) is of order \( o(k_n|W_n|^{-2}) \) as \( n \to \infty \). Combining all these results, uniformly in \( m_n \in A_n \), we find that the first term on the r.h.s. of (4.112) is equal to

\[
\left( \frac{4\lambda^2(s)\theta^2|W_n|^4}{\tau^2k_n^4} + o\left( \frac{|W_n|^4}{k_n^4} \right) \right) \left( \frac{k_n}{\theta^2|W_n|^2} + o\left( \frac{k_n}{|W_n|^2} \right) \right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2k_n^3} + o\left( \frac{|W_n|^2}{k_n^3} \right),
\]

as \( n \to \infty \). By (4.114), the second term on the r.h.s. of (4.112) does not exceed

\[
O\left( \frac{|W_n|^6}{k_n^6} \right) E\tilde{Z}_{k_n:m_n}^4 = O\left( \frac{|W_n|^2}{k_n^4} \right) = o\left( \frac{|W_n|^2}{k_n^3} \right),
\]

as \( n \to \infty \), since \( k_n \to \infty \) as \( n \to \infty \). An application of Cauchy-Schwarz inequality also shows that the third term on the r.h.s. of (4.112) is of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \). Hence we have

\[
\frac{1}{k_n^6} \left| \tilde{Z}_{k_n:m_n} \right| \leq \epsilon_n \frac{k_n}{m_n} \right| X(W_n) = m_n) = o\left( \frac{|W_n|^2}{k_n^3} \right),
\]

(4.115)

as \( n \to \infty \), uniformly in \( m_n \in A_n \).

Next we consider the second term of (4.109). First note that (4.79) remains valid if \( |\tilde{s}(k_n) - s| \) replaced by \( |\tilde{s}(k_n) - s| \). Then, by (4.79) with \( |\tilde{s}(k_n) - s| \) replaced by \( |\tilde{s}(k_n) - s| \) and (4.76), by noting that the r.h.s. of (4.76) is of order \( o(k_n^{-1}) \) as \( n \to \infty \), and in combination with an argument like the one used to prove (4.81), we have

\[
E \left( \frac{1}{|\tilde{s}(k_n) - s|} \right) I \left( |\tilde{Z}_{k_n:m_n}| \geq \epsilon_n \frac{k_n}{m_n} \right) X(W_n) = m_n \right) = o\left( \frac{|W_n|^2}{k_n^3} \right),
\]

as \( n \to \infty \). This implies the second term of (4.109) is of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \). The third term of (4.109) is equal to zero. Combining all of our results, uniformly in \( m_n \in A_n \), we have that

\[
Var \left( \frac{1}{|\tilde{s}(k_n) - s|} \right) X(W_n) = m_n \right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2k_n^3} + o\left( \frac{|W_n|^2}{k_n^3} \right),
\]

(4.116)

as \( n \to \infty \). By (4.116), the quantity in the first term of (4.108) can be computed as follows

\[
\left( \frac{4\lambda^2(s)|W_n|^2}{\tau^2k_n^3} + o\left( \frac{|W_n|^2}{k_n^3} \right) \right) P (X(W_n) \in A_n)
\]

(4.117)
as \( n \to \infty \), because by (4.84) we have \( \Pr(X(W_n) \in A_n) = 1 - o(1) \) as \( n \to \infty \).

Next we consider the second and third terms of (4.108). Sum of these two terms does not exceed

\[
\sum_{m=k_n}^{C_{1,n}-1} \left\{ \mathbb{E} \left( \left( \frac{1}{|\hat{s}(k_n) - s|} \right)^2 I(X(W_n) = m) \right) \right\} \Pr(X(W_n) = m) + \sum_{C_{2,n}+1}^{\infty} \left\{ \mathbb{E} \left( \left( \frac{1}{|\hat{s}(k_n) - s|} \right)^2 I(X(W_n) = m) \right) \right\} \Pr(X(W_n) = m).
\]

By a similar argument as the one used to prove (4.101) and (4.102), but with (4.87) now replaced by (4.86), and also by noting that (4.88) remains valid if we replace \( |\hat{s}(k_n) - s| \) by \( |\bar{s}(k_n) - s| \), we have (4.101) and (4.102) with \( |\hat{s}(k_n) - s| \) replaced by \( |\bar{s}(k_n) - s| \). Now we look at the upper bound for \( \Pr(X(W_n) \in A_n) \) as given in (4.83). By (4.83), we can write the following

\[
\Pr(X(W_n) \in A_n) \leq C \left( \frac{1}{k_n^2} \right) \exp \left( - \frac{a_n^2}{2 + o(1)} + 2\log k_n \right),
\]

as \( n \to \infty \). By choosing now the sequence \( a_n \) such that \( a_n^2/3 \to \infty \) faster than \( 2\log k_n \), we then have that

\[
\Pr(X(W_n) \in A_n) = o(k_n^{-2}), \tag{4.118}
\]

as \( n \to \infty \), which implies

\[
\Pr(X(W_n) \in A_n) = 1 - o(k_n^{-2}), \tag{4.119}
\]

as \( n \to \infty \). Then, by a similar argument as the one used to handle the second and third term of (4.96), but with \( |\hat{s}(k_n) - s| \) in (4.101) and (4.102) now replaced by \( |\bar{s}(k_n) - s| \) and also now we use (4.118) as the upper bound for \( \Pr(X(W_n) \in [k_n, C_{1,n}]) \) and \( \Pr(X(W_n) > C_{2,n}) \) so that \( \Pr^{1/2}(X(W_n) > C_{2,n}) = o(k_n^{-1}) \) as \( n \to \infty \), we find that the second and third term of (4.108) are of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \).

Since the first term of (4.108) is equal to the r.h.s. of (4.117), while its second and third terms are of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \), we then have

\[
\mathbb{E} \left( Var \left( \frac{1}{|\bar{s}(k_n) - s|} I(X(W_n) \geq k_n) \right) \right) = \frac{4\lambda^2(s)|W_n|^2}{\tau^2k_n^3} + o \left( \frac{|W_n|^2}{k_n^3} \right),
\]

as \( n \to \infty \). Since the second term of (4.106) is equal to zero, we then have the first term of (4.105) is equal to \( \lambda^2(s)k_n^{-1} + o(k_n^{-1}) \) as \( n \to \infty \).
Next, we prove that the second term of (4.105) is of order $o(k_n^{-1})$ as $n \to \infty$. To do this, it suffices to check
\[
E \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 I(X(W_n) \geq k_n) \right) = o \left( \frac{|W_n|^2}{k_n^3} \right), \tag{4.120}
\]
as $n \to \infty$. The l.h.s. of (4.120) can be computed as follows
\[
E \left\{ \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 I(X(W_n) \geq k_n) \bigg| X(W_n) = m \right) \right\} \\
= \sum_{m_n \in A_n} \sum_{C_{1,n} - 1} E \left\{ \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 I(X(W_n) = m_n) \bigg| X(W_n) = m \right) \right\} \mathbb{P}(X(W_n) = m_n) \\
+ \sum_{m_n \in A_n} \sum_{C_{2,n} + 1} E \left\{ \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 I(X(W_n) = m) \bigg| X(W_n) = m \right) \right\} \mathbb{P}(X(W_n) = m).
\tag{4.121}
\]

First consider the second and third term on the r.h.s. of (4.121). Recall that, by a similar argument as the one used to prove (4.101) and (4.102), we have (4.101) and (4.102) with $|\tilde{s}(k_n) - s|$ replaced by $|\tilde{s}(k_n) - s|$. By (4.101) and (4.101) with $|\tilde{s}(k_n) - s|$ replaced by $|\tilde{s}(k_n) - s|$, we obtain
\[
E \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 \bigg| X(W_n) = m \right) = O \left( \frac{|W_n|^2}{k_n^3} \right), \tag{4.122}
\]
as $n \to \infty$, uniformly for all $m \in [k_n, C_{1,n})$. By (4.102) and (4.102) with $|\tilde{s}(k_n) - s|$ replaced by $|\tilde{s}(k_n) - s|$, we obtain
\[
E \left( \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 \bigg| X(W_n) = m \right) = O \left( \frac{m^2}{k_n^3} \right), \tag{4.123}
\]
as $n \to \infty$, uniformly for all $m \in (C_{2,n}, \infty)$. Then, by a similar argument as the one used to handle the second and third term of (4.96) (cf. also the argument following (4.102)), provided we are now using (4.118) as the upper bound for $\mathbb{P}(X(W_n) \in [k_n, C_{1,n}))$ and $\mathbb{P}(X(W_n) > C_{2,n})$ so that $\mathbb{P}^{1/2}(X(W_n) > C_{2,n}) = o(k_n^{-1})$ as $n \to \infty$, we find that the second and third term of (4.121) are of order $O(|W_n|^2 k_n^{-3})$ as $n \to \infty$.

Next we consider the first term on the r.h.s. of (4.121). The expectation appearing in this term can be written as
\[
E \left\{ \left( \frac{1}{|\tilde{s}(k_n) - s|} - \frac{1}{|\tilde{s}(k_n) - s|} \right)^2 I \left( \frac{Z_{k_n : m_n}}{m_n} \leq \epsilon_n \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right\}
\]
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\[ + \mathbb{E} \left\{ \left( \frac{1}{|\hat{\sigma}(k_n) - s|} - \frac{1}{|\tilde{\sigma}(k_n) - s|} \right)^2 I \left( |\tilde{Z}_{k_n:m_n} | > \epsilon_n \frac{k_n}{m_n} \right) \bigg| X(W_n) = m_n \right\} \]  \hspace{1cm} (4.124)

First we consider the second term of (4.124). By a similar argument as the one used to prove (4.99), we have

\[ E \left( \left( \frac{1}{|\hat{\sigma}(k_n) - s|} \right)^4 \bigg| X(W_n) = m_n \right) = o \left( \frac{|W_n|^4}{k_n^4} \right), \]  \hspace{1cm} (4.125)

as \( n \to \infty \). Similarly, we also have (4.125) with \( |\hat{\sigma}(k_n) - s| \) replaced by \( |\tilde{\sigma}(k_n) - s| \). Then, by Cauchy-Schwarz inequality, (4.125), (4.125) with \( |\hat{\sigma}(k_n) - s| \) replaced by \( |\tilde{\sigma}(k_n) - s| \), and by using the fact that the r.h.s. of (4.76) is \( o(k_n^{-2}) \) as \( n \to \infty \), we obtain that the second term of (4.124) is of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \).

Next we consider the first term of (4.124). By Lemma 4.10 with condition (4.55) replaced by (4.58), conditionally given \( X(W_n) = m_n \), we have

\[ |\hat{\sigma}(k_n) - s| I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \\
= \left| |\hat{\sigma}(k_n) - s| + O \left( \delta_n k_n^{-1/2} |W_n|^{-1} \right) \right| I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \\
= |\hat{\sigma}(k_n) - s| \left( 1 + |\hat{\sigma}(k_n) - s|^{-1} O \left( \delta_n k_n^{-1/2} |W_n|^{-1} \right) \right) I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right), \]  \hspace{1cm} (4.126)

as \( n \to \infty \). By (4.74), conditionally given \( X(W_n) = m_n \), we have with probability 1

\[ |\hat{\sigma}(k_n) - s|^{-1} I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n k_n m_n^{-1} \right) = O \left( |W_n|^k_n^{-1} \right) I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n k_n m_n^{-1} \right), \]

which implies \( |\hat{\sigma}(k_n) - s|^{-1} O \left( \delta_n k_n^{-1/2} |W_n|^{-1} \right) I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n k_n m_n^{-1} \right) \)

\( = o(1) I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n k_n m_n^{-1} \right) \) as \( n \to \infty \). Then, by (4.74) and (4.126), conditionally given \( X(W_n) = m_n \), we obtain the following stochastic expansion

\[ \frac{1}{|\tilde{\sigma}(k_n) - s|} I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \\
= \left\{ \left( \frac{1}{|\tilde{\sigma}(k_n) - s|} - \frac{1}{|\tilde{\sigma}(k_n) - s|} I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \right)^2 \right\} I \left( \tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \\
= \left\{ \left( \frac{1}{|\tilde{\sigma}(k_n) - s|} - \frac{1}{|\tilde{\sigma}(k_n) - s|} \right)^2 \right\} I \left( \tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \]  \hspace{1cm} (4.127)

as \( n \to \infty \). By (4.127), conditionally given \( X(W_n) = m_n \), we have

\[ \left( \frac{1}{|\tilde{\sigma}(k_n) - s|} - \frac{1}{|\tilde{\sigma}(k_n) - s|} \right)^2 I \left( |\tilde{Z}_{k_n:m_n} | \leq \epsilon_n \frac{k_n}{m_n} \right) \]
with probability 1 as \( n \to \infty \), which implies the first term of (4.124) is of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \). Combining this order bound with the order bound of the second term of (4.124), we obtain

\[
E \left\{ \left( \frac{1}{|\hat{s}_{(k_n)} - s|} - \frac{1}{|\tilde{s}_{(k_n)} - s|} \right)^2 \bigg| X(W_n) = m_n \right\} = o \left( \frac{|W_n|^2}{k_n^3} \right), \quad (4.128)
\]
as \( n \to \infty \), uniformly for all \( m_n \in A_n \). Substituting (4.128) in to the first term on the r.h.s. of (4.121), we find that this term is of order \( o(|W_n|^2k_n^{-3}) \), as \( n \to \infty \). Since the second and third term on the r.h.s. of (4.121) are also of order \( o(|W_n|^2k_n^{-3}) \) as \( n \to \infty \), we have (4.120), which implies that the second term of (4.105) is of order \( o(k_n^{-1}) \), as \( n \to \infty \).

Next we consider the third term of (4.105). Since the first term of (4.105) is \( O(k_n^{-1}) \) and its second term is \( o(k_n^{-1}) \) as \( n \to \infty \), an application of Cauchy-Schwarz inequality shows that the third term of (4.105) is of order \( o(k_n^{-1}) \) as \( n \to \infty \).

It remains to prove the second and third term on the r.h.s. of (4.104) are of order \( o(k_n^{-1}) \) as \( n \to \infty \). By (4.70) and (4.103), the second term on the r.h.s. of (4.104) does not exceed

\[
E \left( \frac{(\hat{r}_n - \tau)k_n}{2|W_n||\hat{s}_{(k_n)} - s|} I(X(W_n) \geq k_n) \right)^2 \leq E \left( \frac{C\delta_n k_n^2}{2|W_n||\hat{s}_{(k_n)} - s|} I(X(W_n) \geq k_n) \right)^2 = C^2\delta_n^2 k_n^4 E \left( \frac{1}{|\hat{s}_{(k_n)} - s|} I(X(W_n) \geq k_n) \right)^2 = C^2\delta_n^2 k_n^4 O \left( \frac{|W_n|^2}{k_n^2} \right) = o \left( \frac{1}{k_n} \right),
\]
as \( n \to \infty \). Hence, the second term on the r.h.s. of (4.104) is of order \( o(k_n^{-1}) \) as \( n \to \infty \). Since the first term on the r.h.s. of (4.104) is of order \( O(k_n^{-1}) \) as \( n \to \infty \), an application of Cauchy-Schwarz inequality shows that the third term on the r.h.s. of (4.104) is of order \( o(k_n^{-1}) \) as \( n \to \infty \). This completes the proof of Theorem 4.7.

Proof of Theorem 4.8

We will prove this theorem by following the outline of the proof of Theorem 4.5. By Remark 4.3, we can write the l.h.s. of (4.61) as the quantity
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in (4.71). We will show the first term on the r.h.s. of (4.71) can be written as the r.h.s. of (4.61), while its second term is of order \( o(k_n^2|W_n|^{-2}) \) as \( n \to \infty \).

First, we show the second term on the r.h.s. of (4.71) is of order \( o(k_n^2|W_n|^{-2}) \) as \( n \to \infty \). By (4.60), there exists a positive constant \( C \) such that we have with probability 1

\[
|\hat{r}_n - r| \leq C\delta_n k_n^3|W_n|^{-4}.
\]  

(4.129)

Then, by (4.129) and (4.92), the second term on the r.h.s. of (4.71) does not exceed

\[
\frac{C\delta_n k_n^4}{2|W_n|^5} E \left[ \frac{1}{|\delta(k_n) - s|} I(X(W_n) \geq k_n) \right] = C\delta_n k_n^4 \mathcal{O} \left( \frac{|W_n|}{k_n^2} \right)
\]

as \( n \to \infty \).

It remains to show the first term on the r.h.s. of (4.71) can be written as the r.h.s. of (4.61). Recall that the expectation appearing in this term can be written as the one in (4.72). First we will show that the second and third term on the r.h.s. of (4.72) multiplied by \( k_n|W_n|^{-1} \) are of order \( \mathcal{O}(k_n^{-1}) \), as \( n \to \infty \). By (4.89), (4.90), and (4.118) (cf. also the argument following (4.89) and (4.90), but now we use (4.118)), we find that the second and third term on the r.h.s. of (4.72) multiplied by \( k_n|W_n|^{-1} \) are respectively of order \( o(k_n^{-2}) \) and \( o(k_n^{-1}) \), which are \( \mathcal{O}(k_n^{-1}) \), as \( n \to \infty \).

It remains to show the first term on the r.h.s. of (4.72) multiplied by \( (\tau k_n)/(2|W_n|) \), which is

\[
\frac{\tau k_n}{2|W_n|} \sum_{m_n \in A_{n}} \left\{ E \left[ \frac{1}{|\delta(k_n) - s|} \right] I(X(W_n) = m_n) \right\} P(X(W_n) = m_n)
\]

\[
= \frac{\tau k_n}{2|W_n|} \sum_{m_n \in A_{n}} \left\{ E \left[ \frac{1}{|\delta(k_n) - s|} I(\sum_{j=1}^{n} \mu_j < m_n \sum_{j=1}^{n} \mu_j) \right] I(X(W_n) = m_n) \right\}
\]

\[
P(X(W_n) = m_n),
\]  

(4.130)

can be written as the r.h.s. of (4.61). Here, as before, \( \epsilon_n \) is a sequence of positive real numbers converging to zero and \( \epsilon_n^{-1} = o(k_n^{1/4}) \) as \( n \to \infty \).

First we show that the second term on the r.h.s. of (4.130) is \( \mathcal{O}(k_n^{-1}) \) as \( n \to \infty \). By (4.79), (4.76), and an argument similar to the one used to prove (4.81), but now we use the fact that the r.h.s. of (4.76) is \( o(k_n^{-2}) \)
as \( n \to \infty \), we obtain

\[
\mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \mathbb{I} \left( |\tilde{Z}_{k_n,m_n}| > \epsilon_n \frac{k_n}{m_n} \right) \left| X(W_n) = m_n \right. \right) = o \left( \frac{|W_n|}{k_n^2} \right) \quad (4.131)
\]

as \( n \to \infty \). Substituting (4.131) into the second term on the r.h.s. of (4.130), we find that the second term on the r.h.s. of (4.130) is \( o(k_n^{-1}) \) as \( n \to \infty \).

It remains to show that the first term on the r.h.s. of (4.130) can be written as the r.h.s. of (4.61). By a similar argument as the one used to prove (4.127), but we now use condition (4.60) instead of (4.58), conditionally given \( X(W_n) = m_n \), we obtain the following stochastic expansion

\[
\frac{1}{|\hat{s}(k_n) - s|} \mathbb{I} \left( |\tilde{Z}_{k_n,m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right) = \left\{ \frac{1}{|\hat{s}(k_n) - s|} + o \left( \frac{k_n}{|W_n|} \right) \right\} \mathbb{I} \left( |\tilde{Z}_{k_n,m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right), \quad (4.132)
\]

as \( n \to \infty \). By (4.132), we have that the expectation appearing in the first term on the r.h.s. of (4.130) is equal to

\[
\mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \mathbb{I} \left( |\tilde{Z}_{k_n,m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right) \left| X(W_n) = m_n \right. \right) + o \left( \frac{k_n}{|W_n|} \right) \quad (4.133)
\]

as \( n \to \infty \), uniformly for all \( m_n \in A_n \). Substituting (4.133) into the first term on the r.h.s. of (4.130), this term reduces to

\[
\frac{\tau k_n}{2|W_n|} \sum_{m_n \in A_n} \left( \mathbb{E} \left( \frac{1}{|\hat{s}(k_n) - s|} \mathbb{I} \left( |\tilde{Z}_{k_n,m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right) \left| X(W_n) = m_n \right. \right) \right)
\]

\[
\cdot \mathbb{P}(X(W_n) = m_n) + o \left( \frac{k_n^2}{|W_n|^2} \right), \quad (4.134)
\]

as \( n \to \infty \).

It remains to show that the first term of (4.134) can be written as the r.h.s. of (4.61). Recall that \( |\hat{s}(k_n) - s| \) has the same distribution as \( H_{n}^{-1}(Z_{k_n,m_n}) \) and consider a modified stochastic expansion of \( H_{n}^{-1}(Z_{k_n,m_n}) \) as given in (4.33), but with \( \mathbb{I}(|\tilde{Z}_{k_n,m_n}| \leq \epsilon_n) \) replaced by \( \mathbb{I}(|\tilde{Z}_{k_n,m_n}| \leq \epsilon_n k_n m_n^{-1}) \). By (4.32), the sum of the second and third term on the r.h.s. of the modified (4.33) is equal to \( \tilde{Z}_{k_n,m_n} O(1) \), as \( n \to \infty \). In order to have an appropriate expansion of the first term on the r.h.s. of the modified (4.33), we need to compute the second and third (right hand) derivative \( H^{-1''}(0) \) and \( H^{-1'''}(0) \), besides \( H^{-1}(0) \) and \( H^{-1'}(0) \), which from the proof of (4.34) we already know that \( H^{-1}(0) = 0 \) and
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\( H^{-1'}(0) = \theta \tau (2\lambda(s))^{-1} \). \( H^{-1''}(0) \) can be computed as follows

\[
H^{-1''}(0) = \frac{d}{dt} \left( H^{-1'}(t) \right)_{t=0} = \frac{d}{dt} \left( \frac{1}{h(H^{-1}(t))} \right)_{t=0} = - \frac{h'(0)}{h^3(0)} = 0,
\]

since \( h'(0) = 0 \) while \( h(0) = 2\lambda(s)(\theta \tau)^{-1} \neq 0 \). A simple calculation shows that \( h''(0) = 2\lambda''(s)(\theta \tau)^{-1} \). Then we can compute \( H^{-1'''}(0) \) as follows

\[
H^{-1'''}(0) = \frac{d}{dt} \left( - \frac{h'(H^{-1}(t)) \left( \frac{H^{-1''}(t)}{h^2(H^{-1}(t))} \right)}{t=0} = - \frac{h''(0)}{h^4(0)}
\]

\[
= - \left( \frac{2\lambda''(s)}{\theta \tau} \right) \left( \frac{\theta \tau}{2\lambda(s)} \right)^4 = - \frac{\theta^3 \tau^3 \lambda''(s)}{8\lambda^4(s)}.
\]

Note here that \( h'(0) \) and \( h''(0) \) denote respectively the first and second (right hand) derivative of \( h \) at 0. Then we can write an expansion of the first term on the r.h.s. of the modified (4.33) as follows

\[
H^{-1} \left( \frac{k_n}{m_n + 1} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right) = H^{-1} \left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right) = 0 + \left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right) H^{-1'}(0) + 0
\]

\[
+ \frac{1}{6} \left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right)^3 H^{-1'''}(0)(1 + \mathcal{O}(1))
\]

\[
= \left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right) \frac{\theta \tau}{2\lambda(s)}
\]

\[
- \frac{\theta^3 \tau^3 \lambda''(s)}{48\lambda^4(s)} \left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right)^3 (1 + \mathcal{O}(1)),
\]  

(4.135)

as \( n \to \infty \). The first term on the r.h.s. of (4.135) can be written as

\[
\left( \frac{k_n}{\theta |W_n|} + \mathcal{O} \left( \frac{a_n k_n}{|W_n|^{3/2}} \right) + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right) \frac{\theta \tau}{2\lambda(s)}
\]

\[
= \frac{\tau k_n}{2\lambda(s)|W_n|} + \mathcal{O} \left( \frac{a_n k_n}{|W_n|^{3/2}} \right) + \mathcal{O} \left( \frac{1}{|W_n|} \right),
\]

(4.136)

as \( n \to \infty \). To simplify the second term on the r.h.s. of (4.135) we argue as follows. Since \( a_n = \mathcal{o}(|W_n|^{1/2}) \) as \( n \to \infty \), we have that

\[
\left( \frac{k_n}{m_n} + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right)^3 = \left( \frac{k_n}{\theta |W_n|} + \mathcal{O} \left( \frac{a_n k_n}{|W_n|^{3/2}} \right) + \mathcal{O} \left( \frac{1}{|W_n|} \right) \right)^3
\]

\[
= \left( \frac{k_n}{\theta |W_n|} + \mathcal{O} \left( \frac{k_n}{|W_n|} \right) \right)^3 = \frac{k_n^3}{\theta^3 |W_n|^3} + \mathcal{o} \left( \frac{k_n^3}{|W_n|^3} \right),
\]
as \( n \to \infty \). Hence, the second term on the r.h.s. of (4.135) can be written as

\[
- \frac{\theta^3 \tau^3 \lambda''(s)}{48 \lambda^4(s)} \left( \frac{k_n^3}{|W_n|^3} + o \left( \frac{k_n^3}{|W_n|^3} \right) \right) (1 + o(1))
\]

\[
= - \frac{\tau^3 \lambda''(s) k_n^3}{48 \lambda^4(s) |W_n|^3} + o \left( \frac{k_n^3}{|W_n|^3} \right),
\]

(4.137)
as \( n \to \infty \). By (4.135) with its first and second term replaced respectively by the r.h.s. of (4.136) and (4.137), in combination with the fact that the sum of the second and third term on the r.h.s. of the modified (4.33) is equal to \( O(1) \tilde{Z}_{k_n:m_n} \) as \( n \to \infty \), we can write the modified (4.33) as the following stochastic expansion

\[
H_n^{-1}(Z_{k_n:m_n}) I \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right)
\]

\[
= \left\{ \frac{\tau k_n}{2\lambda(s)|W_n|} - \frac{\tau^3 \lambda''(s) k_n^3}{48 \lambda^4(s) |W_n|^3} + o \left( \frac{k_n^3}{|W_n|^3} \right) \right\} I \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right)
\]

\[
+ O(1) \tilde{Z}_{k_n:m_n} \right) I \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right),
\]

(4.138)
as \( n \to \infty \). By (4.138), we can compute the following conditional expectation

\[
E \left( \frac{1}{\tilde{s}(k_n) - s} I \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right) \middle| X(W_n) = m_n \right)
\]

\[
= E \left( \frac{1}{H_n^{-1}(Z_{k_n:m_n})} I \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right) \right)
\]

\[
= E \left\{ \frac{2\lambda(s)|W_n|}{\tau k_n} \left\{ 1 + \frac{\tau^2 \lambda''(s) k_n^2}{24 \lambda^3(s)|W_n|^2} + o \left( \frac{k_n^2}{|W_n|^2} \right) + O \left( \frac{k_n^4}{|W_n|^4} \right) \right\}
\]

\[
+ O \left( \frac{a_n}{|W_n|^2} + \frac{1}{k_n} \right) + O \left( \frac{|W_n|}{k_n} \right) \tilde{Z}_{k_n:m_n} + O \left( \frac{|W_n|^2}{k_n^2} \tilde{Z}_{k_n:m_n}^2 \right)
\]

\[
\right) \cdot 1 \left( |\tilde{Z}_{k_n:m_n}| \leq \epsilon_n \frac{k_n}{m_n} \right)
\]

\[
= \left\{ \frac{2\lambda(s)|W_n|}{\tau k_n} + \frac{\tau^2 \lambda''(s) k_n}{12 \lambda^2(s)|W_n|^2} + o \left( \frac{k_n}{|W_n|^2} \right) + O \left( \frac{a_n |W_n|^{1/2}}{k_n} + \frac{|W_n|}{k_n^2} \right) \right\}
\]

\[
\cdot \left( 1 - o \left( \frac{1}{k_n^2} \right) \right)
\]

\[
= \frac{2\lambda(s)|W_n|}{\tau k_n} + \frac{\tau^2 \lambda''(s) k_n}{12 \lambda^2(s)|W_n|^2} + o \left( \frac{k_n}{|W_n|^2} \right) + O \left( \frac{a_n |W_n|^{1/2}}{k_n} + \frac{|W_n|}{k_n^2} \right),
\]

(4.139)
as \( n \to \infty \), uniformly for all \( m_n \in A_n \). Note that, to get the r.h.s. of (4.139) we have used the facts \( \mathbb{P}(|\tilde{Z}_{k_n,m_n}| \leq \epsilon_n k_n^{-1}) = 1 - o(k_n^{-2}) \) (note that the r.h.s. of (4.76) is of order \( o(k_n^{-2}) \) as \( n \to \infty \)), \( \mathbb{E}\tilde{Z}_{k_n,m_n} = 0 \), \( \mathbb{E}\tilde{Z}_{k_n,m_n}^2 = \mathcal{O}(k_n |W_n|^{-2}) \) so that \( \mathcal{O}(|W_n|^{3} k_n^{-3}) \mathbb{E}\tilde{Z}_{k_n,m_n}^2 = \mathcal{O}(|W_n| k_n^{-2}) \), and the term of order \( \mathcal{O}(k_n^3 |W_n|^{-3}) \) can be written as \( o(k_n |W_n|^{-1}) \) as \( n \to \infty \), because of (4.4). Choose now \( a_n = |W_n|^\epsilon_0 \) for arbitrary small \( \epsilon_0 > 0 \). Substituting (4.139) into the first term of (4.134), this term reduces to

\[
\frac{\tau k_n}{2|W_n|} \left\{ \frac{2\lambda(s)|W_n|}{\tau k_n} + \frac{\tau \lambda''(s) k_n}{12 \lambda^2(s)|W_n|} + o\left(\frac{k_n}{|W_n|}\right) \right\} P(X(W_n) \in A_n)
\]

\[
= \lambda(s) + \frac{\tau^2 \lambda''(s) k_n^2}{24 \lambda^2(s)|W_n|^2} + o\left(\frac{k_n^2}{|W_n|^2}\right) + \mathcal{O}\left(\frac{1}{|W_n|^{1/2 - \epsilon_0}} + \frac{1}{k_n}\right), \quad (4.140)
\]

as \( n \to \infty \), since by (4.119) we have \( P(X(W_n) \in A_n) = 1 - o(k_n^{-2}) \), as \( n \to \infty \). This completes the proof of Theorem 4.8. \( \Box \)

4.4 Comparison of nearest neighbor and kernel type estimators

Consider a special case of the kernel type estimator \( \hat{\lambda}_{n,K} \) studied in chapter 3 namely the one with a uniform kernel, i.e., \( K(u) = 1/2 \) for all \( u \in [-1,1] \), and zero otherwise (cf. (3.5) ). In this case, the asymptotic approximations to the variance, bias, and MSE of \( \hat{\lambda}_{n,K} \) as well as the optimal choice of \( h_n \) (cf. (3.57), (3.59), (3.61) and (3.62) ) can be simplified as below.

Suppose that \( \lambda \) is periodic and locally integrable, and \( K \) is the uniform kernel on \([-1,1]\).

(i) If \( h_n \downarrow 0 \), \( |W_n|h_n \to \infty \), and \( |W_n||\hat{\tau}_n - \tau| = \mathcal{O}(\delta_n |W_n|^{-1}) \) with probability 1 as \( n \to \infty \), for some fixed sequence \( \delta_n \downarrow 0 \) as \( n \to \infty \), then we have

\[
\text{Var} \left( \hat{\lambda}_{n,K}(s) \right) = \frac{\tau \lambda(s)}{2|W_n|h_n} + o\left(\frac{1}{|W_n|h_n}\right), \quad (4.141)
\]

as \( n \to \infty \), provided \( s \) is a Lebesgue point of \( \lambda \).

(ii) If \( h_n \downarrow 0 \) and \( |W_n||\hat{\tau}_n - \tau| = \mathcal{O}(\delta_n h_n^3) \) with probability 1 as \( n \to \infty \), for some fixed sequence \( \delta_n \downarrow 0 \) as \( n \to \infty \), and \( \lambda \) has finite second
 derivative \lambda'' at \( s \), then
\[
E\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{\lambda''(s)}{6} h_n^2 + o(h_n^2) + O(|W_n|^{-1}),
\]
for \( n \to \infty \).

(iii) If conditions in (i) and (ii) are satisfied, then we have
\[
MSE\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau \lambda(s)}{2|W_n| h_n} + \frac{1}{36} \left(\lambda''(s)\right)^2 h_n^4
+ O\left(|W_n|^{-1} h_n^{-1}\right) + o\left(h_n^4\right),
\]
for \( n \to \infty \).

(iv) By minimizing the leading term of the \( MSE(\hat{\lambda}_{n,K}(s)) \) (cf. (4.143)), we obtain the optimal choice of \( h_n \), which is given by
\[
h_n = \left[ \frac{9\tau \lambda(s)}{2 \left(\lambda''(s)\right)^2} \right]^{\frac{1}{2}} |W_n|^{-\frac{1}{5}}.
\]

Next note that our nearest neighbor estimator \( \hat{\lambda}_n(s) \), with optimal choice of \( k_n \) given in (4.64), yields the following approximation to the variance and bias,
\[
Var\left(\hat{\lambda}_n(s)\right) = \frac{\tau^{4/5} \left(\lambda''(s)\right)^{2/5} \left(\lambda(s)\right)^{4/5}}{(144)^{1/5}} |W_n|^{-4/5} + o\left(|W_n|^{-4/5}\right),
\]
and
\[
E\hat{\lambda}_n(s) = \lambda(s) + \frac{\tau^{2/5} \left(\lambda''(s)\right)^{1/5} \left(\lambda(s)\right)^{2/5}}{24(144)^{-2/5}} |W_n|^{-2/5} + o\left(|W_n|^{-2/5}\right),
\]
as \( n \to \infty \), provided \( \lambda \) has finite second derivative \( \lambda'' \) at \( s \), \( \lambda(s) > 0 \), and (4.65) holds true. Similarly, the uniform kernel estimator \( \hat{\lambda}_{n,K}(s) \), with optimal choice of \( h_n \) given in (4.144), yields the following approximation to the variance and bias,
\[
Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau^{4/5} \left(\lambda''(s)\right)^{2/5} \left(\lambda(s)\right)^{4/5}}{2(9/2)^{1/5}} |W_n|^{-4/5} + o\left(|W_n|^{-4/5}\right),
\]
and
\[
E\hat{\lambda}_{n,K}(s) = \lambda(s) + \frac{\tau^{2/5} \left(\lambda''(s)\right)^{1/5} \left(\lambda(s)\right)^{2/5}}{6(9/2)^{-2/5}} |W_n|^{-2/5} + o\left(|W_n|^{-2/5}\right),
\]
as \( n \to \infty \), provided \( \lambda \) has finite second derivative \( \lambda'' \) at \( s \) and (4.65) holds true. Note that \( (144)^{1/5} = 2(9/2)^{1/5} \) and \( 24(144)^{-2/5} = 6(9/2)^{-2/5} \),
4.4 Comparison of nearest neighbor and kernel

i.e. \( \hat{\lambda}_n \) and \( \hat{\lambda}_{n,K} \) have the same asymptotic approximations to the variance and bias, which also implies that the two estimators have the same asymptotic approximation to the MSE. This is in agreement with the comparison made by Mack and Rosenblatt (1979) for the density estimation case. Note also that the estimator \( \hat{\lambda}_n \) requires condition \( \lambda(s) > 0 \) which is not needed for \( \hat{\lambda}_{n,K} \).

Note also that, if we compare (4.59) and (4.61) with (4.141) and (4.142), we see that the role of \( \tau \) and \(|W_n|\) is different in the asymptotic approximations to the variance and bias of \( \hat{\lambda}_n \) compared to those of \( \hat{\lambda}_{n,K} \). For the nearest neighbor estimate, the bias is proportional to \( \tau^2|W_n|^{-2} \) while the variance does not depend on either \( \tau \) or \(|W_n|\). In the case of the kernel estimate we have the opposite situation, i.e. the variance is proportional to \( \tau|W_n|^{-1} \), while the bias does not depend on either \( \tau \) or \(|W_n|\).