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Beliefs supported by binary arguments

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ABSTRACT

In this paper, we explore the relation between an agent's doxastic attitude and her arguments in support of a given claim. Our main contribution is the design of a logical setting that allows us reason about binary arguments which are either in favour or against a certain claim. This is a setting in which arguments and propositions are the basic building blocks so that the concept of argument-based belief emerges in a straightforward way. We work against the background of Dung's abstract argumentation framework, extending it to a new setting in which we can study the formal properties of binary arguments as well as the larger structures they establish. This paper introduces a formal 'two-dimensional' language to talk about propositions and arguments, for which a sound and complete axiom system is provided.

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1. Introduction

The literature on argumentation theory contains a number of different views on what a single argument is. The proposed definitions range from Toulmin's six interrelated components (claim, data, warrant, backing, rebuttal, qualifier; Toulmin, 2003) to Dung's abstract argumentation framework (Dung, 1995) where an argument is simply an abstract undefined entity, and what matters is the way it interacts with others (i.e. which ones it 'attacks' and which ones it 'defends'). These abstract argumentation frameworks have proved to be useful for analysing different notions of acceptability of arguments. Intuitively, the arguments that matter are those that will 'survive' the full argumentation process, and the abstract argumentation frameworks provide different concepts and tools to identify them.

This process of identifying the 'acceptable' sets of arguments, evidence or justifications has historically played an important role in the definition of epistemic notions such as knowledge and belief. There is an extensive literature in (formal) epistemology on what ingredients are needed to provide a solid foundation for rational belief and knowledge. In terms of arguments, we provide a quote from Dung (1995, p. 323):

[...] a statement is believable if it can be argued successfully against attacking arguments. In other words, whether or not a rational agent believes in a statement depends on whether or not [an] argument supporting this statement can be successfully defended against the counterarguments.

While Dung talks explicitly about the relationship between beliefs and arguments, it is surprising that no further details are provided in the referred paper.¹ Neither has this topic received much attention within the literature on argumentation theory with a few exceptions including [Schwarzentruher, Vesic, and Rienstra \(2012\)](#) and [Grossi and van der Hoek \(2014\)](#). One of the reasons for this might be that the abstract argumentation framework abstracts away too many elements. Among these elements, two play an essential role when attempting to define the agent's belief. The first is the content of each argument; the second is the issue on which an argument attacks another. Without these ingredients, we lack the details to decide on whether one argument's attack on another one is actually relevant to the claim at stake, or whether all the 'acceptable' arguments can back up the given claim.

This paper picks up this thread proposing a formal definition of the notion of beliefs in terms of the arguments the agent has in support of or against a given claim. In an argumentation on whether P , we stipulate that all the arguments are in support or against P . In this sense, we call them binary arguments. We start (Section 2) by recalling the basic definitions of abstract argumentation theory. Then (Section 3), we extend them with two elements. With the first we unveil a crucial characteristic of each (so far abstract) argument by indicating the propositions it supports; with the second we relativise the attack relation by making explicit the proposition on which a given argument attacks another. Then, we introduce a formal definition of an agent's beliefs. The paper continues (Section 4) introducing a formal language for exploring the properties of this new notion, together with a sound and complete axiom system to reason about binary arguments. The work ends (Section 5) with a brief summary and further lines for research. The proofs of most propositions and theorems are provided in the Appendix A.1.

Related work Our logical setting in this paper ties in with the modal logic analysis of abstract argumentation notions provided in [Grossi \(2010b\)](#), [Grossi \(2010\)](#) and [Grossi \(2012\)](#). Following that line, the work by [Grossi and van der Hoek \(2014\)](#) comes closest to this paper as the authors also present a setting that combines both abstract arguments and an agent's beliefs. Yet different from our approach is the fact that their notion of belief is taken as a basic component which is *not* defined in terms of the arguments involved. In a different but related direction, [Schwarzentruher et al. \(2012\)](#) aims at describing not only the argumentation structure but also the information the agents have about it. Intuitively, an agent's belief is based on what she knows/believes about the argumentation structure, so it is important to make the agent's informational state explicit. Also related is the work in [Dyrkolbotn \(2013\)](#), which uses graph theory tools (kernels and local kernels; cf. the *propositional discourse logic* of [Dyrkolbotn and Walicki \(2014\)](#)) and modal languages for dealing with the acceptability of formulas that might 'argue' against each other, and also introduce three-valued models of belief induced by argumentation frameworks.

Other logic-technical approaches that can contribute to the study of argument-based beliefs includes not only the dialogical logic tradition ([Lorenzen, 1958](#); [Lorenz, 1961](#); [Lorenzen & Lorenz, 1978](#), whose main motivations can be traced back to ancient Greece when logic was conceived as the systematic study of dialogues in which two parties exchange arguments over a central claim), but also the logics for reasoning about the (soft or hard) evidential basis of different attitudes ranging from knowledge to belief ([Baltag, Renne, & Smets, 2014](#); [Baltag, Renne, & Smets, 2012](#)). The later direction builds further on the justification logic tradition ([Artemov & Nogina, 2005](#); [Artemov, 2008](#)) in which reasons are

explicitly represented in justification terms and on the work in dynamic epistemic logic (DEL) on representing different epistemic and doxastic attitudes and their dynamics (van Benthem, 2007; van Ditmarsch, van Der Hoek, & Kooi, 2007; Baltag & Smets, 2008; van Benthem, 2011). The work in Baltag, Fiutek, and Smets (2016) enhances these investigations by focussing on different types of models that connect evidence and beliefs, relating it to both the investigations in neighbourhood structures in van Benthem, Fernández-Duque, and Pacuit (2012) and van Benthem, Fernández-Duque, and Pacuit (2014) where beliefs are defined in terms of maximally consistent sets of evidence, and the recent work by Baltag, Bezhanishvili, Özgün, and Smets (2016a) on a topological semantics for evidence, evidence-based justifications, belief and knowledge.

2. Background on abstract argumentation theory

The argumentation framework of Dung (1995) is known to abstract away from the internal structure of an argument, focusing instead of the way the arguments interact with one another. Here is the formal definition of its basic component, the so-called argumentation graph:

Definition 2.1 (Argumentation Graph Dung (1995)): An *argumentation graph* is a pair $\mathcal{G} = (AR, \leftarrow)$ where AR is a non-empty set of abstract arguments and $\leftarrow \subseteq (AR \times AR)$ is a binary relation on AR .

The relation \leftarrow is interpreted as an *attack* relation, with $s \leftarrow s'$ read as ‘argument s' attacks argument s ’. This notion of attack can be extended to sets of arguments: a set $X \subseteq AR$ attacks an argument s , denoted by $s \leftarrow X$, when there is an argument $s' \in X$ that attacks s , $s \leftarrow s'$. We say that an argument $s \in AR$ is *defended* by $X \subseteq AR$ when every argument attacking s is in turn attacked by an argument in X (i.e. when $s \leftarrow s'$ implies $\exists s'' \in X$ such that $s' \leftarrow s''$).

This level of abstraction gives us insights about the position of each argument with respect to other arguments in the same structure. For example, in the graph whose diagram appears below, the arguments in $X = \{s_1, s_3\}$ do not attack each other (in Dung’s terminology, the set is called *conflict-free*). Also each member of X which is under attack is defended by X itself (the set is called *self-acceptable*), as s_2 attacks $s_1 \in X$, but $s_3 \in X$ attacks s_2 .



The following characteristic functions are useful for defining argumentative-theoretic notions that allow us to formalise the reasons for ‘accepting’ a given set of arguments.

Definition 2.2 (Defence and neutrality functions Grossi (2012)): Let $\mathcal{G} = (AR, \leftarrow)$ be an argumentation graph.

- Given a set $X \in 2^{AR}$, the *defence* function $d : 2^{AR} \rightarrow 2^{AR}$ returns the set of arguments in AR defended by X :

$$d(X) := \{s \in AR \mid \forall s' \in AR : s \leftarrow s' \text{ implies } s' \leftarrow X\}.$$

X is conflict-free	iff	$X \subseteq n(X)$
X is self-acceptable	iff	$X \subseteq d(X)$
X is admissible	iff	$X \subseteq n(X)$ and $X \subseteq d(X)$
X is a stable extension	iff	$X = n(X)$
X is a complete extension	iff	$X \subseteq n(X)$ and $X = d(X)$
X is a preferred extension	iff	X is a maximal complete extension
X is the grounded extension	iff	X is the least fixed point of d

Figure 1. Some key notions in abstract argumentation theory (Dung, 1995) (Table from Grossi, 2012).

- Given a set $X \in 2^{AR}$, the *neutrality* function $n : 2^{AR} \rightarrow 2^{AR}$ returns the set of arguments in AR which are not attacked by X :

$$n(X) := \{s \in AR \mid \nexists s' \in X : s \leftarrow s'\}.$$

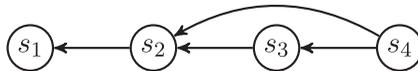
Figure 1 shows some of the key notions discussed in Dung (1995). Note that a stable extension is also a preferred extension (Lemma 15 in Dung (1995)) and, by definition, a preferred extension is also complete, a complete extension is also admissible, and an admissible set is conflict-free and acceptable, but not vice versa.

In this paper, a special class of argumentation graphs called *uncontroversial* (Dung, 1995, Definition 32) will play an important role.

Definition 2.3 (Uncontroversial argumentation graphs): Given an argument $s \in AR$ in an argumentation graph \mathcal{G} , let $\text{Def}_{\mathcal{G}}(s)$ (resp., $\text{Att}_{\mathcal{G}}(s)$) be the set of arguments in \mathcal{G} such that $s' \in \text{Def}_{\mathcal{G}}(s)$ ($s \in \text{Att}_{\mathcal{G}}(s')$) if and only if there is a sequence of arguments s_0, s_1, \dots, s_n where $s = s_0, s' = s_n$ and $s_i \leftarrow s_{i+1}$ for any $0 \leq i < n$ and n is even (odd).

An argument s' in \mathcal{G} is said to be *controversial* with respect to another argument s if s' both attacks and defends s (i.e. $s' \in \text{Def}_{\mathcal{G}}(s) \cap \text{Att}_{\mathcal{G}}(s)$). An argumentation graph is *uncontroversial* if none of its arguments is controversial with respect to any argument.

As an example, consider the argumentation graph below: argument s_4 is controversial with respect to both s_1 and s_2 , as it attacks them (it attacks s_1 by attacking one of s_1 's defender, namely s_3 , and it attacks s_2 directly) but also defends them (by attacking one of their attackers, s_2 and s_3 respectively).



It has been proved (Dung, 1995, Theorem 33) that every uncontroversial argumentation graph is not only *coherent* (each preferred extension is also stable), but also *relatively grounded* (the intersection of all preferred extensions is the grounded extension). Here are two further results about uncontroversial argumentation graphs that will be useful for this work.

Proposition 2.1: Given an uncontroversial argumentation graph \mathcal{G} and an argument s in \mathcal{G} , there is an admissible set of arguments X with $s \in X$ if and only if $s \in \text{GFP}.d$, with $\text{GFP}.d$ the greatest fixed point of the function d .

Corollary 2.1: *Given any uncontroversial argumentation graph \mathcal{G} and any argument s in \mathcal{G} , the union of all the preferred extensions is the greatest fixed point of the defence function d .*

3. Making explicit the supported/attacked propositions

The abstract argumentation graphs provides us with tools for defining different notions of argument acceptability. However, as each argument is itself an abstract undefined entity, it is not possible to say whether an argument supports a given claim; similarly, it is also not possible to decide whether an attack between arguments is relevant to the claim at stake. Thus, there is not enough information to decide whether the argumentation structure as a whole ‘supports’ a given proposition or its negation.

3.1. Argumentation-support frame

Our new semantic structure extends an argumentation graph by providing the missing information about the main propositional content of the arguments. Moreover, it also adds a set of possible worlds to the structure, in order to define not only the propositions involved, but also the relationship between them. The result, called an *argumentation-support frame*, is defined below.

Definition 3.1 (Argumentation-Support Frame (ASF)): *An argumentation-support frame is a structure $\mathcal{F} = (W, AR, \{\leftarrow^P\}_{P \in 2^W}, f)$ where*

- W is a non-empty set of possible worlds and AR a non-empty set of arguments;
- $\leftarrow^P \subseteq AR \times AR$ is an attack relation labelled by propositions $P \in 2^W$;
- $f : AR \rightarrow 2^W$ is a function assigning to each argument $s \in AR$ a subset of W such that, for any $s \in AR$, $f(s) \neq \emptyset$.

Intuitively, the set AR in \mathcal{F} contains all the arguments available to our implicit agent, with each argument s supporting those propositions that are true in all worlds in $f(s)$; thus, the set of propositions supported by $s \in AR$ is given by $C_s := \{P \in 2^W \mid f(s) \subseteq P\}$. The restriction $f(s) \neq \emptyset$ simply states that no argument supports a direct contradiction. With respect to attacks between arguments, there are now several attack relations, each one of them labelled by propositions $P \in 2^W$.² For the interpretation, $s \leftarrow^P s'$ indicates that s is attacked by s' on *whether* P is the case.

In order to model the way a rational agent may reason about binary arguments, we impose the following conditions on the given frame structure:

- (1) $s \leftarrow^P s'$ if and only if $s \leftarrow^{\bar{P}} s'$;³
- (2) if $s \leftarrow^P s'$, then
 - (a) either $f(s) \subseteq P$ or $f(s) \subseteq \bar{P}$; and
 - (b) $f(s) \subseteq P$ implies that $f(s') \subseteq \bar{P}$;
- (3) if $s \leftarrow^P s'$ and $f(s) \subseteq Q \subseteq P$, then $s \leftarrow^Q s'$;

The first condition captures the intuition that a ‘discussion’ on whether P is the case is also a ‘discussion’ on whether \bar{P} is the case. The second is the crucial one for *binary* arguments: while (2)(a) asks for any argument attacked on the issue P to take a stance either in support of P or in support of \bar{P} , (2)(b) says (together with (1) and (2)(a)) that, when one argument

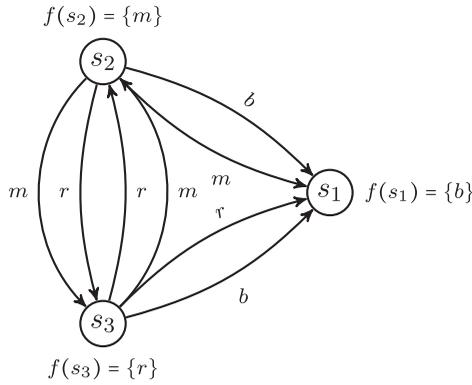


Figure 2. ASF for Example 3.1.

attacks another, they should hold opposite stands on the issue at hand. The last condition states that if s' attacks s on its claim of P , then s' should also attack s on any stronger claim Q . All argumentation-support frames discussed in the rest of this paper will be assumed to satisfy these (natural, we think) conditions. Still, as a consequence of the second (see Proposition 3.1 below), the underlying argumentation graph for every proposition $P \in 2^W$ will be *uncontroversial* (Definition 2.3).

Note that the interpretation of the labels on the attack relations indicates that only the binary argumentative reasoning of an agent is considered in our framework. Of course, we do not claim that the agent’s argumentative reasoning can only be on binary issues; still, for simplicity, those more general cases are left outside the range of our framework.

Here is an example of an argumentation-support frame.

Example 3.1: Concerning an issue of ‘what kind of animal is the one in the picture’, suppose the agent has available the following three arguments (s_1, s_2, s_3), each one of them supporting a possible answer (*bird, mammal, reptile*).

- s_1 : The animal in the picture has wings, so it is a bird.
- s_2 : The animal looks like a bat, so it is not a bird, it is a mammal.
- s_3 : The animal looks like a pterosaur, so it is neither a bird nor a mammal, it is a reptile.

We assume that s_1 is attacked by both s_2 and s_3 , while s_2 and s_3 attack each other.⁴ This attack relationship between these arguments can be represented by the ASF $(\{b, m, r\}, \{s_1, s_2, s_3\}, \{\leftarrow^P\}^{P \subseteq W}, f)$ with the attack relation given by

$$\begin{aligned} \leftarrow^{\{b\}} = \leftarrow^{\{m,r\}} &:= \{(s_1, s_2), (s_1, s_3)\}, & \leftarrow^{\{m\}} = \leftarrow^{\{b,r\}} &:= \{(s_1, s_2), (s_3, s_2), (s_2, s_3)\} \\ \leftarrow^{\emptyset} = \leftarrow^{\{b,m,r\}} &:= \emptyset & \leftarrow^{\{r\}} = \leftarrow^{\{b,m\}} &:= \{(s_1, s_3), (s_3, s_2), (s_2, s_3)\} \end{aligned}$$

and the support function as

$$f(s_1) = \{b\}, \quad f(s_2) = \{m\}, \quad f(s_3) = \{r\}.$$

Figure 2 shows the ASF, with some labels of the attack relation omitted. The reader can verify that it indeed satisfies the three conditions listed above.

3.2. Beliefs supported by arguments

Within an argumentation-support frame, it is possible to define a notion of belief in terms of the global behaviour of the involved arguments (which may be in conflict with one another). First, some useful definitions and observations.

3.2.1. Acceptable argument for P

A given ASF \mathcal{F} contains a set of argumentation graphs, one for each proposition $P \in 2^W$:

$$C_{\mathcal{F}} := \{\mathcal{G}^P = (AR, \leftarrow^P) \mid P \subseteq W\}.$$

As advanced, the ‘binary argument’ requirements discussed above (Page 5) have an important consequence: the set $C_{\mathcal{F}}$ contains only uncontroversial argumentation graphs.

Proposition 3.1: *Given an ASF \mathcal{F} , every $\mathcal{G}^P \in C_{\mathcal{F}}$ is an uncontroversial argumentation graph.*

Proof: It follows from the fact that, due to the properties \mathcal{F} should satisfy, each attack relation \leftarrow^P in \mathcal{F} (and thus the attack relation of each \mathcal{G}^P) relates arguments that hold opposite stands on P ’s truth value. Hence, given any argument s in \mathcal{G}^P , while the set $\text{Def}_{\mathcal{G}}(s)$ contains only arguments that agree with s on P ’s truth value (they are at an even distance), the set $\text{Att}_{\mathcal{G}}(s)$ contains only arguments that have s ’s opposite views about P (they are at an odd distance). As no argument can support both P and \bar{P} (recall that this is excluded by our frame condition $f(s) \neq \emptyset$), no argument can be in the intersection of $\text{Def}_{\mathcal{G}}(s)$ and $\text{Att}_{\mathcal{G}}(s)$, and thus \mathcal{G}^P is uncontroversial. \square

Thus, the conditions of the attack relation imply the uncontroversiality of our argumentation framework. The readers can judge whether the conditions and its consequences are reasonable. For those who take controversiality to be an important feature of arguments, notice how our setting does not rule it out completely; the phenomena can occur, as long as it with respect to different issues.

Note how the only difference of the support-argumentation based graphs from the ones in Definition 2.1 is that each $\mathcal{G}^P \in C_{\mathcal{F}}$ specifies what issue the argumentation is about. For example, the argumentation graph \mathcal{G}^P represents a discussion on the issue of whether P is the case. This, together with the function f in \mathcal{F} , allows us to define notions that are similar to those in abstract argumentation theory (see Figure 1), now relative to specific propositions. Thus, we can provide the following definition.

Definition 3.2 (Acceptable argument for P): Given an ASF \mathcal{F} and an argument s in \mathcal{F} , s is an acceptable argument for P if and only if $f(s) \subseteq P$ (i.e. it supports P) and there is an admissible set of arguments X in \mathcal{G}^P such that $s \in X$ (i.e. s belongs to a set of arguments that do not attack each other and defend themselves).

This is one of the key definitions of this paper: an argument is acceptable for a proposition P whenever the argument belongs to an admissible set in the argumentation graph for P . Then, by Proposition 2.1 and the uncontroversiality of support-argumentation based graphs, an argument is acceptable for P whenever the argument belongs to the greatest fixed point of the defence function d , $\text{GFP}.d$.⁵ This gives us an advantage that Section 4 will make use of: within a formal language, it is possible to express this notion of acceptable

argument in terms of truth-conditions involving a *greatest fixed point*, a concept for which there are formal tools available in the literature.

The proposition below reveals logical properties of the notion of acceptable argument.

Proposition 3.2: *Given any ASF \mathcal{F} and any argument s in \mathcal{F} ,*

- (a) s is an acceptable argument for W ;
- (b) if s is an acceptable argument for $P \subseteq W$, then for any $Q \supseteq P$, s is an acceptable argument for Q ;
- (c) it does not hold that 'if the given s is an acceptable argument for $P \subseteq W$ and $Q \subseteq W$, then s is an acceptable argument for $P \cap Q$ '.

Proof:

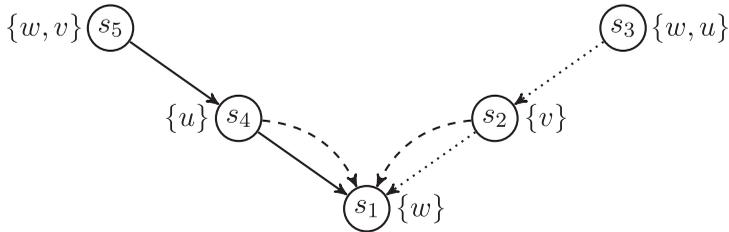
(a) No argument can attack another on W , as otherwise one of involved arguments would support \emptyset (conditions (1) and (2)), which is impossible ($f(s) \neq \emptyset$ for every s). Hence, by definition of acceptability (i.e. admissibility), s is an acceptable argument for W .

(b) Since s is an acceptable argument for P , there is an admissible set X in \mathcal{G}^P such that $s \in X$. Now take any $Q \subseteq W$ with $P \subseteq Q$; to show that implies there is an admissible set Y in \mathcal{G}^Q such that $s \in Y$, define $Y := X \cap \text{Def}^Q(s)$, with $\text{Def}^Q(s)$ the set of defenders of s in \mathcal{G}^Q . We will show that Y is indeed an admissible set in \mathcal{G}^Q .

For any $s' \in Y$, if there is $x \in AR$ such that $s' \leftarrow^Q x$, by condition (3) and $f(s) \subseteq P \subseteq Q$, it follows that $s' \leftarrow^P x$. Since $s' \in Y \subseteq X$ and X is an admissible set in \mathcal{G}^P , there is $s'' \in X$ such that $x \leftarrow^P s''$. By conditions (1) and (2) and $f(s) \subseteq Q$, for any $y \in \text{Def}^Q(s)$, $f(y) \subseteq Q$. Since $s' \in Y \subseteq \text{Def}^Q(s)$, it follows that $f(s') \subseteq Q$. By condition (2)(b) and $s' \leftarrow^Q x$, $f(x) \subseteq \bar{Q} \subseteq \bar{P}$. Together with $x \leftarrow^P s''$ (by condition 1) and $f(x) \subseteq \bar{Q} \subseteq \bar{P}$, it follows by condition 3 that $x \leftarrow^Q s''$ and by condition 1 that $x \leftarrow^Q s''$. Hence $s'' \in \text{Def}^Q(s')$. For $s' \in \text{Def}^Q(s)$, $s'' \in \text{Def}^Q(s)$

Therefore, we have proved $s'' \in Y$, which shows that for any $s' \in Y$, if there is $x \in AR$ such that $s' \leftarrow^Q x$, we can find $s'' \in Y$ such that $x \leftarrow^Q s''$, i.e. $Y \subseteq d^Q(Y)$, with d^Q the defence function for \mathcal{G}^Q . Moreover, because \mathcal{G}^Q is uncontroversial, $\text{Def}^Q(s) \subseteq n^Q(\text{Def}^Q(s))$, which implies that $Y \subseteq n^Q(Y)$, with n^Q the neutrality function for \mathcal{G}^Q . Since $s \in Y$, we are done.

(c) The diagram below (with the set of worlds supported by each argument listed next to each node) shows a counterexample, with s_1 an acceptable argument for $P = \{w, u\}$ (dotted arrow for \leftarrow^P) and for $Q = \{w, v\}$ (solid arrow for \leftarrow^Q), but not an acceptable argument for $P \cap Q$ (dashed arrow for $\leftarrow^{P \cap Q}$).



□

Note that ‘if the given s is an acceptable argument for $P \subseteq W$ and $Q \subseteq W$, then s is an acceptable argument for $P \cap Q$ ’ is not the same as the statement ‘if there is an acceptable argument for P and an acceptable argument for Q , there is an acceptable argument for $P \cap Q$ ’. The counterexample we give in the proof serves as a counterexample against both statements, since s_1 is the only argument supporting $\{w\}$.

3.2.2. Belief

With the notion of acceptable argument for a proposition P already defined, we propose the following argument-based definition for the notion of belief.

Definition 3.3 (Belief): Given an ASF \mathcal{F} , the agent *believes* P in \mathcal{F} if and only if there is an acceptable argument for P and there is no acceptable argument for \bar{P} .

Because of its definition, the notion of belief inherits some properties (Proposition 3.2) from the notion of acceptable argument.

Corollary 3.1: *Given any ASF \mathcal{F} and any $P, Q \subseteq W$,*

- (a) *the agent believes W ;*
- (b) *if the agent believes P , then she also believes Q , for any $Q \supseteq P$;*
- (c) *even if the agent believes P and believes Q , she may not believe $P \cap Q$.*

Note how, in particular, there might be different reasons for the fact that beliefs in this setting are not closed under intersection (i.e. conjunction). A simple one is the fact that there may not be arguments supporting $P \cap Q$, even if there are arguments supporting P and arguments supporting Q . Moreover, even if there are arguments supporting both P and Q , the acceptability for arguments for $P \cap Q$ is evaluated neither in the argumentation graph \mathcal{G}^P nor in \mathcal{G}^Q , but rather in $\mathcal{G}^{P \cap Q}$, a structure that might not contain acceptable arguments (for $P \cap Q$). The counterexample presented for item (c) in the proof of Proposition 3.2 illustrates this situation and it also shows how there might be argumentation-support frames in which the agent believes $\bar{P} \cup \bar{Q}$ while also believing both P and Q . In the philosophical literature a number of discussions can be found on whether beliefs should be closed under conjunction, and arguments against it are often motivated by paradoxical situations such as e.g. the famous lottery paradox.

For a systematic way of exploring the properties of the defined notion of belief as well as other related argumentative theoretic notions, the following section proposes a logic to describe and reason about argumentation-support frameworks.

4. Argument-based belief logic

In this section, we present the argument-based belief logic. There have been some studies of abstract argumentation theory from the perspective of modal logic (Grossi, 2010b; Grossi, 2010; Grossi, 2012; Grossi, 2013; Grossi & van der Hoek, 2014). Especially in Grossi and van der Hoek (2014), the relation between belief and argument is studied in a two-dimensional modal logic. However, in this two-dimensional modal logic, even when the authors study the interaction between beliefs and arguments, the very notion of belief itself is still defined by the doxastic relation rather than in terms of arguments. This aspect is different in the present work where belief emerges as a derived modality. Our logic also has a flavor of two-

dimensional logic which is blended with ingredients from modal μ -calculus and configured to fit the need of characterising the notion of argument-based belief.

4.1. Syntax

Definition 4.1: Let $\text{Prop} = \{p, q, r, \dots\}$ be a non-empty set of atomic propositions. \mathcal{L} is the language generated by the following grammar:

$$\alpha ::= \top \mid p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \boxplus\alpha \mid \boxminus\beta \quad \beta ::= \top \mid \Box\alpha \mid \neg\beta \mid \beta \wedge \beta \mid [\alpha]\beta \mid \text{Gfp}^\alpha$$

where $p \in \text{Prop}$. Symbols $\Diamond, \diamond, \langle\alpha\rangle$ and \perp are abbreviations of $\neg\Box\neg, \neg\Box, \neg\Box\neg, \neg[\alpha]\neg$ and $\neg\top$, respectively.

The language is divided into two parts. While α -formulas (the α part of the language) are used to state facts about possible worlds, β -formulas (the β part) are dedicated to the description of arguments. When there is no need to make distinction, φ is used to denote formulas in the whole language \mathcal{L} .

As it will be seen, \boxplus is a universal operator quantifying over possible worlds; it can be taken as an S5-knowledge operator. Analogously, \boxminus is a universal operator quantifying over arguments, so $\boxminus\beta$ states that, for all arguments, β is the case.⁶ Formulas of the form $\Box\alpha$ state that the current argument supports α , and those of the form $[\alpha]\beta$ state that all arguments which directly attack the current one on α satisfy β . Finally, Gfp^α states that the current argument is acceptable in the argumentation on α .

Remark 4.1: The language allows interaction between α - and β -formulas. For example, $\Diamond\Box p$ expresses that there is an argument supporting p . However, the interaction between these two types of formulas is limited. For example, strings as $\boxplus\beta, \Box[\alpha]\beta$ are not formulas of our language. In the first, $\boxplus\beta$ expresses a fact about possible worlds (it is an α -formula), so we cannot use it to describe arguments; in the second, $[\alpha]\beta$ describes a property of certain arguments (it is a β -formula), and as such it is not a fact about possible worlds that can be supported by arguments.

4.2. Semantics

By adding a valuation function to the argumentation-support frame, we get the argumentation-support model, where formulas in \mathcal{L} can be evaluated.

Definition 4.2: An argumentation-support model is a tuple $\mathcal{M} = (\mathcal{F}, V)$ where \mathcal{F} is an argumentation-support frame and $V : \text{Prop} \rightarrow 2^W$ is an valuation function which assigns to each atomic proposition a set of possible worlds (those in which it is true).

Let \mathcal{M} be an argumentation-support model, and define $\llbracket\alpha\rrbracket_{\mathcal{M}} := \{w \in W \mid \mathcal{M}, (w, s) \models \alpha \text{ for any argument } s\}$ (the subscript \mathcal{M} will be omitted whenever possible).⁷ The truth of $\varphi \in \mathcal{L}$ is defined as follows:

Definition 4.3: Given an argumentation-claim model $\mathcal{M} = (W, AR, \{\leftarrow^P\}^{P \in 2^W}, f, V)$,

- $\mathcal{M}, (w, s) \models \top$
- $\mathcal{M}, (w, s) \models p$ **iff** $w \in V(p)$
- $\mathcal{M}, (w, s) \models \neg\varphi$ **iff** $\mathcal{M}, (w, s) \not\models \varphi$
- $\mathcal{M}, (w, s) \models \varphi \wedge \varphi'$ **iff** $\mathcal{M}, (w, s) \models \varphi$ and $\mathcal{M}, (w, s) \models \varphi'$

- $\mathcal{M}, (w, s) \models \exists \alpha$ **iff** for all $w' \in W$, $\mathcal{M}, (w', s) \models \alpha$
- $\mathcal{M}, (w, s) \models \exists \beta$ **iff** for all $s' \in AR$, $\mathcal{M}, (w, s') \models \beta$
- $\mathcal{M}, (w, s) \models \Box \alpha$ **iff** $f(s) \subseteq \llbracket \alpha \rrbracket$
- $\mathcal{M}, (w, s) \models [\alpha] \beta$ **iff** for any $s' \in AR$ such that $s \leftarrow^{\llbracket \alpha \rrbracket} s'$, $\mathcal{M}, (w, s') \models \beta$.
- $\mathcal{M}, (w, s) \models \text{Gfp}^\alpha$ **iff** s is in an admissible set of arguments relative to $\llbracket \alpha \rrbracket$.

We say a formula φ is satisfied in an argumentation-support model \mathcal{M} if there is a pair (w, s) in \mathcal{M} such that $\mathcal{M}, (w, s) \models \varphi$. A formula φ is valid in \mathcal{M} ($\mathcal{M} \models \varphi$) if for any pair (w, s) in \mathcal{M} we have $\mathcal{M}, (w, s) \models \varphi$. Finally, φ is valid in the whole class of argumentation-support models ($\models \varphi$) if it is valid in every argumentation-support model.

Note how the \mathcal{L} -formula $\Diamond(\Box \alpha \wedge \text{Gfp}^\alpha)$ expresses that the agent has an acceptable argument for a given α . Therefore, the notion of belief can be defined in \mathcal{L} as follows:

$$B\alpha := \Diamond(\Box \alpha \wedge \text{Gfp}^\alpha) \wedge \neg \Diamond(\Box \neg \alpha \wedge \text{Gfp}^{\neg \alpha}).$$

As we have shown in Corollary 3.1, our notion of belief satisfies the following properties in the class of argumentation-support models:

Fact 4.1:

$$\models BT, \quad \models B\alpha \rightarrow B(\alpha \vee \alpha'), \quad \not\models B\alpha \wedge B\alpha' \rightarrow B(\alpha \wedge \alpha').$$

Thus, although the agent's beliefs are closed upward (the validity of $B\alpha \rightarrow B(\alpha \vee \alpha')$), they are not closed under conjunction introduction ($B\alpha \wedge B\alpha' \rightarrow B(\alpha \wedge \alpha')$ is not valid). Moreover, beliefs are consistent and contain all validities.

Fact 4.2:

$$\models \neg B\perp, \quad \models \alpha \text{ implies } \models B\alpha.$$

Note that the language allows us to express higher-order beliefs: $BB\alpha$ and $B\neg B\alpha$ are formulas in \mathcal{L} . In fact, for our notion of belief, the following two properties hold:

Fact 4.3:

$$\models B\alpha \rightarrow BB\alpha, \quad \models \neg B\alpha \rightarrow B\neg B\alpha.$$

It is also worthwhile to notice how, by Propositions 2.1 and 3.1, we have an equivalent truth condition for Gfp^α in the argumentation-support models:

$$\mathcal{M}, (w, s) \models \text{Gfp}^\alpha \text{ iff } s \in \text{GFP}.d^{\llbracket \alpha \rrbracket}.$$

This paves the way for looking for a sound and complete axiom system for our logic, which will be presented next.

4.3. Axiom system

The following is an axiom system (called AB) for the argument-based belief logic. Axioms for Gfp^α indicate that this operator amounts to the greatest fixed point of d .

- All the propositional tautologies.
- Modus ponens
- S5 and Necessitation rule for \boxplus
- For \boxplus :
 - K $\vdash \boxplus(\beta \rightarrow \beta') \rightarrow (\boxplus \beta \rightarrow \boxplus \beta')$
 - D $\vdash \neg \boxplus \perp$
 - N If $\vdash \beta$, then $\vdash \boxplus \beta$
- For \Box :
 - K $\vdash \Box(\alpha \rightarrow \alpha') \rightarrow (\Box \alpha \rightarrow \Box \alpha')$
 - D $\vdash \neg \Box \perp$
 - N If $\vdash \alpha$, then $\vdash \Box \alpha$
- For $[\alpha]$:
 - K $\vdash [\alpha](\beta \rightarrow \beta') \rightarrow ([\alpha]\beta \rightarrow [\alpha]\beta')$
 - N $\vdash \beta$ implies $\vdash [\alpha]\beta$
- For Gfp^α :
 - Unfold $\vdash \text{Gfp}^\alpha \rightarrow [\alpha]\langle \alpha \rangle \text{Gfp}^\alpha$
 - R $\vdash \beta \rightarrow [\alpha]\langle \alpha \rangle \beta$, then $\vdash \beta \rightarrow \text{Gfp}^\alpha$
- Interaction between \boxplus and \boxminus
 - $\boxplus 1 \vdash \boxplus \beta \rightarrow \boxminus \boxplus \beta$
 - $\boxplus 2 \vdash \neg \boxplus \beta \rightarrow \boxminus \neg \boxplus \beta$
- Interaction between \boxplus , \boxminus , \Box and $[\alpha]$
 - I1 $\vdash \boxplus \alpha \leftrightarrow \boxminus \Box \alpha$
 - I2 $\vdash \boxplus \beta \rightarrow \boxminus [\alpha] \beta$
- interaction between \Box and $[\alpha]$
 - 1 $\vdash [\alpha] \beta \leftrightarrow [\neg \alpha] \beta$
 - 2a $\vdash \langle \alpha \rangle \top \rightarrow \Box \alpha \vee \Box \neg \alpha$
 - 2b $\vdash \Box \alpha \rightarrow [\alpha] \Box \neg \alpha$
 - 3 $\vdash \Box \boxplus (\alpha \rightarrow \alpha') \wedge \Box \alpha \wedge \langle \alpha' \rangle \beta \rightarrow \langle \alpha \rangle \beta$

Note that axioms **T**, **4** and **5** do not hold for \boxplus , as $\boxplus \beta \rightarrow \beta$ and $\boxplus \boxplus \beta$ are not expressible in our language. This is also why we need axioms $\boxplus 1$, $\boxplus 2$, **I1** and **I2** to characterise the relationship between \boxplus as a universal operator and other operators. Axioms **1**, **2a**, **2b** and **3** correspond to frame conditions 1, 2(a), 2(b) and 3 on argumentation-support models, respectively. Finally, the axioms for Gfp^α are special cases of the general greatest fixed point operator (Kozen, 1983): the **unfold** axiom says that Gfp^α is a fixed point of $[\alpha]\langle \alpha \rangle$, and rule **R** says that Gfp^α is the greatest postfix point.⁸

Theorem 4.1: *The system AB is sound and weakly complete for the argument-based belief logic.*

Soundness follows from the fact that axioms and rules in AD are valid and validity preserving, respectively. We prove the validity of Axiom **3** as an example.

Proposition 4.1 (Validity of Axiom 3):

$$\models \Box \boxplus (\alpha \rightarrow \alpha') \wedge \Box \alpha \wedge \langle \alpha' \rangle \beta \rightarrow \langle \alpha \rangle \beta$$

Proof: Given an argumentation-claim model \mathcal{M} , take a pair (w, s) where $\Box \boxplus (\alpha \rightarrow \alpha') \wedge \Box \alpha \wedge \langle \alpha' \rangle \beta$ holds. The first conjunct tells us that $\llbracket \alpha \rrbracket \subseteq \llbracket \alpha' \rrbracket$, the second implies that $f(s) \subseteq \llbracket \alpha \rrbracket$, and the third indicates that there is an argument s' such that $s \leftarrow^{\llbracket \alpha' \rrbracket} s'$ and $f(s') \subseteq \llbracket \beta \rrbracket$. These three facts, together with condition (3), entail $s \leftarrow^{\llbracket \alpha \rrbracket} s'$, which implies that $\mathcal{M}, (w, s) \models \langle \alpha \rangle \beta$ \square

For (weak) completeness, the general strategy is, as usual, to show that every consistent formula is satisfiable in a model. The outline of the proof is similar to the standard proof of completeness in the sense of using maximal consistent sets to build the model. However, the process and details of the whole proof are not fully straightforward, because the model is two-dimensional and the attack relations are labeled by different subsets of possible

worlds which satisfy certain conditions. Moreover, the language includes an operator for the greatest fixed point of the defence function. Therefore, we stress some key points in our proof here. The full proof can be found in the appendix, but here are some important details.

First, notice that although the argumentation-support model is two-dimensional, the syntax restricts the interaction between these two dimensions. For example, strings as $\Box\beta \rightarrow \beta$ are not formulas in our language: $\Box\beta$ is an α -formula, but β is not. So $\{\Box\beta, \neg\beta\}$ is a consistent set in our logic (with consistency defined, in the standard way, as the non-derivability of the always false \perp), even though no pair (w, s) in an argumentation-support model satisfies both $\Box\beta$ and $\neg\beta$. This is a two-edged sword. On the one hand, it gives us the flexibility to construct maximal consistent sets for α -formulas and β -formulas separately and put them together in the model. On the other hand, we have to use some devious ways to ensure that the constructed model respects satisfiability.

Second, note that we need to construct the attack relations labelled by all subsets of possible worlds. However, we may not have enough information about them given the formulas in our hands, which are generated from the subformulas of a given formula ϕ . This means we can only construct the model partially. Therefore, we have to prove that any such kind of partial model (we call them *quasi argumentation-support models* in the proof as defined in Definition A) can be extended to a full and real argumentation-support model. Moreover, the extended model should be modally equivalent to the original model with respect to all the formulas we care about.

Third, the proof for the case of Gfp^α in the truth lemma is worth some attention. It is not as straightforward as other cases. We have to prove that the information about the greatest fixed point in the *quasi argumentation-support models* given by those maximally consistent sets matches the information given by the *quasi argumentation-support models* itself. The readers can find the details of how axiom **Unfold** and the inference rule **R** are applied in the proof.

Fourth, note that the model we build in the proof is finite, both in its set of worlds and in its set of arguments: thus, our logic has the finite model property. This, together with (i) the decidability of the 'being a model' property (the 'binary argument' requirements of Page 5 are decidable on finite models), (ii) the enumerability of models (\leftarrow^P is a binary relation on arguments for each subset of worlds P , and f is a function from arguments to sets of worlds), and (iii) the decidability of the semantic satisfiability conditions (the fixed point condition is decidable), shows that our logic is decidable.

4.4. Discussion

For space reasons, an extensive discussion on both concrete applications of the presented framework and, more generally, the idea of argument-based beliefs, are left out. Still, for applications, one can briefly mention Shi (2016), where the logic is used not only to discuss the notion of distributed belief, but also to analyse a 'two-player argumentation game', illustrating how a single agent's belief and the group's distributed belief are decided by the corresponding argumentation. For a deeper discussion about using arguments (and, more generally, using evidence/justifications) for supporting beliefs, the reader is referred to Baltag et al. (2014), Baltag, Bezhanishvili, Özgün, and Smets (2016b) and Shi, Smets, and Velázquez-Quesada (2017).

5. Conclusion

We have provided the first steps in the analysis of beliefs based on binary arguments which either support a given propositional claim or its negation. We offered a complete axiom system for this logic and have shown that this logic is decidable. The proposition-labeled attack relation in our setting does come with a specific interpretation when we apply it to real argumentation scenarios. In particular the attack relation on whether proposition P holds, is to be interpreted as a direct attack on the conclusion of the reasoning that establishes P . If we refine the setting and allow indirect attacks that undermine for instance a given premise that leads up to a conclusion P , the setting we have presented needs to be further refined. Another aspect we have not explored in this paper deals with combined arguments of claims that together provide the support in favor or against their conjunction. The study of combined arguments ties in nicely with an ongoing project by the authors on a topological setting that provides a semantics for arguments (Shi et al., 2017), very much in the spirit of the topological semantics for evidence-based beliefs in Baltag et al. (2016a). Moreover, a multi-agent extension of some of the ideas presented in this paper has been provided in Shi (2016), in order to deal with argumentation games and the notion of distributed belief.

Notes

1. Still, as one reviewer correctly pointed out, Dung (1995) addresses this connection indirectly, by showing that many of the major approaches to nonmonotonic reasoning in AI and logic programming, which can be understood as a form of reasoning that creates beliefs, are different forms of argumentation.
2. The idea of working with accessibility relations labelled by propositions is used in the literature on conditional logic and in particular in the context of logics for belief revision (Baltag & Smets, 2008; Baltag & Smets, 2006a; Baltag & Smets, 2006b).
3. The set \bar{P} denotes, as usual, the complement of P (i.e. $\bar{P} := W \setminus P$).
4. This interpretation can be justified by the fact that, while s_1 only states its ‘positive’ conclusion (‘it is a bird’), s_2 and s_3 take the time to explicitly reject other alternatives. Of course, changes in the paraphrasing might produce different interpretations. The point of the example is not to establish a unique formalisation of the given natural language scenario, but rather to show how our setting can be used to represent this and other similar situations.
5. Note how, by Corollary 2.1, $GFP.d$ is equal to the union of all the preferred extensions. This could suggest an alternative definition of this notion of acceptability, not in terms of the union of all preferred extensions (the greatest fixed point), but rather in terms of the *intersection* of them (which, by Dung, 1995, Theorem 33, is equal to the least fixed point of d). However, such intersection might be empty, and therefore no argument would be acceptable for P , even in the presence of preferred extensions containing arguments supporting P . For this reason, this paper will work with the notion of acceptability as defined above (Definition 3.2).
6. Still, $\Box\beta$ expresses facts about possible worlds, as the (non-)existence of arguments with certain features (what $\Box\beta$ expresses) is not a feature of any specific argument.
7. A notational alternative for the worlds in which an α -formula holds, $\llbracket\alpha\rrbracket_{\mathcal{M}}$, would be to use $\llbracket\cdot\rrbracket_{\mathcal{M}}$ to represent a subset of $W \times AR$ (as formulas are evaluated on *pairs* (w, s)), and then use projection functions to extract its W - and AR -components. We stick to the use of $\llbracket\alpha\rrbracket_{\mathcal{M}}$ for denoting directly a subset of W , as the AR -component of the set or pairs is not used.
8. The completeness proof is not related to the completeness proof for the μ -calculus (Walukiewicz, 2000); it is rather close to the completeness proofs for the *until/since* operators in temporal logics (Burgess, 1982) and the common knowledge operator in epistemic logics (Fagin, Moses, Vardi, & Halpern, 1995).

9. An element $x \in D$ is a *fixed point* of a function $f : D \rightarrow D$ if and only if $f(x) = x$. If a partial order \leq on D is also provided, x is a *postfix point* if $x \leq f(x)$, and it is the *greatest* fixed point if it is greater or equal than any other fixed point. When f is defined over the powerset of a given set D , as the functions d and n above, a fixed point is defined as before, and the subset order over 2^D can be used for defining the greatest fixed point. For more about fixed points, the reader is referred to Chapter 1 of [Arnold and Niwinski \(2001\)](#).
10. Note that $\{\|\alpha\| \mid [\alpha]_{\top} \in Cls(\phi)\}$ will play the role of the labels of attack relations in the quasi argumentation-support model.

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Appendix 1.

A.1. Proof of Proposition 2.1

Proof: (\Rightarrow) The Knaster-Tarski fixpoint theorem (Knaster, 1928; Tarski, 1955) states that $\text{GFP } .d$ is the union of all d 's postfix points, $\text{GFP } .d = \bigcup \{X \subseteq AR \mid X \subseteq d(X)\}$.⁹ Hence, as any admissible set X is, by definition, a postfix point of d , we have $X \subseteq \text{GFP } .d$ and thus $s \in X$ implies $s \in \text{GFP } .d$.

(\Leftarrow) Take any $s \in \text{GFP } .d$ and define $X := \text{Gfp } .d \cap \text{Def } (s)$; it is enough to show that X is an admissible set containing s , i.e. $X \subseteq n(X)$, $X \subseteq d(X)$ and $s \in X$. For the first, observe that any two arguments $x, y \in X$ are, by definition, also in $\text{Def } (s)$. Thus, each one of them is at an even distance from s , and hence cannot attack each other (as otherwise the attacker would be at an odd distance from s , which is impossible as \mathcal{G} is uncontroversial). Therefore, $X \subseteq n(X)$. For the second, take any argument x attacking some $s' \in X$. Since $s' \in \text{GFP } .d$, there must be $s'' \in \text{GFP } .d$ such that $x \leftarrow s''$ (recall, $\text{GFP } .d$ is a fixed point of d , so $\text{GFP } .d = d(\text{GFP } .d)$); hence, $s'' \in \text{Def } (s')$. But then, as $s' \in \text{Def } (s)$ (by X 's definition) and $s'' \in \text{Def } (s')$, it follows that $s'' \in \text{Def } (s)$, that is, $s'' \in X$ (by X 's definition). Summarising, every argument x attacking some argument (s') in X is in turn attacked by an argument (s'') that belongs to X ; thus, $X \subseteq d(X)$. For the third, since clearly $s \in \text{Def } (s)$, $s \in \text{GFP } .d$ implies $s \in X$. \square

A.2. Completeness of the AB system

In this section we show that for any $\varphi \in \mathcal{L}$, $\models \varphi$ implies $\vdash \varphi$. We first define the quasi argumentation-support model and then extend it to an argumentation-support model.

Definition A: [Quasi Argumentation-Support Model] A quasi argumentation-support model \mathcal{M}^X is a structure $(W, AR, \{\leftarrow^P\}_{P \in X \subseteq 2^W}, f, V)$ where W, AR, \leftarrow^P, f and V are defined as in the argumentation-support model, and X is closed under complement, i.e. if $P \in X$, then $\bar{P} \in X$. Moreover, it satisfies a restricted version of the conditions we impose on the argumentation-support model:

1. For any $P \in X$, $s_1 \leftarrow^P s_2$ if and only if $s_1 \leftarrow^{\bar{P}} s_2$.
2. For any $P \in X$, if $s_1 \leftarrow^P s_2$,
 - (a) either $f(s_1) \subseteq P$ or $f(s_1) \subseteq \bar{P}$; and
 - (b) $f(s_1) \subseteq P$ implies $f(s_2) \subseteq \bar{P}$.
3. For any $P, Q \in X$, if $s_1 \leftarrow^P s_2$ and $f(s_1) \subseteq Q \subseteq P$, then $s_1 \leftarrow^Q s_2$.

In words, a quasi argumentation-support model \mathcal{M}^X is simply an argumentation-support model that only 'discusses' propositions in X , and therefore only needs to satisfy the frame conditions relative elements of X . Clearly, any argumentation-support model is a quasi argumentation-support model; for the other direction, we have the follow proposition

Lemma A: Any quasi argumentation-support model \mathcal{M}^X can be extended to an argumentation-support model.

Proof: We define the relation \leftarrow^P for any $P \notin X$ as follows:

$$s \leftarrow^P s' \text{ if and only if there is a set } Q \in X \text{ such that } s \leftarrow^Q s' \text{ and } f(s) \subseteq P \subseteq Q \text{ or } f(s) \subseteq \bar{P} \subseteq Q$$

We claim that the model generated by adding these attacking relations into \mathcal{M}^X is an argumentation-support model. To prove this claim, we only need to check that it satisfies the four frame conditions.

- 1:** $s_1 \leftarrow^P s_2$ if and only if $s_1 \leftarrow^{\bar{P}} s_2$. We only need to prove that if $s_1 \leftarrow^P s_2$, then $s_1 \leftarrow^{\bar{P}} s_2$.
 - If $P \in X$, it follows immediately from the restricted version of condition 2 and the fact that X is closed under complement.
 - If $P \notin X$, the definition of $s_1 \leftarrow^P s_2$ also implies that $s_1 \leftarrow^{\bar{P}} s_2$, since the definition is symmetric.
- 2(a):** if $s_1 \leftarrow^P s_2$, either $f(s_1) \subseteq P$ or $f(s_1) \subseteq \bar{P}$. For $P \in X$, by the restricted version of condition 2(a), \leftarrow^P satisfies condition 2(a). For $P \notin X$, the definition of \leftarrow^P implies $f(s_1) \subseteq P$ or $f(s_1) \subseteq \bar{P}$

2(b): if $s_1 \leftarrow^P s_2$ and $f(s_1) \subseteq P$, then $f(s_2) \subseteq \bar{P}$. For $P \in X$, by the restricted version of condition 2(b), \leftarrow^P satisfies condition 2(b). For $P \notin X$, the definition of \leftarrow^P implies that there is $Q \in X$ such that $f(s_1) \subseteq P \subseteq Q$. By 2(b) for $Q \in X$, $f(s_2) \subseteq \bar{Q} \subseteq \bar{P}$.

3: if $s_1 \leftarrow^P s_2$ and $f(s_1) \subseteq Q \subseteq P$, then $s_1 \leftarrow^Q s_2$.

- If $P, Q \in X$, this condition follows from its restricted version of condition 3.
- If $P \in X$ and $Q \notin X$, take P as the set required by the definition of \leftarrow^Q , it follows immediately that $s_1 \leftarrow^Q s_2$.
- If $P \notin X$ and $Q \in X$, by definition of \leftarrow^P , there is a set $R \in X$ such that $s_1 \leftarrow^R s_2$ and $f(s_1) \subseteq P \subseteq R$ or $f(s_1) \subseteq \bar{P} \subseteq R$. (Note that the second case is not possible since $f(s_1) \subseteq P$ by assumption.) By the restricted version of condition 3, since $Q \in X$ and $f(s_1) \subseteq Q \subseteq P \subseteq R$, we have $s_1 \leftarrow^Q s_2$.
- If $P, Q \notin X$, by the definition of \leftarrow^P , there is a set $R \in X$ such that $s_1 \leftarrow^R s_2$ and $f(s_1) \subseteq P \subseteq R$ or $f(s_1) \subseteq \bar{P} \subseteq R$. Take the set R , since $f(s_1) \subseteq Q \subseteq P \subseteq R$, it follows that $s_1 \leftarrow^Q s_2$ by the definition of $s_1 \leftarrow^Q s_2$.

□

Hence, for any AB-consistent formula $\varphi \in \mathcal{L}$, we first build a quasi argumentation-support model in which it is satisfied by using maximal consistent sets. When doing so, we take care of the fact there are two dimensions in an argumentation-support model. Corresponding to these two dimensions, the language \mathcal{L} is divided into two parts – α -formulas and β -formulas. Note that given an AB-consistent set of α -formulas A and an AB-consistent set of β -formulas B , their union $A \cup B$ is AB-consistent, for suppose otherwise; then there must be $\alpha_1, \dots, \alpha_n \in A$ and $\beta_1, \dots, \beta_m \in B$ such that $\vdash \bigwedge_{i=1}^n \alpha_i \rightarrow \neg \bigwedge_{i=1}^m \beta_i$. However, this is impossible since $\bigwedge_{i=1}^n \alpha_i \rightarrow \neg \bigwedge_{i=1}^m \beta_i$ is not a formula in \mathcal{L} . This fact gives us the flexibility to construct the pairs of possible worlds and arguments on which we evaluate formulas.

Given a formula φ , we define \sim in the following formula:

$$\sim \varphi := \begin{cases} \psi & \text{if } \varphi \text{ is of the form } \neg \psi, \\ \neg \varphi & \text{otherwise} \end{cases}$$

A set of formulas X is closed under single negation if and only if $\sim \varphi$ belongs to X whenever $\varphi \in X$.

Definition B: Let X be a set of formulas. The set X is *FL-closed* if and only if it is closed under subformulas (e.g. if $[\alpha]\beta \in X$, then $\alpha, \beta \in X$) and it satisfies the following additional constraints:

- $\top, \perp, \sqcup \perp, \sqcap \perp \in X$
- if $[\alpha]\beta \in X$, then $[\sim \alpha]\beta, \square \alpha, \langle \alpha \rangle \top \in X$;
- if $\text{Gfp}^\alpha \in X$, then $[\alpha]\langle \alpha \rangle \text{Gfp}^\alpha \in X$.

Next, we build the canonical model starting by constructing maximal consistent sets. Given our two kinds of formulas, α -formulas and β -formulas, an intricate coordination between them is required during the construction.

Definition C: Let Σ be a set of formulas. We define $\text{Sub}(\Sigma)$ as the smallest set containing Σ which is FL-closed and closed under single negations. And $\text{Sub}^+(\Sigma)$ is the smallest set containing $\text{Sub}(\Sigma)$ which satisfies the following two condition:

- if $\boxplus \alpha \in \text{Sub}(\Sigma)$, then $\boxplus \square \boxplus \alpha \in \text{Sub}^+(\Sigma)$ and $\boxplus \square \neg \boxplus \alpha \in \text{Sub}^+(\Sigma)$;
- if $\boxminus \beta \in \text{Sub}(\Sigma)$, then $\boxminus \square \boxminus \beta \in \text{Sub}^+(\Sigma)$ and $\boxminus \square \neg \boxminus \beta \in \text{Sub}^+(\Sigma)$;

The set $\text{Cls}(\Sigma)$, the closure of Σ , is the the smallest set containing $\text{Sub}^+(\Sigma)$ which is FL-closed and closed under single negations.

Definition D: (Atoms) Let Σ be a set of formulas.

- A set of formulas Γ is an atom over Σ if it is a maximal consistent subsets of $\text{Cls}(\Sigma)$. The set $\text{At}(\Sigma)$ contains all the atoms over Σ .
- A set of formulas A is an α -atom over Σ if it is a maximal consistent subsets of $\text{Cls}(\Sigma) \cap \mathcal{L}^\alpha$. The set $\text{At}^\alpha(\Sigma)$ contains all α -atoms over Σ .

- A set of formulas B is an β -atom over Σ if it is a maximal consistent subsets of $Cls(\Sigma) \cap \mathcal{L}^\beta$. The set $At^\beta(\Sigma)$ contains all β -atoms over Σ .

Fact A:

- If A is an α -atom over Σ and B is a β -atom over Σ , then $A \cup B$ is an atom over Σ ;
- If Γ is an atom over Σ , then $\Gamma \cap \mathcal{L}^\alpha$ is an α -atom and $\Gamma \cap \mathcal{L}^\beta$ is a β -atom.

Proof: For the first, we know that $A \cup B \subseteq Cls(\Sigma)$ is AB-consistent; thus, we only need to prove that any X satisfying $A \cup B \subseteq X \subseteq Cls(\Sigma)$ is inconsistent. This is obvious, since any $\psi \in Cls(\Sigma)$ with $\psi \notin A \cup B$ is either an α -formula or β -formula. Without loss of generality we can assume it is an α -formula; since A is an α -atom over Σ , the set $A \cup \{\psi\}$ must be AB-inconsistent. Thus, X is also AB-inconsistent.

For the second, suppose $\Gamma \cap \mathcal{L}^\alpha = A$ is not an α -atom. Since A is consistent, there is an α -formula α' not in A such that $A \cup \{\alpha'\}$ is still consistent, and thus $\Gamma \cup \{\alpha'\}$ is also consistent. But $\alpha' \notin A$ implies $\alpha' \notin \Gamma$, and thus this contradicts the maximality of Γ . Therefore, $\Gamma \cap \mathcal{L}^\alpha$ must be an α -atom. The argument for $\Gamma \cap \mathcal{L}^\beta$ is similar. \square

Lemma B: *If $\Phi \subseteq Cls(\Sigma)$ and Φ is consistent, then there is a $\Gamma \in At(\Sigma)$ such that $\Phi \subseteq \Gamma$.*

The proof of Lemma B, an analogue of Lindenbaum's Lemma, follows the same argument as the proof of Lemma 4.83 of Blackburn, de Rijke, and Venema (2001). Together with the second fact of Fact A, it implies the following lemmas:

Lemma C:

1. *If $X \subseteq Cls(\Sigma)$ and α is consistent, then there is a $A \in At^\alpha(\Sigma)$ such that $X \subseteq A$.*
2. *If $Y \subseteq Cls(\Sigma)$ and β is consistent, then there is a $B \in At^\beta(\Sigma)$ such that $Y \subseteq B$.*

We can now fix a formula $\phi \in \mathcal{L}$ and construct the canonical model for it, which will be proved to be a quasi argumentation-support model. Before doing so, here is first some useful notation.

Notation A: Let X and Σ be sets of formulas.

- If X is finite, define $\widehat{X} := \bigwedge_{\varphi \in X} \varphi$.
- For any $\circ \in \{\exists, \boxplus, \square, [\alpha]\}$, define the sets

$$X^\circ := \{\varphi \in \mathcal{L} \mid \circ\varphi \in X\}, \quad \circ X^\circ := \{\circ\varphi \in \mathcal{L} \mid \circ\varphi \in X\},$$

If X is finite, define also

$$X_\circ^\Sigma := \{\varphi \in Cls(\Sigma) \mid \vdash \widehat{X}^\circ \rightarrow \varphi\}$$

When Σ is a singleton $\{\sigma\}$, the set $X_\circ^{\{\sigma\}}$ will be abbreviated as X_\circ^σ .

Second, the following proof shows a property required by the definition of our canonical model.

Lemma D: *Given a consistent ϕ , there is $\Delta \in At(\phi)$ such that $\phi \in \Delta$ and $\Delta_\boxplus^\phi \subseteq \Delta$.*

Proof: The proof is divided into two cases.

First, if ϕ is an α -formula then, by Lemma C (a), there is an α -atom A such that $\phi \in A$, due to its consistency. Note that A^\boxplus is consistent (otherwise, $\boxplus\perp \in A$, contradicting axiom D). Thus, by this and the fact that $A^\boxplus \subseteq Cls(\phi)$ (Lemma C (b)), there is a β -atom B such that $A^\boxplus \subseteq B$, which implies that $A_\boxplus^\phi \subseteq B$. By Fact A, $A \cup B \in At(\phi)$.

Second, if ϕ is a β -formula, suppose that there is no $\Delta \in At(\phi)$ such that $\phi \in \Delta$ and $\Delta_\boxplus^\phi \subseteq \Delta$. Then there is no α -atom A such that $A_\boxplus^\phi \cup \{\phi\}$ is consistent. (Otherwise, $A_\boxplus^\phi \cup \{\phi\}$ can be extended to a β -atom by Lemma C (b), and by Fact A $A \cup B$ is an atom over ϕ : a contradiction.)

Now take the α -formula $\boxplus\phi$. It can be proved that $\boxplus\phi$ is consistent. (Otherwise, $\vdash \neg \boxplus\phi$, which implies that $\vdash \boxplus\phi \leftrightarrow \boxplus\perp$. Since $\vdash \boxplus(\perp \rightarrow \neg\phi)$ and $\vdash \boxplus(\perp \rightarrow \neg\phi) \wedge \boxplus\perp \rightarrow \boxplus\neg\phi$, it follows that $\vdash \boxplus\phi \rightarrow \boxplus\neg\phi$, which implies that ϕ is inconsistent; a contradiction.) Since $\boxplus\phi$ is an α -formula, from the first auxiliary result it follows that there is an atom over $\boxplus\phi$, say Γ , such that $\Gamma_\boxplus^{\boxplus\phi} \subseteq \Gamma$ and $\boxplus\phi \in \Gamma$.

Next, take $\Delta := \Gamma \cap Cls(\phi)$. It can be shown that Δ is an atom over ϕ (i.e. $\Delta \in At(\phi)$). So take the α -atom $\Theta := \Delta \cap \mathcal{L}^\alpha$. Our supposition implies the inconsistency of $\Theta_\boxplus^\phi \cup \{\phi\}$, as we have shown at

the beginning. However, this implies that $\widehat{\Box(\Theta_{\Box}^{\phi} \cup \{\phi\})}$ is inconsistent. $\Box\Theta^{\Box} \cup \{\Box\phi\} \subseteq \Gamma$ implies that

$\vdash \widehat{\Gamma} \rightarrow \widehat{\Box(\Theta^{\Box} \cup \{\phi\})}$. By the definition of Θ_{\Box}^{ϕ} , $\vdash \widehat{\Theta^{\Box}} \rightarrow \widehat{\Theta_{\Box}^{\phi}}$. Thus, we have $\vdash \widehat{\Gamma} \rightarrow \widehat{\Box(\Theta_{\Box}^{\phi} \cup \{\phi\})}$. Since

$\widehat{\Box(\Theta_{\Box}^{\phi} \cup \{\phi\})}$ is inconsistent, it follows that Γ is inconsistent, contradicting the Γ 's consistency. \square

Here is, then, the definition of the canonical model for ϕ .

Definition E: [Canonical Model over ϕ] Take any $\Delta \in At(\phi)$ satisfying both $\phi \in \Delta$ and $\Delta_{\Box}^{\phi} \subseteq \Delta$ (by Lemma D, such Δ exists). The *canonical model* over ϕ is the structure

$$\mathfrak{M}_{\Delta}^{\{\|\alpha\| \mid [\alpha] \top \in Cls(\phi)\}} = (W, AR, f, \{\leftarrow^{\|\alpha\|} \mid [\alpha] \top \in Cls(\phi)\}, V)^{10}$$

given by

- $W := \{A \in At^{\alpha}(\phi) \mid \Box A^{\Box} \cap Sub(\Sigma) = \Box \Delta^{\Box} \cap Sub(\Sigma) \text{ and } \Box A^{\Box} \cap Sub(\Sigma) = \Box \Delta^{\Box} \cap Sub(\Sigma)\};$
- $\|\alpha\| := \{A \in W \mid \alpha \in A\};$
- $AR := \{B \in At^{\beta}(\phi) \mid \Delta_{\Box}^{\phi} \subseteq B\};$
- $A \in f(B)$ if and only if $B \wedge \widehat{\Delta A}$ is consistent;
- for any $\|\alpha\| \in \{\|\alpha\| \mid [\alpha] \top \in Cls(\phi)\}$, $B \leftarrow^{\|\alpha\|} B'$ if and only if $B \wedge \langle \alpha \rangle B'$ is consistent;
- $V(p) := \{A \in At^{\alpha}(\phi) \mid p \in A\}$

We first prove four existence lemmas for \Box , \Box , \Box and $[\alpha]$. During the proof, we will use the following abbreviations:

$$\begin{aligned} DY &:= \{\Box \alpha \in Sub(\phi) \mid \Box \alpha \in \Delta\}, & DN &:= \{\neg \Box \alpha \in Sub(\phi) \mid \neg \Box \alpha \in \Delta\}, \\ EY &:= \{\Box \beta \in Sub(\phi) \mid \Box \beta \in \Delta\}, & EN &:= \{\neg \Box \beta \in Sub(\phi) \mid \neg \Box \beta \in \Delta\}, \\ YN &:= DY \cup DN \cup EY \cup EN. \end{aligned}$$

Lemma E: *In the canonical model, for any $\Box \alpha \in Cls(\phi)$, $\Box \alpha \notin A$ if and only if there is an α -atom $A' \in W$ such that $\neg \alpha \in A'$.*

Proof: Assume that $\Box \alpha \notin A$ where $\Box \alpha \in Cls(\phi)$. We show that $\{\neg \alpha\} \cup YN$ is consistent. Suppose not; then $\vdash \widehat{YN} \rightarrow \alpha$. By applying the necessitation rule, we have $\vdash \widehat{\Box YN} \rightarrow \Box \alpha$. By axiom 4 and 5 for \Box and $A^{\Box} = \Delta^{\Box}$, we have $\vdash \widehat{A} \rightarrow \Box(\widehat{DY} \wedge \widehat{DN})$. By Axioms $\Box 1$, $\Box 2$ and $A^{\Box} = \Delta^{\Box}$, we have $\vdash \widehat{A} \rightarrow \Box(\widehat{EY} \wedge \widehat{EN})$. Together with $\vdash \widehat{\Box YN} \rightarrow \Box \alpha$, it implies $\Box \alpha \in A$, a contradiction. Therefore, we can extend $\{\neg \alpha\} \cup YN$ to an α -atom over ϕ , A' , such that $\neg \alpha \in A'$. Moreover, $A'^{\Box} = \Delta^{\Box}$ and $A'^{\Box} = \Delta^{\Box}$, so $A' \in W$.

For the other direction, assume there is a α -atom $A' \in W$ such that $\neg \alpha \in A'$. It follows immediately that $\Box \alpha \notin A$. (Otherwise, by $A^{\Box} = \Delta^{\Box} = A'^{\Box}$ and axiom T for \Box , we have $\alpha \in A'$, which contradicts the assumption $\neg \alpha \in A'$.) \square

Lemma F: *In the canonical model, $\Delta_{\Box}^{\phi} = \bigcap AR$*

Proof: As the \subseteq direction is obvious, we focus on the \supseteq direction. Assume $\beta \in Cls(\phi)$ but $\beta \notin \Delta_{\Box}^{\phi}$; we will show that there is a $B \in AR$ such that $\beta \notin B$. This amounts to show that $\{\neg \beta\} \cup \Delta_{\Box}^{\phi}$ is consistent. Suppose not; then $\vdash \widehat{\Delta_{\Box}^{\phi}} \rightarrow \beta$. Since for any $\beta \in \Delta_{\Box}^{\phi}$, we have $\vdash \widehat{\Delta^{\Box}} \rightarrow \beta$ so $\vdash \widehat{\Delta^{\Box}} \rightarrow \widehat{\Delta_{\Box}^{\phi}}$. By $\vdash \widehat{\Delta_{\Box}^{\phi}} \rightarrow \beta$, it follows that $\vdash \widehat{\Delta^{\Box}} \rightarrow \beta$, which implies that $\beta \in \Delta_{\Box}^{\phi}$ by the definition of Δ_{\Box}^{ϕ} . However, this is contradictory to the assumption that $\beta \notin \Delta_{\Box}^{\phi}$. So there must be a $B \in AR$ such that $\beta \notin B$. Therefore, $\beta \notin \bigcap AR$. \square

Lemma G: [Existence Lemma for α -formulas] In the canonical model, for any $\Box\alpha \in Cls(\phi)$, $\Box\alpha \notin B$ if and only if there is a α -atom $A \in W$ such that there is $A \in f(B)$ such that $\neg\alpha \in A$.

Proof: Assume $\Box\alpha \notin B$; we will show that $\{\neg\alpha\}$ can be extended to an α -atom $A \in W$ such that $\widehat{B} \wedge \widehat{\Delta}$ is consistent. For this we follow the argument of Blackburn et al. (2001, Lemma 4.86) and construct an appropriate α -atom A by forcing choices. So, enumerate the formulas in $Clis(\phi)$ as $\alpha_1, \dots, \alpha_m$, and define A_0 as $\{\neg\alpha\} \cup YN$, with YN as above.

We first prove that $\widehat{B} \wedge \widehat{\Delta}_0$ is consistent. Suppose otherwise; then $\vdash \widehat{B} \rightarrow \Box\neg\widehat{\Delta}_0$ and $\neg\widehat{\Delta}_0 = \alpha \vee \neg\widehat{YN}$. Thus, $\vdash \widehat{B} \rightarrow \Box(\alpha \vee \neg\widehat{YN})$, which implies that $\vdash \widehat{B} \rightarrow \Box(\widehat{YN} \rightarrow \alpha)$. It follows that $\vdash \widehat{B} \rightarrow (\neg\Box\alpha \rightarrow \neg\Box\widehat{YN})$ and, since $\neg\Box\alpha \in B$, we have $\vdash \widehat{B} \rightarrow \neg\Box\widehat{YN}$.

In order to get a contradiction, we now proceed to prove that we also have $\vdash \widehat{B} \rightarrow \Box\widehat{YN}$. The proof can be divided into four parts specified as follows.

1. By Axiom **I1**, $\vdash \widehat{DY} \rightarrow \Box\Box\widehat{DY}$. Together with the construction of DY and Δ_{\Box}^{ϕ} , this implies that $\{\Box\Box\alpha \in Cls(\Sigma) \mid \Box\alpha \in \Delta \cap Sub(\Sigma)\} \subseteq \Delta_{\Box}^{\phi} \subseteq B$. Thus, $\vdash \widehat{B} \rightarrow \Box\Box\widehat{DY}$.
2. By Axiom **I1** and axiom **5** for \Box , $\vdash \widehat{DN} \rightarrow \Box\Box\widehat{DN}$. Together with the construction of DN and Δ_{\Box}^{ϕ} , this implies that $\{\Box\neg\Box\alpha \in Cls(\Sigma) \mid \neg\Box\alpha \in \Delta \cap Sub(\Sigma)\} \subseteq \Delta_{\Box}^{\phi} \subseteq B$. Thus, $\vdash \widehat{B} \rightarrow \Box\Box\widehat{DN}$.
3. By Axiom **I1** and Axiom **11**, $\vdash \widehat{EY} \rightarrow \Box\Box\widehat{EY}$. Together with the construction of EY and Δ_{\Box}^{ϕ} , this implies that $\{\Box\Box\beta \in Cls(\Sigma) \mid \Box\beta \in \Delta \cap Sub(\Sigma)\} \subseteq \Delta_{\Box}^{\phi} \subseteq B$. Hence, $\vdash \widehat{B} \rightarrow \Box\Box\widehat{EY}$.
4. By Axiom **I1** and Axiom **12**, $\vdash \widehat{EN} \rightarrow \Box\Box\widehat{EN}$. Together with the construction of EN and Δ_{\Box}^{ϕ} , this implies that $\{\Box\neg\Box\beta \in Cls(\Sigma) \mid \neg\Box\beta \in \Delta \cap Sub(\Sigma)\} \subseteq \Delta_{\Box}^{\phi} \subseteq B$. Hence, $\vdash \widehat{B} \rightarrow \Box\Box\widehat{EN}$.

Therefore, $\vdash \widehat{B} \rightarrow \Box\widehat{YN}$. This is a contradiction, so $\widehat{B} \wedge \widehat{\Delta}_0$ must be consistent.

Now, in order to extend the consistent $\widehat{B} \wedge \widehat{\Delta}_0$, suppose as an inductive hypothesis that A_n is defined such that $B \wedge \Delta_{A_n}$ is consistent ($1 \leq n \leq m$). Then

$$\vdash \widehat{\Delta}_n \leftrightarrow \Diamond((\widehat{A}_n \wedge \alpha_{n+1}) \vee (\widehat{A}_n \wedge \sim \alpha_{n+1}))$$

and thus $\vdash \widehat{\Delta}_n \leftrightarrow (\Diamond(\widehat{A}_n \wedge \alpha_{n+1}) \vee \Diamond(\widehat{A}_n \wedge \sim \alpha_{n+1}))$. Therefore, either for $A' = A_n \cup \{\alpha_{n+1}\}$ or for $A' = A_n \cup \{\sim \alpha_{n+1}\}$, we have $B \wedge \Delta_{A'}$ is consistent. By choosing A_{n+1} to be the consistent expansion, and by letting A be A_m , we have that $\widehat{B} \wedge \widehat{\Delta}$ is consistent.

Thus, suppose there is an α -atom $A \in W$ such that $\neg\alpha \in A \in f(B)$; then $\widehat{B} \wedge \widehat{\Delta}$ is consistent, which implies that $\widehat{B} \wedge \Diamond\neg\alpha$ is consistent. Since $\Box\alpha \in Cls(\phi)$ and B is β -atom over ϕ (and hence maximal consistent in $Clis(\phi)$), we must have $\neg\Box\alpha \in B$. \square

Lemma H: [Existence Lemma for β -formulas] In the canonical model, for any $[\alpha]\beta \in Cls(\phi)$, $[\alpha]\beta \notin B$ if and only if there is a β -atom $B' \in AR$ such that $\neg\beta \in B'$ and $B \leftarrow \|\alpha\| B'$.

Proof: Assume $[\alpha]\beta \notin B$; we will show that $\{\neg\beta\}$ can be extended to an β -atom $B' \in W$ such that $\widehat{B} \wedge \langle \alpha \rangle \widehat{B}'$ is consistent. We construct an appropriate β -atom B' by forcing choices. Enumerate the formulas in $Clis(\phi)$ as β_1, \dots, β_m ; define B_0 as $\{\neg\beta\} \cup \Delta_{\Box}^{\phi}$.

Let $\lambda = \Delta_{\Box}^{\phi}$. We first prove that $\widehat{B} \wedge \langle \alpha \rangle \widehat{B}_0$ is consistent. Suppose not. By an argument similar to that in Lemma G, we can get $\vdash \widehat{B} \rightarrow \neg[\alpha]\lambda$. By rule **N** for \Box , we have (1) $\vdash \Box\widehat{B} \rightarrow \Box\neg[\alpha]\lambda$. However, $\vdash \Box\widehat{B} \rightarrow \Box\lambda$, since $\Delta_{\Box}^{\phi} \subseteq B$. By Axiom **I2**, $\vdash \Box\lambda \rightarrow \Box[\alpha]\lambda$, which implies that (2) $\vdash \Box\widehat{B} \rightarrow \Box[\alpha]\lambda$. But (1) and (2) lead to a contradiction, since $\neg\Box\perp \in B$. Therefore, $\widehat{B} \wedge \langle \alpha \rangle \widehat{B}_0$ is consistent.

The induction part to extend $\widehat{B} \wedge \langle \alpha \rangle \widehat{B}_0$ is similar to that in the proof of previous lemma, and so is this lemma's other direction. \square

Now we are ready to prove the truth lemma.

Lemma I: [Truth Lemma] Let $\mathcal{M}_{\Delta}^{\{\|\alpha\|[\alpha] \top \in Cls(\phi)\}}$ be the canonical model over ϕ . For any $\varphi \in Cls(\phi)$,

$$\mathcal{M}_{\Delta}^{\{\|\alpha\|[\alpha] \top \in Cls(\phi)\}}, (A, B) \models \varphi \text{ if and only if } \varphi \in A \cup B$$

Proof: We proceed by induction on the degree of φ .

- For \top , $(A, B) \models \top$ and $\top \in A$ and $\top \in B$, so it is trivially satisfied. For atomic propositions p , we have $(A, B) \models p$ if and only if $A \in V(p)$ if and only if $p \in A$ if and only if $p \in A \cup B$.
- For the Boolean cases, there are two cases: one for α -formula and one for β -formula. In both, the proof is routine.
- For the four modal operators, the proof uses their respective existence lemmas. Here, as an example, we only deal with the universal modality on the set of arguments. Assume that $(A, B) \models \Box\beta$; by its truth definition, for any $B' \in AR$, $(A, B') \models \beta$. By induction hypothesis, $\beta \in B'$ for any $B' \in AR$, which implies that $\beta \in \bigcap AR \subseteq \Delta_{\Box}^{\phi} = A_{\Box}^{\phi}$. (Note that $\exists A^{\Box} \cap Sub(\Sigma) = \exists \Delta^{\Box} \cap Sub(\Sigma)$ and $\Box A^{\Box} \cap Sub(\Sigma) = \Box \Delta^{\Box} \cap Sub(\Sigma)$ imply $A_{\Box}^{\phi} = \Delta_{\Box}^{\phi}$ by Axiom **I1**, Axiom **\Box1** and the constructions of both $Cls(\Sigma)$ and Δ_{\Box}^{ϕ} , using similar arguments specified in (1)-(4) in Lemma A.6). Since $\Box\beta \in Cls(\phi)$, we have $\Box\beta \in A$. For the other direction, use $\bigcap AR \supseteq \Delta_{\Box}^{\phi}$.
- For the case of Gfp^{α} , the proof goes as follows. For the first direction, assume that $(A, B) \models Gfp^{\alpha}$; define $GFP := \{X \in \mathcal{AR} \mid (A, X) \models Gfp^{\alpha}\}$ and $D := GFP \cap Def^{\llbracket \alpha \rrbracket}(B)$. Let δ denote $\bigvee_{X \in D} \widehat{X}$, and define $E := \{X \in \mathcal{AR} \mid \delta \wedge \langle \alpha \rangle \widehat{X} \text{ is consistent}\}$. We will use ε to denote $\bigvee_{X \in E} \widehat{X}$.
Claim one: $X \in E$ if and only if there is a $Y \in D$ such that $Y \leftarrow^{\llbracket \alpha \rrbracket} X$.

$$\begin{aligned}
X \in E &\Leftrightarrow \delta \wedge \langle \alpha \rangle \widehat{X} \text{ is consistent} \\
&\Leftrightarrow \text{there is a } Y \in D \text{ such that } \widehat{Y} \wedge \langle \alpha \rangle \widehat{X} \\
&\Leftrightarrow \text{there is a } Y \in D \text{ such that } Y \leftarrow^{\|\alpha\|} X \\
&\Leftrightarrow \text{there is a } Y \in D \text{ such that } Y \leftarrow^{\llbracket \alpha \rrbracket} X \\
&\quad (\text{by the induction hypothesis } \llbracket \alpha \rrbracket = \|\alpha\|)
\end{aligned}$$

Claim two: for any $ar \subseteq \mathcal{AR}$ we have $\vdash \bigvee_{X \in ar} \widehat{X} \rightarrow [\alpha] \Delta_{\Box}^{\phi}$. Suppose not. Then $\bigvee_{X \in ar} \widehat{X} \wedge \neg[\alpha] \Delta_{\Box}^{\phi}$ is consistent, which implies that $\bigvee_{X \in ar} \widehat{X} \wedge \langle \alpha \rangle \neg \Delta_{\Box}^{\phi}$ is consistent. So there must be one $\beta \in \Delta_{\Box}^{\phi}$ such that $\bigvee_{X \in ar} \widehat{X} \wedge \langle \alpha \rangle \neg \beta$ is consistent. Furthermore, there must be one $X \in ar$ such that $\widehat{X} \wedge \langle \alpha \rangle \neg \beta$ is consistent. By the existence lemma for $[\alpha]$, there must be one $Y \in \mathcal{AR}$ such that $X \leftarrow^{\|\alpha\|} Y$ and $\neg \beta \in Y$. However, this is impossible, since $\beta \in \Delta_{\Box}^{\phi} \subseteq Y$ by the definition of \mathcal{AR} . Therefore, $\vdash \bigvee_{X \in ar} \widehat{X} \rightarrow [\alpha] \Delta_{\Box}^{\phi}$.

Claim three: $\vdash \varepsilon \rightarrow \langle \alpha \rangle \delta$. Suppose not. Then $\varepsilon \wedge \neg \langle \alpha \rangle \delta$ is consistent. Thus there must be an $X \in E$ such that $\widehat{X} \wedge \neg \langle \alpha \rangle \delta$ is consistent, which implies that there is no $Y \in D$ such that $\widehat{X} \wedge \langle \alpha \rangle \widehat{Y}$ is consistent. Hence, there is no $Y \in D$ such that $X \leftarrow^{\|\alpha\|} Y$. However, by Claim one and $X \in E$, there is a $Z \in D$ such that $Z \leftarrow^{\llbracket \alpha \rrbracket} X$. Since $Z \in D \subseteq GFP$, for any $X \in AR$ such that $Z \leftarrow^{\llbracket \alpha \rrbracket} X$, there is $Y \in GFP$ such that $X \leftarrow^{\llbracket \alpha \rrbracket} Y$. Because $X \in Att^{\llbracket \alpha \rrbracket}(B)$, $Y \in Def^{\llbracket \alpha \rrbracket}(B)$. Hence $Y \in D$ and $X \leftarrow^{\|\alpha\|} Y$, which implies that $\widehat{X} \wedge \langle \alpha \rangle \widehat{Y}$ is consistent by $\llbracket \alpha \rrbracket = \|\alpha\|$. Contradiction!

Claim four: $\vdash \delta \rightarrow [\alpha] \varepsilon$. Suppose not. Then $\delta \wedge \langle \alpha \rangle \neg \varepsilon$ is consistent, which implies that there is $X \in D$ such that $\widehat{X} \wedge \langle \alpha \rangle \neg \varepsilon$ is consistent. By Claim two, $\vdash \widehat{X} \rightarrow [\alpha] \Delta_{\Box}^{\phi}$. By $\vdash [\alpha] \beta \wedge \langle \alpha \rangle \beta' \rightarrow \langle \alpha \rangle (\beta \wedge \beta')$, the consistency of $\widehat{X} \wedge \langle \alpha \rangle \neg \varepsilon$ implies that $\widehat{X} \wedge \langle \alpha \rangle (\Delta_{\Box}^{\phi} \wedge \neg \varepsilon)$ is consistent. Thus there must be a $Y \in \mathcal{AR} \setminus E$ such that $\widehat{X} \wedge \langle \alpha \rangle \widehat{Y}$ is consistent. However, this means that $X \leftarrow^{\|\alpha\|} Y$. Since $X \in D$, by Claim one, it follow that $Y \in E$, contradictory to the fact that $Y \in \mathcal{AR} \setminus E$. Therefore, $\vdash \delta \rightarrow [\alpha] \varepsilon$.

Claim five: $\vdash \delta \rightarrow [\alpha] \langle \alpha \rangle \delta$. By rule **N** for $[\alpha]$ on Claim three, we have $\vdash [\alpha] \varepsilon \rightarrow [\alpha] \langle \alpha \rangle \delta$. Together with Claim four, it implies that $\vdash \delta \rightarrow [\alpha] \langle \alpha \rangle \delta$. By rule **R** on Gfp^{α} we have $\vdash \delta \rightarrow Gfp^{\alpha}$. Since $B \in Def^{\llbracket \alpha \rrbracket}(B)$ by the definition of Def and $B \in GFP$ by the definition of GFP and the assumption that $(A, B) \models Gfp^{\alpha}$, we have $B \in D = Def^{\llbracket \alpha \rrbracket}(B) \cap GFP$. Recall that $\delta := \bigvee_{X \in D} \widehat{X}$. By $B \in D$, we have $\vdash \widehat{B} \rightarrow \delta$. Together with $\vdash \delta \rightarrow Gfp^{\alpha}$, it follows that $\vdash \widehat{B} \rightarrow Gfp^{\alpha}$, which gives us that $Gfp^{\alpha} \in B$.

For the other direction, assume $Gfp^{\alpha} \in A \cup B$ and take $G := \{X \in \mathcal{AR} \mid Gfp^{\alpha} \in X\}$. We first show that G is a postfix point for $d^{\|\alpha\|}$, with $d^{\|\alpha\|}(G) = \{X \in \mathcal{AR} \mid \text{for any } Y \text{ such that } X \leftarrow^{\|\alpha\|} Y, \text{ there is } Z \in G \text{ such that } Y \leftarrow^{\|\alpha\|} Z\}$.

Take any $X \in G$. Since $Gfp^{\alpha} \in X$, by **Unfold** we get $[\alpha] \langle \alpha \rangle Gfp^{\alpha} \in X$. If there is no Y such that $X \leftarrow^{\|\alpha\|} Y$, then $X \in d^{\|\alpha\|}(G)$. If there is Y such that $X \leftarrow^{\|\alpha\|} Y$, take any of them. Since

$\langle \alpha \rangle \langle \alpha \rangle \text{Gfp}^\alpha \in X, \neg \langle \alpha \rangle \text{Gfp}^\alpha \notin Y$. By maximality and $\langle \alpha \rangle \text{Gfp}^\alpha \in \text{Cls}(\phi)$ we have $\langle \alpha \rangle \text{Gfp}^\alpha \in Y$ so, by the existence lemma for $\langle \alpha \rangle$, there must be a set Z such that $Y \leftarrow^{\|\alpha\|} Z$ and $\text{Gfp}^\alpha \in Z$. Thus, $X \in d^{\|\alpha\|}(G)$, and therefore $G \subseteq d^{\|\alpha\|}(G)$.

By the definition of $\text{GFP}.d^{\|\alpha\|}$, $G \subseteq \text{GFP}.d^{\|\alpha\|}$. By the inductive hypothesis, $\text{GFP}.d^{\|\alpha\|} = \text{GFP}.d^{\llbracket \alpha \rrbracket}$. Thus, $G \subseteq \text{GFP}.d^{\llbracket \alpha \rrbracket}$, which means that $B \in \text{GFP}.d^{\llbracket \alpha \rrbracket}$. Therefore, by the truth condition of Gfp^α , $(A, B) \models \text{Gfp}^\alpha$. □

The next step is to prove that the canonical model over ϕ is indeed a quasi argumentation-support model.

Lemma J: For any $B \in \mathcal{AR}$ in $\mathfrak{M}_\Delta^{\{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}}$, we have $f(B) \neq \emptyset$.

Proof: For any $B \in \mathcal{AR}$ we have $\Box \perp \notin B$. By the existence lemma for \Box , there is a α -atom $A \in W$ such that $\top \in A \in f(B)$. Therefore, for any $B \in \mathcal{AR}$, $f(B) \neq \emptyset$. □

Lemma K: For any $\|\alpha\| \in \{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}$ and $B_1, B_2 \in \mathcal{AR}$, we have $B_1 \leftarrow^{\|\alpha\|} B_2$ if and only if $B_1 \leftarrow^{\llbracket \alpha \rrbracket} B_2$.

Proof: This follows directly from axiom 1 and the fact that $\{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}$ is closed under complement. □

Lemma L: For any $\|\alpha\| \in \{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}$ and $B_1, B_2 \in \mathcal{AR}$, if $B_1 \leftarrow^{\|\alpha\|} B_2$ then either $f(B_1) \subseteq \|\alpha\|$ or else $f(B_1) \subseteq \|\neg \alpha\|$.

Proof: From $B_1 \leftarrow^{\|\alpha\|} B_2$ it follows that $\widehat{B}_1 \wedge \langle \alpha \rangle \widehat{B}_2$ is consistent. Thus, $\widehat{B}_1 \wedge \langle \alpha \rangle \top$ is consistent, which implies that $\langle \alpha \rangle \top \in B_1$ and hence $\vdash \widehat{B}_1 \rightarrow \langle \alpha \rangle \top$. Together with axiom 2a, $\vdash \langle \alpha \rangle \top \rightarrow \Box \alpha \vee \Box \neg \alpha$, it implies that $\vdash \widehat{B}_1 \rightarrow (\Box \alpha \vee \Box \neg \alpha)$.

Now take any $A \in f(B_1)$, we will show that $A \in \|\alpha\|$ implies $f(B_1) \subseteq \|\alpha\|$. Assume both $A \in f(B_1)$ and $A \in \|\alpha\|$; from $A \in f(B_1)$ it follows that $\widehat{B}_1 \wedge \diamond \widehat{A}$ is consistent, so $\widehat{B}_1 \wedge \diamond \alpha$ is consistent by $\vdash \widehat{A} \rightarrow \alpha$. By $\Box \alpha \in \text{Cls}(\phi)$ and $\vdash \widehat{B}_1 \rightarrow (\Box \alpha \vee \Box \neg \alpha)$, we have $\Box \alpha \in B_1$, and therefore, $f(B_1) \subseteq \|\alpha\|$.

It can be shown by similar argument that if $A \notin \|\alpha\|$ then $f(B_1) \subseteq \|\neg \alpha\|$. □

Lemma M: For any $\|\alpha\| \in \{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}$ and $B_1, B_2 \in \mathcal{AR}$, if $B_1 \leftarrow^P B_2$ and $f(B_1) \subseteq \|\alpha\|$, then $f(B_2) \subseteq \|\alpha\|$.

Proof: Assume both $B_1 \leftarrow^{\|\alpha\|} B_2$ and $f(B_1) \subseteq \|\alpha\|$. It follows that $\Box \alpha \in B_1$. Thus, by axiom 2b we have $\vdash \widehat{B}_1 \rightarrow [\alpha] \Box \neg \alpha$ and hence, for any B' such that $B_1 \leftarrow^{\|\alpha\|} B'$, we have $\not\vdash \widehat{B}' \rightarrow \neg \Box \neg \alpha$. Since $\Box \neg \alpha \in \text{Cls}(\phi)$, we also have $\Box \neg \alpha \in B_2$. This implies that, for any $A \in f(B_2)$, $\not\vdash \widehat{A} \rightarrow \alpha$. Thus, for any $A \in f(B_2)$, $\neg \alpha \in A$ and therefore $f(B_2) \subseteq \|\neg \alpha\|$. □

Lemma N: For any $P, Q \in \{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}$ and $B_1, B_2 \in \mathcal{AR}$, if $B_1 \leftarrow^{\|\alpha\|} B_2$ and $f(B_1) \subseteq Q \subseteq P$, then $B_1 \leftarrow^Q B_2$.

Proof: Take $P = \|\alpha'\|$ and $Q = \|\alpha\|$. We claim that for any $A \in W$, $\vdash \widehat{A} \rightarrow \Box(\alpha \rightarrow \alpha')$. Suppose not. Then $\widehat{A} \wedge \diamond(\alpha \wedge \neg \alpha')$ is consistent, and thus $\{\alpha, \neg \alpha'\}$ can be extended to an α -atom Y over ϕ belonging to W . However, this contradicts $Q \subseteq P$. Hence for any $A \in W$, $\vdash \widehat{A} \rightarrow \Box(\alpha \rightarrow \alpha')$. This implies that $\vdash \widehat{B}_1 \rightarrow \Box \Box(\alpha \rightarrow \alpha')$.

Now, $B_1 \leftarrow^P B_2$ implies that $\widehat{B}_1 \wedge \langle \alpha' \rangle \widehat{B}_2$ is consistent, and $f(B_1) \subseteq Q$ implies that $\vdash \widehat{B}_1 \rightarrow \Box \alpha$. Together with $\vdash \widehat{B}_1 \rightarrow \Box \Box(\alpha \rightarrow \alpha')$ and Axiom 3, they imply that $\widehat{B}_1 \wedge \langle \alpha \rangle \widehat{B}_2$ is consistent. Therefore, $B_1 \leftarrow^Q B_2$. □

We have proved that the canonical model over ϕ , $\mathcal{M}_\Delta^{\{\|\alpha\| \mid [\alpha] \top \in \text{Cls}(\phi)\}}$, is indeed a quasi argumentation-support model. By Lemma A, this structure can be extended to an argumentation-support model. It is only left to prove that this extension, denoted by \mathcal{M}_Δ^F , indeed preserves the behaviour of the formulas we are interested in.

Lemma O: For any $\varphi \in Cls(\phi)$,

$$\mathcal{M}_{\Delta}^F, (A, B) \models \varphi \quad \text{if and only if} \quad \mathcal{M}_{\Delta}^{\{\|\alpha\| \mid [\alpha]_{\top} \in Cls(\phi)\}}, (A, B) \models \varphi$$

Proof: The proof proceeds by induction on the degree of φ . The basic case is trivial, since the extension does not change V . The proof for other cases is also routine, as neither the support functions f nor the attack relations $\leftarrow^{\llbracket \alpha \rrbracket}$ for $[\alpha]_{\top} \in Cls(\phi)$ are changed when building the extension. \square

Thus,

Theorem 1.1: For any $\varphi \in \mathcal{L}$, $\models \varphi$ implies $\vdash \varphi$.

Since by construction the set $Cls(\phi)$ is finite for any $\phi \in \mathcal{L}$, it follows that both W and \mathcal{AR} in $\mathcal{M}_{\Delta}^{\{\|\alpha\| \mid [\alpha]_{\top} \in Cls(\phi)\}}$ are finite. Therefore, our completeness proof also gives us the finite model property, which implies that our logic is decidable.