Dynamics of Price Formation in Financial Markets

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2.1 Introduction

Prices in financial markets should obviously reflect information. However, to what extent prices reflect information, has been and still is the cause of debate among financial economists. It is understood, nowadays, that the price-information relation is not as simple as contented by the efficient market hypothesis, or the neoclassical work on price formation. Besides the empirical support for a more subtle nature of the information role of prices, theoretical work on price formation has yielded many insights. The mechanics of most of these theoretical models are driven by the rational expectations paradigm. This concept captures the flow of information from agents to prices and vice versa. Since its introduction to the area of finance by Grossman[1976,1977], this notion has become the primary modeling tool for trade under differential or asymmetric information. In particular, the CARA-Gaussian subclass of noisy rational expectations models has proven to be useful in the characterization of price formation. Within this thesis we adopt this approach as well, and additionally try to capture the dynamics of trade under information asymmetry. The extension toward multi-period variations is not a trivial task however. As Huang and Litzenberger stated in 1988,

'We also know very little about the rational expectations equilibria in multi-period economies [...] the multi-period extension is a formidable task. This extension, when successful, will give rise to a much richer model. Questions such as to what extent historical prices contain information about future prices and whether volume of trade can play any informational role can only be answered in models of a multi-period economy.'

In spite of the apparent technical difficulty, over the last decade, the literature has seen many successful attempts. This literature is the main focus of this survey. It provides a background against which our efforts in this area are to be evaluated.
It should be noted that this survey does not have the intention to be complete or cover the whole strand of literature concerning the topic\textsuperscript{1}. The literature has grown to be extremely rich in this area. Hence, we abstain from elaborate discussions on the variety of subfields that have arisen, and we restrain ourselves to the competitive noisy rational expectations approach. Within this area our attention is drawn to the more recent multi-period models, where we have tried to include the most relevant contributions.

This chapter is set up as follows. In section two, we start with the illustration of the difference between the conventional Walrasian market-clearing mechanism and the rational expectations paradigm. We discuss the problem with fully revealing equilibria, and motivate the need for noise. We proceed by solving explicitly a static noisy rational equilibrium. The generic structure of the model allows us to subsequently derive the Diamond and Verrecchia\textsuperscript{[1982]} model, and the Hellwig\textsuperscript{[1980]} limiting economy. We conclude the section with discussion of the Admati\textsuperscript{[1985]} multi-asset extension. In section three, we focus on multi-period rational expectations models. We start the section with a discussion of the general problem involved in solving for these equilibria. Subsequently, we discuss the various multi-period models that have been suggested, starting with the early two period models, and concluding with the infinite period models by Wang. The final section of this chapter elaborates on the differences between the models discussed, and the approach that is adopted in this thesis.

2.2 The Informational Feedback from Prices: Rational Expectations

In a market where assets are traded whose future payoff is uncertain, agents need to make forecasts for an optimal allocation decision. The expectations formed by agents thus play a crucial role in the formation of prices. If information is dispersed throughout the economy the prior expectations of agents will generally differ. Prices may, however, aggregate individual pieces of information through the demands of agents and consequently provide a signaling function to investors. Hayek \textsuperscript{[1945]} considered this informational role of prices. In his view, the price system should serve as a communication device, resolving informational differences between agents. Agents’ investment decisions are then optimal, as if their actions are coordinated by an invisible hand.

The neoclassical framework is insufficiently rich to capture this role of the price system. To

\textsuperscript{1}There are excellent reviews available that do encompass a larger scope of the literature on trade under differential information. The reader is referred to O’Hara\textsuperscript{[1994]} for an overview of the market microstructure literature, and to Brunnermeier\textsuperscript{[1998]} for more emphasis on competitive rational expectations models.
illustrate this, consider how the classical Walrasian framework would cope with differentially informed agents. Assume that an asset is traded with uncertain payoff \( \bar{u} \). \( N \) agents are present, with agent \( i \) initially endowed with \( z_i \) of the asset. Information dispersion is present through the assumption that each agent receives a private signal \( \bar{y}_i \) regarding \( \bar{u} \). Given a utility function \( U(W) \), agents choose their demand such that they maximize their expected utility conditional on their private information. Their demand schedule, \( d_i(P, \bar{y}_i) \), is formally represented by

\[
d_i(P, \bar{y}_i) = \max_d E[U(W(d, \bar{u}, z_i)) | \bar{y}_i], \text{ s.t. } z_i P = d_i P + M
\]

where \( M \) is the amount of consumption good invested in the riskless asset. If agents have CARA utility functions with Arrow-Pratt risk measure \( \rho_i \), the solution to this maximization problem is straightforward, and given by \(^2\)

\[
d_i(P, \bar{y}_i) = \frac{E[\bar{u} | \bar{y}_i] - P}{\rho_i \text{var}[\bar{u} | \bar{y}_i]}
\]

The equilibrium price \( \bar{P} \) then follows by demanding that the market clears. That is, \( \bar{P} \) is the solution to

\[
\frac{1}{N} \sum_i d_i(P, \bar{y}_i) = \frac{1}{N} \sum_i z_i \equiv Z
\]

Using (2.1), one finds that the market-clearing condition can be written as

\[
\bar{P} = \left[ \frac{1}{N} \sum_i \frac{1}{\rho_i \text{var}[\bar{u} | \bar{y}_i]} \right]^{-1} \left[ \frac{1}{N} \sum_i \frac{E[\bar{u} | \bar{y}_i]}{\rho_i \text{var}[\bar{u} | \bar{y}_i]} - Z \right]
\]

This equilibrium price may at first sight be appealing. It indeed aggregates information. In fact, each individual influences the price with his expectation of \( \bar{u} \). The magnitude of this impact is determined by both the precision of his information, and his risk aversion level; individuals with more precise information have a stronger impact. We also observe a risk premium: if the excess supply \( Z \) is large, price will be low, and hence the expected excess return will be high. Moreover, this price discount is determined by the both the risk aversion and the information precision level of the economy. If the average individual is less risk averse, or has better private information, the risk adjustment factor is smaller. An equilibrium such as the above was studied by Lintner (1969).

\(^2\)This solution can be simply found as follows. Since

\[
E[U(M + d, \bar{u}) | \bar{y}_i] = -E[\exp(-\rho_i(M + d, \bar{u})) | \bar{y}_i] \\
= -\exp(-\rho_i M - \rho_i d, E[\bar{u} | \bar{y}_i]) + \frac{1}{2} \rho_i^2 d^2 \text{var}[\bar{u} | \bar{y}_i])
\]

Maximizing with respect to \( d_i \) yields the expression.
However, there is a problem with this formulation. Although agents know the market-clearing price upon submitting their demands, they do not use the information reflected in prices. In fact, with hindsight agents would generally want to re-contract, in light of this additional information. Hence, this equilibrium is not market clearing conditional on the information contained in prices. The rational expectations concept resolves this difficulty. In a rational expectations framework, the requirement is that agents optimally use all information available, and, in particular, the information displayed through prices. Agents' demand schedules are thus formed conditional on the market-clearing price in conjunction with private and/or public information. Returning to our example, this implies that the maximization problem is replaced by

$$d_i(\hat{P}, \tilde{y}_i) = \arg\max_d E[U(M + d\hat{u})|\tilde{y}_i, \hat{P}] \text{ subject to } z_i\hat{P} = d_i\hat{P} + M.$$ 

Hence, in contrast to the classical Walrasian models, market prices not only influence the demand of each agent through their budget constraint, but also through their conditional expected utility function. Accordingly, the market-clearing condition changes to

$$\frac{1}{N} \sum_i \frac{\mathbb{E}[\hat{u}|\tilde{y}_i, \hat{P}] - \hat{P}}{\text{var}[u|\tilde{y}_i, \hat{P}]} = Z.$$ 

Indeed, the solution to this pricing problem depends on the functional relation between $\hat{P}$ and the expectation of $\hat{u}$. The rational expectations approach demands that all agents conjecture the same pricing functional, and, that agents' (subjective) beliefs about the price function and the probability of outcomes coincides with the actual market-clearing price and probabilities. The advantage of this approach, first proposed by Muth[1960,1961], is that systematic errors or biases in the inference of agents, are ruled out, leading to a plausible means to the handling of expectations.

The determination of equilibrium under rational expectations can be considered as a fixed-point problem in the space of functions that map supplies and information signals to prices. Denote this space of functions by $P$. For an economy with a set $\mathcal{N}$ of agents, each endowed with information set $\mathcal{I}_i$, and endowment $z_i$, we have $\forall f \in P, f : (\mathcal{I}_i^i, \{z_i\}) \rightarrow \mathbb{R}_+$. Given a realization of the price $\hat{P}$, agents can derive a conditional distribution regarding the signals that span the information set of each agent. Consequently, agents' demands are formed using this conditional distribution. If agents conjecture a price function $f$, their demand $d_i(\cdot)$ can therefore be written as $d_i = d(P, y_i; f)$. The market-clearing price $P$ is then implied through the requirement that the market clears, i.e. $P$ solves $\hat{Z} = \Sigma_i d_i(\hat{P}, y_i; f)$. Denote the solution of this problem by $f'$, i.e. $\hat{Z} = \Sigma_i d_i(f', y_i; f)$. Next, define the transformation $T$ that transforms
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Let $f$ to $f'$, i.e. $Tf = f'$. A rational expectations equilibrium is then defined as a fixed point of this transformation, that is $Tf_{eq} = f_{eq}$. The functional form that is usually considered, certainly in the CARA-Gaussian framework, imposes linearity in the state variables. In the above framework, one would for instance assume that $\bar{P}_t = \pi_0 + \pi_t \sum y_t$.

Grossman\textsuperscript{3}[1976] was the first\textsuperscript{4} to use the rational expectations concept to consider the aggregation of information through prices. He shows that this aggregation can be perfect, and ultimately lead to a Pareto optimal equilibrium. In fact, a social planner with complete knowledge of the economy could not achieve a better allocation than in the fully revealing rational expectations equilibrium. This result is exactly what Hayek\textsuperscript{5}[1945] had contended. Agents need not to look further than market prices to know all they need to know to act optimally. This type of equilibrium is called informationally efficient. The signal provided by the price is a sufficient statistic for individual private signals. Beside the information aggregation function of markets, another informational role for prices is in the form of a transmitter of information. Typically, this could occur, when one group of traders possesses superior information compared to the other group. Prices then transmit information from better-informed agents to uninformed agents, rather than to aggregate individual pieces of information. Grossman\textsuperscript{7}[1977] explores such a situation explicitly within the context of futures markets for commodities (wheat). A so-called fully revealing equilibrium results, where the private signals of informed agents are completely revealed through the market-clearing price. As Grossman\textsuperscript{7}[1977] notes 'spot prices act like a xerox machine, freely distributing the information of the informed firms to the uninformed firms'.

This type of full revelation of private information always occurs if the uncertainty about fundamentals is spanned by the information signals available to agents. In the Grossman\textsuperscript{7}[1977] model, the single source of uncertainty is revealed through the observation of the price realization. Generally, if there are as many assets as sources of noise, a rational expectations equilibrium exists that resolves all uncertainty\textsuperscript{5} (Allen\textsuperscript{1982}). Though Grossman\textsuperscript{1976,1977} showed that such revelation may lead to an optimal allocation, fully revealing equilibria have several conceptual problems. In case of the aggregation of information, prices’ being sufficient

\textsuperscript{3}For an overview of the contributions of Sanford Grossman to the rational expectations literature, see Admati\textsuperscript{1991}.

\textsuperscript{4}Another early contribution to this literature is Green\textsuperscript{1977}.

\textsuperscript{5}Apart from the number of assets, the type of assets also influences its informational content. As such, financial innovations may be driven by enhancing informational efficiency of financial markets (see for instance Boot and Thakor\textsuperscript{1993}, Duffie and Rahi\textsuperscript{1995}, Allen and Gale\textsuperscript{1994}).
statistics for the aggregate knowledge in the market, makes agents' demands independent of their private signal. A paradox is the result: if agents' demands do not depend on private signals, how can information be incorporated in prices? In case of disclosure of private information, a problem arises when information is costly. If prices are fully revealing, agents will never have an incentive to acquire costly information, because they cannot exploit their initial comparative advantage if prices freely distribute all information. Grossman and Stiglitz [1980] point out this problem in their seminal work 'On the Impossibility of Informationally Efficient Markets'. To resolve the problem of fully revealing prices, Grossman and Stiglitz [1980] assume that another source of uncertainty is present that distorts the signal displayed through prices. The result is that prices are only partially revealing. This gives informed traders the opportunity to exploit their private information, such that they can offset the cost of information acquisition. Grossman and Stiglitz [1980] also consider the interior equilibrium that arises when agents have the choice to acquire costly information. Given the diminishing returns when the number of informed traders grows, they show that an interior equilibrium may exist. The equilibrium allows them to consider the informativeness of the price system, and to elaborate on several conjectures. The most important observation concerns the efficient market hypothesis. Grossman and Stiglitz [1980] argue that

'Efficient market theorists seem to be aware that costless information is a sufficient condition for prices to fully reflect all available information (..); they are not aware that it is a necessary condition. But this is a reductio ad absurdum, since price systems and competitive markets are important only when information is costly'.

The implication is that a price system should only partially reveal private information in order to reveal information at all. This has come to be known as the Grossman-Stiglitz paradox. Full revelation of information can be precluded through the incorporation of additional noise. That noise in an economy may play a profound role in general is emphasized by

\footnote{A related problem with fully-revealing equilibria is the so-called Hirshleifer effect. Hirshleifer [1971] shows how full resolution of uncertainty may preclude trade, that otherwise would have improved risk sharing. If risk averse agents face uncertainty, where one group of traders is better off in one state and the other group is better off in the other, trade may improve risk sharing. However, under full revelation of uncertainty the group that is better off will not want to trade anymore. Hence, no trade occurs in the full revelation equilibrium. Moreover, with the anticipation of no-trade, trade will not take place, leading to no resolution of uncertainty, while pertaining the inefficient allocation of risk.}
Black[1986] in his infamous ‘Noise’ paper. Noise introduces an additional degree of freedom rendering the market incomplete. In the standard CARA-Gaussian framework, noise typically enters the market-clearing condition in the following fashion

\[ \sum d_t \propto X \]

where \( X \) represents the noise term. In words, the excess demand scales with noise. The existence of such noisy excess supply can be motivated through the presence of liquidity traders, who trade on the basis of non-informational reasons, perhaps motivated by private liquidity shocks, or a hedging need given some random endowment with the risky asset. An alternative is proposed by Wang[1994], who introduces noise through informed agents who have the availability over a private production technology that causes their demand for the public risky asset include a non-informational component. In Campbell, Grossman and Wang[1993] noise is incorporated through the assumption of a stochastic average risk aversion level of the economy. From a strict mathematical viewpoint, all of these approaches are equivalent. More complex specifications of additional uncertainty (such as in Romer[1993]) often impair the CARA-Gaussian elegance by introducing non-linearities. In these cases, the only means to extract results is by numerical procedures.

A criticism specific to the CARA-Gaussian framework is that the distributional properties of the state variables allows for unrealistic outcomes in the form of negative stock prices and risk aversion levels. In spite of this flaw, the normality assumption is common in this strand of literature. One justification is that it should be considered a first order approximation to a more complex distribution. Alternatively, one can argue that the CARA-Gaussian framework is equivalent to a model where more complex distributions are allowed, and agents have mean-variance utility functions (see Campbell, Grossman and Wang[1993]).

Though the inclusion of additional noise by Grossman and Stiglitz[1980] resolves the conceptual difficulties with informationally efficient equilibria, their model implies a schizophrenia of rational investors. Though agents rationally anticipate the complex relation between information and prices, they neglect their individual impact on the market-clearing price. In Hellwig[1980], this problem is resolved under the assumption of a large market, rendering individuals with an infinitesimally small impact on prices. Another novel feature in Hellwig[1980] is that it studies the information aggregation ability of financial markets, endowing each agent with a private piece of information. Another approach that considers an economy with dispersed information can be found in Diamond and Verrecchia[1981]. In the following we consider a generic framework that incorporates both of these approaches.
Assume that there are $N$ traders, which we index by $i \in \mathcal{N} = \{1, 2, ..., N\}$. Each individual is endowed with a private endowment given by $z_i$. Investors can trade a riskless asset yielding a fixed gross return of unity, and a risky asset with uncertain payoff which we denote by $\bar{u}$, with $\bar{u} \sim N[w_0, h_0^{-1}]$. They do so with the objective of maximizing the expectation of their exponential utility at the consumption date that coincides with the liquidation date. We denote the risk aversion coefficient of investor $i$ by $\rho_i$. Each trader receives a private information signal $\tilde{y}_i$, regarding $\bar{u}$. These information signals are imperfect through the perturbation of a normally distributed error term unique to each investor, i.e. $\tilde{y}_i = \bar{u} + \tilde{\xi}_i$, with $\tilde{\xi}_i \sim N[0, s_i^{-1}]$.

The initial endowments $\{z_i\}_{i\in\mathcal{N}}$ are distributed normal with mean $z_0$ and variance $V = t^{-1}$. Additionally, there is a group of liquidity traders that adds noise to the per capita excess supply. This extra per capita supply noise is represented by $\tilde{X}$, with $\tilde{X} \sim N[0, x]$. We assume that the random vector $(\tilde{X}, z_0, ..., z_N, \tilde{\xi}_0, ..., \tilde{\xi}_N)$ has a non-singular variance-covariance matrix.

Given the distribution of imperfect information signals across investors, the collective knowledge in the economy is superior to any individual information set. Therefore, the signal present in the market-clearing price can provide agents with information beyond their private signals. Rational expectations enters through the requirement that agents extract information from prices, and do so optimally. Their demand is based on their conjecture of a price function, assumed to be linear in the state variables. Explicitly, we assume that agents conjecture a price function of the form

$$\tilde{P} = \pi_0 + \sum_{i=1}^{N} \pi_i \tilde{y}_i - \gamma \tilde{Z} = \pi_0 + \pi \bar{u} + \sum_{i=1}^{N} \pi_i \tilde{\xi}_i - \gamma \tilde{Z}$$  \hspace{1cm} (2.2)

where $\tilde{Z} \equiv \frac{1}{N} \sum z_i + \tilde{X}$, and $\pi \equiv \sum_{i=1}^{N} \pi_i$. The form of the demand schedule of agent $i$ is standard and given by

$$d_i(\tilde{P}, \tilde{z}_i, \tilde{y}_i) = \frac{E[\bar{u}|\tilde{P}, \tilde{z}_i, \tilde{y}_i] - \tilde{P}}{\rho_i \text{var}[\bar{u}|\tilde{P}, \tilde{z}_i, \tilde{y}_i]}$$  \hspace{1cm} (2.3)

The equilibrium is determined by the solution to the market-clearing condition

$$\frac{1}{N} \sum_{i=1}^{N} d_i(\tilde{P}, \tilde{z}_i, \tilde{y}_i) = \tilde{Z}$$

The next step in the usual approach for solving rational expectations equilibria is the derivation of the conditional expectations and variances of each investor concerning $\bar{u}$. In this case, we

\footnote{Note that we have made a simplifying assumption in this structure. We assume that the individuals' endowment $z_i$ affects prices in a manner independent of his preferences. This is good enough for the derivation of both Hellwig[1980] and Diamond and Verrecchia[1982]. Generally, one should replace the term $\gamma Z$ by $\gamma L X + \sum_i \gamma_i z_i$ in order to find a solution. This extension is straightforward, though more tedious.}
do so by means of the projection theorem. First note that the vector \((\bar{u}, \bar{z}, \bar{y}, \bar{P})\) has mean 
\[(u_0, z_0, u_0, \pi_0 + \pi u_0 - \gamma z_0)\] 
and variance-covariance matrix
\[
M_i = \begin{pmatrix}
    h_{0i}^{-1} & 0 & 0 \\
    0 & h_{0i}^{-1} & 0 \\
    h_{0i}^{-1} & 0 & h_{0i}^{-1} + s_i^{-1} \\
    \pi h_{0i}^{-1} & -\gamma_i/t & \pi h_{0i}^{-1} + \pi_i s_i^{-1} \\
end{pmatrix}
\]

This expression can be used to determine the conditional expectation of trader \(i\). Applying
the projection theorem (see appendix A), the conditional expectation of investor \(i\) is linear in
his information signals,
\[
E[\bar{u}|\bar{P}, \bar{y}_i, \bar{z}_i] = \alpha_{0i} + \alpha_{1i}\bar{y}_i + \alpha_{2i}\bar{P} + \alpha_{3i}\bar{z}_i, 
\]
with the coefficients \(\alpha_{ij}\) given by
\[
\alpha_{0i} = b_i^{-1}u_0h_0 \left( \left( \sum_{j=1}^{N} \frac{\pi_j^2}{s_j} \right) + \gamma^2 x^{-1} - s_i^{-1}\pi_i^2 + \gamma^2(1 - N^{-1})/tN \right) 
+ b_i^{-1}(\gamma z_0 - \pi_0)(\pi - \pi_i) 
\]
(2.5)
\[
\alpha_{1i} = b_i^{-1} \left( s_i \gamma^2(1 - N^{-1})/tN + s_i \gamma x^{-1} + s_i \left( \sum_{j=1}^{N} \frac{\pi_j^2}{s_j} \right) - \pi \pi_i \right) 
\]
(2.6)
\[
\alpha_{2i} = b_i^{-1}(\pi - \pi_i) 
\]
(2.7)
\[
\alpha_{3i} = b_i^{-1}\gamma N^{-1}(\pi - \pi_i) 
\]
(2.8)
where
\[
b_i = \left[ \left( \sum_{j=1}^{N} \frac{\pi_j^2}{s_j} \right) + \gamma^2(1 - N^{-1})/tN + \gamma^2 x^{-1} - \pi_i^2 s_i^{-1} \right] (s_i + h_0) + (\pi - \pi_i)^2 
\]
(2.9)

The individual \(i\)'s conditional variance of the liquidation value is represented by \(\beta_i\)
\[
\beta_i = \operatorname{var}[\bar{z}|\bar{P}, \bar{z}_i, \bar{y}_i] 
= b_i^{-1}s_i^{-1} \left[ \left( \sum_{j=1}^{N} \frac{\pi_j^2}{s_j} \right) + \gamma^2 x^{-1} - \pi_i^2 s_i^{-1} + \gamma^2(1 - N^{-1})/tN \right] 
\]
(2.10)

Inserting (2.11) and (2.4) in the demand schedule of investor \(i\) (2.3), the market-clearing
condition gives us the following equation specifying the equilibrium:
\[
\frac{1}{N} \sum_{i=1}^{N} \alpha_{0i} + \alpha_{1i}\bar{y}_i + \alpha_{3i}\bar{z}_i + (\alpha_{2i} - 1)\bar{P} = \bar{z} 
\]
(2.12)
This implies that the market-clearing price has the structure
\[
\tilde{P} = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1 - \alpha_{2i}}{\rho_i \beta_i} \right]^{-1} \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\alpha_0 + \alpha_i \tilde{y}_i + \alpha_{3i} \tilde{z}_i}{\rho_i \beta_i} - \tilde{Z} \right]
\]

Rational expectations requires that the conjecture of agents concerning the pricing functional, given by relation (2.2), coincides with this functional form\(^8\). Moreover, this conjecture should be valid for all realizations of the random vector \((\tilde{P}, \{\tilde{y}_i\}_{i \in \mathcal{N}}, \{\tilde{z}_i\}_{i \in \mathcal{N}}, \tilde{X})\). Imposing these requirements leads to the following specification for the constants of the pricing functional
\[
\pi_0 = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1 - \alpha_{2i}}{\rho_i \beta_i} \right]^{-1} \left[ \frac{\alpha_0}{N} \sum_{i=1}^{N} \frac{1}{\rho_i \beta_i} \right]
\]
\[
\pi_i = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1 - \alpha_{2i}}{\rho_i \beta_i} \right]^{-1} \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\alpha_i \tilde{y}_i}{\rho_i \beta_i} \right]
\]
\[
\gamma = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1 - \alpha_{2i}}{\rho_i \beta_i} \right]^{-1} \left[ 1 - \frac{1}{N} \sum_{i=1}^{N} \frac{\alpha_{3i}}{\rho_i \beta_i} \right]
\]

These relations are the starting point for the discussion of two special cases of this model that can be found in Diamond and Verrecchia[1981] and Hellwig[1980].

To enhance tractability, Diamond and Verrecchia[1981] make some simplifying assumptions. First, each trader is assumed to have a risk tolerance of unity, i.e. \(\rho_i = 1 \ \forall_{i \in \mathcal{N}}\). Second, traders receive information signals that, though different, have the same precision, i.e. \(s_i = s \ \forall_{i \in \mathcal{N}}\). Finally, the exogenous liquidity component is absent, i.e. \(x^{-1} = 0\). Given the equivalence across traders, these assumptions imply that the pricing functional should be symmetric in \(\pi_i\). In accord with this, we impose that \(\pi_i = \pi/N\). Observe that within this setup, each trader has four sources of information: the prior that he shares with all other traders, his private information signal, the market-clearing price, and his own endowment. The latter also benefits the precision of his estimates, as it can explain part of the error in the market-clearing price.

Diamond and Verrecchia[1981] solve the pricing problem explicitly using the standard approach (i.e. by means of the projection theorem). We can apply the above relations to derive the equilibrium immediately. It is easily shown\(^9\) that upon implementing the specifics of their assumptions, the following expressions for the equilibrium pricing coefficients are

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\(^8\)Note that this requirement can only be met, if we let \(\alpha_{3i}/(\rho_i \beta_i)\) be independent of \(i\). In both Diamond and Verrecchia[1982] and Hellwig[1980] this is indeed the case. See previous footnote.

\(^9\)Using the expressions (2.5)-(2.11), the update rule of each trader is characterized by the following coefficients
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found:

\[ \pi_0 = \frac{u_0 h_0 (ts + 1) + ts z_0 (N - 1)}{ts (Ns + h_0) + s + h_0}, \]

\[ \sigma = \frac{s}{ts (Ns + h_0) + s + h_0}, \]

\[ \gamma = \frac{s}{ts (Ns + h_0) + s + h_0}. \]

The corresponding pricing functional is given by

\[ \tilde{P} = K^{-1} \left[ u_0 h_0 (ts + 1) + ts z_0 (N - 1) + s (st N + 1) \frac{1}{N} \sum y_i - (st N + 1) \frac{1}{N} \sum \tilde{z}_i \right] \]

where

\[ K = ts (Ns + h_0) + s + h_0 \]

Observe that the price is a weighted average of the common prior of the market \( u_0 \) and the information signals of each agent, \( \frac{1}{N} \sum y_i \), distorted by the uncertain per capita excess supply \( \frac{1}{N} \sum \tilde{z}_i \).

Recall that the ex ante information of each trader is given by \( h_0 \) with mean \( u_0 \). After observing the market-clearing price, each trader now has a better estimate with mean

\[ \mu_i = u_0 + \frac{st x_i + s (\tilde{y}_i - u_0)}{st (Ns + h_0) + (h_0 + s)} + \frac{N s t}{N s t + 1} (\tilde{P} - u_0) \]

(Dropping the indices)

\[ \alpha_0 = b^{-1} u_0 h_0 \left( N - 1 \right) \frac{\pi^2 t + \gamma^2 s}{N^2 s t} + b^{-1} (\gamma z_0 - \pi_0) (\pi - \pi/N) \]

\[ \alpha_1 = b^{-1} \gamma^2 \left( N - 1 \right) \frac{s}{N^2 s t} \]

\[ \alpha_2 = b^{-1} (\pi - \pi/N) \]

\[ \alpha_3 = b^{-1} \gamma N^{-1} (\pi - \pi/N) \]

where \( b = (N - 1) s^{-1} t^{-1} (h_0 (\pi^2 t + s \gamma^2) + s (\pi^2 N t + s \gamma^2)) \) and \( \beta = b^{-1} s^{-1} (N - 1) \frac{s^2 t + s \gamma^2}{N^2 s t} \). Indeed, all \( \alpha \)s are independent of \( i \). We can thus safely drop the subscripts \( i \). The equations that pin down the equilibrium, are then given by

\[ \gamma = \frac{1}{N} \frac{\gamma s t \pi - \pi^2 t - s \gamma^2}{N (h_0 (\pi^2 t + s \gamma^2) + s (\pi^2 N t + s \gamma^2)) - st \pi} \]

\[ \pi = \frac{1}{N} \frac{s^2 \gamma^2}{N (h_0 (\pi^2 t + s \gamma^2) + s (\pi^2 N t + s \gamma^2)) - st \pi} \]

\[ \pi_0 = \frac{u_0 h_0 (\pi^2 t + s \gamma^2) + st (\gamma z_0 - \pi_0) \pi}{N (h_0 (\pi^2 t + s \gamma^2) + s (\pi^2 N t + s \gamma^2)) - st \pi} \]

Though of third order, the solution to the above equation is simple (and unique), and given by the coefficients presented in the text.
and precision $h \equiv \sigma_i^{-2} = h_0 + \frac{s(Ns+1)}{1+sN}$. Indeed, the updated expectation is a weighted average of the common prior, the trader's private signal, and the prevailing price $\hat{P}$, corrected with the trader's knowledge of the prices' error. Observe that the precision of the ex post update diminishes with increasing noise, and increases with the number of traders and the precision of the private signal. In the limiting economy, where the number of traders approaches infinity, uncertainty is resolved completely. The reason is that supply shocks are uncorrelated: upon applying the law of large numbers, the variance of the supply noise vanishes. In Grundy and McNichols[1989], agents are also endowed with random supply shocks. They, however, assume that as the limit of traders grows, additionally the variance of individual supply shocks grows. This results in an equilibrium in which the variance of the per capita excess supply remains non-zero.

Diamond and Verrecchia[1981] report two limiting cases that are of interest. First, consider the limit in which the variance of supply noise, $\sigma^2$, goes to infinity. In that case, the ex post information precision becomes, $\lim_{\sigma^2 \to 0} h = h_0 + s$ which equals the ex ante level. In other words, there is no information contained in the price, and agents can only use their private information signal. Second, consider the limit of no supply noise. The ex post information precision now becomes $\lim_{\sigma^2 \to 0} h = h_0 + Ns$, which equals the information precision of the combined knowledge in the market. Indeed, here, prices convey information perfectly, as if each trader additionally receives the signals of all other traders. However, as Diamond and Verrecchia[1981] note, this result can only be seen as a limiting case of a partially revealing equilibrium. In fact, it is impossible to exist. In their words, 'But when price is fully revealing, individuals' beliefs are fixed by aggregate information. This precludes beliefs from depending on private data, which in turn precludes price from depending on private data'.

Although Diamond and Verrecchia[1981] show how an economy may aggregate information, the price taking behavior of agents contrasts with their knowledge of their impact on price. This so-called schizophrenia of investors, is resolved in Hellwig[1980], by considering the actions of traders within the context of a very large (infinitely large) market. Individual agents can consequently not influence the price due to their mere smallness. Though Hellwig[1980] also considers existence of a finitely sized economy, his limiting economy has been used widely in the literature, and is the focus of the following discussion.

Hellwig[1980] does not include shocks to private endowments of investors. In our generic economy, an identical situation occurs when we impose $\alpha^{-1} = a_M = 0$. The equations speci-
fying the regression coefficients of investor \( i \) then become

\[
\alpha_{0i} = b_i^{-1} \alpha_0 h_0 \left( \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) + \gamma^2 x^{-1} - s_i^{-2} \pi_i^{-2} \right) + b_i^{-1} (\gamma \alpha_0 - \pi_i) (\pi - \pi_i)
\]

\[
\alpha_{1i} = b_i^{-1} \left( s_i \gamma^2 x^{-1} + s_i \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) - \pi \pi_i \right)
\]

\[
\alpha_{2i} = b_i^{-1} (\pi - \pi_i)
\]

where

\[
b_i = \left[ \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) + \gamma^2 x^{-1} - \pi_i^{-2} s_i^{-1} \right] (s_i + h_0) + (\pi - \pi_i)^2
\]

and

\[
\beta_i = b_i^{-1} s_i^{-2} \left[ \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) + \gamma^2 x^{-1} - \pi_i^{-2} s_i^{-1} \right]
\]

Again we can identify our parameters

\[
\gamma = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1 - \alpha_{2i}}{\rho_i \beta_i} \right]^{-1}
\]

\[
\pi_i = \gamma \left[ \frac{\alpha_{1i}}{N \rho_i \beta_i} \right]
\]

\[
\pi_0 = \gamma \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\alpha_{0i}}{\rho_i \beta_i} \right]
\]

Explicitly the relation for \( \pi_i \) becomes

\[
\pi_i = \frac{\gamma s_i s_i \gamma^2 + s_i \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) x - x \pi \pi_i}{\rho_i N s_i \left( \sum_{j=1}^{N} \frac{s_j^2}{s_j^2} \right) x + s_i \gamma^2 - x \pi^2_i}
\]  \(\text{(2.13)}\)

Of interest to us are the asymptotic properties of this price coefficient. As Hellwig shows, for large \( N \), the fraction \( Q_i \equiv \pi_i / \gamma \) is given by\(^{10}\) \( Q_i(N) = \frac{s_i}{\rho_i N} + o(1/N^2) \). Armed with these
asymptotic properties of $\pi_i$, we can approximate the relations for $\alpha_{ij}$ and $\beta_i$, up to zeroth order in $N^{-1}$. It follows that we can write

$$\alpha_{0i} = b_i^{-1}u_0h_0\gamma^2x^{-1} + b_i^{-1}(\gamma z_0 - \pi_0)\pi + o(N^{-1})$$

$$\alpha_{1i} = b_i^{-1}s_i\gamma^2x^{-1} + o(N^{-1})$$

$$\alpha_{2i} = b_i^{-1}\pi + o(N^{-1})$$

with

$$b_i = \gamma^2x^{-1}(s_i + h_0) + \pi^2 + o(N^{-1})$$

and

$$\beta_i = b_i^{-1}s_i^{-1}\gamma^2x^{-1} + o(N^{-1})$$

Hence, the equations specifying the pricing coefficients become

$$\pi_i = \frac{\gamma \sqrt{s_i}}{\rho_i} + o(1/N)$$

$$\gamma = \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{\gamma^2 s_i + \gamma^2 h_0 + x\pi^2 - x\pi}{\rho_i \gamma^2} + o(1/N) \right]^{-1}$$

$$\pi_0 = \frac{\gamma}{N} \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{u_0 h_0 \gamma^2 + x\gamma z_0 \pi - x\pi \pi}{\rho_i \gamma^2} + o(1/N) \right]$$

Before we proceed to the limiting case, note that for large $N$, the fraction $\pi_i/\pi_j \rightarrow \frac{s_i\beta_j}{\rho_i s_j}$. In other words, the relative importance of each agents' information is proportional to his risk tolerance and the precision of his information. Indeed, where in the Grossman[1977] model of information aggregation, this weight was only determined by the precision of each agents' information signal, here, his risk tolerance also contributes to the impact on the pricing functional. Observe that this means that for finite $N$, information is aggregated in a suboptimal way, since the most efficient estimator is given by $\bar{y} = \sum s_i y_i$.

Moreover, we also have that

$$Q_i = \frac{s_i s_j \left( \sum_{j=1}^{N} \frac{Q_j^2}{s_j} \right)}{\rho_i N} x + s_i - xQ_i^2 - xQ_j \sum_{j \neq i} Q_j > \frac{s_i}{\rho_i N} \left( 1 - \frac{xQ_i \sum_{j \neq i} Q_j}{s_i \sum_{j} Q_j} \right) > \frac{s_i}{\rho_i N} \left( 1 - \frac{x}{\rho_i N} \frac{s_i}{s_j} \frac{s_j}{\rho_j} \right) > \frac{s_i}{\rho_i N} \left( 1 - \frac{x}{\rho_i N} \frac{s_i}{s_j} \frac{s_j}{\rho_j} \right)$$

Hence, for large $N$, $Q_i(N) = \frac{s_i}{\rho_i N} + o(1/N^2)$. 
The economy that results, taking the limit of \( N \to \infty \), is characterized by the relations

\[
\begin{align*}
\pi &= \frac{(xs^2r^2 + s)}{s + h_0 + xs^2r^2} \\
\gamma &= \frac{xsr + r^{-1}}{s + h_0 + xs^2r^2} \\
\pi_0 &= \frac{u_0h_0 + xz_0sr}{s + h_0 + xs^2r^2}
\end{align*}
\]

where we defined \( r \equiv \int \frac{1}{\rho_i} \, dt_i \), and \( s \equiv \int \frac{s_i}{\rho_i} \, dt_i \). These constants can be interpreted as the average risk tolerance and a proxy for the average information precision in the economy, respectively. Next, consider the pricing functional

\[
\tilde{P} = \pi_0 + \pi \tilde{u} + \sum_{i=1}^{N} \pi_i \tilde{e}_i - \gamma Z
\]

Applying the law of large numbers, in the limit \( N \to \infty \), the term \( \sum_{i=1}^{N} \pi_i \tilde{e}_i \) has zero mean, and since \( \gamma \) converges to zero when taking the limit, also has zero variance. Hence, the pricing functional converges in probability to

\[
\tilde{P} = \pi_0 + \pi \tilde{u} - \gamma Z
\]

A very simple and elegant structure indeed. The private errors of individuals do not appear in this equilibrium price, given their infinitesimally small impact on the price. However, prices only imperfectly transmit information. This follows from inspecting the ex post informedness of agent \( i \), as measured through

\[
\text{var}^{-1}[\tilde{u}_i|\tilde{y}_i, \tilde{P}],
\]

These definitions are standard. The aggregation of a random variable \( z^i (i \in \mathcal{N}) \) over \( \mathcal{N} \) is denoted by the integral \( \int z^i \, dt \). A formal definition of this aggregation requires the definition of the triple \( (\mathcal{N}, g(\mathcal{N}), \mu) \), where \( g(\mathcal{N}) \) denotes the collection of all subsets of \( \mathcal{N} \) and \( \mu : g(\mathcal{N}) \to \mathbb{R}_+ \) is a finitely additive measure (Jordan measure) with the property that \( \mu(A) = \lim_{N \to \infty} \#(A \cap \{1, 2, \ldots, N\}) \forall A \subseteq \mathcal{N} \) for which the limit exists. The integral is then defined through

\[
\int z^i \, dt = \int_{i \in \mathcal{N}} z^i \, d\mu(i) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} z^i.
\]

11 We define \( \bar{r} = \sup_{i \in \mathcal{N}} s_i \) and \( \bar{\rho} = \inf_{i \in \mathcal{N}} \rho_i \).
which remains finite. Furthermore, note how the informativeness of prices (represented through $s^2 r^2 x$) depends on the preferences of agents in the economy. It implies that the larger $rs$, the more reliable the market is in terms of communication. In words, the more risk tolerant, and the higher the precision of private information, the better the information that is revealed through trading.

The conditional estimate of the future payoff is given by

$$E[\bar{u}|\tilde{y}, \tilde{P}] = \alpha_0 + \alpha_1 \tilde{y} + \alpha_2 \tilde{P} = b \gamma x^{-1} (u_0 h_0 + s \tilde{y}) + b \gamma^{-1} \pi (\tilde{P} - \pi_0 - \gamma z_0).$$

Another appealing feature arises: agents also use their own signal when forming their demand. This resolves the conceptual problem of prices being sufficient statistics. Many researchers have adopted the Hellwig[1980] framework. It allows for a very generic structure of information distribution across investors, yet, at the same time it results in an elegant and tractable equilibrium.

In Hellwig[1980]'s model all agents are endowed with a certain information precision independent of their preferences and initial endowments. If the information acquisition efforts are determined endogenously, however, it is likely that information precision levels and preferences are correlated. Verrecchia[1982] addresses this issue. As he indicates, the decisions of each agent will not only depend on his own preferences, but also on the precision of information revealed through prices. Since the latter quantity depends on the information acquisition efforts of all other agents, the information acquisition problem depends on the degree of information revelation and vice versa. Hence, these problems have to be solved simultaneously. Verrecchia adopts the model developed by Hellwig[1980], and considers the maximization problem for each investor given the cost for each precision level, represented by $c(s)$. By means of a fixed-point argument he shows that an equilibrium exists in which the amount of information acquisition is endogenously determined. Additionally, Verrecchia[1982] shows that the level of precision a trader acquires is a non-decreasing function of his risk tolerance. This result can be understood by realizing that more risk tolerant traders will generally have larger positions, and hence are inclined to acquire more information to protect their riskier positions with higher accuracy. Verrecchia[1982] also shows that the informativeness of price increases as noise decreases. This result does not coincide with the findings of Grossman and Stiglitz[1980]. In their model, a decrease in noise leads to a reduction in the information gathering efforts of agents, such that the two effects exactly offset one another. In Verrecchia's model, this is not the case. Instead, the informativeness of price increases as noise decreases. This happens because the direct impact of the decrease in noise on the informativeness is
always larger than the impact of the corresponding reduction in information acquisition\(^\text{13}\).

The foregoing analysis discussed economies in which a single risky asset is traded. Of course, the reality is that many assets can be traded, and that investors hold portfolios of assets. Admati[1985] extends the elegant framework of Hellwig[1980] to a multi-asset securities market. Her work is a straightforward, though tedious, extension of the Hellwig[1980] model. Admati, as Hellwig, first studies the finite agent economy, and then considers explicitly its limiting special case. The latter is the focus of the following discussion. As in Admati[1985], assume that \(n\) assets are traded. The vector of asset payoffs is denoted by \(F\). Each agent receives a signal vector, regarding this quantity, of the following form: \(Y_i = F + \varepsilon_i\). Each asset has a liquidity component, which we represent through a vector of per capita excess supplies, denoted by \(Z\). Furthermore, we denote the means of \(F\) and \(Z\) by \(\mu_F\) and \(\mu_Z\), and the variance-covariance matrices of \(F\), \(Z\), and \(\varepsilon_i\), by \(H_0^{-1}\), \(X^{-1}\), and \(S_i^{-1}\) respectively. We conjecture an equilibrium in which the pricing functional is linear in \(F\) and \(Z\), i.e. the vector of prices is given by

\[
\tilde{P} = \pi_0 + \pi_1 \tilde{F} - \gamma \tilde{Z}
\]

Additionally, define the following constants:

\[
r = \int_0^1 r_i \, di
\]

where \(r_i\) is the risk tolerance level of trader \(i\), and

\[
Q = \int_0^1 r_i S_i \, di
\]

a proxy for the average risk tolerance weighted information precision in the economy.

Observe that conditional on their private information and the pricing functional, agent \(i\) updates his beliefs regarding the vector of future payoffs according to

\[
E[\tilde{F}|Y_i, \tilde{P}] = B_{0i} + B_{1i} Y_i + B_{2i} \tilde{P}
\]

Denoting the variance-covariance matrix of this estimate by \(V_i\), it follows immediately that the demand of agent \(i\) is given by

\[
d_i(Y_i, \tilde{P}) = r_i V_i^{-1}(B_{0i} + B_{1i} Y_i + (B_{2i} - RL) \tilde{P})
\]

\(^\text{13}\)If information can be bought, an information seller is obviously necessary. Admati and Pfleiderer[1986,1988] show how information may be exploited by an information owner, if the latter has the possibility to sell it directly, use it himself to trade, or sell it indirectly through a mutual fund. The optimal strategy of a monopolistic information owner depends on his preferences. Be they risk averse, they tend to sell information. If instead they are risk neutral, information owners will trade on their own account.
The market-clearing condition thus becomes

\[ \int r_i V_i^{-1}(B_{0i} + B_{1i} \bar{y}_i + (B_{2i} - RI) \bar{P})di = \bar{Z} \]

This is a rational expectations equilibrium if the following conditions hold:

\[ \gamma^{-1} = \int r_i V_i^{-1}(RI - B_{2i})di \]
\[ \pi_1 = \gamma \int r_i V_i^{-1}B_{1i}di \]
\[ \pi_0 = \gamma \int r_i V_i^{-1}B_{0i}di \]

and if

\[ \int r_i V_i^{-1}B_{1i} \epsilon_i di = 0 \text{ a.s.} \]

Note that the latter equation is guaranteed by the law of large numbers. The derivation of equilibrium needs some tedious algebra, which can be found in the appendix to chapter 3. The equilibrium price coefficients that ultimately follow this effort, are given by

\[ \pi_0 = \frac{
}{\bar{R}} (rH_0 + rQXQ + Q)^{-1} \left( H_0 \bar{F} + QX \bar{Z} \right) \]
\[ \pi_1 = \frac{1}{R} (rH_0 + rQXQ + Q)^{-1} (Q + \rho QXQ) \]
\[ \gamma = \frac{1}{R} (rH_0 + rQXQ + Q)^{-1} [I + \rho QX] \]

Indeed, the expressions are very similar to the ones derived by Hellwig[1980]. However, the multi-asset market does have properties that cannot be directly captured by the single-asset market.

In the multi-asset market under information dispersion given certain conditions, interesting phenomena can be observed. As Admati[1985] shows, the prediction of an asset’s payoff may be decreasing with its price, and assets can be Giffen14 goods. These counter-intuitive dependencies derive from the regression analysis agents apply in order to update their beliefs. In fact, these results hold only ceteris paribus. Due to correlation between payoffs and supplies, agents use other price movements to update their beliefs about a certain asset. If one asset provides a strong information signal, but does not change value, while another low informative asset increases in value, agents are inclined to infer that this movement is caused by a supply innovation, and hence assume that the fundamental has decreased. Another interesting observation concerns the comparison with the conventional homogenous multi-asset markets.

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14 A Giffen good has the property that the demand for it is increasing with its price.
2.3 Multi-period Noisy Rational Expectations Models

The models described in the previous sections are static; trading takes place in only one period. A natural extension includes a multi-period setting that permits an examination of the intertemporal behavior of prices under information asymmetry. Unfortunately, extending the rational expectations paradigm to multi-period environments has been proven to be very difficult. The informational feedback from prices introduces a complexity that limits the tractability and transparency of multi-period studies. In spite of this, many researchers have been successful in performing this extension.

A primary difficulty in the derivation of multi-period equilibria, concerns the maximization problem of investors. The simplicity of the demand function encountered in the static NREE

\[ \text{In Admati and Ross}[1985] \text{ the model is used to consider the issue of performance evaluation. Measuring the performance of portfolio managers is important given the amount of capital that is affected through their investment decisions. Hence, many performance criteria have been put forward. Examples are the reward-to-variability ratio or the reward-to-volatility ratio. Admati and Ross}[1985] \text{ utilize the Admati}[1985] \text{ model to consider how a fund manager who receives private information is evaluated under such criteria. They show that these performance criteria may yield incorrect results. For instance, the reward-to-variability ratio may be a decreasing function of precision of the information of the fund manager. The reason is the more aggressive trading actions of better informed agents, leading to more volatile return patterns. With the failure of conventional performance measures the need arises for alternative statistics. The only true measure of superior performance is the precision matrix of the private information of the fund manager. Admati and Ross}[1985] \text{ suggest a few guidelines that may help extracting this precision matrix. Related work can be found in Dybvig and Ross}[1985a,b]. \]
framework generally does not carry over to multi-period environments. The early work on multi-period models by Brown and Jennings[1989] and Grundy and McNichols[1989], therefore, assumes myopia of agents to derive the properties of the equilibria. In the past years, however, much progress has been made, through the work of Wang[1993], Slezak[1994], Vives[1995] and Brennan and Cao[1996], who allow agents to rationally take into account the possibility of intermediate trade.

Before we discuss the models proposed in the recent literature, we first consider a generic approach to the dynamic pricing problem under information asymmetry assuming a CARA-Gaussian structure. Though the equations that specify the equilibrium cannot be solved explicitly, it allows us to demonstrate a few basic properties of this framework.

For ease of exposition, we adopt the following notation. We introduce the state variable $\Psi_t$ which is a sufficient statistic for all information up to and including time $t$, including investors' signals, liquidity shocks and private endowments, i.e. $\Psi_t \in \Omega_t = \mathbb{R} \times \left( \prod_{i \in \mathcal{N}} I_t^{(i)} \right) \times I_t^{(p)} \times \mathcal{N}$, $I_t^{(i)}$ represents the information set private to investor $i$, $I_t^{(p)}$ represents the information public to all investors, and $\mathcal{N}$ represents the noise space. Note that we include a constant dimension to ease notation. The state variable is assumed Gaussian-Markov. Specifically, it is to evolve in time according to

$$\Psi_{t+1} = H_{t+1} \Psi_t + L_{t+1} \eta_{t+1}$$

where $\eta_{t+1}$ is normal with mean zero, and covariance matrix $\Sigma_{t+1}$. The matrix $L$ projects the (smaller) noise space $\mathcal{N}$ onto the (larger) state space $\Omega_t$. A risky asset can be traded, that does not generate any dividends. The true value of the asset depends solely on fundamentals. Its price varies stochastically, due to the arrival of new information, and the occurrence of liquidity trades. We assume that the price is measurable with respect to $\Psi_t$. Additionally, we assume that it is affine in the state variables. Hence, the price at time $t$ is a projection of $\Omega_t$ onto $\mathbb{R}_+$, i.e.

$$P_t = p_t \Psi_t, \forall \Psi_t \in \Omega_t$$

Furthermore, a riskless asset is present that yields a gross return of $R$ in each period. Both assets are infinitely divisible.

A continuum of investors is present, which we index by $i \in \mathcal{N} = \{1, 2, \ldots\}$, who can take positions in the assets and act competitively. Each investor has a information set $T_t^i$ which includes all public signals, his private signals, and prices including $P_t$, i.e. $T_t = T_t^{(i)} \times T_t^{(p)}$. It is measurable with respect to $\Omega_t$, and can be expressed as a projection of $\Psi_t$ onto a subspace.
Given the Gaussian-Markov property of $\Psi_t$, conditional on this information, the investor has a belief regarding the future state of the economy, $\Psi_{t+1}$, characterized by

$$E[\Psi_{t+1} | \mathcal{F}_t] = \Psi_{t+1} | \mathcal{F}_t = b_t' \Psi_t$$

(2.14)

where we implicitly defined $b_t'$. Define the uncertainty vector $\tilde{\xi}_{i,t}$ of investor $i$ implicitly as follows

$$\Psi_{t+1} - \Psi_{t+1} | \mathcal{F}_t = M_i \tilde{\xi}_{i,t+1}$$

where $M_i$ maps noise onto the state space. The variance-covariance matrix of $\tilde{\xi}_{i,t}$ is required to be non-singular and given by $O_i$. Note that this implies that for any inner product $A' \Psi_{t+1}$, the uncertainty of this inner product for investor $i$ is given by $\text{cov}[A' \Psi_{t+1} | \mathcal{F}_t] = A' M_i O_i + M_i A$.

Regarding the preferences of the investors, there are basically two possibilities that are of interest. If we allow for intertemporal consumption, we end up with a model like Wang (1993, 1994). A more common assumption, however, is that agents maximize their utilities over some future date. This is the approach we discuss here. Consider therefore agent $i$ who maximizes his expected utility at time $T_i$. Denote the current time by $t$. Given a total wealth $W_i^t$ at time $t$, agent $i$ has a maximization problem of the form

$$V_i(W_i^t; \Psi; t) = \max_{d_t} E[V_i(W_i^{t+1}; \Psi_{t+1}; t)] = \max_{d_t} \left[ -\rho_i \left( W_i^t + \sum_{n=t+1}^{T_i} \Delta W_n \right) \right]$$

where $\Delta W_i = (p_t - R p_{t-1}) d_{t-1}$, with $d_{t-1}$ the demand of the investor at time $t - 1$. Note that we can write

$$\Delta W_i = (p_t' \Psi_i - R p_{t-1}' \Psi_{t-1}) d_{t-1}$$

The maximization problem can also be represented through a Bellman equation, i.e. the value function $V_i(W_i^t; \Psi; t)$ should solve

$$0 = \max_{d_t} \left[ E[V_i(W_i^{t+1}; \Psi_{t+1}; t)] - V_i(W_i^t; \Psi; t) \right]$$

under the constraints that

$$W_i^{t+1} = W_i^t + (p_{t+1}' \Psi_{t+1} - R p_t' \Psi_t) d_t$$

and $V_i(W_i^t; \Psi; t) = -e^{-\rho_i W_i^t}$.

By backward induction it can be shown that the following theorem applies to the solution.

**Theorem 2.1** The value function of investor $i$ at time $t$ can be written as a exponent of quadratic form in $\Psi_t$, i.e.

$$V_i(W_i^t; \Psi; t) = -C_i \exp \left[ -\rho_i W_i^t - \frac{1}{2} \Psi_i' \Psi_t \right]$$

$C_i$ a constant, independent of $\{\Psi_t\}_{t \leq T}$. 

In the following we prove this theorem. Assume that the theorem holds for $t+1$. At $t$, the investor is faced with the optimization problem

$$d'_i = \arg \max_{d'_i} -C'_t E_t\left\{ \exp \left[ -\rho_t \Psi_t - \rho_t (\mu'_{t+1} \Psi_{t+1} - R p'_t \Psi_t) d'_t - \frac{1}{2} \Psi_{t+1} \gamma'_{t+1} \Psi_{t+1} \right] \right\}$$

This expectation can be calculated readily. In terms of the uncertainty vector of investor $i$, $\varepsilon_{i,t+1}$, we obtain

$$\phi'_{t+1} = \left( p'_{t+1} \Psi_{t+1} | - R p'_t \Psi_t \right) d'_t + \frac{1}{2 \rho_t} \Psi_{t+1} \gamma'_{t+1} \Psi_{t+1} d'_t$$

Define

$$G'_{t+1} \equiv \left( (O_t^i)^{-1} + M_t \gamma'_{t+1} M'_t \right)^{-1}$$

Then consider the expectation of $- \exp \left[ -\rho_t \phi'_{t+1} \right]$. Using a standard formula (see appendix), one obtains

$$E[- \exp \left[ -\rho_t \phi'_{t+1} \right] | T_t] =$$

$$-C'_t \exp \left[ -\rho_t \left\{ \left( p'_{t+1} \Psi_{t+1} | - R p'_t \Psi_t \right) d'_t + \frac{1}{2 \rho_t} \Psi_{t+1} \gamma'_{t+1} \Psi_{t+1} d'_t \right\} \right]$$

with

$$C'_t = |O_t^i|^{-\frac{1}{2}} \left| (O_t^i)^{-1} + M_t \gamma'_{t+1} M'_t \right|^{-\frac{1}{2}}$$

The maximum value is attained for

$$d'_t = \frac{(p'_{t+1} \Psi_{t+1} | - R p'_t \Psi_t) - p'_{t+1} G'_{t+1} \gamma'_{t+1} \Psi_{t+1}}{\rho_t p'_{t+1} G'_{t+1} \mu_t}$$

where we used (2.14). Upon substitution, the expression in the theorem follows, with $\gamma'_t$ recursively determined by

$$\gamma'_t = b'_t \gamma'_{t+1} \left( 1 - G'_{t+1} \gamma'_{t+1} \right) b'_t$$

$$+ \frac{(p'_{t+1} \Psi_{t+1} | - R p'_t) \left( p'_{t+1} \Psi_{t+1} \right)}{p'_{t+1} G'_{t+1} \mu_t}$$
The boundary condition at the consumption horizon of the investor is satisfied by imposing that $\gamma_{f_t} = 0$.

Note that even in this generic setup, due to the linearity assumptions, the value function keeps a relatively simple form. Also observe that the relative increase in utility due to trading is independent of the risk aversion level.

Next consider how the pricing coefficients could be determined. Armed with the optimal demand schedule of each agent, we next need market clearing. Assume that at time $t$ the per capita supply is given by $Z_t$. The market-clearing condition is given by

$$\int d_t \psi_t \, di = Z_t$$

Hence, the pricing coefficients are recursively determined through

$$p_t = R^{-1} \left[ \int \frac{1}{\rho_t G_{t+1} p_{t+1}} \, dt \right]^{-1} \left[ \int \frac{p_{t+1} (1 - G_{t+1} \gamma_{t+1}^\mu) b_t}{\rho_t G_{t+1} p_{t+1}} \, dt - \nu_t \right]$$

This expression somewhat downplays the complexity involved in solving for an equilibrium. The regression coefficients, $b_t$, will generally depend on $p_{t+1}$. Similarly, the uncertainty matrix $G_{t+1}$ will depend on the informativeness of prices, and consequently on $p_t$ as well.

This approach obviously is too generic to extract meaningful implications. The successful multi-period models derived in the literature impose additional structure to maintain tractability. Especially, the dependency of agents’ beliefs on the complete history of state variables is cumbersome. Most multi-period models therefore impose an informational structure that keeps the updating of beliefs tractable.

In the following subsections, we discuss the main contributions to the multi-period rational expectations literature. We start with the two-period models and, in particular, the dynamic models by Brown and Jennings[1989] and Grundy and McNichols[1989]. More recently, multi-period models have been proposed that allow for an arbitrary number of periods. Examples are the Brennan and Cao[1996] model, and the He and Wang[1996] model. We conclude our overview with the infinite horizon models proposed in Wang[1993,1994].

### 2.3.1 Two Period Models

Grundy and McNichols[1989] and Brown and Jennings[1989] were the first that extended the noisy rational expectations approach to a multi-period environment. In these models, it is shown that the consideration of past price realizations may yield additional information above the information reflected in current price levels. As such, they provide a rationale
for technical analysis. Agents can enhance the precision of their prediction by including past market statistics in their estimation of the ultimate payoff. Both models extend the Hellwig[1980] model to two periods. The structure of noise and information signals differs significantly however.

In Brown and Jennings[1989] the focus is on the determination of the value of technical analysis. As in Hellwig[1980], prior to the first trading round, agents all share the same belief about the distribution of the liquidation value of the risky asset. The supply of the asset experiences a shock in each period, \( z_t \), which is randomly distributed. These liquidity shocks are persistent, and correlated between the two trading periods. Initially, Brown and Jennings[1989] derive the maximization problem explicitly when agents rationally foresee the possibility of intermediate trade in the second period. They show that the optimal demand of investor \( i \) in the second trading round is given by

\[
d_i^2 = \frac{E[\tilde{u}|T_2] - \tilde{P}_2}{\text{var}[\tilde{u}|T_2]}
\]

and in the first trading period by

\[
d_i^1 = \frac{E[\tilde{P}_2|T_1] - \tilde{P}_1}{\rho_i G_{11}} + \frac{E[d_i^2|T_1]|G_{12} - G_{11}|}{\rho_i G_{11}}
\]

where \( G_{ij} \) are elements of the matrix \( G = (2N + M^{-1}) \), where

\[
N = \frac{1}{\text{var}(\tilde{u}|T_2)} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\text{, and } M = \begin{pmatrix}
\text{var}(\tilde{P}_2|T_1) & \text{cov}(\tilde{P}_2, \tilde{u}|T_1) \\
\text{cov}(\tilde{P}_2, \tilde{u}|T_1) & \text{var}(\tilde{u}|T_1)
\end{pmatrix}
\]

Indeed, the demand function in the first period already reflects the anticipated optimal demand in the second trading round. Brown and Jennings[1989] use the fact that the expectations are linear in information signals and the price realization, to show that the demand is linear in the information signals and the supply of the asset. This allows them to infer that in each trading round a linear equilibrium price may exist.

Due to the intractability of the demand functions, Brown and Jennings[1989] consider the simpler myopic investor economy, in which agents maximize their utility of the next period. The demand function in the first trading period simplifies to

\[
d_i^1 = \frac{E[\tilde{P}_2|T_1] - \tilde{P}_1}{\rho_i \text{var}[\tilde{P}_2|T_1]}
\]

Brown and Jennings[1989] use the equilibrium that arises to show that, under mild assumptions, technical analysis always has value in the linear, myopic investor economy. Additionally, they consider the magnitude of the added value of technical analysis as a function of some
exogenous parameters. An interesting discussion on the informational efficiency hypotheses follows. According to the original definition of weak-form efficiency by Fama[1970] this market is not efficient. Fama[1970] defines a market weak-form efficient if all historical information, including past prices, is fully reflected in current prices. In the Brown and Jennings[1989] model, current prices are not sufficient statistics for past prices, and hence, it violates this definition. Brown and Jennings[1989] discuss several alternative definitions of weak-form efficiency. They indicate that, according to the definition provided by Verrecchia[1982], this market is weak form efficient. In Verrecchia[1982] a market is weak-form efficient if conditional on noise in prices, past prices do not provide additional information above current prices. From that perspective, weak form efficiency does not imply that technical analysis is of no value. This is an important observation. Especially when taking into account that financial economics textbooks often take the equivalence between weak-form efficiency and irrelevance of technical analysis for granted.

The Grundy and McNichols[1989] model differs from the Brown and Jennings[1989] on several accounts. Investors receive private information signals with an error term that has a common component, $\bar{w}$. The implication is that even the aggregate information in the economy cannot reveal the ultimate payoff of the asset with perfect precision. Investors also differ in the way their endowment is modeled. This endowment is assumed to contain a random component that has infinite variance in the limiting economy. Upon taking the limit, supply variance remains while agents cannot use the observation of their private endowment to predict the aggregate supply. In the limiting economy, the impact of each investor's endowment on the aggregate supply is infinitesimally small.

Grundy and McNichols[1989] consider a variety of implications that follow from their model. An important part of the paper is devoted to one particular equilibrium. They assume that in the second trading round no additional shocks impact the economy. Intuitively, one would expect that under these circumstances, in the second trading round no new trade be initiated. However, Grundy and McNichols[1989] show that there is an alternative equilibrium in which additional trade does occur. To understand why, consider the price conjectures in round 1 and 2:

$$\tilde{P}_1 = \pi_{0,1} + \pi_{1,1} \bar{Y} - \gamma_{1,1} Z$$
$$\tilde{P}_2 = \pi_{0,2} + \pi_{1,2} \bar{Y} - \gamma_{1,2} Z$$

where $\bar{Y}$ is the aggregate information signal, i.e. $\bar{Y} = \bar{u} + \bar{w}$. Observe that these are in fact two equations in the two unknowns $\bar{Y}$ and $Z$. If this system is non-degenerate, these unknowns can
be solved for. The non-degeneracy holds if $\pi_{1,1}/\gamma_{1,1} \neq \pi_{1,2}/\gamma_{1,2}$. Hence, if agents conjecture this non-degeneracy, in the second trading round the aggregate knowledge of the market is revealed, which leads to additional trade in the second round. Grundy and McNichols[1989] show that this equilibrium exists as long as the variance of the common signal error is not too large. The interesting implication is that even in an eventless period, trade may occur. They relate this possibility to the crash of October 1987, which typically was void of any new information and seemingly came out of the blue. As Grundy and McNichols[1989] show, even in absence of news, price changes need not to be trivial. Another important point concerns the Milgrom-Stokey[1982] No-Speculation theorem. This theorem establishes that if allocations are Pareto optimal and investors hold essentially concordant beliefs, investors cannot agree to any non-null trade. As Grundy and McNichols[1989] show, the allocation after the first round is indeed Pareto optimal. The trade in the second round, therefore, seems to violate the No-Speculation theorem. However, this is not the case. In the second round, a public signal in the form of $P_2$, is released. This signal does not resolve all uncertainty due to the common error term. Conditional on this public signal, allocations are no longer Pareto optimal. In light of this, it is derived that beliefs are not essentially concordant in this environment. Hence, agents still want to trade. Note that the common error term is necessary, for else, the Hirshleifer[1971] effect prevails, inhibiting further trade in the risky asset. Grundy and McNichols[1989] continue by examining other properties of their model. In particular they consider how a public information signal and an additional supply shock in the second round affect the equilibrium. They also stress the important point, as Brown and Jennings[1989], that rational agents are chartists that learn from the observation of price sequences.

Brown and Jennings[1989] and Grundy and McNichols[1989] were among the first to address the issue of technical analysis within a rational expectations model. Following them, other authors have argued that the study of past market statistics may indeed enhance the quality of trading decisions. Given the many sources of uncertainty present in financial markets, the inclusion of other statistics, as well as past statistics, may lead to a better or more complete spanning of the state of economy, and reveal additional information. One of the contributions that is of interest is by Blume et.al. [1994]. They deviate from the common approach by considering a market in which the aggregate supply is fixed. Though ceteris paribus this would lead to fully revealing prices, they introduce another source of uncertainty in the form of the stochasticity of the quality of information that agents receive. A problem arises,

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16 If agents hold concordant beliefs, they agree on the conditional likelihood of an event given a realization of an information signal.
since there are as many sources of uncertainty as signals (i.e., price and volume). One equilibrium is guaranteed, the no-trade equilibrium. The observation of both variables reveals all private information leading to no trade at all\textsuperscript{17}. To circumvent this problem Blume et al. [1994] therefore adopt the Hellwig [1982] strategy and assume that prices and volume can only be observed ex post. The complexity of the model introduces a non-linearity that makes explicit closed formula impossible to derive. Using simulations Blume et al. [1994] show how rational agents engage in technical analysis using both volume and price sequences for their trading decisions. Technical analysis is necessary for all agents to learn about some underlying uncertainty in the economy. They also show that if the information precision of price sequences is high, watching market data is less valuable. The reverse applies to low informative price sequences. This implies that technical analysis may be appropriate for especially "small, less widely" followed stock.

An interesting paper by Romer [1993] also incorporates uncertainty of information about the information precision of agents. He proposes two situations for which such an uncertainty arises: uncertainty about the precision of the economy, or uncertainty about the relative precision of private information. Though his model cannot be solved explicitly, he shows, using simulations to illustrate his findings, that even in the absence of additional information shocks, agents may want to revise their beliefs, due to the fact that they learn about the relative precision of their own information. This can lead to large price shocks, even if news is absent. Using this result, Romer [1993] motivates a rational explanation for the 1987 crash.

\textbf{2.3.2 Multi-period Models}

In this section we focus our attention on models that allow for an arbitrary number of trades before the asset is liquidated. We start our description with Slezak [1994] who extended the Brown and Jennings [1989] model. Following this effort, we discuss the Vives [1995] model that, though incorporating a risk neutral market-making sector, provides a structure that can be solved explicitly. Brennan and Cao [1996] provide a multi-period extension of the Hellwig [1980] model that is both very rich, and can be solved explicitly as well. We end with a discussion of He and Wang [1995] who apply a similar framework, but include a correlation between subsequent supply shocks. Explicit solutions cannot be found in this environment.

\textsuperscript{17}If agents have common preferences and endowments and additionally have the same information, they want to make the same trade. The only consistent outcome is no trade.
Many striking empirical observations have been reported concerning patterns in volatility and volume. Examples are the U-shapes of intra-day volatility and volume, and the abnormally low variances of weekend or holiday returns. Slezak[1994] motivates these regularities using a noisy rational expectations model. In his model, in each period informed investors receive a private information signal about the risky asset. The uninformed learn this signal only just before the next trading round. As in Vives[1995] and Brennan and Cao[1996] it is assumed that the asset is ultimately liquidated, and investors maximize their CARA utility at the liquidation date. Slezak[1994] however allows the per capita excess supply to be mean-reverting. The latter gives rise to correlations between subsequent price changes. This feature has a major impact on agents' resulting demand schedules. As such, his model has a similar feature as the Brown and Jennings[1989] model that also includes a correlation between subsequent supply shocks. As Slezak[1994] shows, the demand function of each agent can be written as follows

$$d_{i,t} = (\rho \Sigma_t)^{-1} \left( (E_t^i(P_{t+1}) - P_t + \gamma_t E_t^i(d_{i,t+1})) \right)$$

The demand function of each agent consists of two parts, an unconditional holding in the asset represented through the second term on the left hand side, and a conditional holding that enters through the first term. The whole demand is weighted with the inverse of $\Sigma_t$, which measures the effective uncertainty of investor $i$. This variance matrix also takes into account the dynamic diversification ability of investor $i$ through the possibility of re-trade, and in the period prior to liquidation equals the conditional variance of the liquidation value of the asset.

Slezak[1994] considers the situation in which the financial market is closed in some periods. The implication is that uninformed investors have to estimate both the current signal and the signal informed received during the market close. A closure in particular affects the risk faced by agents. After the closure, uninformed risk has increased due to the accumulation of uncertainty during the market close. Informed on the other hand, do not face a change in risk after the closure. Both informed and uninformed agents, however, face greater uncertainty prior to the closure. The increase in risk is larger for the informed, given that their advantage over the uninformed is smaller through the less predictive power of private information in this situation. The result is a complicated change in risk characteristics of the market, leading to innovations in both mean and variance of price returns. Slezak[1994] shows that many of the reported empirical patterns are consistent with a market where relatively few informed investors are active. If the fraction of informed is low, liquidity costs are mainly determined through the uncertainty of the uninformed agents. The return variance during closure is smaller in that case, due to the reduced sensitivity of price to news, given the greater uncertainty of
the informed agents. The mean return over market closures is relatively low. The cause is the increase in uncertainty in the post-closure period, which magnifies liquidity costs, and hence depresses the post-closure price.

Vives [1995] considers \( N \) trading periods before at \( N+1 \) the asset is liquidated at its fundamental value \( \bar{v} \). Apart from rational traders, there are noise traders whose aggregate demands, \( \{ \tilde{z}_t \}_{t=1}^N \), follow an i.i.d. normal process. These demands add, so there is no mean reversion and liquidity follows a pure random walk. At any period, agent \( i \) receives a private signal \( \tilde{s}_{it} = \bar{v} + \tilde{e}_{it} \) for which the usual assumptions apply. The precision of the signals \( \tau_{ei} \) is the same across agents, but may differ across periods. There is a market maker who sets prices, conditional on current and past order flow, to the expected liquidation value. Note that as such we have a Kyle type of market-clearing mechanism combined with a Hellwig type of investor. To ease our discussion, define the best estimator of \( \tilde{u} \) conditional on all signals up until time \( n \), by

\[
\tilde{s}_{in} = (\sum_{t=1}^n \tau_{ei})^{-1} \sum_{t=1}^n \tau_{ei} \tilde{s}_{it}.
\]

Note that this quantity is a sufficient statistic for all previous signals. This allows us to significantly reduce the state space.

The aggregate demand \( L_n \) in period \( n \), observed by the market maker is given by the sum of the changes in the aggregate desired holdings of the rational and noise trader community. Hence,

\[
L_n(.) = \int_0^1 d_n di - \int_0^1 d_{m-1} di + \tilde{z}_n
\]

Vives uses the fact that agents only differ in the realization of their private signal, \( \tilde{s}_m \), and proposes a demand function of the form

\[
d_n(\tilde{s}_m, \tilde{P}^n) = a_n \tilde{s}_m + \zeta_n(\tilde{P}^n)
\]

where \( a_n \) measures the trading aggressiveness at time \( n \). Using this expression, the aggregate order flow follows as

\[
L_n(.) = \tilde{g}_n + \zeta_n(\tilde{P}^n) - \zeta_{n-1}(\tilde{P}^{n-1})
\]

where \( \tilde{g}_n = \Delta a_n \tilde{u} + \tilde{z}_n \), and \( \Delta a_n = a_n - a_{n-1} \). The market maker observes the sequence of prices \( \{ P_1, ..., P_n \} \) which is equivalent to the observation of the sequence \( \tilde{g}^n = \{ \tilde{g}_1, ..., \tilde{g}_n \} \). He sets prices competitively, yielding each price a sufficient statistic for all public information. Hence, the market-clearing price is a function of the previous price realization and the realization of \( \tilde{g} \) only. As Vives shows, this implies that the price can be written as

\[
\tilde{P}_n = \mathbb{E}[\tilde{u}|\tilde{g}^n] = \lambda_n \tilde{g}_n + (1 - \lambda_n \Delta a_n) \tilde{P}_{n-1}
\]

where \( \lambda_n = \tau_n \Delta a_n / \tau_n \), and \( \tau_n = \tau_0 + \tau_u \sum_{i=1}^{n}(\Delta a_i)^2 \).
Armed with this expression, Vives first focuses on the properties of the equilibrium that prevails if investors have short time horizons. As usual, the demand of each trader can be written as

\[ d_t^i = \frac{E[\bar{P}_{t+1} - \bar{P}_t | I_t]}{\rho \text{var}[\bar{P}_{t+1} - \bar{P}_t | I_t]} \]

In the myopic investors case, an elegant expression can be found, in the form of

\[ d_t^i = a_t (\bar{s}_t - \bar{P}_t) \]

where \( a_t = \rho^{-1} (\sum_{i=1}^{T} \tau_{i,t})^{-1} + (\tau_{t+1})^{-1})^{-1} \). This expression can be understood as follows. The expected price of the next period is given by a linear combination of the fundamental value \( \bar{u} \) and the current market-clearing price \( \bar{P}_t \). The optimal estimator is given by \( \bar{s}_t \). Hence, investors submit a demand proportional to the difference between \( \bar{s}_t \) and \( \bar{P}_t \). The quantity \( a_t \), the trading aggressiveness, is simply the precision of the information of the investor regarding the future price weighted with his risk tolerance. The following remarks can be made regarding the equilibrium. First, the trading intensity strictly increases with \( t \). This can be understood from the fact that there is resolution of uncertainty regarding the fundamental value, which in turn follows from the increasing informativeness of prices with \( t \). This increase in trading intensity is similar to the findings of Dow and Gorton[1994].

In the long-term investment case, agents are assumed to maximize their expected utility at the liquidation time \( T \). Interestingly, the demand schedule in this case is simpler than in the short-term investors case, since it allows for an explicit solution (a similar solution is found by Brennan and Cao[1996]). The demand function in this case is given by

\[ d_t^i = a_t (\bar{s}_t - \bar{P}_t) \]

with \( a_t = \rho^{-1} \sum_{i=1}^{T} \tau_{i,t} \).

Note that this demand is exactly the same as if the agent can only trade once and hold his position until liquidation. The reason is, that the noise introduced by the liquidity traders is persistent. Because of this, agents cannot profit from temporary distortions. From the demand function it can be directly assessed that agents trade more aggressively compared to the myopic case. The reason is that agents do not suffer from the additional risk introduced by the liquidity traders which affects the price in each period.

Vives[1995] also considers the informativeness of prices for the two different cases. He shows that it depends on the way information arrives which of the two economies exhibits a higher informativeness. Specifically he shows that with concentrated information arrival the long-term investors economy is more informative, while with diffuse informational arrival
short-term investors enhance the informativeness of prices.

An interesting contribution concerning multi-period noisy rational expectations models is by Brennan and Cao[1996]. It is unique in that it captures the intertemporal aspects of information asymmetry while allowing for an explicit closed form representation of the equilibrium. Their framework is a direct extension of the Hellwig[1980] framework, with information dispersion across investors. Additionally, they explicitly allow for a term structure of public signals, private information signals, and supply shocks. Given the connection with the Hellwig[1980] approach, and the closed form specification of the equilibrium, we elaborate on some of the technicalities of the model.

Let us introduce a slightly modified setup of the model. Adopting the notation of the previous sections, we assume that there is a single risky asset which is liquidated at an uncertain value \( \hat{u} \sim N[\mu_0, h_0^{-1}] \). Agents can trade the asset in the \( T \) trading sessions that proceed the liquidation period. As in the limiting economy of Hellwig[1980], a continuum of agents is present indexed by \( i \in [0,1] \). Each agent maximizes the expectation of a constant relative risk aversion function that is determined through his final wealth at the liquidation date. In each period, prior to trading, a public information signal is revealed denoted by \( \hat{v}_t = \hat{u} + \eta_t \), with \( \eta_t \sim N[0,\sigma_t^{-1}] \). Furthermore, agents receive a private information signal \( \hat{y}_it = \hat{u} + \varepsilon_{it} \), where \( \varepsilon_{it} \sim N[0, s_i^{-1}] \). Finally, in each period a per capita supply shock \( \tilde{z}_t \) affects the market, where the innovation in supply is given by \( \tilde{z}_t \sim N[0, \sigma_t^{-1}] \). Brennan and Cao[1996] incorporate this element through the endowment of investors. This allows them to consider the Pareto optimality of the equilibria that are derived. In the following we present a heuristic derivation of the equilibrium.

For expositional purposes, we introduce some additional notation. We define \( I_t^i \) as the total, pre-trading, information set of investor \( i \) at time \( t \). Denoting by \( \mathcal{A}_t \equiv (\hat{A}_1,\ldots,\hat{A}_t) \) the history of realizations of a stochastic process \( \{\hat{A}_t\} \) up to and time \( t \), this information set can be written as \( I_t^i = \{Z_0, P_{t-1}, \hat{u}_t, \hat{y}_it\} \). Denote the information signal in prices by \( I_t \), i.e. \( \{Z_0, P_{t-1}, \hat{u}_t\} \leftrightarrow \{Z_0, L_{t-1}, \hat{y}_it\} \), and the precision of \( I_t \) by \( p_t \).

Assume now that agents choose their demand in each period as if they are only allowed a single trade. Under this assumption, the pricing kernel in each period should reduce to the Hellwig[1980] type of pricing kernel. This pricing kernel can be written as follows:

\[
\tilde{P}_t = \left[ \int \frac{r_t}{\text{var}(\hat{u}|I_t)} d\mu(i) \right]^{-1} \left[ \int r_t \frac{E[\hat{u}|I_t]}{\text{var}(\hat{u}|I_t)} d\mu(i) - \left( \sum_{j=0}^{t} \tilde{z}_j \right) \right]
\]
To explicitly determine price, we need the expectations of each agent. It readily follows that
\[
E[u_{t}] = \frac{hu_{0} + \sum_{j=0}^{t} h_{j} \tilde{v}_{j} + \sum_{j=0}^{t} P_{j} \tilde{I}_{j} + \sum_{j=0}^{t} s_{ij} \tilde{y}_{ij}}{h + \sum_{j=0}^{t} (h_{j} + p_{j} + s_{ij})}
\]
\[
\text{var}^{-1}[u_{t}] = h + \sum_{j=0}^{t} (h_{j} + p_{j} + s_{ij})
\]
Substituting these expressions, we have, using the definitions \( r = \int r_{i} d\mu(i) \) and \( s_{j} = \int r_{i} s_{ij} d\mu(i) \),
\[
\hat{P}_{i} = K_{t}^{-1} \left[ rh u_{0} + r \sum_{j=0}^{t} h_{j} \tilde{v}_{j} + r \sum_{j=0}^{t} s_{j} (s_{j}^{-1} P_{j} \tilde{I}_{j} + \tilde{u} - r^{-1} s_{j}^{-1} \tilde{z}_{j}) \right]
\]
where \( K_{t} = \left( h + \sum_{j=0}^{t} (h_{j} + p_{j} + s_{ij}) \right) \).

Our next goal is to pin down the signal \( I_{t} \), which we left unspecified so far. For expositional purposes, define \( \tilde{w}_{t} = \tilde{u} - r^{-1} s_{t}^{-1} \tilde{z}_{t} \). The pricing functional at time \( t \), maps information signals into price space, i.e. \( \hat{P}_{i} : \{u_{0}, \tilde{u}_{t}, \tilde{w}_{t-1}\} \rightarrow \mathbb{R}_{+} \). The innovation signal in price \( \tilde{I}_{t} \) must be orthogonal to the public information set at time \( t \) represented through \( \{u_{0}, \tilde{u}_{t}, \tilde{w}_{t-1}\} \). Necessarily, therefore \( \tilde{I}_{t} \) is spanned by \( \tilde{w}_{t} \). By definition, this signal is an unbiased estimator of \( \tilde{u} \). Hence, it follows that \( \tilde{I}_{t} = \tilde{w}_{t} = \tilde{u} - r^{-1} s_{t}^{-1} \tilde{z}_{t} \), with corresponding precision \( p_{j} = x_{j} r^{2} s_{j}^{2} \).
Substituting this expression for \( I_{t} \) in the price function(2.16), we obtain
\[
\hat{P}_{i} = K_{t}^{-1} \left[ rh u_{0} + r \sum_{j=0}^{t} h_{j} \tilde{v}_{j} + r \sum_{j=0}^{t} s_{j} (x_{j} r^{2} s_{j} + 1) (\tilde{u} - r^{-1} s_{j}^{-1} \tilde{z}_{j}) \right]
\]
with \( K_{t} = \left( h + \sum_{j=0}^{t} (h_{j} + x_{j} r^{2} s_{j}^{2} + s_{ij}) \right) \). This is indeed exactly the pricing functional found by Brennan and Cao[1996]. The loose end that remains is the optimization problem of investors, who we have assumed to behave as if only one trading round is available. Brennan and Cao[1996] show by backward induction, however, that this trading strategy is optimal even if agents anticipate the possibility of intermediate trade.

The demand of investor \( i \) can be written as
\[
d_{it} = r_{i} [hu_{0} + \sum_{j=0}^{t} (h_{j} \tilde{v}_{j} + x_{j} r^{2} s_{j}^{2} \tilde{I}_{j} + s_{ij} \tilde{y}_{ij})] - (h + \sum_{j=0}^{t} (h_{j} + x_{j} r^{2} s_{j}^{2} + s_{ij})) \hat{P}_{i}
\]
\[
= r_{i} [\sum_{j=0}^{t} (s_{ij} \tilde{y}_{ij} - s_{j} (\tilde{u} - \tilde{z}_{j}/r)) - (s_{ij} - s_{j}) \hat{P}_{i}]}
\]
Here, an interesting result can be observed. Agents who are less well informed than the representative agent \( (s_{ij} < s_{j}) \), tend to increase their demand with increasing price, while
better-informed agents do the opposite. Brennan and Cao[1996] use this feature to motivate why many investors are trend-followers.

Brennan and Cao[1996] focus their attention to the impact of multiple trading sessions on the welfare of the economy. They show that generally all market participants gain from additional market sessions. Moreover, the limiting continuous time economy is shown to be even Pareto efficient.

They also make an interesting case regarding the implication of the trading strategies of investors. To illustrate this, consider the change in demand of an investor in the absence of additional supply shocks. Assume that there are no additional private information signals following the first period. The investors’ demand can then be written as

\[
d_{it} = r_i[s_iy_i - s(\bar{u} - \bar{z}/r)] - (s_i - s)\bar{P}_t
\]

The change in demand is given by \(\Delta d_{it+1} = -r_i(s_i - s)\Delta \bar{P}_{t+1}\), and allows us to write the demand as

\[
d_{it} = d_{i0} - \sum_{j=1}^{t} r_i(s_i - s)\Delta \bar{P}_j
\]

Clearly the trading activities of agents are predetermined after the initial position in the asset. Agents apply a dynamic trading strategy that resembles an option replication strategy. Indeed, taking the continuous time limit, one obtains the following demand

\[
d_{iT} = d_{i0} - \int_{0}^{T} r_i(s_i - s)d\bar{P}_t dt = d_{i0} - \int_{P_{0}}^{P_{T}} r_i(s_i - s)PdP
\]

Hence, agents replicate a quadratic option through their dynamic trading strategy. This fascinating result motivates Brennan and Cao[1996] to consider a market in which additionally options are traded. They show that a single trading session with the availability of quadratic options achieves the same effect as a market in which the security can be traded continuously: Pareto optimality. Brennan and Cao[1996] further discuss some generalizations of their model, as well as other market statistics such as volume and market depth.

In Brennan and Cao[1997] a multi-asset extension of the Brennan and Cao[1996] model is considered. This extension is, given the single asset version, relatively straightforward, and underscores the elegance and tractability of this approach. Brennan and Cao[1997] use the model to capture the dynamics of international portfolio flows. The motivation for this study stems from the numerous empirical reports on so-called home-country biases in equity portfolios. Such biases seem to neglect the international diversification ability, and hence violate optimality against conventional optimal portfolio theory. The basic assumption that Brennan
and Cao[1997] invoke to rationalize the home-country bias, is that domestic investors have an information advantage over foreign investors about their domestic market. In order to find empirical support for this hypothesis, features derived in the Brennan and Cao[1996] model are used. In the discussion of Brennan and Cao[1996], it was already shown that investors who are less well informed relative to the market average tend to be trend-followers. This result is used in Brennan and Cao[1997] to demonstrate how domestic investors tend to be contrarians in their domestic market, while foreign investors are trend-followers. Additionally, they demonstrate that under information asymmetry, portfolio flows depend on all market indices, while under symmetric information the portfolio flow only depends on the host market return. Having stated this idea in terms of implications for the regression coefficients between shifts in portfolio holdings and market index returns, Brennan and Cao[1997] next undertake an empirical study of portfolio flows between countries. They indeed find evidence for the asymmetric information hypothesis. Their results also indicate that while US investors have an information disadvantage compared to the domestic investors in foreign markets, foreign investors seem to be equally informed about the US market.

In He and Wang[1995] a model similar to Brennan and Cao[1996] is used to consider volume and its relation to information flow. The model differs in the more complex process that is imposed on liquidity supply. An interesting point to note is that all of the above models assume that shocks to liquidity are persistent. This presents no problem if the asset is ultimately liquidated. The virtue of this assumption is the simple solution it creates to the long time horizon problem. However, these types of models cannot be implemented in a stationary infinite time economy. Ultimately, the liquidity level should always revert to some mean for else it grows without bounds. Though He and Wang[1995] also assume liquidation of the asset, they do allow supply to follow a process to more accords to reality. Adopting the notation of previous section, liquidity supply follows the process

$$\tilde{Z}_{t+1} = a \tilde{Z}_t + \tilde{\eta}_t$$

where $-1 < a < 1$, and $\tilde{\eta}_t$ is i.i.d. normal. The drawback of this assumption is that it complicates the analysis due to the correlation between subsequent supply shocks. The result is that the demand functions do not collapse to the simple myopic form derived in Vives[1995] and Brennan and Cao[1996]. Hence, the analysis that follows does not allow for an explicit solution as in Brennan and Cao[1996]. He and Wang[1995] point out that with heterogeneously informed agents in the spirit of Hellwig[1980], in an intertemporal setting, the problem of higher order expectations arises. Higher order expectations concern expectations about other
agents' expectations. They arise through price realizations that are a combination of an average belief of agents and the true value of the asset. The belief of each agent in turn depends on the history of public and private signals. The problem is that each agent has to form expectations about the average expectations of agents, which generally leads to an infinite regress problem (Townsend[1983]). However, as He and Wang[1995] show, higher order expectations (in the form of an expectation about the market average expectation) are spanned by the common information and private information set of each investor. Hence, the potentially infinite dimensional state space collapses to a two-dimensional one. Ultimately this result can be ascribed to the continuum of investors that are present in the market. The law of large numbers then takes care of the collision into the two-dimensional information space.

He and Wang[1995] use numerical procedures to derive implications for the dependency of volume and volatility on information flow. Several interesting results are presented. First, they show that even in absence of new information, trading persists until the liquidation date. The cause is the supply shocks that keep on entering the market. The non-informational trade that occurs in each period discloses some of the private information of investors, inducing further trade. In fact, volume can even display an uni-modal time-pattern while no additional information enters the market. Two effects accumulate to establish this pattern. On the one hand investors tend to trade more aggressively given their higher information precision regarding the liquidation value. On the other hand, the shorter the time to liquidation, the less they can dynamically diversify their holding in the asset, decreasing the aggressiveness of investors. Additionally, it is shown that prior to public announcements, investors increase their positions to speculate on the outcome, and subsequently close their positions right after the announcement. The result is a peak in volume around announcement dates. The authors further argue that release of public information causes both high volume and large price shocks, while release of private information may generate high volume combined with small price changes.

2.3.3 INFINITE-PERIOD MODELS

An alternative means to model the intertemporal behavior prices under information asymmetry is presented by Wang[1993,1994]. Instead of assuming investors with a finite consumption horizon that coincides the assets liquidation date, Wang[1993] assumes agents with infinite horizons. Accordingly, the asset is infinitely long lived, and generates a continuous stream of dividends. Specifically, Wang[1993] assumes that the dividend rate $D$ follows the diffusion
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process

\[ dD = (\Pi - kD)dt + b_D dw \]

where \( \Pi \) follows a mean reversion process given by

\[ d\Pi = a_\Pi(\Pi - \bar{\Pi})dt + b_\Pi dw \]

and \( w \) is a three dimensional vector of Wiener processes. Another source of uncertainty stems from the varying supply of the risky asset. The supply is given by \( 1 + \Theta \), where \( \Theta \) is a stochastic variable that follows a mean reversion process

\[ d\Theta = -a_\Theta \Theta dt + b_\Theta dw \]

The investors maximization problem is now replaced by

\[ \max A E \left[ -\int_t^\infty e^{-\rho s - c(s)} ds | \mathcal{I}_t \right] \]

where \( c(s) \) is the consumption of investor \( i \) at time \( s \) and \( \rho \) the discount factor (time-impatience parameter). The variable \( X^i \) represents the holding in the risky asset. The maximization problem is solved under the budget constraint

\[ dW^i = (rW^i - c^i)dt + X^idQ \]

where \( W^i \) is the agents wealth and \( Q \) is return on the risky asset.

The information structure is assumed hierarchical, with uninformed investors who observe only public signals displayed through the price \( P_t \) and the instantaneous dividend rate \( D_t \), and informed investors who additionally observe the state variable \( \Pi \). The uninformed investors are thus faced with the problem of estimating \( \Pi \). Agents do so by means of a Kalman filter. The usage of a Kalman filter replaces the need of agents to consider the whole history of prices and dividends to estimate \( \Pi \). Instead, the Kalman filter produces an equivalent representation of the information structure, and agents only need to consider its innovation process to optimally update their beliefs. The nature of agents’ maximization problem allows Wang[1993] to use the Bellman equations to derive the optimal demands of investors. Conform what we derived in the beginning of the section, this demand is a quadratic form in the state variables. An explicit closed form solution is however not possible given the complexity of the market-clearing condition. Wang[1993] therefore uses numerical procedures to consider the impact of information asymmetry on market statistics. He shows that the innovation variance of prices is strictly larger under information asymmetry. Also, the risk premium on stock is increasing in the fraction of informed investors. The reason is that the absolute value of the stock decreases
with higher informativeness about fundamentals. Wang[1993] also considers the correlation in stock returns. Within his model, due to the mean reverting nature of supply shocks, price returns exhibit negative auto-correlation. Only for high persistence of liquidity combined with high interest rate levels, autocorrelations become positive. Another interesting feature is that, under information asymmetry, uninformed investors may exhibit trading characteristics that are quite different from homogeneously informed economies. Under information asymmetry uninformed investors can behave as trend-followers. This result is due to the different impact of public signals on uninformed investors' updates of beliefs and prices. In homogeneously informed economies the informative part of prices is identical to the update of investors' beliefs. With informed investors present, the informative part of prices combines both private and public information. Hence, public information has a smaller impact on prices than on uninformed investors' beliefs. Uninformed investors' demands reflect the public signal, and so do prices. Hence follows the trend-following behavior of uninformed investors in this model.

Wang[1994] applies a similar, yet discrete, framework to consider how volume is impacted in the presence of information asymmetry. Additionally, an interesting alternative for the source of noise is presented. Instead of the usual liquidity driven supply noise, he endows agents with private production technologies. The return on investment in these private technologies follows an AR(1) process. The key is that the shocks to these returns are correlated with the dividend generation process. Hence, agents have the ability to hedge their exposure in their private investment opportunity through position taking in the risky asset. This gives informed agents an additional non-information based demand component that enters the market-clearing price as if liquidity noise is present. The advantage of this setup is however that it allows for welfare analysis.

The equilibrium is solved in a similar manner as in Wang[1993]. Again, the equilibrium cannot be solved explicitly, and Wang[1994] uses numerical procedures to extract results. The main implications of his analysis concern trading volume. He shows that trading volume is decreasing in information precision of the informed. A result that is in part due to the dual character of informed traders. They are both liquidity as well as informed traders. Trade thus always takes place between informed traders and uninformed traders. The higher the information precision of informed investors, the higher the adverse selection effect between uninformed and informed. The informed character of informed traders then becomes more dominant, increasing information asymmetry, and ultimately leading to a decrease in the attractiveness of trade for the uninformed investors. Additionally, Wang[1994] considers how volume relates to price movements and unanticipated dividend changes. Volume is shown to
be positively correlated with both absolute price changes and dividend changes. The reason is that public information, through the observation of information signals or dividend pay-outs, affects the updates of informed and uninformed differently, leading to additional trade compared to the symmetric information economy.

2.4 Concluding Remarks

In this survey we focused on the competitive rational expectations paradigm, and in particular the multi-period extensions. Though not exhaustive, the main contributions in this area have been described. It should be noted, however, that within the literature on the dynamics of trading under differential information, alternative approaches have been developed. In particular, the Kyle[1985,1989] type of structure, which differs fundamentally from the competitive models, is often used. Examples of these imperfect competition models in a dynamic setting include Kyle[1985], Palomino[1996], and Dow Gorton[1994]. Another approach that has inspired others can be found in Glosten and Milgrom[1985] who consider price formation in the presence of a monopolistic risk neutral market-marking sector.

The approach we adopt in this thesis differs on several accounts from the models that we have discussed in this survey. In particular, this applies to the way in which the asset is modeled. Most models, with the exception of Wang[1993,1994], assume that the asset is ultimately liquidated, and that agents' consumption horizon coincides with the corresponding liquidation date. However, shares are rarely liquidated. Moreover, in these models there is resolution of uncertainty regarding the liquidation value. This introduces a time-dependency in the properties of the equilibrium. For instance, in Vives[1995] agents trade more aggressively when they near the liquidation date. A result that is solely due to the resolution effect, since generally a short time till liquidation implies that agents trade less aggressively. Additionally, when one considers the results of He and Wang[1995], one has to conclude that some results\(^\text{18}\) can be ascribed to the liquidation date that is imposed.

If one wants to study technical trading rules, or derive implications for unconditional moments of price changes, a steady state equilibrium is needed. We therefore adopt an approach

\(^{18}\)In particular the uni-modal pattern of volume discussed in He and Wang[1995]. Two effects accumulate to achieve this pattern: agents on the one hand trade less aggressively nearing their consumption horizon, on the other hand the resolution of uncertainty makes them trade more aggressively. Both effects disappear in a steady state economy. There is no resolution of uncertainty, and the time independence implies that the average consumption horizon is constant through time.
in which assets are infinitely long lived, which allows for such a stationary solution. A consequence of this assumption is that the inclusion of multi-horizon investors, who are outlived by the asset, introduces a dimension that is not present in all of the models discussed. The economy necessarily consists of agents with different time horizons. This contrasts previous approaches, including Wang\(^{19}\)[1993,1994], in which investors are homogenous in the length of their time horizon. Additionally, an infinite period model imposes constraints on the way in which supply can be modeled. Persistence of supply shocks cannot be maintained in such an environment. It would create bubbles, which violates the rationality requirement. Consequently, one needs to incorporate a mean reversion component in the level of supply. As in Brown and Jennings[1989], Slezak[1994] and He and Wang[1995], an explicit solution can therefore not be found. Consequently one needs to resort to the usage of numerical procedures to extract implications.

\(^{19}\)In Wang[1993,1994] investors are all infinitely long lived.
2. A Relations Useful to the Noisy Rational Expectations Approach

This appendix repeats some basic mathematical results that are of use in many of the problems that are encountered when applying the rational expectations approach.

A. The Projection Theorem

The conditional expectation is of particular importance for asymmetric information models. A central theorem in the calculation of these expectations is the projection theorem. Denote by \( X \) and \( Y \) random vectors that are distributed normally with mean \( \mu = (\mu'_x, \mu'_y)' \) and variance-covariance matrix

\[
\text{cov}(X, Y) = \Sigma = \begin{pmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy}
\end{pmatrix}
\]

Then conditional on the observation of \( Y, X \) is distributed normally with mean

\[
E[X|Y] = \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(Y - \mu_y)
\]

and variance-covariance matrix

\[
\text{cov}(X, X|Y) = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}
\]

These expressions can also be derived using the Kalman filter equations (see further in the appendix).

B. Expectation of an exponential quadratic form

The following formula is convenient when calculating expected utilities. Assume that the vector \( Z \) is normal with mean 0 and covariance matrix \( \Sigma \). Define the quadratic form \( \phi \) as

\[
\phi = a + b'Z + Z'cZ
\]

Then the expectation \( E[\exp(-\phi)] \) is given by

\[
E[\exp(-\phi)] = |\Sigma|^{-\frac{1}{2}}|\Sigma^{-1} + 2c|^{-\frac{1}{2}} \exp \left[ -a + \frac{1}{2}b' \left( \Sigma^{-1} + 2c \right)^{-1}b \right]
\]

For unconditional utilities, the following special case can be of help. Using the above expression it can be derived immediately that

\[
E[e^{-tZ^2}] = \frac{1}{\sqrt{1 + 2t}} \exp \left[ -\frac{(E[Z])^2}{1 + 2t} \right]
\]

if \( Z \) has zero mean and variance-covariance of 1.
C. Inverse of a Partitioned Matrix

Another result that often comes handy in this line of field is the inverse rule for partitioned matrices. Given a partitioned matrix $A$, where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

we have that (if all necessary inverses exist) its inverse is given by

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

D. The Kalman Filter Equations

The Kalman filter equations prescribe how to estimate the first two momentums of the distribution of an unobservable vector $X_t$ that follows an AR process conditional on the observation of a correlated signal $Y_t$ that follows an AR process as well. The importance of these equations lies in the fact that they are recursive. As such, though even observations that lie an arbitrary number of periods in the past contribute to the estimate of $X_t$, one only needs to use the last estimate and the current observation of $Y_t$, to generate an optimal prediction for $X_t$. We present the Kalman equations in case $X_t$ and $Y_t$ follow an AR(1) process. Assume that $X_t$ and $Y_t$ follow the AR(1) processes

$$X_t = A X_{t-1} + \varepsilon_t,$$
$$Y_t = B X_{t-1} + \eta_t.$$

The quantities $\varepsilon_t$ and $\eta_t$ are white noise, i.e. distributed normal with zero mean and variance-covariance matrix which we denote by

$$\text{cov}(\varepsilon_t, \eta_t) = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma'_{xy} & \Sigma_{yy} \end{pmatrix}.$$

Further, we denote by $\hat{X}_t$ the conditional expectation of $X_t$ and by $V_t$ the conditional variance-covariance matrix of $X_t$. Denote by $I_t$ the information set at time $t$, i.e. $I_t = \{Y_t, Y_{t-1}, Y_{t-2}, \ldots\}$. Hence, we have the definitions

$$\hat{X}_t = \mathbb{E}[X_t | I_t],$$
$$V_t = \text{cov}(X_t | I_t).$$

\footnote{The analysis easily extended to the more general case where the variance-covariance matrix of the noise and the AR coefficients are time-dependent. Also extensions toward higher order processes can be done easily by extending the state space with additional variables that represent lagged realizations of $X_t$ and $Y_t$.}
The Kalman filter equations then tell us that $\tilde{X}_t$ and $\tilde{V}_t$ are updated according to

$$\tilde{X}_t = A\tilde{X}_{t-1} + H_t(Y_t - B\tilde{X}_{t-1}), \text{ and}$$

$$\tilde{V}_t = \Sigma_{xx} + AV_{t-1}A' - H_t(\Sigma_{xy} + BV_{t-1}B').$$

where

$$H_t = (\Sigma_{xy} + AV_{t-1}B')(\Sigma_{yy} + BV_{t-1}B')^{-1}.$$

Some variations that are useful for the models which are employed in this thesis are considered next.

**Extension I**

Consider the case in which the observable is dependent on the state of the vector $X$ at time $t$, i.e. $Y_t = BX_t + \epsilon_t$ where $\epsilon_t \sim N(0, \Sigma_{ee})$. Then we can write

$$Y_t = BAX_t + B\epsilon_t + \eta_t = DX_{t-1} + \zeta_t$$

The covariance matrix then becomes

$$\text{cov}(\epsilon_t, \zeta_t) = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xx}B' + \Sigma_{xy} \\ \Sigma_{xy}' + B\Sigma_{xx}' & \Sigma_{yy} + B\Sigma_{xx}B' \end{pmatrix}$$

Using the above relations, the adjusted Kalman filter is given by

$$\tilde{X}_t = A\tilde{X}_{t-1} + H_t(DX_{t-1} + \zeta_t), \text{ and}$$

$$\tilde{V}_t = \Sigma_{xx} + AV_{t-1}A' - H_t(\Sigma_{xy} + BX_{t-1}B')$$

with

$$H_t = (\Sigma_{xx}B' + \Sigma_{yy} + AV_{t-1}A'B')(\Sigma_{yy} + B\Sigma_{xx}B' + BAV_{t-1}A'B')^{-1}.$$

**Extension II**

Most rational expectations model assume that observation and process noise are uncorrelated, i.e. $\Sigma_{xy} = 0$. Using the above, the Kalman equations are

$$\tilde{X}_t = A\tilde{X}_{t-1} + H_t(Y_t - B\tilde{X}_{t-1}), \text{ and}$$

$$\tilde{V}_t = \Sigma_{xx} + AV_{t-1}A' - H_tB(\Sigma_{xx} + AV_{t-1}A')$$

with

$$H_t = (\Sigma_{xx} + AV_{t-1}A')B'(\Sigma_{yy} + B(\Sigma_{xx} + AV_{t-1}A')B')^{-1}.$$

This is the form which is found in Wang[1993, 1994] and is used in the chapters 6 and 7 of this thesis to determine the optimal demand of technical analysts.