Class fields by Shimura reciprocity
Gee, A.C.P.

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
SINGULAR VALUES OF THE
ROGERS-RAMANUJAN CONTINUED FRACTION

ALICE GEE AND MASCHA HONSBEEK

ABSTRACT. Let $z \in \mathbb{C}$ be imaginary quadratic in the upper half plane. Then the Rogers-Ramanujan continued fraction evaluated at $q = e^{2\pi iz}$ is contained in a class field of $\mathbb{Q}(z)$. Ramanujan showed that for certain values of $z$, one can write these continued fractions as nested radicals. We use the Shimura reciprocity law to obtain such nested radicals whenever $z$ is imaginary quadratic.

1. Introduction

The Rogers-Ramanujan continued fraction is a holomorphic function on the complex upper half plane $\mathbb{H}$, given by

$$R(z) = q^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{\lambda}{q}\right)}, \quad \text{with } q = e^{2\pi iz} \text{ and } z \in \mathbb{H}.$$

Here $\left(\frac{\lambda}{q}\right)$ denotes the Legendre symbol. The function $R$ owes part of its name to the expansion

$$R(z) = \frac{q^{\frac{1}{2}}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{\ddots}}}},$$

as a continued fraction. In their first correspondence of 1913, Ramanujan astonished Hardy with the assertion

$$\frac{e^{-\frac{2\pi}{\lambda}}}{1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{\ddots}}}} = \sqrt{5 + \sqrt{5}} - \frac{\sqrt{5 + 1}}{2}.$$

1991 Mathematics Subject Classification. Primary 11Y65; Secondary 11Y40.

Key words and phrases. Rogers-Ramanujan continued fraction, Shimura reciprocity.

The first author thanks Heng Huat Chan for providing the opportunity to enjoy his kind hospitality in Singapore. She is indebted to the lively discussions that initiated this work. This research was funded in part by the Academic Research Fund of the National University of Singapore, project number RP3981645.
Hardy was unaware of the product expansion (1) that Ramanujan had used to compute identity (3), which is none other than the evaluation of $R$ at $i$. In the same correspondence, Ramanujan expressed the equality

$$-R\left(\frac{5+i}{2}\right) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2}$$

with a similar dramatic flair. The radical symbol in (3) and (4) should be interpreted as the real positive root on $\mathbb{R}$. Ramanujan communicated radical expressions for $R(\sqrt{-5})$ and $-R\left(\frac{5+\sqrt{-5}}{2}\right)$ in his second letter to Hardy, and several other values of $R$ at imaginary quadratic arguments are recorded in his notebooks. The other name connected to the function $R$ is that of L.J. Rogers, who proved the equality of (1) and (2) in 1894. This was discovered by Ramanujan after his arrival in England.

In this paper, we evaluate singular values of the Rogers-Ramanujan continued fraction. These are the function values of $R$ taken at imaginary quadratic $\tau \in \mathbb{H}$. As $R$ is a modular function of level 5—a classical fact for which we furnish a proof—these values generate abelian extensions of $\mathbb{Q}(\tau)$. Exploiting the Galois action given by the Shimura reciprocity law, we give a method for constructing a nested radical for $R(\tau)$ that works whenever $\tau$ is imaginary quadratic. Our systematic approach extends the results of [1], [10] and [6], which only apply to individual examples.

By way of example, we provide nested radicals for $R(\sqrt{-n})$ for $n = 1, 2, \ldots, 16$ when $n \not\equiv 3 \pmod{4}$. Writing down nested radicals for $R(\tau)$ becomes increasingly unwieldy as the discriminant of $\tau$ grows, so in the case $n = 3 \pmod{4}$, where $\mathbb{Q}$ and $R(\frac{5+\sqrt{-3}}{2})$ generate a subfield of $\mathbb{Q}(R(\sqrt{-3}))$, we evaluate $R(\frac{5+\sqrt{-3}}{2})$ instead of $R(\sqrt{-3})$. In the classical literature, the notation $S(z) = -R(\frac{5+z}{2})$ is frequently used.

The authors thank Heng Huat Chan and Peter Stevenhagen for several helpful discussions.

### 2. The Modular Function Field of Level 5

A modular function of level $N$ is a meromorphic function on the extended complex upper half plane $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ that is invariant under the natural action of the modular group $\Gamma(N) = \ker[\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})]$ of level $N$. As such functions are invariant under $z \mapsto z + N$, they admit a Fourier expansion in the variable $q^{\frac{1}{N}} = e^{2\pi i z}$. The modular functions of level $N$ with Fourier expansion in $\mathbb{Q}(\zeta_N)(\langle q^{\frac{1}{N}} \rangle)$ form a field $F_N$, the function field of the modular curve $X(N)$ over $\mathbb{Q}(\zeta_N)$. 
The extension $F_N$ is Galois over $F_1$ with group $GL_2(Z/NZ)/\{\pm 1\}$. For a proof, see [5, p. 66, Thm 3]. One can describe the action of $GL_2(Z/NZ)$ on $F_N$ explicitly. The group $(Z/NZ)^*$ acts as a group of automorphisms of $F_N$ over $F_1$, by restricting its natural cyclotomic action on $Q(\zeta_N)((q^{1/6}))$. The natural action of $\Gamma(1) = SL_2(Z)$ on $H$ induces a right action of $\Gamma(1)/\Gamma(N) = SL_2(Z/NZ)$ on $F_N$ which leaves $F_1$ invariant. The homomorphisms

$$(Z/NZ)^* \to \text{Gal}(F_N/F_1) \quad \text{and} \quad SL_2(Z/NZ) \to \text{Gal}(F_N/F_1)$$

can be combined into an action of the semi-direct product

$$(Z/NZ)^* \ltimes SL_2(Z/NZ) \simeq GL_2(Z/NZ)$$
on $F_N$. For the isomorphism (5), we identify $d \in (Z/NZ)^*$ with the element $(1 \ 0 \ \ 0 \ d) \in GL_2(Z/NZ)$. The resulting sequence

$$(6) \quad 1 \to \{\pm 1\} \to GL_2(Z/NZ) \to \text{Gal}(F_N/F_1) \to 1$$
is exact.

The modular invariant $j$ generates $F_1$ over $Q$, and induces the isomorphism $X(1) \simeq \mathbb{P}^1(Q)$. In a similar fashion, the curve $X(5)$ has genus 0, thus its function field $F_5$ can be generated by a single function over $Q(\zeta_5)$. The Rogers-Ramanujan continued fraction $R$ is such a generator. There are several ways to prove this classical fact. Our proof is based upon Watson's formulas [13]

$$(7') \quad \frac{1}{R(z)} - R(z) - 1 = \frac{\eta(z/5)}{\eta(5z)},$$

$$(8') \quad \frac{1}{R^5(z)} - R^5(z) - 11 = \left(\frac{\eta(z)}{\eta(5z)}\right)^6,$$

which relate $R$ to Dedekind's $\eta$-function

$$\eta(z) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m), \quad q^{1/24} = e^{2\pi i z/24}.$$ 

Watson's formulas will prove useful in section 4, where we evaluate singular values of $R$. We define functions

$$h_0 = \frac{\eta \circ (\begin{smallmatrix} 1 & 0 \\ 0 & 5 \end{smallmatrix})}{\eta} \quad \text{and} \quad h_5 = \sqrt{5} \cdot \frac{\eta \circ (\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix})}{\eta},$$
so that equations (7') and (8') become

\begin{align}
(7) \quad \frac{1}{R} - R - 1 &= \sqrt{5} \cdot \frac{h_0}{h_5}, \\
(8) \quad \frac{1}{R_5} - R^5 - 11 &= \frac{5^3}{h_5^5}.
\end{align}

Before we show that $R$ is modular, we first prove that the functions appearing on the right hand side of (7) and (8) are modular of level 5. This is well known for $h_5^5$ [9, p. 619], but for lack of a reference in the case of (7), we provide a proof that works in both cases.

In order to compute the action of $SL_2(\mathbb{Z})$ on $h_0$ and $h_5$, we begin by observing that the generating matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $SL_2(\mathbb{Z})$ act on the Dedekind $\eta$-function as

\begin{equation}
\eta \circ S(z) = \sqrt{-iz} \eta(z) \quad \text{and} \quad \eta \circ T(z) = \zeta_{24} \eta(z).
\end{equation}

The radical sign in (9) stands for the holomorphic branch of the square root on $-i\mathbb{H}$ that is positive on the real axis. The observation $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \cdot S = S \cdot \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ gives $h_0 \circ S = h_5$ with the help of (9). Let $\Delta_5$ denote the set of $2 \times 2$ matrices with coefficients in $\mathbb{Z}$ that have determinant 5. The matrices

\[ M_i = \begin{pmatrix} 1 & i \\ 0 & 5 \end{pmatrix}, \quad i = 0, 1, \ldots, 4 \quad \text{and} \quad M_5 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \]

form a set of representatives for $\Gamma \backslash \Delta_5$. For any $A \in SL_2(\mathbb{Z})$ and $i = 0, 1, \ldots, 5$, we can find $B \in SL_2(\mathbb{Z})$ and $j \in \{0, 1, \ldots, 5\}$ such that $M_i \cdot A = B \cdot M_j$ holds. We put

\begin{equation}
(10) \quad h_5 = \sqrt{5} \cdot \frac{\eta \circ M_i}{\eta} \quad \text{and} \quad h_i = \frac{\eta \circ M_i}{\eta} \quad \text{for} \quad i = 0, 1, \ldots, 4.
\end{equation}

Using (9) one computes

\begin{equation}
(11) \quad \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix} \circ S = \begin{pmatrix} h_5 \\ \zeta_{24}^{-3} h_4 \\ h_3 \\ h_2 \\ \zeta_{24}^{-1} h_1 \\ h_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix} \circ T = \begin{pmatrix} \zeta_{24}^{-1} h_1 \\ \zeta_{24}^{-1} h_2 \\ \zeta_{24}^{-1} h_3 \\ \zeta_{24}^{-1} h_4 \\ h_0 \\ \zeta_{24} h_5 \end{pmatrix}.
\end{equation}
Lemma 1. The functions $h_i^6$ and $h_i/h_j$ with $i, j \in \{0, 1, \ldots, 5\}$, are modular of level 5.

Proof. From [7] we know that $\Gamma(5)$ is the normal closure of $(T^5)$ in $SL_2(\mathbb{Z})$. This means that $\Gamma(5)$ is generated by matrices of the form $AT^5A^{-1}$ with $A \in SL_2(\mathbb{Z})$.

From (9) we observe

$$h_i \circ T^5 = \zeta_5^{-1} \cdot h_i \quad \text{for } i = 0, 1, \ldots, 5.$$ 

For $A \in SL_2(\mathbb{Z})$ and $j \in \{0, 1, \ldots, 5\}$, the equations (11) show that all $h_j \circ A$ are of the form $h_j \circ A = \zeta \cdot h_i$ for some $i \in \{0, 1, \ldots, 5\}$ and some root of unity $\zeta$. Similarly

$$h_j \circ AT^5 = \zeta_5^{-1} \cdot h_j \circ A$$

holds for every $A \in SL_2(\mathbb{Z})$. Thus $h_i^6$ is invariant under $AT^5A^{-1}$ for all $A \in SL_2(\mathbb{Z})$, as well as every quotient $h_i/h_j$. \hfill \Box

Lemma 2. The Rogers-Ramanujan continued fraction $R$ is modular of level 5.

Proof. As we know from (1) that $R$ is holomorphic on $\mathbb{H}$, it suffices to show that $R \circ AT^5A^{-1} = R$ for all $A \in SL_2(\mathbb{Z})$. From Watson's formula (7) one derives

$$(X - R)(X + \frac{1}{R}) = X^2 + \left(\sqrt{5} \cdot \frac{h_0}{h_5} + 1\right)X - 1.$$ 

As $AT^5A^{-1}$ acts trivially on $\sqrt{5}h_0/h_5$, it maps $R$ to either $R$ or $-1/R$. Suppose the latter to be true. Then $R \circ AT^5 = -1/(R \circ A)$ holds. As the translation $T^5$ fixes the cusp $i\infty$, we have

$$R \circ A(i\infty) = R \circ AT^5(i\infty) = \frac{-1}{R \circ A(i\infty)},$$

which implies $R \circ A(i\infty) = \pm i$. Then Watson's formula (8) yields

$$\frac{5^3}{(h_5 \circ A(i\infty))^6} = \frac{1}{(\pm i)^6} - (\pm i)^6 - 11 = \pm 2i - 11.$$ 

On the other hand, we can evaluate $h_5 \circ A(i\infty)$ by considering the product expansion for $h_5 \circ A$ at $q = 0$. By (11), one has $h_5 \circ A = \zeta \cdot h_j$ for some root of unity $\zeta$ and some $j \in \{0, 1, \ldots, 5\}$. For $j = 0, 1, \ldots, 4$, we compute

$$h_j(i\infty) = \lim_{N \to \infty} \frac{e^{2\pi i (\frac{jN}{15} + \frac{1}{2})}}{e^{2\pi i (\frac{5N}{15})}} = 0.$$ 

A similar calculation shows that $h_5$ has a pole at $i\infty$. Contradiction with (13). \hfill \Box
Proposition 3. The minimum polynomial of $R^5$ over $F_1 = \mathbb{Q}(j)$ is

$$P(X) = X^{12} + 1 + (j - 684)(X^{11} - X) + (55j + 157434)(X^{10} + X^2)$$

$$+(1205j - 12527460)(X^9 - X^3) + (13090j + 77460495)(X^8 + X^4)$$


The minimum polynomial of $R$ over $\mathbb{Q}(j)$ is $P(X^5)$, with $P$ as above.

Proof. Weber shows [14, p. 256] that $h_6^2$ is a zero of $X^6 + 10X^3 - \gamma_2X + 5$, with $\gamma_2$ a cube root of $j$. Another zero is $h_6^2 = (h_0 \circ S)^2$ because $S$ fixes $\gamma_2$. We obtain

$$j = \left(\frac{h_6^{12} + 10h_6^4 + 5}{h_6^6}\right)^3.$$  

Rewriting (8) gives the identity

$$h_6^6 = \frac{5^3 \cdot R^5}{-R^{10} - 11R^5 + 1}.$$  

Substituting the above relation for $h_6^6$ into (15), we have

$$j = \frac{(1 + 228R^5 + 494R^{10} - 228R^{15} + R^{20})^3}{(-R + 11R^6 + R^{11})^5},$$

which readily yields $P(R^5) = 0$, with $P$ as in (14). To see that $P$ is irreducible in $\mathbb{Z}[X, j]$, compose the evaluation map $\mathbb{Z}[X, j] \rightarrow \mathbb{Z}[X]$ defined by $j \mapsto 1$ with reduction modulo 2. We obtain a homomorphism $\mathbb{Z}[X, j] \rightarrow \mathbb{F}_2[X]$ that sends $P$ to the cyclotomic polynomial $\Phi_{13} \in \mathbb{F}_2[X]$, which is irreducible because 2 is a primitive root modulo 13. As $P$ is a monic polynomial in $X$, we conclude that it is the minimum polynomial of $R^5$ over $\mathbb{Q}(j)$.

In order to see that $\mathbb{Q}(R)$ has degree 5: $[\mathbb{Q}(R^5) : \mathbb{Q}(j)] = 60$ over $\mathbb{Q}(j)$, it suffices to observe that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \text{SL}_2(\mathbb{Z})$, which acts as $R(z) \mapsto R(z + 1) = \zeta_5R(z)$ induces an automorphism of order five of $\mathbb{Q}(R)$ over $\mathbb{Q}(R^5)$. Thus $P(X^5)$ is the minimum polynomial of $R$ over $\mathbb{Q}(j)$.

Theorem 4. The Rogers-Ramanujan continued fraction $R$ generates $F_5$ over $\mathbb{Q}(\zeta_5)$.

Proof. As $R$ has rational Fourier coefficients, the subfields $\mathbb{Q}(R) = F_1(R)$ and $F_1(\zeta_5)$ of $F_5$ are linearly disjoint extensions of $F_1$ having degree 60 and 4, respectively. Their composite, which has degree 240 = $\#(\text{GL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\})$ over $F_1$ is therefore equal to $F_5$.  

□
The rational function on the right hand side of (16) appears in Klein’s study of the finite subgroups of \( \text{Aut}(\mathbb{P}^1(\mathbb{C})) \). His icosahedral group \( A_5 \) is isomorphic to \( \text{SL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\} \), and the natural map from \( \mathbb{P}^1(\mathbb{C}) \) to the orbit space of the icosahedral group ramifies above 3 points. The relation (16) defines a generator [4, p. 61, 65] for the field of functions invariant under the icosahedral group. In our situation the natural map \( X(5) \to X(1) \) ramifies over 3 points and the Galois group of \( \mathbb{C}(R) \) over \( \mathbb{C}(j) \) is \( \text{SL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\} \).

The subgroups of \( \text{GL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\} \) that stabilize the functions appearing in the equations (7) and (8) are given in the diagram below. The stabilizers of \( h_5^\phi \) and \( \sqrt{5} \cdot h_0/h_5 \) in \( \text{SL}_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\} \) can be determined using (11).

\[ (17) \]

\begin{verbatim}
F_5 = \mathbb{Q}(R, \zeta_5)

Q(R)

|\langle (\frac{1}{0}, \frac{1}{2}) \rangle |

Q(J)

\langle (\frac{1}{0}, \frac{1}{2}) \rangle

SL_2(\mathbb{Z}/5\mathbb{Z})/\{\pm 1\}

Q(h_5^\phi)

Q(\sqrt{5} \cdot h_0/h_5)

Q(\sqrt{5})

\langle (\frac{1}{0}, \frac{1}{2}) \rangle

\end{verbatim}

3. Galois Theory for Singular Values of Modular Functions

Let \( \mathcal{O} \) be an imaginary quadratic order having \( \mathbb{Z} \)-basis \([\tau, 1]\). Define \( H_N = H_{N,\mathcal{O}} \) to be the field generated over \( K = \mathbb{Q}(\tau) \) by the function values \( h(\tau) \), where \( h \) ranges over the modular functions in \( F_N \) that are pole-free at \( \tau \). The first main theorem of complex multiplication [5] states that \( H_N \) is an abelian extension of \( K \). For \( N = 1 \), the field \( H_1 \) is the ring class field for \( \mathcal{O} \). If \( \mathcal{O} \) is a maximal quadratic order with field of fractions \( K \), then \( H_N \) is the ray class field of conductor \( N \) over \( K \), and \( H_1 \) is the Hilbert class field of \( K \). For ray class fields of non-maximal orders, see for example [12].
Before we can describe the explicit action of \( \text{Gal} (H_N/K) \) on elements of \( H_N \), we first look at \( \text{Gal} (H_N/H_1) \), which fits in a short exact sequence

\[
1 \rightarrow \mathcal{O}^* \rightarrow (\mathcal{O}/\mathcal{NO})^* \xrightarrow{\Delta} \text{Gal} (H_N/H_1) \rightarrow 1.
\]

In order to describe the Artin map \( A \) in (18), we write the elements of \( \mathcal{O}/\mathcal{NO} \) as row vectors with respect to the \( \mathbb{Z}/N\mathbb{Z} \)-basis \([\tau, 1]\). If \( \tau \) has minimum polynomial \( X^2 + BX + C \in \mathbb{Z}[X] \), define the homomorphism

\[
g_\tau : (\mathcal{O}/\mathcal{NO})^* \rightarrow \text{GL}_2(\mathbb{Z}/N\mathbb{Z})
\]

\[
s\tau + t \mapsto (t - B \tau - C \tau).
\]

The matrix \( g_\tau(x) \) represents multiplication by \( x \) on \( \mathcal{O}/\mathcal{NO} \) with respect to the \( \mathbb{Z}/N\mathbb{Z} \)-basis \([\tau, 1]\). For \( h \in F_N \), the Shimura reciprocity law [11] gives the action of \( x \in (\mathcal{O}/\mathcal{NO})^* \) on \( h(\tau) \) as

\[
(h(\tau))^{-1} = h^{g_\tau(x)}(\tau).
\]

Here \( g_\tau(x) \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) acts on \( h \in F_N \) as described in (6). Moreover, if \( h \in F_N \) is a function for which \( \mathbb{Q}(h) \subset F_N \) is Galois, then \( K(h(\tau)) \) is the fixed field of

\[
\{ x \in (\mathcal{O}/\mathcal{NO})^* \mid h^{g_\tau(x)} = h \} \subset (\mathcal{O}/\mathcal{NO})^*.
\]

For any \( h \in F_N \), we aim to compute the conjugates of \( h(\tau) \) with respect to the full group \( \text{Gal} (H_N/K) \). In the case \( N = 1 \), the Galois group of \( H_1 = K(j(\tau)) \) over \( K \) is isomorphic to the ideal class group \( \text{Cl} (\mathcal{O}) \) of \( \mathcal{O} \). The elements of \( \text{Cl} (\mathcal{O}) \) can be represented as primitive quadratic forms \([a, b, c]\) of discriminant \( D = b^2 - 4ac \), where \( D \) is the discriminant of \( \mathcal{O} \). The \( \mathbb{Z} \)-module having basis \([a, -b+\sqrt{D} \, / 2a]\) is an \( \mathcal{O} \)-ideal in the class of \([a, b, c]\), and the class of \([a, -b, c]\) acts on \( j(\tau) \) as

\[
j(\tau)^{[a, -b, c]} = j\left(\frac{-b + \sqrt{D}}{2a}\right).
\]

In the general case for \( N > 1 \), we need the elements of \( \text{Gal} (H_N/K) \) that lift (22) for each representative \([a, b, c]\) in \( \text{Cl} (\mathcal{O}) \). The formula [3, p. 32, Thm. 20] produces one element \( \sigma \in \text{Gal} (H_N/K) \) along with a matrix \( M_N = M_N(a, b, c) \) in \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) such that for all \( h \in F_N \), the relation

\[
h(\tau)^\sigma = h^{M_N}\left(\frac{-b + \sqrt{D}}{2a}\right).
\]
holds. The automorphism \( \sigma \) clearly lifts the action in (22) to \( \text{Gal}(H_N/K) \) because \( M_N \in \text{GL}_2(\mathbb{Z}/\mathbb{N}\mathbb{Z}) \) acts trivially on \( j \in F_1 \). As every automorphism in \( \text{Gal}(H_N/K) \) is obtained by composing elements of \( \text{Gal}(H_N/H_1) \) with one of the coset representatives for \( \text{Gal}(H_1/K) \) in (23), we can determine the conjugates of \( h(\tau) \) under \( \text{Gal}(H_N/K) \) for any \( h \in F_N \).

Given this explicit action on \( H_N \) over \( K \) we can compute representations for singular values of modular functions by minimal polynomials as well as radical expressions over \( \mathbb{Q} \).

The natural way to describe an algebraic number is its minimum polynomial over \( \mathbb{Q} \). Let \( h \in F_N \) be a function for which \( h(\tau) \) is an algebraic integer. The conjugates of \( h(\tau) \) over \( K \) can be approximated numerically when the Fourier expansion for \( h \) is known. One expresses each conjugate in the form \( h^M(\theta) \), with \( M \in \text{GL}_2(\mathbb{Z}/\mathbb{N}\mathbb{Z}) \) and \( \theta \in K \), and then writes \( M = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \cdot A \) with \( x = \text{det}(M) \) and \( A \in \text{SL}_2(\mathbb{Z}/\mathbb{N}\mathbb{Z}) \). After modifying the Fourier coefficients of \( h \) with respect to \( \zeta_N \mapsto \zeta_N^{\bar{x}} \), one evaluates the new expansion at \( \bar{A}(x) \), where \( \bar{A} \in \text{SL}_2(\mathbb{Z}) \) is a lift of \( A \). We calculate the minimum polynomial \( f \) of \( h(\tau) \) over \( \mathbb{Q} \) by approximating the conjugates of \( h(\tau) \) over \( K \). Adjoining complex conjugates gives a full set of conjugates over \( \mathbb{Q} \). In order to determine the polynomial \( f \in \mathbb{Z}[X] \), we need only to approximate its coefficients accurate to the nearest integer.

Because \( H_5 \) is abelian over \( K \), one can also express \( h(\tau) \) as a nested radical over \( \mathbb{Q} \) in the spirit of Ramanujan's evaluations (3) and (4). Unlike the minimum polynomial \( f \), which is unique, many different nested radicals over \( \mathbb{Q} \) exist that all represent \( h(\tau) \). Given any abelian extension \( H/K \) of degree greater than 1 and any \( w \in H \), the following standard procedure expresses \( w \) as a radical expression over a field \( H' \) with the property \([H':K] < [H : K]\). Applying the procedure recursively produces a nested radical for \( w \) over \( K \).

We choose an automorphism \( \sigma \in \text{Gal}(H/K) \) of order \( m > 1 \) and set \( H' = H^\sigma(\zeta_m) \), where \( H^\sigma \) denotes the fixed field of \( \langle \sigma \rangle \). Then \( H'/K \) is an abelian extension of degree

\[ [H' : K] \leq \varphi(m) \cdot [H^\sigma : K] < m \cdot [H^\sigma : K] = [H : H^\sigma][H^\sigma : K] = [H : K]. \]

We write

\[ w = \frac{1}{m} (h_0 + h_1 + h_2 + \cdots + h_{m-1}) , \]

where

\[ h_i = \sum_{k=1}^{m} \zeta_m^{ik} \cdot w(\zeta_m^k) , \quad i = 0, 1, \ldots, m - 1. \]
are the Lagrange resolvents for \( w \) with respect to \( \sigma \). Note that \( h_0 = \text{Tr}_{H/H_\sigma}(w) \) is an element of \( H' \). Every \( \rho \in \text{Gal}(H(\zeta_m)/H') \) acts trivially on \( \zeta_m \) and as some \( \sigma^a \in \langle \sigma \rangle \) on \( H \). For \( i = 1, 2, \ldots, m - 1 \), we have

\[
  h_i^\rho = \sum_{k=1}^n \zeta_m^{i k} \cdot w^{(\sigma^k + \sigma^a)} = \zeta_m^{-ia} \cdot h_i,
\]

which means \( h_1^m, h_2^m, \ldots, h_{m-1}^m \in H' \). As \( h_i = \sqrt[n]{h_i^m} \) for the appropriate choice of the \( m \)-th root, equation (24) represents \( w \) as a radical expression over \( H' \). The recursion step is applied to \( h_0, h_1^m, h_2^m, \ldots, h_{m-1}^m \in H' \).

Suppose \( h \in F_N \) such that \( h(\tau) \) is an algebraic integer. In order to apply the recursive procedure above to \( h(\tau) \), one needs not only the action of \( \text{Gal}(H_N/K) \), but also that of \( \text{Gal}(H_N(\zeta_d)/K) \) for various numbers \( d > 1 \). This is obtained by restricting the action of \( \text{Gal}(H_{dn}/K) \) to \( H_N(\zeta_d) \). The elements computed in the final recursion step are in \( \mathcal{O}_K \), which is a discrete subgroup of \( \mathbb{C} \). An approximation of their coordinates with respect to a \( \mathbb{Z} \)-basis for \( \mathcal{O}_K \), that is accurate to the nearest integer, produces a nested radical for \( h(\tau) \) over \( \mathbb{Q} \).

The methods above can be extended to arbitrary imaginary quadratic numbers \( \theta \in H \) that are not necessarily algebraic integers. In order to compute the conjugates of \( h(\theta) \) over \( K = \mathbb{Q}(\theta) \) we take an integral basis \( \{\tau, 1\} \) for \( K \) and write \( \theta = a \tau + \frac{b}{d} \) with \( a, b, d \in \mathbb{Z} \). One evaluates \( h \circ \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \) at \( \tau \), which is contained in \( H_{dn} \). Again, (20) and (23) allow us to calculate the conjugates of \( h(\theta) \) over \( K \).

### 4. The Ray Class Field \( H_5 \)

We turn our attention back to the functions of level 5 from section 2. In this section, we compute some singular values \( R(\tau) \) of the Rogers-Ramanujan continued fraction. As the singular values of \( j \) are known to be algebraic integers, the same holds true for \( R \) because the polynomial (14) has coefficients in \( \mathbb{Z}[j] \). We fix \( \mathcal{O} = \{\tau, 1\} \) to be an order in some imaginary quadratic number field \( K \). We state a few properties of \( R(\tau) \) before computing some examples.

**Theorem 5.** The class field \( H_5 = H_5,\mathcal{O} \) is generated by \( R(\tau) \) over \( K \).

**Proof.** As we have \( F_5 = \mathbb{Q}(R, \zeta_5) \) by Theorem 4, the extension \( F_5/\mathbb{Q}(R) \) is Galois and we are in the situation for which (21) applies. As \( \mathbb{Q}(R) \) is the subfield of \( F_5 \) fixed by

\[
\left\{ \left( \begin{smallmatrix} 0 & 1 \\ d & 0 \end{smallmatrix} \right) \mid d \in (\mathbb{Z}/5\mathbb{Z})^* \right\} \subset \text{GL}_2(\mathbb{Z}/5\mathbb{Z}) ,
\]

the class field \( K(R(\tau)) \) is the subfield of \( H_5 \) fixed by

\[
G = \{ x \in (\mathcal{O}/5\mathcal{O})^* \mid g_r(x) = \pm \left( \begin{smallmatrix} 0 & 1 \\ d & 0 \end{smallmatrix} \right) \text{ for some } d \in (\mathbb{Z}/5\mathbb{Z})^* \} \subset (\mathcal{O}/5\mathcal{O})^* .
\]
From formula (19) we see that the only diagonal matrices appearing in the image $g_r([O/5O]^*)$ are scalar. We conclude $G = \{\pm 1\}$ and $K(R(\tau)) = H_5$. □

Let $w(z) = \eta(\frac{z}{5})/\eta(5z)$ denote the function that appears on the right hand side of equation (7'). Thus we have

$$\frac{1}{R(z)} - R(z) - 1 = w(z).$$

Proposition 6. The values $R(\tau)$ and $-1/R(\tau)$ are conjugate over $K(w(\tau))$. Furthermore, $H_5$ is generated over $K$ by $\zeta_5$ together with $w(\tau)$.

Proof. The polynomial

$$X^2 + (w(\tau) + 1)X - 1 \in K(w(\tau))[X]$$

derived from (25) has zeroes $R(\tau)$ and $-1/R(\tau)$. To show that (26) is irreducible in $K(w(\tau))[X]$ we consider the homomorphism $g_r : ([O/5O]^* \rightarrow GL_2(Z/5Z)$ in (19). By (20), the group $\text{Gal}(H_5/H_1)$ contains the automorphism $R(\tau) \mapsto R^{g_r(2)}(\tau)$. In order to determine the action action of

$$g_r(2) = \left( \begin{array} {cc} 2 & 0 \\ 0 & 3 \end{array} \right) = \left( \begin{array} {cc} 1 & 0 \\ 0 & 4 \end{array} \right) \left( \begin{array} {cc} 2 & 0 \\ 0 & 3 \end{array} \right) \in GL_2(Z/5Z)$$

on $F_5$, we recall that $R$ and $w$ have rational Fourier coefficients and thus are fixed by $\left( \begin{array} {cc} 1 & 0 \\ 0 & 4 \end{array} \right)$. Using (11) one checks that $w$ is stabilized by $\left( \begin{array} {cc} 2 & 0 \\ 0 & 3 \end{array} \right) \in SL_2(Z/5Z)$. Theorem 2.4 together with equation (25) tells us that this matrix $\left( \begin{array} {cc} 2 & 0 \\ 0 & 3 \end{array} \right)$ sends $R$ to $-1/R$, so $R(\tau)$ and $-1/R(\tau)$ are conjugates over $K$. As $K(\tau) = H_5$ contains $\zeta_5$, the situation $R(\tau) = -1/R(\tau) = \pm i$ is impossible. We conclude that (26) is irreducible in $K(w(\tau))[X]$.

We have $[H_5 : K(w(\tau))] = 2$. In fact, $K(w(\tau))$ is the subfield of $H_5$ fixed by the subgroup of $(O/5O)^*$ generated by $2$ and the image of $O^*$. By (26) the action of $2 \in (O/5O)^*$ on $\zeta_5 \in H_5$ is $\zeta_5 \mapsto \zeta_5^{-1}$. We conclude $\zeta_5 \not\in K(w(\tau))$ and $H_5 = K(w(\tau), \zeta_5)$. □

To determine the minimum polynomial of $R(\tau)$ over $\mathbb{Q}$, it is convenient to first compute the polynomial for $w(\tau)$ and then recover $R(\tau)$ with (25). As both values $R(\tau)$ and $1/R(\tau)$ are algebraic integers, it follows that $w(\tau)$ is an algebraic integer too. In particular, the method of section 4 for computing $\int_Q^{w(\tau)}$ works here.

Working with values of $w$ is easier than working with $R$ directly as there are only half as many conjugates over $K$ to compute. More importantly, the
Dedekind $\eta$-function is implemented in several software packages that quickly compute $\eta(z)$ to a high degree of accuracy. These routines make use of $\text{SL}_2(\mathbb{Z})$-transformations to ensure that the imaginary part of $z$ is sufficiently large to guarantee rapid convergence of the Fourier expansion of $\eta(z)$. One obtains the minimal polynomial of $R(\tau)$ over $\mathbb{Q}$ from $f^w_\mathbb{Q}(\tau)$ by writing

$$w = \frac{1 - R - R^2}{R}$$

using (7'). Then $R(\tau)$ is a zero of the monic polynomial

$$x^{\deg.} f^w_\mathbb{Q}(\tau) \left( \frac{1 - X - X^2}{X} \right) \in \mathbb{Q}[X]$$

with $\deg = \deg(f^w_\mathbb{Q}(\tau))$. According to proposition 6, the resulting polynomial is irreducible because its degree is $2 \cdot \deg$.

An inspection of product expansion (1) shows $R(z) \in \mathbb{R}$ whenever the real part of $z \in \mathbb{H}$ is an integer multiple of $\frac{\pi}{2}$. For the singular arguments

$$\tau_n = \begin{cases} \sqrt{-n} & \text{if } n \not\equiv 3 \mod 4 \\ \frac{5 + \sqrt{-n}}{2} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

the value $R(\tau_n)$ is a real, and its minimum polynomial over $K$ is contained in $\mathbb{Z}[X]$ because complex conjugation acts as

$$f^R_{K(\tau_n)} = f^R_{K(\tau_n)} = f^R_{K(\tau_n)}.$$ 

Proposition 6 implies that $f^R_\mathbb{Q}(\tau) = \sum_{i=0}^{2d} c_i X^i$ is the minimum polynomial for both $R(\tau)$ and $-1/R(\tau)$, thus the coefficients satisfy $c_i = (-1)^i c_{2d-i}$. For this reason we only list the first half of the coefficients $c_{2d}, c_{2d-1}, \ldots, c_d$ in table 1, where we give the minimum polynomials for $R(\tau_n)$ with $1 \leq n \leq 16$.

A nice way of generating $H_5 = K(w(\tau_n), \zeta_5)$ comes from proposition 6. The subfield $K(w(\tau_n))$ of $H_5$ is the fixed field for the subgroup generated by 2 and $\mathcal{O}^*$ in $(\mathcal{O}/5\mathcal{O})^*$. Because $\sqrt{5}$ is invariant under $g_{\tau_n}(2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we conclude $\sqrt{5} \in K(w(\tau_n))$. Thus for the function

$$\tilde{w} = \frac{w}{\sqrt{5}} = \frac{h_0}{h_5}$$

we have $\tilde{w}(\tau_n) \subset K(w(\tau_n))$ and $H_5 = K(\tilde{w}(\tau_n), \zeta_5)$. 
SINGULAR VALUES OF THE ROGERS-RAMANUJA CONTINUED FRACTION 65

Table 1. The minimum polynomials of $R(\tau_n)$ over $\mathbb{Q}$

<table>
<thead>
<tr>
<th>n</th>
<th>first half of coefficients $c_{2d}, c_{2d-1} \ldots c_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4, 1, 2, -6</td>
</tr>
<tr>
<td>2</td>
<td>12, 1, 6, -1, 0, 50, -14, 16</td>
</tr>
<tr>
<td>3</td>
<td>4, 1, -3, -1</td>
</tr>
<tr>
<td>4</td>
<td>8, 1, 14, 22, 22, 30</td>
</tr>
<tr>
<td>5</td>
<td>20, 1, 10, -90, 280, -730, 1022, -2410, 2540, -3330, 1730, -2006</td>
</tr>
<tr>
<td>6</td>
<td>16, 1, 28, 140, 60, -365, 264, 482, 340, 2035</td>
</tr>
<tr>
<td>7</td>
<td>12, 1, -4, -1, -25, -25, -14, 31</td>
</tr>
<tr>
<td>8</td>
<td>24, 1, 32, -96, 268, 51, -328, -1446, -5112, 996, 3972, 10594, 4208, -6924</td>
</tr>
<tr>
<td>9</td>
<td>16, 1, 38, -240, -300, -235, -726, 92, -1840, -675</td>
</tr>
<tr>
<td>10</td>
<td>20, 1, 60, 360, -120, 120, -1728, 3540, 840, 4320, -7620, -1006</td>
</tr>
<tr>
<td>11</td>
<td>8, 1, -6, -13, -28, 5</td>
</tr>
<tr>
<td>12</td>
<td>24, 1, 82, 329, -282, -74, 3672, -3846, 4238, 13521, -9028, 7844, 2408, 43651</td>
</tr>
<tr>
<td>13</td>
<td>24, 1, 82, -996, 968, 1051, 1422, -96, -24912, 7896, 16722, 28844, 13658, -114024</td>
</tr>
<tr>
<td>14</td>
<td>32, 1, 116, 614, -3040, 25230, 17988, -103372, 184292, 207725, -409400, -323390, -129140, 2879690, 3515800, -5057000, -4838560, 7624315</td>
</tr>
<tr>
<td>15</td>
<td>20, 1, -15, 60, -270, 720, -1353, 2115, -2610, 2970, -1095, 3119</td>
</tr>
<tr>
<td>16</td>
<td>16, 1, 148, -670, 240, 1570, -2616, 302, 1180, -1610</td>
</tr>
</tbody>
</table>

Proposition 7. The value $\tilde{w}(\tau_n)$ is an algebraic integer. If $5 \nmid n$ then $\tilde{w}(\tau_n)$ is a unit in $H_5$, the ray class field of conductor 5 over $\mathcal{O} = [\tau_n, 1]$.

Proof. Hasse and Deuring [2, p. 43] determine exactly the ideals generated by singular values of the lattice functions

$$\varphi_M(z) = \frac{\Delta(M(z))}{\Delta(z)} ,$$

with $M$ a $2 \times 2$ matrix having coefficients in $\mathbb{Z}$. Our functions $h_0$ and $h_5$ were defined in (10) as

$$h_0 = \frac{\eta \circ M_0}{\eta} , \quad h_5 = \sqrt{5} \cdot \frac{\eta \circ M_5}{\eta} \quad \text{with} \quad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} , \quad M_5 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} .$$

Thus we have

$$\varphi_{M_0}\left(\begin{array}{c} z \\ 1 \end{array}\right) = h_0(z)^{24} \quad \text{and} \quad \varphi_{M_5}\left(\begin{array}{c} z \\ 1 \end{array}\right) = h_5(z)^{24} ,$$

and

$$\tilde{w}(\tau_n)^{24} = \frac{\varphi_{M_0}\left(\begin{array}{c} \tau_n \\ 1 \end{array}\right)}{\varphi_{M_5}\left(\begin{array}{c} \tau_n \\ 1 \end{array}\right)} .$$
If \( n \) is not divisible by 5, then

\[
M_0\left(\frac{\tau_n}{1}\right) = [\tau_n, 5] \quad \text{and} \quad M_5\left(\frac{\tau_n}{1}\right) = [5\tau_n, 1]
\]

are both proper ideals of \( \mathcal{O} = [\tau_n, 1] \). Deuring's theorem [2, p. 42] shows that \( \varphi_{M_0}\left(\frac{\tau_n}{1}\right) \) and \( \varphi_{M_5}\left(\frac{\tau_n}{1}\right) \) are associate elements in the ring of integral algebraic numbers; one writes

\[
\varphi_{M_0}\left(\frac{\tau_n}{1}\right) \approx \varphi_{M_5}\left(\frac{\tau_n}{1}\right).
\]

It follows that the quotient \( \bar{w}(\tau_n)^{24} \) is a unit.

If \( 5 \mid n \) but \( 25 \nmid n \), then \( \varphi_{M_0}\left(\frac{\tau_n}{1}\right) \) is again a proper \( \mathcal{O} \)-ideal. However, the multiplier ring for \( M_5\left(\frac{\tau_n}{1}\right) \) is not \( \mathcal{O} \), but \( [5\tau_n, 1] \). Deuring's formulas [2, p. 43] yield

\[
\varphi_{M_0}\left(\frac{\tau_n}{1}\right) \approx 5^6 \text{ in } \mathcal{O} \text{ and } \varphi_{M_5}\left(\frac{\tau_n}{1}\right) \approx 5^{6/5} \text{ in } [5\tau_n, 1].
\]

We find \( \bar{w}(\tau_n) \approx 5^{1/5} \).

When \( n \) is divisible by 25, the multiplier rings for \( M_0\left(\frac{\tau_n}{1}\right) \) and \( M_5\left(\frac{\tau_n}{1}\right) \) are \( \mathcal{O} \) and \( [1, \tau_n/5] \) respectively. In this case, the formulas [2, p. 43] show that \( \bar{w}(\tau_n) \) is again associated to a positive rational power of 5.

When \( n \in \mathbb{Z} \) is not divisible by 5, the Galois action (6) for the matrix \( g_{\tau_n}\left(\tau_n\right) \) of (19) sends \( \bar{w} \) to \( (\frac{n}{5}) \cdot \bar{w}^{-1} \). We define

\[
(28) \quad v(\tau_n) = \bar{w}(\tau_n) + \left(\frac{n}{5}\right) \bar{w}(\tau_n)^{-1}.
\]

Clearly we have \( \bar{w}(\tau_n) = v(\tau_n) \) when \( n \) is divisible by 5. However, if \( n > 1 \) with \( 5 \nmid n \) then \( \bar{w}(\tau_n) \) has degree 2 over \( v(\tau_n) \). In these cases the minimum polynomial for \( \bar{w}(\tau_n) \) satisfies

\[
(29) \quad f_{Q}(\bar{w}(\tau_n)) = X^{\deg} \cdot f_{Q}(v(\tau_n)) \left(\frac{X^2 + (n/5)}{X}\right)
\]

with \( \deg = \deg(f_{Q}(v(\tau_n))) \). Table 2 lists the minimum polynomials over \( \mathbb{Q} \) for \( v(\tau_n) \) for \( 1 \leq n \leq 16 \).

5. Nested Radicals

In order to obtain nested radicals for \( R(\tau_n) \) over \( \mathbb{Q} \) it is suffices to have a radical for \( \bar{w}(\tau_n) \). On the imaginary axis, \( R(\tau) \) and \( w(\tau) \) and \( \bar{w}(\tau) \) take positive real values, and when \( \text{Re}(\tau) = \frac{\gamma}{2} \), each of the values \( R(\tau) \) and \( w(\tau) \) and \( \bar{w}(\tau) \) are real.
Table 2. The minimum polynomials over \( \mathbb{Q} \) for \( v(\tau_n) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>degree</th>
<th>coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1, 2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1, -2, 3, -4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1, 1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1, -6, 4</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>1, -10, 25, -30, 25, -10, 25, 0, 25, 0, 25</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>1, -8, -16, 28, 31</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1, 2, 3, 9</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>1, -16, 20, -100, 25, -156, -124</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>1, -22, 54, 62, -59</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>1, -20, -75, -60, -75, -20, -25, 0, -25, 0, -25</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>1, 4, -1</td>
</tr>
<tr>
<td>12</td>
<td>6</td>
<td>1, -34, -5, -150, -75, -144, -99</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>1, -46, 210, -290, 905, -456, 576</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>1, -44, -238, 88, 520, -2508, -4978, -176, 2711</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>1, 5, 0, 15, 0, 5, -25, 0, -25, 0, -25</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>1, -68, 14, 328, -284</td>
</tr>
</tbody>
</table>

negative numbers. As the conjugate of \( R(\tau) \) over \( K(w) \) is \(-1/R(\tau)\), equation (12) gives

\[
R(\tau) = \begin{cases} 
-\frac{1+w(\tau)}{2} + \sqrt{\left(\frac{1+w(\tau)}{2}\right)^2 + 1} & \text{if } n \not\equiv 3 \pmod{4} \\
-\frac{1+w(\tau)}{2} - \sqrt{\left(\frac{1+w(\tau)}{2}\right)^2 + 1} & \text{if } n \equiv 3 \pmod{4} 
\end{cases}
\]

where \( \sqrt{\cdot} \) is always the positive square root of a positive real number.

When \( n > 1 \) and \( 5 \nmid n \), the algebraic number \( \tilde{w}(\tau_n) \) has degree 2 over \( \mathbb{Q}(v(\tau_n)) \). As the absolute value of \( \tilde{w}(\tau_n) \) satisfies \(|\tilde{w}(\tau_n)| > 2\) when \( n > 1 \), one recovers \( \tilde{w}(\tau_n) \) from \( v(\tau_n) \) as

\[
2\tilde{w}(\tau_n) = \begin{cases} 
v(\tau_n) + \sqrt{v(\tau_n)^2 - 4\left(\frac{n}{5}\right)} & \text{if } n \not\equiv 3 \pmod{4} \\
v(\tau_n) - \sqrt{v(\tau_n)^2 - 4\left(\frac{n}{5}\right)} & \text{if } n \equiv 3 \pmod{4} 
\end{cases}
\]

Note that the radicands in (30) are positive real numbers. This is obvious for \( \left(\frac{5}{n}\right) = -1 \). When \( \left(\frac{5}{n}\right) = 1 \), we have \(|v(\tau_n)| = |\tilde{w}(\tau_n) + 1/\tilde{w}(\tau_n)| > 2\). One easily recovers \( R(\tau_n) \) from \( \sqrt{5} \cdot \tilde{w}(\tau_n) = w(\tau_n) \). In the case \( n \) is divisible by 5, one simply has \( v(\tau_n) = \tilde{w}(\tau_n) \).

Below, we give nested radicals for \( v(\tau_n) \) with \( 1 \leq n \leq 16 \). In many cases the radicals below have undergone some cosmetic modifications made by factorizing elements in real quadratic orders of class number one. Every root appearing in
(30) and in our examples should be interpreted as the real positive root of its real positive argument. Our computation \( v(\tau_1) = 2 \) for example, leads to \( \tilde{w}(\sqrt{-1}) = 1 \) and \( w(\sqrt{-1}) = \sqrt{-5} \), which gives Ramanujan’s formula (3). The value \( v(\tau_3) \) also gives a trivial extension of \( \mathbb{Q} \).

\[
v(\tau_1) = 2 \quad v(\tau_2) = -1
\]

For \( n = 4, 6, 9, 11, 14, 16 \) the degree \( [\mathbb{Q}(v(\tau_n)) : \mathbb{Q}] \) is a power of 2. In these cases we opt for a tower of quadratic extensions in solving for \( v(\tau_n) \).

\[
v(\tau_4) = 3 + \sqrt{5} \\
v(\tau_6) = \frac{1}{2}(4 + \sqrt{10} + \sqrt{20} + \sqrt{50}) \\
v(\tau_9) = \frac{1}{2}(11 + 5\sqrt{2} + 3\sqrt{5} + 3\sqrt{15}) \\
v(\tau_11) = -2 - \sqrt{5} \\
v(\tau_{14}) = \left( \frac{1+\sqrt{2}}{2} \right)^2 \left( 6 + \sqrt{2} + 5\sqrt{-2} + 4\sqrt{2} + 2\sqrt{5\left(21 - 10\sqrt{2} + (15 - 2\sqrt{2})\sqrt{-11 + 8\sqrt{2}}\right)} \right) \\
v(\tau_{16}) = \frac{1}{2}(34 + 25\sqrt{2} + 11\sqrt{10} + 14\sqrt{5}).
\]

For \( n \equiv \pm 2 \mod 5 \), the group \( (\mathcal{O}/5\mathcal{O})^* \) is cyclic of order 24. If the discriminant \( D \) of \( \mathcal{O} = [\tau_n, 1] \) satisfies \( D < -4 \), then \( v(\tau_n) \) generates a degree 3 extension over the ring class field \( H_{\mathcal{O}} \). In the examples below, we choose the field tower \( H_{\mathcal{O}}(v(\tau_n)) \supset H_{\mathcal{O}} \supset \mathbb{Q}(\sqrt{-n}) \) to solve for \( v(\tau_n) \).

\[
v(\tau_2) = \frac{1}{3}(2 + \sqrt{35 + 15\sqrt{6}} - \sqrt{-35 + 15\sqrt{6}}) \\
v(\tau_7) = \frac{1}{3}(4 - \sqrt{20(-41 + 9\sqrt{21})} - \sqrt{20(41 + 9\sqrt{21})}) \\
v(\tau_9) = \frac{1}{3}\left(8 + 5\sqrt{2} + \sqrt{\frac{3}{2}(782 + 565\sqrt{2} + 3\sqrt{6(7771 + 5490\sqrt{2})}} \right) \\
+ \sqrt{\frac{3}{2}(782 + 565\sqrt{2} - 3\sqrt{6(7771 + 5490\sqrt{2})}} \right) \\
v(\tau_{12}) = \frac{1}{3}(17 + 10\sqrt{3} + 4\sqrt{260 + 150\sqrt{3}} + \sqrt{23975 + 13875\sqrt{3}}) \\
v(\tau_{13}) = \frac{1}{3}\left(23 + 5\sqrt{13} + \sqrt{\frac{3}{2}(14123 + 3905\sqrt{13} + 9\sqrt{274434 + 76110\sqrt{13}}}) \right) \\
+ \sqrt{\frac{3}{2}(14123 + 3905\sqrt{13} - 9\sqrt{274434 + 76110\sqrt{13}}}) \right)
\]

When \( n \) is divisible by 5, the value of \( v \) at \( \tau_n \) generates a field extension of degree 5 over the ring class field for \( \mathcal{O} = [\tau_n, 1] \). In applying the algorithm of section 3, our first step solves for \( v(\tau_n) \) over \( H_{\mathcal{O}} \).

\[
v(\tau_5) = 1 + \frac{1}{\sqrt{5}} (\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4}), \quad \text{where}
\]
SINGULAR VALUES OF THE ROGERS-RAMANUJAN CONTINUED FRACTION

\[ a_1, a_2 = 10 \left( 55 + 25\sqrt{5} \pm \sqrt{8050 + 2258\sqrt{5}} \right) \]
\[ a_3, a_4 = \frac{5}{2} \left( 55 + 25\sqrt{5} \pm \sqrt{50 + 22\sqrt{5}} \right) \]
\[ v(\tau_{10}) = \frac{1}{5} \left( 5 + 2\sqrt{5} + \sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4} \right), \text{ where} \]
\[ a_1, a_2 = 20 \left( 5(3 + \sqrt{5})(16 + \sqrt{5})(9 + 4\sqrt{5}) \pm 51 \left( \frac{1 + \sqrt{5}}{2} \right)^6 \sqrt{2(5 + 2\sqrt{5})} \right) \]
\[ a_3, a_4 = 5 \left( \frac{1 + \sqrt{5}}{2} \right)^{12} (22 - 3\sqrt{5}) \pm 3 \left( \frac{1 + \sqrt{5}}{2} \right)^6 \sqrt{2(5 + 2\sqrt{5})} \]
\[ v(\tau_{15}) = -\frac{1}{5} \left( \frac{1}{5} + 5\sqrt{5} + \sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4} \right), \text{ where} \]
\[ a_1, a_2 = \frac{125}{4} \left( 5(25 + 13\sqrt{5}) \pm 14\sqrt{\frac{15}{2}(25 + 11\sqrt{5})} \right) \]
\[ a_3, a_4 = \frac{125}{4} \left( 5(15 + 7\sqrt{5}) \pm 2\sqrt{\frac{15}{2}(25 + 11\sqrt{5})} \right) \]

REFERENCES

3. A.C.P. Gee, This thesis.
8. D.S. Kubert and S. Lang, Modular Units, Springer Grundlehen 244, 1981.

Submitted June 13, 1999