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Signatures of rare states and thermalization in a theory with confinement

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There is a dichotomy in the nonequilibrium dynamics of quantum many body systems. In the presence of integrability, expectation values of local operators equilibrate to values described by a generalized Gibbs ensemble, which retains extensive memory about the initial state of the system. On the other hand, in generic systems such expectation values relax to stationary values described by the thermal ensemble, fixed solely by the energy of the state. At the heart of understanding this dichotomy is the eigenstate thermalization hypothesis (ETH): individual eigenstates in nonintegrable systems are thermal, in the sense that expectation values agree with the thermal prediction at a temperature set by the energy of the eigenstate. In systems where ETH is violated, thermalization can be avoided. Thus establishing the range of validity of ETH is crucial in understanding whether a given quantum system thermalizes. Here we study a simple model with confinement, the quantum Ising chain with a longitudinal field, in which ETH is violated. Despite an absence of integrability, there exist rare (nonthermal) states that persist far into the spectrum. These arise as a direct consequence of confinement: pairs of particles are confined, forming new ‘meson’ excitations whose energy can be extensive in the system size. We show that such states are nonthermal in both the continuum and in the low-energy spectrum of the corresponding lattice model. We highlight that the presence of such states within the spectrum has important consequences, with certain quenches leading to an absence of thermalization and local observables evolving anomalously.

I. INTRODUCTION

Understanding the nonequilibrium dynamics of quantum many body systems has become one of the central goals of physics in recent years.\(^1\) This has been motivated by groundbreaking progress in experiments on cold atomic gases,\(^2\) which realized unprecedented control and isolation of quantum systems. Experiments have highlighted a lack of understanding of fundamental issues, such as thermalization: How and when does a quantum system thermalize? What does thermalization mean in an isolated quantum system undergoing unitary time evolution?

The eigenstate thermalization hypothesis (ETH)\(^3–8\) plays a central role in answering such questions. It states conditions that matrix elements of an operator \(\hat{A}\) must satisfy in order for its expectation value on an eigenstate to agree with the microcanonical (thermal) prediction. These conditions can be summarized as

\[
\hat{A}_{\alpha,\beta} = A(E)\delta_{\alpha,\beta} + e^{-S(E)/2}f_A(E,\omega)R_{\alpha,\beta}, \tag{1}
\]

where \(\hat{A}_{\alpha,\beta} = \langle E_\beta | \hat{A} | E_\alpha \rangle\) are matrix elements in the basis of eigenstates \(|E_\alpha\rangle\), \(E = (E_\alpha + E_\beta)/2\) and \(\omega = E_\alpha - E_\beta\). ETH tells us that the diagonal matrix elements \(\hat{A}_{\alpha,\alpha}\) are controlled by a smooth function \(A(E)\).\(^9\) The off-diagonal elements are suppressed by the thermodynamic entropy \(S(E)\) and depend on both a smooth function \(f_A(E,\omega)\) and the random variable \(R_{\alpha,\beta}\), which has zero mean and unit variance.

It is generally assumed that nonintegrable quantum many body systems satisfy ETH, in the sense that matrix elements of local observables obey Eq. (1) and hence expectation values are thermal (see, e.g., the brief argument of Ref. [10]). It is, however, known that finite size systems can have rare (nonthermal) eigenstates which violate ETH.\(^11–13\) Such states are often called nontypical, see Ref. [14] for a recent example.

The presence of rare states in the finite volume leads to two interpretations of ETH. The so-called “weak ETH” supposes that the fraction of rare to thermal states vanishes in the infinite volume limit. It has been shown that this is not sufficient to imply that the system thermalizes, and applies equally well to nonthermalizing integrable models.\(^11,15,16\) On the other hand, the “strong ETH” postulates that all rare states must vanish in the infinite volume. This much stronger condition then implies thermalization, with expectation values of local operators coinciding with the microcanonical ensemble (MCE) average.

In nonequilibrium scenarios, such as following a quantum quench, thermalization is signalled by the diagonal ensemble (DE) prediction agreeing with the MCE constructed at the appropriate energy density.\(^5\)

Distinguishing the weak and strong ETH scenar-
ios is a challenging problem. Evidence for the weak ETH has been seen in many nonintegrable systems, primarily through exact diagonalization of small systems.\textsuperscript{10,12,13,15,17–24} Generally the rare states are observed at the very edges of the spectrum, which is not entirely surprising with many cases of low energy emergent integrability being known (see, e.g., Ref. \cite{25}). Evidence consistent with strong ETH, on the other hand, is less well established. Numerical studies of lattice hard-core bosons with next-neighbor and next-next-neighbor interactions, as well as the quantum Ising chain with certain values of transverse and longitudinal fields, suggest that it is valid in some contexts.\textsuperscript{26}

One significant issue encountered when tackling such problems is the lack of available techniques. Almost all studies are confined to the exact diagonalization of small systems ($L \lesssim 20$ sites), which results in large finite size effects. This makes it hard to extrapolate results to the infinite volume and hence make concrete statements about strong or weak ETH. Some progress has been made in the last few years using tools from typicality,\textsuperscript{14,27,28} with which one can push to slightly larger system sizes ($L \lesssim 35$). Other techniques that can compute real-time dynamics \textsuperscript{15} such as time evolving block decimation (TEBD) and related algorithms,\textsuperscript{19,29–33} numerical renormalization group;\textsuperscript{34} equations of motion;\textsuperscript{35–42} and Boltzmann equations\textsuperscript{35,36,41–45} allow one to establish whether expectation values of operators approach their thermal values following a quench from a given state, but have little to say about ETH.

With this in mind, it is desirable to develop techniques that can study system sizes beyond those accessible to exact diagonalization, or in scenarios that exact (full) diagonalization cannot study. Here, to investigate thermalization and the validity of ETH in a large nonintegrable system—in both equilibrium and nonequilibrium scenarios—we focus on a continuum model and use recently developed extensions of the truncated spectrum method.\textsuperscript{33,46} We will see that these results are consistent with lattice simulations away from the scaling limit (in the low energy limit).

Before introducing the model that we study herein, it would be remiss of us not to mention another example in which ETH is violated: many body localization. Many body localization can arise in interacting disordered models, as shown in the seminal work of Basko, Aleiner and Altshuler\textsuperscript{47} (see also the recent review articles\textsuperscript{48–53}). Much like the scenarios discussed above, study of such models is almost entirely restricted to exact diagonalization, where violation of ETH has been established for numerous examples (see, e.g., Ref. \cite{53} and references therein). The violation of ETH reflects an emergent integrability in the localized phase, with models possessing an extensive number of local conserved quantities.\textsuperscript{48,54}

### A. The perturbed Ising field theory

We consider the perturbed Ising field theory, which arises as the continuum limit of the quantum Ising chain\textsuperscript{55,56} described by the Hamiltonian

\[
H(m,g) = \int dx \left[ i(\bar{\psi} \partial_x \psi - \psi \partial_x \bar{\psi} + m \psi \bar{\psi}) + g \sigma \right].
\]

Here $\psi (\bar{\psi})$ is a left (right) moving Majorana fermion, $m$ is the fermion mass, and $g$ is equivalent to a longitudinal field in the lattice model. In the ordered phase, the fermions can (loosely) be thought of as domain wall excitations in the ferromagnetic spin configuration. The longitudinal field $g$ acts as a \textit{nonlocal} confining potential for the domain walls.\textsuperscript{55} While the field theory is nonintegrable for generic values of the parameters $m$ and $g$, there exist two special lines in parameter space ($m = 0$ and $g = 0$) along which the model is integrable.\textsuperscript{57,58}

Herein we will mostly focus on the ordered phase of the model, corresponding to $m > 0$. Within this phase, the spectrum of the model depends intimately on the longitudinal field: when $g = 0$ a flipped spin fractionalizes into two independent domain walls, which can freely propagate though the system. Thus at low energies, when $g = 0$, there is a two particle continuum of excitations, separated from the ground state by an energy gap of $2m$. On the other hand, when $g \neq 0$ there are profound changes in the spectrum. The presence of the longitudinal field, which is nonlocal in terms of the domain wall fermions, induces a linear potential between domain wall excitations, leading to confinement.\textsuperscript{55,59,60} This is very much reminiscent of the formation of mesons in quantum chromodynamics (analogies between magnetic systems and quantum chromodynamics have recently been emphasized in Ref. \cite{61}). The low energy spectrum now completely restructures: the two domain wall continuum at $g = 0$ collapses into well-defined meson excitations for $g \neq 0$, with a new multimeson continuum forming above energies $E \sim 4m$. We sketch this schematically in Fig. 1. We note that this restructuring of the continuum has been observed in the quasi-one-dimensional Ising ferromagnet CoNb$_2$O$_6$\textsuperscript{62,63} and the XXZ antiferromagnet SrCo$_2$V$_2$O$_8$.\textsuperscript{64,65}

Working directly with a continuum theory might, at first glance, seem more difficult than working on the lattice. However, we will see that low energy eigenstates of the Hamiltonian (2) can be constructed, and are representative of the behavior in the infinite volume limit. We will do so using truncated spectrum methods,\textsuperscript{33} a well-established toolbox in the study of the Ising field theory (2). One question that it is natural to ask before starting is: What is the status of ETH in a field theory? Is it expect to hold? Here one can turn to the original work of Deutsch,\textsuperscript{3} which explicitly considered a continuum model and argued that it would be expected to thermalize. Srednicki’s later paper\textsuperscript{6} considered a continuum system with a bounded spectrum, i.e. the theory

\[
H(m,g) = \int dx \left[ i(\bar{\psi} \partial_x \psi - \psi \partial_x \bar{\psi} + m \psi \bar{\psi}) + g \sigma \right].
\]
is not absent a cutoff. We will see that expectation values of local operators in the vast majority of low energy eigenstates of (2) exhibit features consistent with ETH, while there are a subset of rare states that appear to violate it.

With regards to ETH in (2), there is one additional point worth noting. We can split the Hamiltonian (2) into two pieces: a noninteracting part \( H(m,0) \) and an interaction term \( g \int dx \sigma(x) \). The interaction term is strongly renormalization group relevant, with the scaling dimension of the operator \( \sigma \) being \( 1/8 \). Defining the theory with an explicit cutoff \( \Lambda \) and performing a weak-coupling \( (g \ll 1) \) renormalization group analysis, the interaction strength \( g(\Lambda) \) flows to zero in the ultraviolet \( \Lambda \to \infty \). As a result, the high energy eigenstates of the theory (2) should correspond to noninteracting \( (g=0) \) eigenstates. Concomitantly the high energy sector of the theory (2) has an emergent integrability and hence we should not expect ETH to be valid there. Hence we will concentrate our efforts on the low energy sector, where the interaction term has a strong effect, and ask whether ETH is valid in this subspace.

This work complements results presented by the authors in Ref. [66], in particular extending the one-dimensional aspect of that work.

### B. Layout

This work proceeds as follows. In Sec. II, we examine the matrix elements of the local spin operator between eigenstates constructed with truncated spectrum methods. We will see clear signatures of rare states that exist in the low energy spectrum of the field theory (2). We discuss ETH and behavior of these rare states with increasing system size. In Sec. III we establish the nature of the rare states, relating them to the two-fermion “meson” confined states of model. We study the lifetime of the meson excitations to understand their stability and influence on ETH in the infinite volume limit.

Following this, in Sec. IV we turn our attention to the nonequilibrium dynamics following a quench of the longitudinal field strength \( g \). The rare states established in the equilibrium spectrum will also be apparent in the DE, signalling a lack of thermalization after certain quenches. In Sec. V we related our work to the analogous lattice problem and discuss the presence of rare states there, comparing the finite size scaling analysis on the lattice with that in the continuum. We follow with our conclusions in Sec. VI and cover a number of technical points in the appendices.

### II. TESTING ETH IN A THEORY WITH CONFINEMENT

In order to test ETH within the field theory (2), we must be able to construct eigenstates. We do so using truncated spectrum methods, which we briefly summarize below. A detailed overview of the technique can be found in the recent review [33].

#### A. Truncated spectrum methods

1. **General philosophy**

Truncated spectrum methods (TSMs) are a general approach to treating certain classes of field theory problems. The best known is the truncated conformal space approach (TCSA) where the Hamiltonian can be written in the following form:

\[
H = H_{\text{CFT}} + V. \tag{3}
\]

Here the first part of the Hamiltonian, \( H_{\text{CFT}} \), describes a conformal field theory (CFT), while \( V \) is a renormalization group relevant, but otherwise arbitrary, operator. The central idea of TSMs is to use the eigenstates of a ‘known’ theory (the CFT in Eq. (3)) as a computational basis for constructing the full Hamiltonian (3).

The presence of a relevant operator, \( V \), leads to a strong mixing of the low-energy computational basis states. This ultimately leads to a failure in any perturbative treatment – requiring instead a nonperturbative approach. The TSM is one such method: using the fact that \( V \) does not strongly couple basis states of largely differing energies, one motivates a truncation of the Hilbert space of basis states through the introduction of an energy cutoff \( E_\Lambda \). For low energy eigenstates of \( H \) this is a reasonable approximation. To obtain a finite spectrum, even with an energy cutoff \( E_\Lambda \), one must then place the system in a finite volume \( R \). Subsequently one diagonalizes the (finite) Hamiltonian to obtain approximate energies and eigenvectors of the full problem (3).

More generally, the TSM can be applied to problems of the form

\[
H = H_{\text{known}} + V, \tag{4}
\]

where \( H_{\text{known}} \) is a theory where one knows how to construct eigenstates. As in Eq. (3), this may be a CFT,

![Fig. 1. Schematic of the low energy spectrum of the Ising field theory (2). The longitudinal field \( g \) leads to a complete restructuring of the low energy continuum (shaded) and the appearance of confined ‘meson’ states (solid lines).](image-url)
but more generally one can consider integrable theories. Then, one needs to be able to compute matrix elements of $V$ in the basis of $H_{\text{known}}$, and one can follow the procedure above to obtain approximate eigenstates and eigenvalues. TSMs were first used by Yurov and Zamolodchikov to study the field theory (2) in what they called the truncated-free-fermionic-space approach. These methods can also be extended in order to mitigate truncation effects, for example using numerical renormalization group techniques, which can be a necessity in more complicated theories (we discuss convergence of the TSM for the problem at hand in Appendix A).

Herein we fix our attention on the zero momentum sector of the theory, which contains the system’s ground state. Restricting ourselves to a particular momentum sector incurs no loss of generality [as momentum is conserved by our theory Eq. (2)]. Eigenstates obtained with TSMs (discussed in further detail in the next section) will be used to compute observables and, following recent works, nonequilibrium dynamics.

### 2. As applied to the Ising field theory (2)

Let us briefly recap some details of the TSM for the Ising field theory; a recent summary of known results and this method can be found in Ref. [33]. We work in the finite volume $R$ and begin by separating the Hamiltonian (2) into a ‘known’ piece and a perturbation, cf. Eq. (4),

$$H(m, g) = H_0(m) + gV,$$

$$H_0(m) = \int_0^R dx \left( \bar{\psi} \partial_x \psi - \bar{\psi} \psi + m \bar{\psi} \psi \right),$$  \hspace{1cm} (5)

$$V = \int_0^R dx \sigma(x).$$

Here our known piece, $H_0(m)$, describes a system of free fermions with mass $m$.

Let us now consider the eigenstates of the known theory. The Hilbert space of the model is split into two sectors, known as Neveu-Schwartz (NS, antiperiodic boundary conditions) and Ramond (RM, periodic boundary conditions), in which the momenta of fermions is quantized in a different manner: $p_{NS} = 2\pi n/R$ with $n \in \mathbb{Z} + 1/2$ and $p_{RM} = 2\pi n/R$ with $n \in \mathbb{Z}$, respectively. Eigenstates are obtained by acting on the vacuum (within a given sector) with fermion creation operators

$$\{|k\rangle_{NS} = a^\dagger_k \cdots a^\dagger_{k_N} |0\rangle_{NS}, \quad k_i \in \mathbb{Z} + \frac{1}{2} \rangle,$$

$$\{|q\rangle_{RM} = a^\dagger_q \cdots a^\dagger_{q_N} |0\rangle_{RM}, \quad q_i \in \mathbb{Z}. \rangle$$

Here $\{k\rangle = \{k_1, \ldots, k_N\}$ is a convenient shorthand notation, and henceforth we use $k_i$ ($q_i$) to signify momenta in the NS (RM) sector. The fermion creation operators obey the canonical anticommutation relations

$$\{a_k, a^\dagger_{k'}\} = \delta_{k,k'}, \quad \{a_q, a^\dagger_{q'}\} = \delta_{q,q'}, \quad \{a_k, a^\dagger_q\} = 0. \ (7)$$

Eigenstates of $H_0(m)$ with $N$ particles in the finite volume $R$ have energies

$$E_N(R) = E_0(R) + \sum_{j=1}^{N} \omega_{p_{\nu,j}}(R), \ (8)$$

where $\nu = NS, RM$ and $p_{\nu,j} = k_j (q_j)$ for $\nu = NS (RM)$. The dispersion relation for the fermions is given by

$$\omega_{p_{\nu}}(R) = \sqrt{m^2 + \left( \frac{2\pi}{R} p_{\nu} \right)^2}, \ (9)$$

while the vacuum energy is

$$E_0(R) = \frac{m^2 R}{8\pi} \log m^2 - |m| \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \log \left( 1 \pm e^{-|m|R \cosh \theta} \right), \ (10)$$

with the upper (lower) sign applying for $\nu = NS (RM)$.

With these expressions for the eigenstates and eigenvalues of $H_0(m)$ established, we turn our attention to the full problem (2). For nonzero longitudinal field strength $g$, the spin operator $\sigma(x)$ is present in the Hamiltonian and has to obey periodic boundary conditions $\sigma(x) = \sigma(x + R)$. This imposes a restriction on the fermion states, which varies depending on the sign of $m$ ($m > 0$ corresponds to the ordered phase)

$$m > 0 : \begin{cases} \{|k\rangle_{NS}, \text{with } N \in 2\mathbb{Z}, \} \\ \{|q\rangle_{RM}, \text{with } N \in 2\mathbb{Z}, \} \end{cases}$$

$$m < 0 : \begin{cases} \{|k\rangle_{NS}, \text{with } N \in 2\mathbb{Z}, \} \\ \{|q\rangle_{RM}, \text{with } N \in 2\mathbb{Z} + 1, \} \end{cases} \ (11)$$

The spin operator $\sigma(x)$ has non-zero matrix elements between states in different sectors, thus coupling them. Detailed expressions for these matrix elements can be found in the literature.

With both a computational basis, Eqs. (11), and matrix elements at hand, one proceeds by introducing an energy cutoff $E_\Lambda$ for the basis states, forming the (dense) Hamiltonian matrix, and then diagonalizing it. We illustrate how the number of computational basis states, $N_{\text{states}}$, varies as a function of the energy cutoff $E_\Lambda$ in Table I. Subsequently, the eigenstates obtained from the truncated spectrum procedure can be used to compute observables, as we discuss in the following sections.

### B. Diagonal matrix elements

Let us begin by examining the behavior of diagonal matrix elements. As seen in Eq. (1), under ETH these matrix elements should be smooth as a function of the energy of the eigenstate $\lambda^\alpha$.

$$\tilde{A}_{\alpha, \alpha} = A(E_\alpha), \quad \left| A(E_\alpha + \delta E_\alpha) - A(E_\alpha) \right| \propto e^{-R/R_0}, \ (12)$$
TABLE I. The number of computational basis states \( N_{\text{states}} \) in the zero momentum sector, as a function of energy cutoff \( E_\Lambda \) for a system of size \( R = 35 \).

<table>
<thead>
<tr>
<th>( E_\Lambda )</th>
<th>( N_{\text{states}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2305</td>
</tr>
<tr>
<td>9</td>
<td>6269</td>
</tr>
<tr>
<td>9.5</td>
<td>9809</td>
</tr>
<tr>
<td>10</td>
<td>15309</td>
</tr>
<tr>
<td>10.5</td>
<td>23498</td>
</tr>
</tbody>
</table>

where \( R \) is the system size and \( R_0 > 0 \) is a dimensionful constant. To examine whether Eqs. (12) hold in the perturbed Ising field theory, we compute the expectation value of the local spin operator \( \sigma(0) \) within eigenstates [so-called eigenstate expectation values (EEVs)] in a finite volume, \( R \). Results as a function of energy density \( E/R \) with \( m = 1, g = 0.1 \) for \( R = 25, 35, 45 \) are presented in Figs. 2. It is immediately apparent that in the finite volume EEVs are not a smooth function of the energy: in particular there is a clear band of states lying above the majority with significantly different EEVs.

The unusual nature of this band of states is further highlighted by a comparison to the MCE result for the expectation value, constructed by averaging over an energy window of \( \Delta E = 0.1 \). Error bars on the MCE reflect the standard deviation of the data averaged over. It is clear that numerous states fall outside the MCE prediction (plus one standard deviation), both close to the threshold of the multi-particle continuum \( (E/R \sim 0) \), as well as at energies far within the continuum. The band of states above the continuum, whose EEVs differ significantly from the MCE result, are rare states by definition: their EEVs are nonthermal.

We note that when reading figures such as Fig. 2, one should always regard EEVs at larger energy densities as having truncation effects. This can be seen, for example, in the upturn of the MCE (increase in the gradient of the slope as a function of \( E/R \)) at higher energy densities in Figs. 2(b) and (c) (this also occurs in Fig. 2(a) at higher energy densities than plotted), and the decrease in spread of the continuum at larger energy densities. Convergence checks as a function of energy cutoff are presented in Appendix A; it should be noted that the separation between the rare states and multiparticle continuum is well converged and not a truncation effect. We also note that the MCE converges rapidly in the energy cutoff. In the next section, we examine the issue of finite size scaling: does this difference between rare and thermal states persist to the infinite volume?

Before moving on to the finite size scaling, it is worth briefly commenting on the energy cutoff \( E_\Lambda \) as a function of system size \( R \). As we see in the definition of the single particle dispersion relation, Eq. (9), at fixed energy cutoff \( E_\Lambda \), as \( R \) is varied the maximum value of the integer \( p_\nu \) is increased. Thus to keep a fixed energy cutoff, one needs to consider polynomially more states as \( R \) is increased.

FIG. 2. Comparison of the eigenstate expectation values (EEV, black) of the local magnetization \( \sigma \) with the microcanonical ensemble prediction (MCE, orange) for eigenstates at energy density \( E/R \) in the Ising field theory (2) with \( m = 1, g = 0.1 \) on the ring of size (a) \( R = 25 \); (b) \( R = 35 \); (c) \( R = 45 \). Eigenstates are constructed via the TSM with an energy cutoff of (a) \( E_\Lambda = 13.5 \); (b) \( E_\Lambda = 10.5 \); (c) \( E_\Lambda = 9 \). States containing up to ten particles are considered in the truncated Hilbert space. The MCE is constructed by uniformly averaging over an energy window of size \( \Delta E = 0.1 \), with error bars denoting the standard deviation.
This quickly becomes computationally problematic, so generally $E_A$ has to be decreased as $R$ is increased, as has been done in Fig. 2.

1. Finite size scaling analysis

It is interesting to consider the behavior of the system at different system sizes. One of the major advantages of TSMs is that one can construct thousands of low-energy eigenstates to high precision in rather large systems (up to $R \sim 50|n|^{-1}$, where $|n|^{-1}$ is the correlation length of the unperturbed model). Examples for a number of system sizes are presented in Fig. 2. At each value of the system size, focussing on the the low-energy part of the spectrum (where convergence is best) we see that the same essential features appear: there is a continuum of states where (roughly) the MCE and EEV results agree, and above this there is a separate band of rare states, whose EEVs are distinctly nonthermal.

We see that the majority of EEVs are consistent with Eq. (1): with increasing volume $R$ the thermal continuum narrows (reflecting the extensivity of the thermodynamic entropy $S(E)$) and the MCE encompasses the majority of results. Similar sharpening of the distribution of EEVs, and improved agreement with the MCE, is observed in exact diagonalization of lattice models (see, e.g., Refs. 10, 12, 15, 18, 20, 22, 23, 26, and 27). We note that here the agreement is particularly clear compared to lattice calculations, as TSMs allow us to access many states in the low-energy spectrum at relatively large system size.

The band of rare states persists to the largest volumes we can access, and so it is natural to ask what becomes of these in the infinite volume limit, $R \to \infty$? As we cannot work directly in this limit (the spectrum of the theory becomes continuous and there are an infinite number of states below any non-zero energy cut-off), we infer results from a finite size scaling analysis.

Ordering the rare states by energy, we focus on the five states numbered $n = 11 \to 15$, which have energies well within the multiparticle continuum, cf. Fig. 2. For all system sizes that we access, these states are well converged for the maximum value of the cutoff $E_A$ achieved with reasonable computational resources. We compare the finite size scaling of these EEVs to the MCE constructed at the same average energy density, as shown in Fig. 3. In each case, the MCE prediction for the magnetization of the state is different to the EEV of the rare state, and the finite size scaling analysis suggests that this persists to the infinite volume limit, $1/R \to 0$. This is particularly clear if one computes the average magnetization of the five rare states and compares to the average from the MCE, as shown in Fig. 3(e).

The convergence of the EEV spectrum as a function of system size $R$, is presented in Sec. V A 1, where we compare to similar results on a corresponding lattice model. For the low-energy parts of the spectrum (including the rare states), the results are converged to those of the thermodynamic limit, $R \to \infty$.

C. Off-diagonal matrix elements

As we have already seen, ETH (1) also supposes that off-diagonal matrix elements of an operator have a certain structure,

$$\hat{A}_{\alpha, \beta} = e^{-S(E)/2} f_A(E, \omega) R_{\alpha, \beta}, \quad \alpha \neq \beta. \quad (13)$$

Here $S(E)$ is the thermodynamic entropy, $f_A(E, \omega)$ is a smooth function of both the average $E$ and difference $\omega$ of the energies of the eigenstates $E_\alpha, E_\beta$, $R_{\alpha, \beta}$ is a random variable with zero average and unit variance. The off-diagonal matrix elements are exponentially suppressed by $S(E)$, an extensive quantity. We have already seen signatures of this in the previous section: the variance of the diagonal matrix elements decreases with increasing system size. From Eq. (13), we would expect average off-diagonal matrix elements to be much smaller than the diagonal ones (as these are not suppressed by $S(E)$).
FIG. 4. Off-diagonal matrix elements of the local spin operator $|\langle E_\beta | \sigma(0) | E_\alpha \rangle|$ ($\alpha \neq \beta$) between the first 500 eigenstates constructed with the TSM for $m = 1$, $g = 0.1$ and $R = 35$, cf. Fig. 2(b). Parameters are $m = 1$, $g = 0.1$. Matrix elements satisfying $|\sigma_{\alpha\beta}| < 10^{-7}$ are plotted as white. Note the visible vertical/horizontal lines are not remnants of the plotting, but show eigenstates that are only coupled very weakly to other states by the spin operator $\sigma(0)$.

To examine this, we compute the off-diagonal matrix elements of $\sigma(0)$ in the basis of eigenstates:

$$\sigma_{\alpha\beta} = \langle E_\beta | \sigma(0) | E_\alpha \rangle, \quad \alpha \neq \beta. \quad (14)$$

For $\alpha, \beta = 1, \ldots, 500$ we present these in Fig. 4, where we consider the same set of parameters as in Fig. 2. From inspection of the numerical data, as well as Fig. 4, we see that off-diagonal matrix elements are generally much smaller [$O(10^{-7})$] and smaller than the diagonal elements (although we note that there are some off-diagonal elements that are comparable to the diagonal ones for $R = 35$). It is also apparent that there is significant structure present within the off-diagonal elements (cf. Eq. (13)), with clear lines of zeros in the $(\alpha, \beta)$ plane, as well as regions with larger matrix elements (on average). Many of the vertical/horizontal lines with small matrix elements coincide with the index of the rare states. Prominent examples include the $\alpha = 175$, $\alpha = 319$ and $\alpha = 457$ lines (by symmetry, the same lines along the $\beta$ axis), as well as the many lines of suppressed matrix elements at small $\alpha$ and $\beta$.

### III. THE NATURE OF THE RARE STATES

In the previous section, we have seen that rare states with nonthermal EEVs are present within the model (2). A natural question is then: do these states share common characteristics? Looking at Fig. 2 there is an obvious first guess as to their physical characteristics: the rare states extend in a band from the lowest-energy excitations in the system, which are the well-known ‘meson’ confined states, with wave functions of the form

$$|\psi_n\rangle = \sum_{\nu=\text{NS, RM}} \sum_{p_\nu} \Psi_{n,\nu} (p_\nu) a_{p_\nu}^\dagger a_{-p_\nu}^\dagger |\nu\rangle. \quad (15)$$

These states consist of linearly-confined pairs of domain walls (described by the fermions $a_{p_\nu}^\dagger$ in the $\nu = \text{RM, NS}$ sectors of the Hilbert space, with vacuum $|\nu\rangle$ in each sector, see e.g. Ref. [33]). On the basis of Figs. 2 it...
is easy to suggest that the rare states are simply higher energy meson states.

To ascertain whether this is correct, we can use the information directly accessible to us from the TSM procedure. In the first case, we can check whether the rare states are (majority) two particle in nature by defining the particle weights. To do so, we recall that we construct states in terms of free fermion basis states

$$|E_m^n\rangle = \sum_{N=0,2,4,\ldots} \sum_j c_{N,j}^m \{|p_j\}\rangle_N,$$

where $c_{N,j}^m$ are superposition coefficients and $\{|p_j\}\rangle_N$ are $N$-fermion Fock states with fermions carrying momenta $\{p_j\}_N \equiv \{p_{j1}, \ldots, p_{jN}\}$. Particle weights $w_N$ telling us the $N$-fermion fraction of state are:

$$w_N = \sum_j |c_{N,j}^m|^2.$$

We present a plot of the EEV spectrum for $R = 35$, Fig. 2(b), with superimposed two particle weights in Fig. 6. We see that those EEVs corresponding to rare states are also those which have $w_2 \approx 1$ and so we conclude the rare states are majority two-particle in nature, consistent with Eq. (15).

To further strengthen our evidence that rare states are meson-like, we (semi)analytically compute the energy of the meson states. This can proceed in a number of manners; we consider in particular the energies computed via a semiclassical approximation (the salient points of which are summarized in Appendix C). We thus conclude that the rare states are described by Eq. (15), meson-like confined pairs of domain walls.\cite{79–81}

As previously noted, the rare states exist far beyond the multiparticle continuum (occurring in Fig. 6 at $E/R \sim 0.03$), which is surprising when drawing analogies with mesons in quantum chromodynamics (QCD),\cite{63} in QCD we usually think of high energy mesons as splitting into multiple lower-energy mesons. Indeed, in the case considered here such processes are kinematically allowed, so it is worth spending some time to understand why the meson states (15) exist above the continuum and have EEVs that remain well separated from the thermal result (cf. Fig. 3). To reemphasize: with the TSM we have constructed well-converged eigenstates of the Hamiltonian (2) that are still meson-like above the threshold of the multiparticle continuum.\cite{82}

### A. Stability of meson excitations above the multimeson threshold

To gain some insight into why the rare states persist above the multimeson threshold, we take the mesons defined in Eq. (15),

$$\hat{b}_n^\dagger = \sum_{\nu=NS,RM} \sum_{p_\nu} \Psi_{n,\nu}(p_\nu) a_{p_\nu}^\dagger a_{-p_\nu}^\dagger,$$

which are approximate quasi-particles of the problem—they do not have infinite lifetime because exact eigenstates of (2) contain a finite amount of $N \geq 4$ particle dressing (see Appendix B)—and discuss their hybridization with states with higher fermion number.

We first recap known results for the zero temperature lifetime of meson excitations, before discussing the correction to the meson energies due to hybridization with higher fermion number states. We will see that the simple meson, Eq. (18), has very long lifetime at zero temperature and only very weakly hybridizes with states containing four (or more) fermions. This is consistent with the idea that with a small amount of dressing the meson excitations (18) become absolutely stable.

#### 1. Meson lifetime

Let us consider the results of Rutkevich\cite{80} for the zero temperature meson lifetime. Firstly, the lowest energy meson excitations (below the two-meson threshold, $E \lesssim 4m$) are absolutely stable, with no decay channels. Above the two meson threshold energy, the $n$th meson state has decay width $\Gamma_n$, which can be estimated to leading order
using Fermi’s golden rule
\[
\Gamma_n = 2\pi \sum_{E_{\text{out}}} \left| \langle E_{\text{out}} | V | \psi_n \rangle \right|^2 \delta(M_n - E_{\text{out}}).
\] (19)

Here \( M_n \) is the meson mass (energy of the zero momentum meson state), \(|\psi_n\rangle\) is the meson wave function (cf. Eq. (15) and Appendix D), \(|E_{\text{out}}\rangle\) are eigenstates of the full Hamiltonian with energy \( E_{\text{out}} \) (measured relative to the ground state energy), and \( V \) is the interaction term
\[
V = g \int dx \sigma(x).
\] (20)

Under a semiclassical analysis, the decay width \( \Gamma_n \) in the infinite volume has been computed by Rutkevich. This leads to a lifetime, above the two-meson threshold energy, for the meson states (15) that varies as
\[
\tau \propto g^{-3},
\] (21)

which is very large at sufficiently small \( g \). We note that this formula was derived for large meson number. More generally one faces a challenge in computing the lifetime using Fermi’s golden rule. The matrix elements of the spin operator generically have IR divergences, which reflect the fact that the matrix elements represent both connected and disconnected diagrams. To make sense of a lifetime computation in a fully quantum setting, one must be able to sensibly separate out only the connected part [83]. Alternatively, one can extract the life time from form factor perturbation theory, see also Refs. [84]. Results can also be compared to computations from finite-volume TSMs as shown in Ref. [83]. We will do neither here – we delay a detailed study of the lifetime (as well as the meson Green’s function) to a later work.

2. Corrections to the meson energy: energy dependence

We instead will content ourselves here by considering the correction to the energy of the mesons (18) due to hybridization with zero and four fermion states. From this quantity we will be able to deduce that the mesons as constructed from two-particle states (i.e. Eq. (18)) are only weakly coupled to sectors of the unperturbed theory with different particle number.

To second order in perturbation theory in \( g \), the energy correction for the \( j \)th meson is
\[
\Delta E_{2j} = \frac{g^2}{M_j} \sum_{\nu=\text{RM,NS}} \left| \sum_{p_\nu} \Psi_j(p_\nu) \langle \nu | \sigma(0) | -p_\nu, p_\nu \rangle \right|^2 \delta_0 \sum_{p_\nu} \langle f_{j,(p_\nu)4} \rangle \sum_{p_{\nu_1}, p_{\nu_2} \ldots} \delta_0 \omega_{p_{\nu_1}}(R) \omega_{p_{\nu_2}}(R) \ldots, (22)
\]

where
\[
f_{j,(p_\nu)4} = \sum_{p_\nu} \Psi_j(p_\nu) \langle \nu | \sigma(0) | -p_\nu, p_\nu \rangle \langle f_{j,(p_\nu)4} \rangle.
\] (23)

The matrix elements of the spin operator \( \sigma(0) \) required above are known, see Refs. [33, 74, 76, 77, and 79].

The correction to the energy, \( \Delta E_{2j} \) will take the form
\[
\Delta E_{2j} = \alpha R + \delta E_{2j},
\] (24)

where the term scaling with volume, \( \alpha R \), will be identical to the correction to the vacuum (ground) state due to hybridization with higher fermion number states. This follows from the fact that the energy of a meson excitation will differ only by \( O(1) \) compared to the ground state. Thus \( \delta E_{2j} \) is the correction to the meson mass relative to the ground state energy. We compute \( \Delta E_{2j} \) as a function of \( R \) by numerically evaluating (22) and fitting the result to Eq. (24). We present the results of this in Fig. 7 for two values of the field, \( g = 0.1, 0.2 \).
Some insights into the non-monotonic nature of $\delta E_{2j}$ with bound state number can be gained by looking at the wave functions of successive bound states. We do so in Fig. 8 using the analytic forms of the wave function developed in Appendix D. The bound state wave functions have considerable structure and it is this, and its relative positioning relative the $E = 4m$ threshold, that leads to the non-monotonic behavior of the energy corrections coming from hybridization with four particle states.

We compare the correction to the meson energy to the analogous calculation in the disordered (paramagnetic) phase of the model, $m < 0$, where single particle excitations are spin flips (i.e., we compute the correction to the energy of a single Ramond sector fermion). The calculation is similar to that above, with the explicit form for the correction being

$$\Delta E_{2sf} = \left(\frac{g^2 a^2}{m^2}\right) R + \delta E_{sf},$$

$$\delta E_{sd} = \frac{g^2 a^2}{2mR} \sum_{q \in \text{NS}} \frac{1}{\omega_q(R)^2 |m - 2\omega_q(R)|^2 \tanh^2 \frac{\theta_q}{2}}.$$

Here $\theta_q$ is a rapidity variable defined through the relation $|m| R \sinh \theta_q = 2\pi q$. The result for the correction to the spin flip mass, $\delta E_{sd}$, is also shown in Fig. 7. There we see that the correction to the spin flip energy is between two and four orders of magnitude larger than the corresponding corrections to the meson energies. Combined with the results of Sec. III A 1, we see that the mesons (18) are almost unaffected by hybridization with states of higher fermion number.

3. Corrections to the meson energy: field dependence

We can also consider how the hybridization corrections to the meson energies evolve as a function of field strength $g$. Whilst naively the correction $\Delta E_{2j}$ is a $g^2$ diagram, it also features the meson wave function which evolves with $g$. In Fig. 9 we show the evolution of $\delta E_2$ as a function of $g$ for four meson states: the first, third, fifth and ninth as ordered by energy. One sees that $\delta E_2$ grows with $g$ provided the bound state energy is below the four domain-wall threshold (i.e. $4m$). We show when this threshold is crossed, as a function of $g$, for the four different mesons in Fig. 10. For the lowest meson (the green line in Figs. 9 and 10), we see that for $0 \leq g \leq 0.3$ this meson’s energy never exceeds $4m$, and correspondingly the energy correction increases monotonically. However for the fifth meson (the brown line in these two figures), its energy exceeds $4m$ once $g \gtrsim 0.06$. We further see that while initially $\delta E_2$ increases with increasing $g$ for this meson, it begins to behave non-monotonically (with a decreasing trend) once $g$ exceeds 0.15.

B. Approximate $U(1)$ symmetry at low energy

In the previous section, we have seen that single meson excitations (18) only very weakly hybridize with the multi-meson continuum. This is in spite of the fact that the spin operator $\sigma(x)$ in principle couples states with widely differing numbers of fermions (and hence mesons). One might then suppose that there is an approximate low energy $U(1)$ symmetry, with the Hamiltonian being almost block diagonal in the space of fermion number. Here we present results for the EEV spectrum that show such an approximate block diagonalization works well within the low energy eigenstates. We leave a detailed study of this approximate symmetry to future work.

In Fig. 11, we show the EEV spectrum from the full TSM treatment with energy cutoff $E_A$ (cf. Fig. 2(b)) and contrast it to the EEV spectrum computed from the Hamiltonian with fermion number conservation imposed.
low-energy eigenstates, with either \( w_2 \approx 1, w_4 \approx 0 \) or \( w_2 \approx 0, w_4 \approx 1 \).

## IV. NONEQUILIBRIUM DYNAMICS

In the previous sections we have shown that rare states exist within the spectrum of the perturbed Ising field theory (2). These states yield EEVs that do not conform with the MCE. One question that is natural to ask is whether these states can ever been seen? They exist above a large continuum of states that behave thermally, and hence one may imagine any response from the rare states is ‘washed out’ and they cannot be observed. In this section, we will show that features of these rare states can be brought to bear on nonequilibrium dynamics following a quantum quench.

We first show that at long times after a quench, one can arrive at a stationary state (the DE) that gives expectation values inconsistent with the appropriate MCE. Having ascertained that rare states show up in the long time limit, we then ask whether at finite times the presence of rare states influences nonequilibrium dynamics. We show that, indeed, they do.

We note that the real-time dynamics following a quantum quench of both the fermion mass \( m \) and the longitudinal field \( g \) in the Ising field theory have been recently studied by Rakovszky et al. A similar quench in the closely-related lattice model has recently been shown to lead to a suppression of the light cone spreading of correlations and a back-and-forth motion of domain walls. Similar behavior is also seen in a two dimensional model exhibiting confinement.

### A. Equilibration after a sudden quantum quench

Let us begin by defining our nonequilibrium protocol. We study dynamics that are induced at time \( t = 0 \) by an instantaneous change of the longitudinal field within the Hamiltonian (2)

\[
H_i \equiv H(m, g_t) \rightarrow H_f \equiv H(m, g_f),
\]

restricting our analysis to quenches starting from eigenstates \( |E_m\rangle \) of the initial Hamiltonian. The expectation value of the local spin operator \( \sigma(0) \) in the long-time limit is probed through the DE

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{d}t' \langle E_m | e^{iH_{t'} | \sigma(0) e^{-iH_{t'}} | E_m} \rangle = \sum_n |\tilde{c}_{m,n}|^2 \langle \tilde{E}_n | \sigma(0) | \tilde{E}_n \rangle,
\]

where \( |\tilde{E}_n\rangle \) are the eigenstates of \( H_f \) and \( \tilde{c}_{m,n} = \langle E_m | \tilde{E}_n \rangle \) are the overlap coefficients. The DE predictions are compared to the MCE at the appropriate energy density constructed by averaging over an energy window \( \Delta E \). Agreement between the DE and MCE signals thermalization,
whilst disagreement suggests an absence of thermalization. The convergence of the DE and MCE are discussed in the Appendix A.

Herein we consider quenches in which both the initial and final longitudinal fields are positive – this avoids problems with convergence for sign changing quenches (see also Ref. [46]), which can be understood in terms of projecting onto the ’false vacuum’ in finite size systems.

We present DE and MCE results for the local spin operator $\sigma(0)$ at long-times after the quench $g = 0.1 \rightarrow 0.2$ in Fig. 12. We use a system of size $R = 25$ and fermion mass $m = 1$, cf. Figs. 2. Each plotted point of the DE is constructed starting from a different eigenstate of the initial Hamiltonian; we see that the vast majority of initial states have DE and MCE results that agree (within one standard deviation, represented by the error bars on the MCE). However, much like in the equilibrium spectrum (see Figs. 2, and recall the finite size scaling in Fig. 3) we observe a well-separated band of states above the thermal continuum. We also include the DE ensemble for a smaller value of the TSM cutoff to highlight that these well-separated states are well converged.

Figure 12 leads us to conclude that the rare states in the equilibrium spectrum indeed influence the nonequilibrium dynamics, leading to states that do not thermalize following a quantum quench. This is despite the fact that the Hamiltonian governing the time evolution is nonintegrable and hence generically is expected to lead to thermalization. This is yet more support for ETH: if ETH is absent, thermalization can be avoided even in nonintegrable models.

B. Nonequilibrium real-time dynamics

In the long time limit we see that there are states that do not thermalize. Now we want to understand whether such states have unusual behavior (or anything noticeably different from thermal states) in their short-to-intermediate time dynamics. This time window is of primary interest for experiments on cold atomic gases, and is also accessible in lattice models using numerical methods such as TEBD. Let us first discuss how we compute the real-time dynamics, before presenting results.

### 1. Time evolution by the Chebyshev Expansion

To compute the time evolution of states within TSMs, we use the recently developed formalism of Rakovszky et al. [46] and expand the time evolution operator in terms of Chebyshev polynomials $T_n$. The expansion reads

$$e^{-iH_f t} = J_0(\tilde{t}) \mathbb{1} + 2 \sum_{n=1}^{\infty} (-i)^n J_n(\tilde{t}) T_n(\tilde{H}_f), \quad (27)$$

where we rescale both the time $\tilde{t} = E_{\text{max}} t$ and the Hamiltonian $\tilde{H}_f = H_f / E_{\text{max}}$ by the maximal eigenvalue of the Hamiltonian $H_f$, such that the eigenvalues of $\tilde{H}_f$ lie in the interval $[-1,1]$. $J_n(x)$ are the Bessel functions

$$J_n(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!} \left( \frac{x}{2} \right)^{2l+n}, \quad (28)$$

and the Chebyshev Polynomials are defined through the recursion relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad \text{with \ } T_0(x) = 1, T_1(x) = x. \quad (29)$$

With Eq. (27) at hand, the procedure for computing the time evolution is straightforward. The initial state $|\Psi_i\rangle$ is known in the free fermion basis, as is the final Hamiltonian, so, one generates the Chebyshev vectors $|t_n\rangle = T_n(\tilde{H}_f)|\Psi_i\rangle$ through the recursion relation (29) and then sums them, Eq. (27), to obtain the time-evolved state. Observables, such as the local magnetization $\langle \sigma(0) \rangle$, can then be computed. Herein we refer to this approach as TSM+CHEB.

The Chebyshev expansion of the time evolution operator, Eq. (27), contains an infinite number of terms and so it is necessary to truncate the expansion. This truncation introduces a time scale after which the reported time evolution cannot be trusted. Qualitatively we see that this time scale increases (approximately) linearly with the number of terms kept in the sum. We discuss further convergence of TSM+CHEB in Appendix E.

### 2. Quenches from the ground and first excited states

Let us first develop some intuition by studying quenches from the very lowest states. We present the
time evolution of the local spin operator $\sigma(0)$ following the quench $H(m = 1, g = 0.2) \rightarrow H(1, 0.1)$, when starting from the ground state of the initial $H$, in Fig. 13. Results are well converged already for $E_\Lambda \sim 10$ and the expectation value oscillates about its DE prediction.\(^9\) These oscillations occur at many frequencies, as can be extracted through the power spectrum (Fourier transform) shown in the lower panel. These frequencies coincide with the post-quench meson energies (and their differences), giving an effective route for ‘meson spectroscopy’ (for further details of how we compute these energies, see Appendix C).

Analogous behavior is seen for the quench in which we start from the first excited state, Fig. 14. The power spectrum is now dominated by the frequency $\omega = M_2 - M_1$, implying the initial state projects heavily onto the first and second meson excitations. This is not entirely surprising, as we know in both the initial and final Hamiltonians the meson states are mostly two-particle in nature (see Sec. III).

In both the above cases, Figs. 13–14, oscillations of the expectation value are centered on the DE prediction. For the time-scales that we can reach within the TSM+CHEB procedure, it is not clear that these oscillations are decaying. This is an important question to address, but remains beyond the reach of existing approaches.

3. Quenches from a rare state and proximate thermal state

Having looked at quenches starting from the very lowest states, let us now turn our attention to states with larger energy densities. In particular, we will compare the short time dynamics of a rare state to that of thermal-
TABLE II. The energy \( E_n = \langle \Psi_n | H_f | \Psi_n \rangle \) and diagonal ensemble result for the local spin operator \( \langle \sigma(0) \rangle_{DE} \) for the \( n \)th eigenstate of the initial Hamiltonian \((2)\) \( H_i = H(1,0.1) \), constructed with energy cutoff \( E_\Lambda \) in the TSM procedure. The state is time evolved according to the final Hamiltonian \( H_f = H(1,0.2) \). The \( n = 100 \) state is a rare state, whilst the others are broadly consistent with the microcanonical ensemble, see Figs. 12 and 15. The microcanonical result is \( \langle \sigma(0) \rangle_{MCE} = -0.693097 \pm 0.129259 \), with the uncertainty showing the standard deviation of values averaged over in the energy window of width \( \Delta E = 0.1 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( E_n )</th>
<th>( \langle \sigma(0) \rangle_{DE} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>5.19928</td>
<td>-0.32158</td>
</tr>
<tr>
<td>185</td>
<td>5.21496</td>
<td>-0.550455</td>
</tr>
<tr>
<td>333</td>
<td>5.15805</td>
<td>-0.657668</td>
</tr>
<tr>
<td>352</td>
<td>5.19533</td>
<td>-0.718688</td>
</tr>
<tr>
<td>502</td>
<td>5.19786</td>
<td>-0.83252</td>
</tr>
</tbody>
</table>

To address this question, we consider the time evolution following the quench \( H(m = 1, g = 0.1) \rightarrow H(1,0.2) \) (as in Sec. IV A), and consider eigenstates of the initial Hamiltonian, \( |\Psi_n\rangle \), whose energy in the final basis are very close (computed via \( E_n = \langle \Psi_n | H(1,0.2) | \Psi_n \rangle \)). This in turn implies that the MCE prediction for the long-time expectation values are very close. The precise states that we consider are summarized in Table II, with the \( n = 100 \) initial eigenstate being a rare state (the DE and MCE disagree outside one standard deviation). All other states are thermal, in the sense that the DE agrees with the MCE prediction (within one standard deviation), cf. Fig. 12(a).

Having chosen our initial states with similar energies, we compute their time evolution using the TSM+CHEB explained in Sec. IV B 1. The time evolution of the local magnetization \( \langle 0 \rangle \) is shown in the solid lines of Fig. 15, with dashed lines of the same color corresponding to the DE prediction. The shaded region shows the MCE prediction, with the vertical extent denoting the standard deviation of the data averaged over. As expected, thermalizing states have time evolution compatible with relaxation to the MCE and DE predictions (some of the thermalizing states selected, e.g. \( n = 185 \) and \( n = 502 \), are on the edges of what we call thermal). On the other hand, the rare \( n = 100 \) state is oscillating about a value consistent with the DE result, which is far from the MCE prediction.

In examining Fig. 15, we do see some apparent differences between the time evolution of the rare states and close-in-energy thermal states. The rare state, \( n = 100 \), exhibits large amplitude low frequency oscillations, whilst the thermal states have many more oscillation frequencies, highlighted in the power spectra of Fig. 16. The relaxation towards the DE/MCE is rapid for the thermal states, with only small amplitude oscillations beyond times \( t \sim 30 \). In contrast, the rare state still has large amplitude oscillations at the longest accessible times, and it is not clear that these decay within the time frame \( mt \sim 100 \).

These results suggest that one can diagnose rare states even in the short time dynamics by looking for states which exhibit large (and slow decaying) oscillations in observables. Indeed, this is reminiscent of the behavior observed in lattice simulations with confinement, where the light cone spreading of correlations is linearly suppressed, leading to ‘back and forth’ oscillations of correlations.86

V. RELATION TO THE LATTICE PROBLEM

Up until now we have considered states in the perturbed Ising field theory (2). An important question is whether any of the discussed behavior carries through to the lattice, or if it is an artefact of the scaling limit? To address this, we consider a lattice model whose scaling limit is described by (2), and work away from this limit. Our lattice Hamiltonian reads

\[
H = J \sum_l \sigma_l^x \sigma_{l+1}^x + h^z \sum_l \sigma_l^z + h^x \sum_l \sigma_l^x,
\]

where we set the lattice spacing \( a \) to one, \( \sigma_n^{x,z} \) are the \( x,z \) Pauli matrices at position \( n \), \( J \) is the Ising exchange, and \( h^x,h^z \) are magnetic fields in the \( x,z \) directions, respectively. Equation (30) is a good description.
of CoNb$_2$O$_6$ a quasi-one-dimensional Ising ferromagnet in which linearly confined domain wall excitations have been observed via inelastic neutron scattering and THz spectroscopy. The scaling limit that reproduces (2) is:

$$J \to \infty, \quad a \to 0, \quad h^z \to 1, \quad h^z \ll 1, \quad m = 2J(1-h^z), \quad 2Ja = 1.$$ 

The lattice model (30) has been investigated extensively, with the quench dynamics receiving particular attention. Bänils et al. showed that certain initial states do not appear to thermalize, in spite of the lack of integrability. It is thought that this is linked to the presence of rare states in the spectrum. On the other hand, Ref. 26 studied whether for certain parameter regimes all eigenstates of (30) obey ETH—finding certain sets of parameters where this did appear to be the case from small system exact diagonalization. Putting these results together, it appears the precise details of the parameter regime matter. This is perhaps not surprising, as the physics contained within this simple lattice model is rather rich.

In the following subsection we will show that, indeed, rare states exist within the spectrum of this lattice model, at least for certain values of the parameters.

### A. Eigenstate expectation values

To begin our study of the lattice model, Eq. (30), we consider the EEV spectrum for low-energy states. To do so, we use DMRG to construct eigenstates in a finite (open) chain of $N = 20 - 40$ sites. We first use the standard finite-size DMRG technique to find the ground state, and subsequently construct excited states using projector methods (see, e.g., Ref. [100] for a detailed explanation of this approach). With the projector method, it is important to remember that excited states may not be found in order, but provided one constructs a sufficient number of states (and chooses the ‘weight parameter’ of the procedure carefully) it is possible to correctly construct the low energy spectrum.

In our DMRG simulations, the truncation error was $10^{-10}$ and we allowed up to 20 finite size sweeps (20 left and 20 right sweeps) for each state. As a check of our routines, we compared the spectrum computed for $N = 14$ sites with that obtained from exact diagonalization of the Hamiltonian, finding excellent agreement.

To draw analogies between the lattice model (30) and the continuum theory (2), in particular in order to compare to Fig. 2, we compute the average magnetization across the chain. This mimics a projection onto the zero (quasi-)momentum sector of the theory, which is studied in the field theory. We remind the reader that eigenstates of the field theory are translationally invariant and hence satisfy

$$\langle E | \sigma(x) | E \rangle = \frac{1}{R} \int_0^R dx \langle E | \sigma(x) | E \rangle,$$

where $|E\rangle$ is any eigenstate.

In Fig. 17 we present data from our lattice simulations for (30) with $J = -1, h^x = -0.5$ and $h^z = 0.05$ for three system sizes. This should be compared to analogous Figs. 2 in the field theory; the similarity between lattice and field theory calculations is rather striking. We see a clear band of nonthermal states above a multiparticle ‘continuum’ (this appears rather discrete due to the relatively small number of sites, $N = 20 - 40$, but would broaden into a continuum in the large $N$ limit). It is worth noting that in the DMRG simulations on an open system, the meson excitations are localized in the vicinity of the boundaries due to the decreased energy cost of such excitations there. We plot the local magnetization $\langle \sigma_f \rangle$ in a number of representative states in Fig. 18; the corresponding states are labelled in Fig. 17(c).

It is also worth noting that we are only able to access the low-energy part of the spectrum. We will see in the following section that this part of the spectrum is well converged as a function of system size $N$, being representative of the thermodynamic limit. 

![FIG. 16. Power spectrum $|\sigma(\omega)|^2$ for: (a) the $n = 100$ rare state; (b) the $n = 333, 352$ thermalizing states as computed from data in Fig. 15.](image-url)
FIG. 17. The eigenstate expectation values for the chain-averaged magnetization $\frac{1}{N}\sum_{j=1}^{N}\langle \sigma^z_j \rangle$ in the lowest $\sim 150$ states of the lattice model, Eq. (30), with $J = -1$, $h^x = -0.5$ and $h^z = 0.05$. Eigenstates were computed with DMRG for a chain of (a) $N = 20$; (b) $N = 30$; (c) $N = 40$ sites. We encourage the reader to compare this to the analogous plot in the scaling limit, Fig. 2, and note the striking similarity. In plotting (a), (b), we exclude the false vacuum state. The labels in (c) indicate the states shown in Fig. 18.

FIG. 18. The spin expectation value $\langle \sigma^z_\ell \rangle$, on each site $\ell = 1, 2, \cdots, 40$, for the 0th (ground), 1st, 44th, 48th and 51st states in the spectrum as found using DMRG. These states are marked in Fig. 17(c). The 1st and 48th states are examples of mesons localized on the boundary.

the low-energy meson states remain well-separated from the continuum in the infinite volume limit. We cannot, however, rule out that the meson states melt into the continuum with increasing energy (although, we note, there is no evidence of this for the states that we can construct). Speculatively, such behaviour in the scaling limit could be caused by the presence of marginal or irrelevant operators (neglected in the field theory Hamiltonian (2)), which are difficult to incorporate into a truncated spectrum treatment. Further investigations of the lattice model, which is already known to exhibit anomalous dynamics,86,92,96,97 are undoubtedly warranted.

1. Comparison between the finite size scaling of the EEV spectrum on the lattice and in the continuum

As in the continuum, we can ask to what extent the rare states on the lattice persist to large volumes. To do so, we consider the finite size scaling of the low-energy EEV spectrum, which we also compare to that in the continuum. In order to make a like-for-like comparison between results at different system sizes, in this part of the appendix we focus on the total magnetization of the eigenstates relative to the ground state (i.e., the number of flipped spins in the eigenstate) as a function of energy (relative to the ground state) of the eigenstate. We focus on total magnetization to avoid obvious problems with scaling with volume of local magnetization: the states we construct have $O(1)$ flipped spins compared to the ground state and hence in the thermodynamic limit have the same magnetization density as the ground state.

With this in mind, the results of Fig. 17 are translated into Fig. 19 (the corresponding continuum results, Fig. 2, become Fig. 20). We see that low energy EEV spectrum at different lattice sizes $N$ (system sizes $R$) is well converged: meson state EEVs are well converged and match
VI. CONCLUSIONS

ETH is key to understanding whether quantum systems thermalize. When ETH is valid, and matrix elements of operators in the eigenbasis satisfy Eq. (1), expectation values of operators within an eigenstate are thermal, and dynamics following a quantum quench are expected to lead to thermalization. In this work, we have shown an explicit example of a nonintegrable model, Eq. (2) and its lattice regularization Eq. (30), where ETH is violated. This is signalled through the presence of rare states in the spectrum, with expectation values within these states being nonthermal.

To show this, we used TSMs and explicitly construct the low-energy spectrum of the theory. With the eigenstates at hand, we established the nature of the rare states that violate ETH: they arise as a direct result of confinement, corresponding to the well-known “meson” states: single particle excitations that can have extensive energy ($E \propto R$). These excitations are formed from pairs of linearly confined “domain wall fermions” and remain kinematically stable far above the threshold of the multiparticle continuum. This surprising result can, however, be understood by means of perturbative calculations of the meson lifetime and energy corrections, combining both a standard Bethe-Salpeter analysis for the mesons with a perturbative treatment of the meson-to-four-fermion vertices. This reveals only a very weak hybridization of the meson with the multiparticle continuum. This implies that just slight dressing of the two-fermion excitation can render the excitation absolutely stable. A finite size scaling TSM analysis is consistent with the meson states remaining nonthermal in the infinite volume limit, see Figs. 3 and 20.

With rare states established within the spectrum of the model through the study of EEVs, we turned our attention to understanding their influence on nonequilibrium dynamics. We first showed, through construction of the diagonal ensemble, that certain quantum quenches exhibit an absence of thermalization in the infinite time limit. This is in spite of the fact that our model is non-integrable. Subsequent studies of the real-time nonequilibrium time evolution of observables revealed that rare states have EVs that show large amplitude, low frequency oscillations that appear not to decay on relatively long time scales $t \sim 100|\lambda|^{-1}$. This should be contrasted to thermalizing states, which rapidly relax to their thermal values, showing only small fluctuations about their diagonal ensemble value. These difference may help diagnose rare states in experiments on, e.g., cold atoms.

Finally, we addressed whether the violation of ETH in (2) is related to the scaling limit. To do so, we considered the lattice regularization of the model (e.g., the spin chain whose scaling limit gives the field theory) explicitly away from the scaling limit. Using DMRG we constructed the low-energy spectrum and present the EEV spectrum (at low energies). This has striking similarities to the EEV spectrum in the scaling limit, including the

for different values of $N$ ($R$). The thermal continuum is also qualitatively well converged at low energies, in the sense that the features are correct, but the density of states increases with increasing $N$ ($R$).

In Figs. 19 and 20, it is nevertheless easy to see that the meson states in the low-energy part of the spectrum remain well-separated from the thermal continuum, and results are converged to the thermodynamic limit, being independent of $N$ and $R$. 

FIG. 19. The EEV spectrum obtained by DMRG on the finite lattice of $N$ sites for the Hamiltonian (30) with $J = -1$, $h^x = -0.5$ and $h^z = 0.05$. Here we consider the total magnetization of the eigenstates relative to the ground state, as a function of energy (relative to the ground state energy $E_0$). See Fig. 17 for further details.

FIG. 20. The EEV spectrum for $m = 1$, $g = 0.1$ in systems of sizes $R = 25 - 45$ (see Figs. 2 for details). Here we plot the total magnetization (relative to the ground state) as a function of energy (relative to the ground state). We see that the meson states are well converged as a function of system size $R$, while the thermal continuum is qualitatively converged (in the sense that the features are correct, but the density of state increases with increasing $R$). Note the striking similarity to Fig. 19.
presence of a band of rare states above the multiparticle continuum. This finding suggests that confinement-induced rare states are not a remnant of the scaling limit, and supports previous results (see e.g. Ref. [86]) that ascribed anomalous nonequilibrium dynamics to the presence of rare states in the spectrum. They may also be responsible for a lack of transport observed in time-evolution of the domain wall initial state.\(^\text{97}\) We do note, however, that we were unable to probe energies far into the spectrum and could not rule out that on the lattice such states melt into the continuum at finite energy densities.

The rare states in the field theory (2) are intimately related to the presence of a linearly confining potential for the domain wall excitations. We note that recent work on confined phases in holographic theories also suggests an absence of thermalization.\(^\text{101}\) which may lead one to speculate that models with confinement in general exhibit a lack of thermalization. This is partially supported by recent results of the authors\(^\text{66}\) that show anomalous nonequilibrium dynamics in a 2D model with confinement that are completely analogous to the 1D problem.\(^\text{86}\) It would be interesting to test this conjecture in other theories, such as the Schwinger model\(^\text{102–109}\) or the \(q\)-state Potts model,\(^\text{110,111}\) with the view that this may have important consequences in the context of quantum chromodynamics (which has some parallels with the quantum magnetism discussed here\(^\text{61}\)).

We finish by noting that recent works suggest that other kinetic constraints, beyond those provided by confinement, can also lead to nonthermal behavior.\(^\text{24,112,113}\) This may have important implications for experiments on Rydberg gases.\(^\text{114}\)

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**Appendix A: Convergence of the TSM**

Truncated spectrum methods can be used to construct approximate eigenstates of a field theory in a nonperturbative manner.\(^\text{33}\) The technique hinges on studying ‘known’ theories that are perturbed by a relevant operator, where one uses the eigenstates of the ‘known’ theory as an efficient computational basis. This basis is truncated by the introduction of an energy cutoff, which is motivated by properties of relevant operators: they strongly mix low energy degrees of freedom, while essentially leaving the high energy ones unperturbed.

At a practical level, this can still remain a difficult problem. Even with the introduction of a cutoff, there is a rapid growth of the Hilbert space with increasing cutoff, limiting the truncation energies that one can reach. In the first part of this appendix, we examine the convergence of results with increasing energy cutoff at fixed system size \(R\). Scaling with \(R\) is discussed in the main body of the text. We finish by examining the convergence of expectation values in the diagonal and microcanonical ensembles.

1. **With cutoff \(E_A\)**

We begin by discussing the convergence of TSM results with increasing energy cutoff \(E_A\) at fixed system size \(R\). Low energy eigenstates of the field theory (2) are con-
structured for \( m = 1, g = 0.1 \) on a ring of size \( R = 35 \). In Fig. 21 we show the EEVs of the local spin operator \( \sigma(0) \) as a function of energy density of the eigenstate \( E/R \) for four values of the cutoff \( E_\Lambda = 8, 9, 10, 10.5 \). We note that \( E_\Lambda = 10.5 \) corresponds to 23500 states in the truncated Hilbert space\(^{118} \).

As expected from the ideas underpinning TSMs, the energies of the low-energy eigenstates are well converged. With increasing energy density, convergence of the eigenstate energies decreases, with the highest constructed operator in an eigenstate of energy density all values of the cutoff is qualitatively converged. At high energies, as with the EEV spectrum, results are dominated by cutoff effects.

When examining the expectation value of the spin operator in an eigenstate of energy density \( E/R \), we see similar behavior to the convergence of the energy. Low energy EEVs are well converged, with convergence becoming poorer at higher energies, and clearly cutoff dependent behavior occurring towards the top of our energy scale. We note, however, that qualitative behavior does not change with increasing cutoff: the multiparticle continuum remains well separated from the band of meson states, with this gap being robust to increasing cutoff. For four values of the energy cutoff \( g_i = 0.1 \rightarrow g_f = 0.2 \) in the field theory (2) with \( m = 1 \) and \( R = 25 \) for four values of the energy cutoff. Free fermion basis states containing up to ten particles are considered.

### 2. The diagonal ensemble

The diagonal ensemble (DE) is assumed to describe the long-time limit of expectation values after a quantum quench. This can be motivated by considering off-diagonal matrix elements: in the long time limit these oscillate rapidly and average to zero. We consider the DE result for the local spin operator \( \sigma(0) \), Eq. (26),

\[
\lim_{t \to \infty} \langle E_m | e^{iHt} \sigma(0) e^{-iHt} | E_m \rangle = \sum_n |c_{m,n}|^2 \langle \tilde{E}_n | \sigma(0) | \tilde{E}_n \rangle. \tag{A1}
\]

Here \( |E_m\rangle \) are the \textit{prequench} eigenstates, \( |\tilde{E}_n\rangle \) are \textit{postquench} eigenstates (eigenvectors of the Hamiltonian performing the time evolution) and \( c_{m,n} = \langle E_m | \tilde{E}_n \rangle \) are the overlap coefficients.

From Eq. (A1), it is not \textit{a priori} obvious that we should expect accurate results when constructing the DE from the TSM: after all, it sums over all eigenstates obtained in the procedure, and we know many of these will not be well converged. Thus it is important to check in what sense the DE from TSMs is correct.

We present the DE result for \( \sigma(0) \) following the quench \( g_i \rightarrow g_f \) with both \( g_i, g_f > 0 \) in Fig. 22. In particular, our interest is in convergence of the result with increasing energy cutoff \( E_\Lambda \) (we fix \( g_i = 0.1, g_f = 0.2 \) and \( R = 25 \)). We see, once again, that the rare states are robustly separated from the multiparticle continuum, and there is qualitatively consistent behavior even at intermediate energy densities, for different values of the cutoff.

### 3. The microcanonical ensemble

With the convergence of the DE established, we now consider the other ensemble used in this work, the microcanonical ensemble (MCE). For a given energy \( E \), the MCE is constructed by averaging over all eigenstates within a given energy window \( (\Delta E) \) of \( E \):

\[
\rho_{MCE}(E) = \frac{1}{W} \sum_{\tilde{E}_i} g(E, \tilde{E}_i) |\tilde{E}_i\rangle \langle \tilde{E}_i|. \tag{A2}
\]

Here \( |\tilde{E}_i\rangle \) are the post-quench eigenstates, \( W \) is a normalization parameter

\[
W = \sum_{\tilde{E}_i} g(E, \tilde{E}_i),
\]

that enforces \( \text{Tr} \rho_{MCE}(E) = 1 \), and

\[
g(E, \tilde{E}) = \begin{cases} 1, & \text{if } \tilde{E} \in [E - \frac{\Delta E}{2}, E + \frac{\Delta E}{2}] \, , \\ 0, & \text{otherwise}. \end{cases}
\]

The convergence of the MCE prediction for \( \sigma(0) \) with increasing energy cutoff \( E_\Lambda \) is shown in Fig. 23 for the field theory (2) with \( m = 1, g = 0.2 \) and \( R = 25 \). We see that the MCE result (plus one standard deviation error bars) is well converged up to relatively high energy densities. This seems to reflect the fact that, even if individual EEVs are moving around a little with energy cut-off, their unweighted average is not significantly changing. Once again, at high energy results are dominated by cutoff effects, as would be expected from the EEV spectrum.
FIG. 23. The MCE prediction for $\langle \sigma(0) \rangle$ at a given energy density in the field theory (2) with $m = 1$, $g = 0.2$ and $R = 25$. Error bars denote the standard deviation of the averaged data for the window $\Delta E = 0.1$.

### Appendix B: $N$-particle weights

Within the ordered phase, $m > 0$, the TSM provides us with (approximate) eigenstates $|E_m\rangle$ of the field theory (2) of the form

$$|E_m\rangle = \sum_{N=0,2,4,...} \sum_j c_{N,j}^m |\{p_j\}_N\rangle,$$

(B1)

where $N$ labels the number of fermions within a basis state, $c_{N,j}^m$ are the superposition coefficients, and we defined the $N$-fermion Fock states

$$|\{p_j\}_N\rangle = a_{p_1}^+ \cdots a_{p_N}^+ |0\rangle.$$

(B2)

This leads to a natural definition of the $N$-particle weights: the sum of the absolute-value squared of the superposition coefficients restricted to a given $N$:

$$w_N = \sum_j |c_{N,j}|^2.$$

(B3)

The quantity $w_N$ tells us what fraction of the state is described by the $N$ fermion basis states.

In the main body of the text, we used $w_2$ to support the assertion that the rare states are meson-like in nature, being built predominantly from linearly confined pairs of fermions. Here we present additional data for completeness for the $N = 0, 4, 6, 8$ particle weights within the constructed eigenstates, for the same eigenstates are presented in Fig. 6. These weights are shown in Fig. 24.

At low energy densities $E/R \lesssim 0.15$, Fig. 24 supports the assertion in the main text that there is an approximate $U(1)$ symmetry at low energies, with the states being predominantly “$N$-particle” in nature. To emphasize this we present a histogram of the zero, two, four and six particle weights, $w_0 - w_6$, restricted to lower energy densities $E/R < 0.15$ in Fig. 24(e). We see that there is a bimodal distribution for both $w_2$ and $w_4$, with strong peaks at $w_N = 0, 1$. Detailed studies of how these results (and $w_6, w_8, \ldots$) evolve with increasing energy cutoff and system size are beyond the scope of this work and will be addressed in future studies.

### Appendix C: Semiclassical energies of the meson states

Here we provide semiclassical expressions for the energies of the meson states.\(^{29-31}\) For weak longitudinal field $g$ and away from the two-particle threshold ($2m$) the energy of a meson state can be expressed as\(^{81}\)

$$M_n = E_0 + 2m \cosh \theta_n, \quad n = 1, 2, \ldots,$$

(C1)

where $E_0$ is the ground state energy and $\theta_n$ is a rapidity that satisfies the nonlinear quantization condition

$$\sinh 2\theta_n - 2\theta_n = 2\pi \lambda \left( n - \frac{1}{4} \right) - \lambda^2 S_1(\theta_n) - O(\lambda^3),$$

(C2)

where $\lambda = 2\tilde{\sigma} g/m^2$ with $\tilde{\sigma} = |m|^{1/8} \tilde{s}$, and $\tilde{s} = 2^{1/2} e^{-1/8} \mathcal{A}^{3/2}$ with $\mathcal{A} = 1.2824271291\ldots$ being Glashier’s constant. We have also defined the function

$$S_1(\theta) = -\frac{1}{\sinh 2\theta} \left[ \frac{5}{24} \cosh^2 \theta + \frac{1}{4} \cosh^2 \theta - \frac{1}{12} \sinh^2 \theta \right].$$

Solutions for $m = 1, g = 0.1$ are shown in Fig. 6 (arrows are drawn at these energies) for the first forty mesons.

### Appendix D: The meson wave function at finite $R$

In this appendix, we consider states of the meson form

$$|\psi_n\rangle = \sum_{v=NS, RM} \sum_{p_v} \Psi_{n,v}(p_v) a_{p_v}^+ a_{-p_v}^+ |v\rangle,$$

(D1)

where $|v\rangle$ is the vacuum state in the NS or RM sector, and $a_{p_v}^+$ creates a fermion of momentum $p_v$ in the appropriate sector. Our aim is to constrain the form of the wave function such that these states are approximate eigenstates of the Hamiltonian (2). To do so, we derive a Bethe-Salpeter equation for the wave function, see also \([81]\).
FIG. 24. The weights (a) $w_0$; (b) $w_4$; (c) $w_6$; (d) $w_8$. Eigenstates were constructed with the TSM on the length $R = 35$ chain with $m = 1$, $g = 0.1$ and an energy cutoff $E_\Lambda = 10.5$, with up to 10 free fermions in the basis states. These plots illustrate the approximate low-energy U(1) symmetry discussed in the main text: below an energy density of $E/R \sim 0.15$, the weight for each state are either $w_N \approx 1$ or $w_N \approx 0$. This is emphasized in (e), where we show the distribution of the weights for eigenstates with energy density $E/R < 0.15$, where bins are of width $\Delta w_N = 0.02$. We see almost all states have weights close to zero or close to one, with the histograms being bimodal (note the logarithmic y-axis scale. The issue of where this approximate U(1) symmetry breaks is tough to address, with cutoff effects dominating results at higher energies.
1. The Bethe-Salpeter Equation

A Bethe-Salpeter equation for the meson wave function (D1) is obtained by restricting the Schrödinger equation to this manifold of states. To do this, we need to evaluate matrix elements of the form $\langle v|a_{-p_o} a_{p_o} H|\psi \rangle$. This gives rise to a restricted Schrödinger equation

$$E_n\Psi_{n,v}(p_o) = 2\omega(p_o)\Psi_{n,v}(p_o) + \frac{gR}{2} \sum_{v',q_o} \Psi_{n,v'}(q_o) \langle v|a_{-p_o} a_{p_o} \sigma(0) a_{q_o}^\dagger a_{-q_o}^\dagger \rangle |v'\rangle.$$  

(D2)

Here $\omega(p) = \sqrt{p^2 + m^2}$ is the dispersion relation for noninteracting fermions, and matrix elements of the spin operator in the finite volume are known.\(^{76,77,79}\) The spin operator only connects states in different sectors (NS and RM) of the Hilbert space, setting $v' = \bar{v}$ (where for $v = \text{NS, RM}$, $\bar{v} = \text{RM, NS}$). The matrix elements in the large but finite volume are given by

$$\sum_{q_o} \langle v|a_{-p_o} a_{p_o} \sigma(0) a_{q_o}^\dagger a_{-q_o}^\dagger |\bar{v}\rangle \Psi_{n,\bar{v}}(q_o) = -\sum_{q_o} \frac{p_o q_o}{R^2} \frac{\bar{\sigma}}{2\omega(p_o)2\omega(q_o)^2} \left( \frac{\omega(p_o) + \omega(q_o)}{\omega(p_o) - \omega(q_o)} \right)^2 \Psi_{n,\bar{v}}(q_o).$$ 

(D3)

The spatial wave function is obtained by Fourier transformation

$$\Psi_{n,\bar{v}}(q_o) = \frac{1}{R} \int_0^R dx \, e^{i q_o x} \Psi_{n,\bar{v}}(x).$$

Fourier transform the wave function in Eq. (D3), we can rewrite this equation as

$$\sum_{q_o} \langle v|a_{-p_o} a_{p_o} \sigma(0) a_{q_o}^\dagger a_{-q_o}^\dagger |\bar{v}\rangle \Psi_{n,\bar{v}}(q_o) = -\frac{\bar{\sigma}}{R^3} \int_0^R dx \, S(x, p_o) \Psi_{n,\bar{v}}(x),$$

(D4)

where we define the Fourier transformed matrix elements

$$S(x, p_o) = \sum_{q_o} e^{i q_o x} \frac{p_o q_o}{\omega(p_o)^2 \omega(q_o)^2} \left( \frac{\omega(p_o) + \omega(q_o)}{\omega(p_o) - \omega(q_o)} \right)^2.$$ 

(D5)

A subsequent Fourier transform of Eq. (D5), where we expand in powers of $p_o$ and $q_o$, leads to

$$\sum_{q_o} e^{-i q_o x} S(x, p_o) = \sum_{q_o} e^{i (q_o x - p_o x')} \left[ \frac{4}{(p_o - q_o)^2} - \frac{4}{(p_o + q_o)^2} + \frac{2p_o q_o}{m^4} + O(p_o q_o^2 + O(q_o^2)) \right].$$ 

(D6)

Here we note that divergences that arise from vanishing denominators are avoided by $p_o$ and $q_o$ being in different sectors of the Hilbert space (and hence the smallest difference being $\pm \pi/R$).

To continue, the sums on the right hand side of Eq. (D6) are evaluated:

$$\sum_{p_{NS}} e^{-i p_{NS} x} = R^2 \left( \frac{R}{2} - |x| \right), \quad \sum_{p_{RM}} e^{i p_{RM} (x - x')} = R \delta(x - x').$$

In turn Eq. (D6) becomes

$$\sum_{p_o} e^{-i p_o x} S(x, p_o) = 2R^2 \left( \frac{R}{2} - |x| \right) \left[ \delta(x - x') - \delta(x + x' - R) \right] + \frac{2R^2}{m^4} \delta'(x)\delta'(x') + \ldots$$

The restricted Schrödinger equation (D2) can now be recast into a differential equation for the real space wave function

$$E_n\Psi_{n,v}(x) = \left( 2m - \frac{1}{m} \partial_x^2 - \frac{\bar{\sigma}}{4m^2} \partial_x^4 - \frac{\bar{\sigma}}{8m^2} \partial_x^6 \right) \Psi_{n,v}(x) - 2\bar{\sigma} \left( \frac{R}{2} - |x| \right) \Psi_{n,v}(x) + \frac{\bar{\sigma}}{m^4} \delta'(x)\Psi_{n,\bar{v}}(0).$$  

(D7)

Here we keep only the leading (small momentum) terms in the expansion of the dispersion relation $\omega(p)$. Furthermore, we will herein assume that $\Psi_{n,\bar{v}}(x) = \Psi_{n,v}(x) \equiv \Psi_{n}(x)$, which will ultimately be justified in an a posteriori manner.
This assumption simplifies Eq. (D7):
\[
\left( E_n + g\bar{\sigma}R - 2m \right) \Psi_n(x) = - \left( \frac{1}{m} \partial_x^2 + \frac{1}{4m^3} \partial_{xx}^4 + \frac{1}{8m^5} \partial_x^6 - g\bar{\sigma}|x| \right) \Psi_n(x) + \frac{g\bar{\sigma}}{m^4} \delta'(x)\Psi'_n(0).
\]
Performing a change of variables, \( y = xmt \) with
\[
t = \left( \frac{2g\bar{\sigma}}{m^2} \right) \frac{1}{4},
\]
the restricted Schrödinger equation is
\[
\epsilon_n \Psi_n(y) = \left( |y| - \partial_y^2 - \mu t^2 \partial_y^4 - \nu t^4 \partial_y^6 \right) \Psi_n(y) - t^4 \delta'(y)\Psi'_n(0),
\]
with \( \epsilon_n m^2 t^2 = (E_n - 2m + g\bar{\sigma}R) \). We call Eq. (D9) the Bethe-Salpeter equation.

2. Solution of the Bethe-Salpeter Equation

Now let us solve Eq. (D9). We consider the general equation
\[
\epsilon_n \Psi_n(y) = \left( |y| - \partial_y^2 - \mu t^2 \partial_y^4 - \nu t^4 \partial_y^6 \right) \Psi_n(y) + \rho t^4 \delta'(y)\Psi'_n(0),
\]
where Eq. (D9) is recovered by setting \( \mu = 1/4, \nu = 1/8 \) and \( \rho = 1/2 \). To begin we construct solutions of
\[
0 = \left( |y| - \partial_y^2 - \mu t^2 \partial_y^4 - \nu t^4 \partial_y^6 \right) F(y).
\]
These can be written in terms of solutions \( A(y) \) of Airy’s equation
\[
(y - \partial_y^2) A(y) = 0.
\]
Neglecting terms higher order in \( t \) than \( t^4 \), the solutions of Eq. (D11) are \( F_A(y) = A(y) + t^2 F_2(y) + t^4 F_4(y) \), where
\[
F_2(y) = - \frac{4\mu}{5} y A(y) - \frac{\mu}{5} y^2 A'(y),
\]
\[
F_4(y) = - \left( 2\mu^2 - \frac{9}{7} \nu \right) y^2 A(y) + \frac{\mu^2}{50} y^5 A(y) + \left( \frac{8\mu^2}{5} - \frac{10\nu}{7} \right) A'(y) + \left( \frac{14\mu^2}{35} - \frac{\nu}{7} \right) y^3 A'(y).
\]
To show that these are solutions, it is useful to remember that \( A'(y) = y A(y) \), allowing one to write all terms as functions of only \( A(y) \) and \( A'(y) \).

Let us now consider the full equation (D10), and make the ansatz that the solution has the form
\[
\Psi_n(y) = \text{sgn}(y) F_n(|y| - \epsilon_n),
\]
with \( F_n(y) \) a solution of Eq. (D11). This will be a solution provided the function \( F \) satisfies the following boundary conditions:
\[
(i) \quad F_n(-\epsilon_n) = O(t^2),
(ii) \quad \mu F_n(-\epsilon_n) + v t^2 F'_n(-\epsilon_n) = O(t^4),
(iii) \quad F_n(-\epsilon_n) + \mu t^2 F''_n(-\epsilon_n) + v t^4 F^{(4)}_n(-\epsilon_n) - \frac{\rho}{2} t^4 F'_n(-\epsilon_n) = O(t^6).
\]
To ensure these are satisfied, we look for a solution of the form \( F_n(y) = F_{Ai}(y) + \alpha_n(\epsilon_n) F_{Bi}(y) \), where \( Ai(y), Bi(y) \) are the linearly independent solutions to Airy’s equation (D12). Furthermore, we assume that one can write an expansion in powers of \( t \) for \( \alpha_n(\epsilon_n) \): \( \alpha_n(\epsilon_n) = \alpha_{0,n}(\epsilon_n) + t^2 \alpha_{2,n}(\epsilon_n) + t^4 \alpha_{4,n}(\epsilon_n) + O(t^6) \). The terms \( \alpha_{i,n}(\epsilon_n) \) are fixed by imposing the boundary conditions (D14), leading to
\[
\alpha_{0,n}(\epsilon_n) = - \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)}, \quad \alpha_{2,n}(\epsilon_n) = - \frac{\mu \epsilon_n^2}{5} \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)} \left( \frac{Ai'(-\epsilon_n)}{Ai(-\epsilon_n)} - \frac{Bi'(-\epsilon_n)}{Bi(-\epsilon_n)} \right),
\]
\[
\alpha_{4,n}(\epsilon_n) = \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)} \left( \frac{Ai'(-\epsilon_n)}{Ai(-\epsilon_n)} - \frac{Bi'(-\epsilon_n)}{Bi(-\epsilon_n)} \right) \left[ \frac{2\mu^2}{5} - \frac{4\nu}{7} + \frac{\rho}{2} + \left( \frac{84\mu^2}{350} - \frac{\nu}{7} \right) \epsilon_n^3 + \frac{Bi'(-\epsilon_n)}{Bi(-\epsilon_n)} \frac{\mu^2 \epsilon_n^4}{25} \right].
\]
To complete the solution of Eq. (D10), we restrict our attention to normalizable solutions $\Psi_n(y)$. Using that \( \lim_{y \to \infty} Bi(y) = \infty \), this forces us to find $\epsilon_n$ such that $\alpha_n(\epsilon_n) = 0$. Combining this condition with the above, we arrive at

$$\epsilon_n = -z_n + \delta_{2,n}t^2 + \delta_{4,n}t^4 + O(t^6), \quad \text{(D15)}$$

where $Ai(z_n) = 0$, and

$$\delta_{2,n} = -\frac{\mu}{5}z_n^2, \quad \delta_{4,n} = \left(\frac{84\mu^2}{350} - \frac{2\mu^2}{25} - \frac{\nu}{7}\right)z_n^3 - \left(\frac{2\mu^2}{5} - \frac{4\nu}{7} + \frac{\rho}{2}\right).$$

### 3. Explicit expressions in original units

To recover normalizable solutions of (D9), we set $\mu = 1/4$, $\nu = 1/8$ and $\rho = 1/2$. The meson energies, $E_n$, are a power series in $t = (2g斗争/m)^{1/3}$

$$E_n - E_0 = 2m\left(1 + a_{2,n}t^2 + a_{4,n}t^4 + a_{6,n}t^6 + O(t^8)\right), \quad \text{(D16)}$$

with $E_0 = -g斗争R$ the energy of the ground state, and dimensionless parameters

$$a_{2,n} = \frac{z_n}{2}, \quad a_{4,n} = \frac{z_n^2}{40}, \quad a_{6,n} = -\frac{57}{280} - \frac{11z_n^3}{2800}.$$  

Equation (D16) agrees with previous calculations by other authors.\(^{79,81}\) The $n$th meson wave function is

$$\Psi_n(x) = \frac{1}{N_n}\text{sgn}(x)F_n\left(mt|x| - \frac{1}{mt^2}\left(E_n - E_0 - 2m\right)\right),$$

where $N_n$ sets the normalization, and $F_n(y) = G_{0,n}(y) + G_{2,n}(y)t^2 + G_{4,n}(y)t^4 + O(t^6)$, with

$$G_{0,n}(y) = Ai(y) - \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)}Bi(y),$$

$$G_{2,n}(y) = \frac{1}{20}\left(-y\left[4Ai(y) + yAi'(y)\right] + \frac{Ai(-\epsilon_n)y}{Bi(-\epsilon_n)}\left[4Bi(y) + yBi'(y)\right] + \epsilon_n^2\frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)}\left[\frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)} - \frac{Bi(-\epsilon_n)}{Bi(-\epsilon_n)}\right]Bi(y)\right),$$

and

$$G_{4,n}(y) = \frac{1}{28}y^2Ai(y) + \frac{1}{800}y^5Ai(y) + \frac{1}{140}(y^3 - 11)Ai'(y) - \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)}\left[\frac{1}{28}y^2Bi(y) + \frac{1}{800}y^5Bi(y) + \frac{1}{140}(y^3 - 11)Bi'(y)\right]$$

$$- \frac{\epsilon_n y Ai(-\epsilon_n)}{400 Bi(-\epsilon_n)}\left[\frac{Ai'(-\epsilon_n)}{Bi(-\epsilon_n)} - \frac{Bi'(-\epsilon_n)}{Bi(-\epsilon_n)}\right]\left[4Bi(y) + yBi'(y)\right] + \frac{Ai(-\epsilon_n)}{Bi(-\epsilon_n)}\left[\frac{Ai'(-\epsilon_n)}{Bi(-\epsilon_n)} - \frac{Bi'(-\epsilon_n)}{Bi(-\epsilon_n)}\right]\left[\frac{127}{280} - \frac{\epsilon_n^3}{350} + \frac{Bi'(-\epsilon_n)\epsilon_n^4}{Bi(-\epsilon_n)25}\right]Bi(y).$$
Appendix E: Convergence of the TSM+CHEB

In this final appendix, we consider the convergence of the TSM+CHEB with expansion order and energy cutoff. We then examine results for the nonequilibrium dynamics at different system sizes.

1. With expansion order $N_{max}$

Let us first consider convergence of the TSM+CHEB with order of the expansion, $N_{max}$. The upper panel of Fig. 25 presents the time evolution of $\sigma(t)$ following the quench $(m, g) = (1, 0.1) \rightarrow (1, 0.2)$ for $R = 25$, with energy cutoff $E_\Lambda = 9$. This is computed following Sec. IV for four values of $N_{max}$; we see that results are well converged up to a time $t_{br}$, where the finite order of expansion leads to a divergence in the result. This breakdown time $t_{br}$ increases approximately linearly with expansion order, and manifests as a divergence of the result from its converged value.

2. With energy cutoff $E_\Lambda$

Next we consider how the TSM+CHEB results change with energy cutoff $E_\Lambda$. In the right panel of Fig. 25(b) we see the time evolution of $\sigma(t)$ for the quench $(m, g) = (1, 0.1) \rightarrow (1, 0.2)$ at fixed order of expansion $N_{max} = 2000$ on the system of size $R = 25$. Results are present for three different cutoff energies $E_\Lambda$, showing that there is very good convergence in the energy cutoff. We note also that $t_{br}$ decreases with increasing $E_\Lambda$ (not shown).
3. With volume $R$

Finally we examine results of the TSM+CHEB at different system sizes $R$. Time evolution is induced by the quench $(m,g) = (1,0.2) \rightarrow (1,0.1)$ with fixed energy cutoff $E_A = 9$ and expansion order $N_{\text{max}} = 2000$. In Fig. 26(a) we start from the ground state of the initial Hamiltonian and compare results for $R = 25, 40$. The two data sets are almost identical and show no signs of finite size revivals. Similarly good agreement is observed when expansion order is felt.

For quenches starting from higher states, we need to be careful with the observable computed. Starting from the first meson, we know that the number of flipped spins, compared to the ordered ground state, will be $O(1)$ and hence local observables will vary as $1/R$. To counter this, we compute the total magnetization by multiplying the local magnetization by $R$ (using translational invariance of the states). We faced similar issues with the finite size scaling analysis of observables in the meson states in the main text. This is shown in Fig. 26(b), where we start from the first meson state. We see similar excellent agreement of the time-evolution of observables at different system sizes $R$ as in the left panel.

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9 For neighboring energy eigenvalues $E$ and $E’ (E’ > E)$ this function satisfies $A(E’) - A(E) \propto e^{-R/\tilde{R}}$, with $\tilde{R}$ being the system size and $R$ being some (possibly $E$-dependent) constant.

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53. F. Alet and N. Laflorencie, “Many-body localization: An
introduction and selected topics,” C. R. Phys. (2018),
https://doi.org/10.1016/j.crhy.2018.03.003.
arXiv:1403.7837 [math-ph].
78. We say that the EEVs and MCE agree when the EEVs fall within one standard deviation of the MCE. The spread of the EEVs reflects the second term in Eq. (1), which features the random variable $R_{a,b}$, which has mean zero and unit variance. We see that the standard deviation decreases with increasing system size $R$, reflecting the extensivity of the thermodynamic entropy $S(E)$.
82. We note that in the context of magnetic systems, one might conclude that the mesons are not present in the spectrum as one does not see signatures in dynamical spin-spin correlation functions: but from Fig. 2 one could conclude that they are indeed there, though their signal is washed out by the thermal signal from the large number of states in the multiparticle continuum.

Interesting, we note that this particular quench has essentially no finite size effects, see the appendix E for further information. A similar lack of finite-size effects was seen in quenches of the lattice Ising chain perturbed by a longitudinal field.96.


