Dynamics of Price Formation in Financial Markets
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A Multi-Period Multi-Asset Noisy Rational Expectations Equilibrium

3.1 Introduction

Financial assets usually have a lifetime that extends beyond the time horizon of the common investor. Shares of company stock, for instance, are rarely seen to get liquidated. Many models of price formation under differential information, however, assume that the time horizon of investors coincides with the liquidation date of the asset at which its value is revealed. Indeed, this assumption seems somewhat unrealistic. More likely, investors conjecture a dependency between their information and the future price of an asset, and try to exploit this information by means of trade and re-trade in the future with the objective of realizing capital gains. One crucial difference is that the price in each period is then not only determined by the conjecture of agents regarding the relation between information and current prices, but also by the information-price conjecture prevailing for the price at which they are going to re-trade, given that the latter partially determines their payoff. Additionally, unless agents have an infinite lifetime, models that assume liquidation of the asset by definition incorporate a time-dependency. The unfortunate consequence is that the properties of the price process are dynamic over time. Hence, unconditional moments of price changes and investors' trading characteristics typically depend on the time to liquidation. These features motivate the need for a model in which agents time horizons are small relative to the lifetimes of assets, and from which a steady state economy can be derived.

In this chapter, we develop such a framework that describes markets in which infinitely long lived assets are traded and agents are allowed to exchange the asset across generations. It is intentionally generic to provide a basis for the models that are studied in the next two chapters. We incorporate information asymmetry, in the form of a distribution of private information signals across investors, a multi-asset market and a multi-period environment with investors whose time horizons extend over multiple periods. As we show, due to the re-trade feature, pricing coefficients are recursively defined through the dependency of investors' uncertainty on future pricing relations. Though of a highly complex nature, the structure of the pricing
functions as well as other quantities are shown to have a natural interpretation. The generality of this approach is emphasized by the fact that it incorporates several models that have been described in the literature as special cases.

Given the complexity of the model, this chapter is devoid of any explicit quantitative or normative statements, but rather has the ambition to contribute on a more fundamental and (rational expectations wise) technical level. Additionally, it provides the basis for the simplified single-asset version that is utilized in the next two chapters, where we subsequently study how information friction affects re-trade economies, and how markets may evolve under costly information acquisition.

The means by which we model the financial market can be seen as a direct extension of Admati[1985]'s multi-asset market. The difference is the re-trade, or overlapping-generations feature in our model, that contrasts the liquidation assumption of Admati[1985] and for instance Brennan and Cao[1997]. A risky asset is traded, whose true value changes in each period due to the arrival of new information. Hence, there is no resolution of uncertainty. However, information itself is persistent, and as such, today's true value is the optimal predictor for the future true value of the asset. All agents observe a public signal in each period about this true value. Information asymmetry arises through the existence of investors who additionally receive a private signal regarding this quantity. The traders are assumed to maximize their CARA expected utility at some future date. Liquidity traders are present who cause the per capita excess supply to vary randomly.

We derive the equilibrium conditions for the coefficients of the pricing functional. These equilibrium conditions are recursion relations, which, due to their highly complex nature, are not explicitly solvable. However, the structure it imposes on pricing coefficients can be given intuitive interpretations. Additionally, it allows us derive certain features of this type of equilibrium. For instance, we show that if agents rationally foresee an ultimate liquidation date of the asset, the price is an unbiased estimator of the fundamental value of the asset. We extract additional insights by considering special cases of our generic model. In particular, we derive the equilibrium for a myopic two-type investor economy. This simplified framework allows us to illustrate the role that is played by information asymmetry in re-trade models. What matters in such economies is not the absolute degree of information precision, but rather the degree of information dispersion across investors. In fact, economies in which either all investors are uninformed or all investors are informed, are show to be equivalent in the sense that the pricing functionals are identical, except for a lead-lag effect due to the lagged information flow in the uninformed economy. Regarding statistics such as liquidity, volume and variance of price changes there is no difference. Hence, any deviation from these two...
extreme economies with respect to these statistics can be ascribed to information frictions between agents. In the next chapter, we exploit this advantage to consider the impact of front-running in financial markets, using a single asset version.

This chapter is organized as follows. Section two introduces the model. Section three derives the equilibrium conditions for the generic model, and discusses some of its features. In section four we consider various special cases of the model. Section five concludes.

3.2 The Model

The model we study is an extension of Admati's [1985] model in which the Hellwig[1980] framework is applied to a multi-asset market. Instead of assuming that the asset only lives for one period though, we assume that the asset is infinitely long lived. As such, given the finiteness of agents' lives, agents eventually need to re-trade to asset to cash capital gains.

A Assets

We assume that \( n \) assets are traded, whose true value at time \( t \) is given by the vector \( \tilde{F}_t \). In each period, this true value experiences a shock \( \delta_t \) which is normally distributed with variance-covariance matrix \( \Sigma \) and mean zero\(^2\). We simplify the model by assuming that the riskfree rate is zero\(^3\).

B. Investors

There is an infinity of agents present, which we index by \( i \in N = \{1, 2, \ldots \} \). We assume that each agent has a certain finite time-horizon at which he maximizes a CARA utility function, and in each period re-allocates resources optimally conditional on the information available. In each period, a new generation of traders enters the market and replaces an old

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1Another extension of Admati’s multi-asset model is found in Brennan and Cao[1997]. Our extension differs in that we study an overlapping generations model of trade, whereas in their model the assets are liquidated at a finite \( T \).

2Note that our assets do not yield any dividends. The model is easily extended though to the case where the assets do pay a dividend.

3This assumption is however not without consequences. Would we want to incorporate a non-zero riskfree rate, we would also have to include a drift in the true value of the asset. Such an extension may lead to different behavior of the time series. For instance in Wang[1993] it is shown that the combination of high interest rates and high correlation between subsequent supply shocks may lead to a positive autocorrelation of returns. This feature will not show up in our model, where returns are always negatively correlated.
A Multi-Period Multi-Asset Noisy Rational Expectations Equilibrium

 generation that has arrived at its consumption horizon. Note that it implies that the economy is heterogeneously endowed with traders that have different time horizons.

We denote the time horizon of agent \( i \) by \( T_i \) and his risk tolerance by \( r_i \). Formally, the maximization problem of investor \( i \) is given by

\[
\mathcal{V}_{i,t}(W^i_t; t) = \max_{d^i_t} \mathbb{E}\left[ -\exp\left(-\rho_t W^i_t \right) | I^i_t \right], \quad \text{subject to} \quad W^i_{t+1} = W^i_t + d^i_t \left( \tilde{P}_{t+1} - P_t \right)
\]

where we defined the value function \( \mathcal{V}_{i,t}(W^i_t; t) \).

Additionally, liquidity traders are present who cause the per capita excess supply, denoted by \( \tilde{Z}_t \), to vary stochastically through time. We assume that this quantity is normally distributed in each period, with mean 0 and variance-covariance matrix \( U_t \).

C. Information structure

In each period, a public signal reveals the previous true value of the asset, \( \tilde{F}_{t-1} \). Additionally (some) agents receive a private noisy signal regarding the current true value, i.e. investor \( i \) observes the signal \( \tilde{Y}_t = \tilde{F}_t + \varepsilon_t \). The variance-covariance of the noise in each signal is denoted by \( S_{i,t} \).

D. Equilibrium

The rational expectations equilibrium in the economy is defined as a price vector \( \tilde{P} \) and demand schedules \( \{d_i(I_{i,t})\}_{i \in N} \), such that (a) \( \tilde{P} \) is measurable with respect to state of the economy, spanned by the information sets of investors \( \{I_{i,t}\}_{i \in N} \) and the per capita excess supply \( \tilde{Z}_t \), (b) for all \( i \in N \), the demand schedule of investor \( i \) maximizes his expected utility conditional on \( I_{i,t} \), i.e. \( d_{i,t} = \arg \max_{d_t} \mathbb{E}[U_{i,T_{i,t}}(W_{iT_{i,t}}); I_{i,t}] \), (c) in each period the market clears, i.e. \( \int_{i \in N} d_{i,t} dt = \tilde{Z}_t, \forall t \), and (d) all agents conjecture the correct pricing functional.

We assume a pricing functional linear in the true value of the asset, \( \tilde{F}_t \), the common knowledge about this true value, \( \tilde{F}_t \), and the liquidity shocks in the economy, \( \tilde{Z}_t \), i.e. at time \( t \)

\[
\tilde{P}_t = \pi_{0,t} \tilde{F}_t + \pi_{1,t} \tilde{F}_t - \pi_{e,t} \tilde{Z}_t
\]

Note that since in each period the previous value of the fundamental is revealed, we have \( \tilde{F}_t = \tilde{F}_{t-1} \forall t \).

E. Additional Remarks

Future wealth is determined by future price realizations, and as such, agents can never perfectly predict this quantity even if they possess all information available. The reason is
that the future price depends on both new information entering the economy and uncertain future liquidity demand. Both are unknown prior to this next period. Hence, even if the aggregation of information is perfect, this aggregate knowledge is not sufficient to predict the future price. This additional uncertainty complicates the model severely as will be shown later\(^4\).

We additionally remark that the revelation of the previous period's fundamental value simplifies this model significantly. However, this assumption is quite common, though usually it is in disguised form by means of a dividend pay-out (see for instance Wang[1993]). If we drop this assumption, uninformed agents would need to use past prices in order to update their common prior. This type of equilibrium in a single-asset market is studied in chapters 6 and 7.

3.3 Equilibrium

In this section we develop the equilibrium conditions for this economy, utilizing the standard approach. First the demand functions for each agent are derived, given the conjectured form of the price function. Next we impose the condition that agents rationally foresee an ultimate liquidation of the asset, which yields an additional constraint for the pricing coefficients. Ultimately, we demand market clearing to derive the equilibrium conditions explicitly.

A Equilibrium

First, we consider the demand function of each agent. In the appendix it is shown that the following lemma applies.

**Lemma 3.1** The demand function of agent \(i\) is characterized through

\[
d_{i,t}(I_{i,t}) = r_i^{-1} \Omega_{i,t}^{-1} \left( E[\tilde{P}_{t+1}|I_{i,t}] - \tilde{P}_t \right)
\]

where \(\Omega_{i,t}\) is determined recursively by

\[
\Omega_{i,t} = (G_{i,t}^{11} - G_{i,t}^{12}(G_{i,t}^{22} + \Omega_{i,t+1}^{-1})^{-1}G_{i,t}^{12})^{-1}
\]

with the boundary condition \(\Omega_{i,T_{-1}} = \text{var}_{T_{-1}}(\tilde{P}_T)\), and \(G_{i,t}^{kl}\) is the \(kl\)-th partition of size

\(^4\)Similar mechanics can be observed in models where agents receive private signals that contain a common error term. Even in the single period, single asset case, this leads to a cubic equation for the coefficients of the pricing functional (see for instance Grundy and McNichols[1989]).
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$n \times n$ of the matrix $G_{t,t}$, defined by

$$
G_{t,t} = \begin{pmatrix}
\text{var}_i(P_{t+1} - \tilde{P}_t) & \text{cov}_i^t(\tilde{P}_{t+1} - \tilde{P}_t, \tilde{\Omega}^{t+1}_{t+2}) \\
\text{cov}_i^t(\tilde{P}_{t+1} - \tilde{P}_t, \tilde{\Omega}^{t+1}_{t+2}) & \text{var}_i^t(\tilde{\Omega}^{t+1}_{t+2})
\end{pmatrix}^{-1}
$$

with $\tilde{\Omega}^{t+1}_{t+2} = E[\tilde{P}_{t+2} - \tilde{P}_{t+1} | \tilde{Z}_{t+1}^t]$, if the following transversality condition holds:

$$(\pi_{0,t} + \pi_{1,t}) = (\pi_{0,t+1} + \pi_{1,t+1})$$

Proof. See Appendix A. ■

This demand function has a similar structure as found in a single-asset multi-period model studied in Slezak[1994]. The main difference is the presence of a non-zero fixed supply level in his model. The result is that the demand functions additionally contain a constant component that reflects this bias. In our model, investors only have conditional holdings in the asset. This is a rather intuitive result, given that the only imbalances in the supply of the asset, which the market needs to accommodate, are due to the liquidity investors. Since this imbalance has an unconditional value of zero in each period, also the unconditional holding of the rational investors equals zero.

The quantity $\Omega_{i,t}$, which plays an important role in the determination of investors $i$'s holding, is of a highly complex nature. It should be interpreted as the effective uncertainty of investor $i$ regarding the future price change. In fact, in the period just prior to consumption, this quantity equals the uncertainty of the future price realization, thereby coinciding with the demand functions found in single period models, as in Hellwig[1980] or Admati[1985] for instance. In earlier periods $\Omega_{i,t}$ additionally incorporates the correlations between subsequent price changes. Hence, this uncertainty matrix measures the ability of agent $i$ to dynamically diversify through time. Indeed, it is found in most models of trading in multi-period environments that this diversification aspect allows agents to trade more aggressively (see for instance Vives[1995]).

The posterior expectation of investor $i$, i.e. conditional on the price realization, of the future price can be written as

$$
E[\tilde{P}_{t+1} | \tilde{Y}_{it}, \tilde{P}_t] = E[\pi_{0,t+1} \tilde{F}_t + \pi_{1,t+1} \tilde{F}_{t+1} - \pi_{2,t+1} \tilde{Z}_{t+1} | \tilde{Y}_{it}, \tilde{P}_t] = (\pi_{0,t+1} + \pi_{1,t+1})E[\tilde{F}_t | \tilde{Y}_{it}, \tilde{P}_t] = (C_{0,t} \tilde{P}_{t-1} + C_{1,t} \tilde{Y}_{it} + C_{2,t} \tilde{P}_t).
$$

\footnote{Note that the expectations and covariances are defined conditional on the observation of the price at time $t$.}
where we used that \( \tilde{F}_{t+1} = \tilde{F}_t + \delta_{t+1}, E_t[\delta_{t+1}] = E_t[\tilde{Z}_{t+1}] = 0 \), and defined the regression coefficients \( C_{it,t} \). Inserting this expectation in the demand schedule (3.3), we obtain
\[
d_{it} = r_t \Omega^{-1}_{it}(C_{0it} \tilde{F}_{t-1} + C_{1it} \tilde{Y}_{it} + (C_{2it} - I) \tilde{P}_t)
\]
for investor \( i \). The aggregation\(^6\) over the individual investors’ demands should equal the per capita excess supply, i.e. in equilibrium we need
\[
\int_{i \in N} r_t \Omega^{-1}_{it}(C_{0it} \tilde{F}_{t-1} + C_{1it} \tilde{Y}_{it} + (C_{2it} - I) \tilde{P}_t) di = \tilde{Z}_t
\]  
(3.7)
The law of large numbers ensures that the errors in signals cancel almost surely\(^7\), i.e. we have that \( \int_{i \in N} r_t \Omega^{-1}_{it} C_{0it} \tilde{F}_{t-1} di = \tilde{F}_t \) a.s.. Hence, the market clearing condition (3.7) becomes
\[
\int_{i \in N} r_t \Omega^{-1}_{it} C_{0it} di \times \tilde{F}_{t-1} + \left( \int_{i \in N} r_t \Omega^{-1}_{it} C_{1it} di \times \tilde{F}_{it} + \int_{i \in N} r_t \Omega^{-1}_{it} (C_{2it} - I) di \right) \times \tilde{P}_t = \tilde{Z}_t
\]
We can relate this equality to the conjectured pricing functional of investors given by (3.2). Applying the rationality requirement, we obtain the following necessary conditions for the existence of a linear rational expectations equilibrium
\[
\pi_{0,t}^{-1} = \int_{i \in N} r_t \Omega^{-1}_{it} (I - C_{2it}) di
\]
\[
\pi_{1,t} = \pi_{2t} \int_{i \in N} r_t \Omega^{-1}_{it} C_{1it} di
\]
\[
\pi_{0,t} = \pi_{2t} \int_{i \in N} r_t \Omega^{-1}_{it} C_{0it} di
\]
Our model can be simplified according to the following lemma, which is proven in the appendix.

**Lemma 3.2** If investors foresee that the asset is liquidated at a certain period in the future, for all previous periods, we have that
\[
\pi_{0,t} + \pi_{1,t} = I
\]

**Proof.** See appendix B. □

Indeed, the informational component is a weighted average of the common prior of the market and the private information in the market. This elegant feature can be found in most

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\(^6\)The aggregation of a variable \( z^i \) over investors, represented by \( \int_{i \in N} z^i di \), is defined as usual, as \( \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} z^i \).

\(^7\)For a more elaborate discussion on how this aggregation is more formally defined, we refer to Admati[1985, p635].
NREE models. In this multi-period setting, it can also be interpreted as a transversality condition. It ensures that unconditionally, prices are an unbiased estimator of the fundamental value of the asset.

The lemma allows us to simplify the pricing functional according to

$$\tilde{P}_t = (I - \pi_{1,t})\tilde{F}_{t-1} + \pi_{1,t}\tilde{F}_t - \pi_{2,t}\tilde{Z}_t$$

Using the market clearing condition, and the expectation of each investor regarding the fundamental value, it is shown in the appendix that the equilibrium is characterized by the following theorem.

**Theorem 3.1** For the economy described in section 3.2, the pricing function that satisfies the equilibrium definition 3.2.D is given by

$$\tilde{P}_t = \tilde{F}_{t-1} + \pi_{1,t}\tilde{F}_t - \pi_{2,t}\tilde{Z}_t$$

(3.8)

The pricing coefficients are given by

$$\pi_{1,t} = \left(Q_tR_tU_t^{-1}R_tQ_t + Q_t + V_t^{-1}\right)^{-1}\left(Q_tR_tU_t^{-1}R_tQ_t + Q_t\right)$$

(3.9)

with $Q_t$ given by $Q_t = R_t^{-1}\Phi_t$ and $R_t$ defined as

$$R_t = \int_{\Omega_{t-1}} \tau_s \Omega_{s,t}^{-1} (V_{t}^{-1} + S_{s,t}^{-1} + \Phi_sU_t^{-1}\Phi_t)^{-1} ds$$

(3.10)

with \( \Phi_t \) the solution to

$$\Phi_t = \int_{\Omega_{t-1}} \tau_s \Omega_{s,t}^{-1} (V_{t}^{-1} + S_{s,t}^{-1} + \Phi_sU_t^{-1}\Phi_t)^{-1} S_{s,t}^{-1} ds$$

(3.11)

The equilibrium exists if a solution to (3.11) exists. The equilibrium is uniquely tied down through the boundary condition $\pi_{1,T} = 1$, $\pi_{2,T} = 0$.

**Proof.** See appendix. \( \blacksquare \)

The above equations (3.10) and (3.11) are recursion relations due to the pseudo-uncertainty matrix $\Omega_{t-1}$ that depends on the structure of future pricing functionals. The specification of $\Omega_{t-1}$ in (3.4) indicates the complicated nature of these relations. Hence, the integral (3.11) from which $\Phi_t$ is determined, also runs over various functional forms for $\Omega_{t-1}$. This general setting does not allow the equilibrium coefficients to be determined explicitly.
We have assumed that the supply bias is zero. Were we to allow a non-zero fixed supply level, we need to impose additional structure on the price function. In that case, an equilibrium can be found by assuming a non-zero risk free rate $r$, and including a constant term in the price functional. This constant would reflect the discount on price, or risk premium, that compensates agents for holding a biased position in the asset. This extension is relatively easy to perform.

B. Interpretation

Note how the coefficient $\pi_{1,t}$ in relation (3.9) is determined through the innovation variance $V_t$. Indeed, larger information shocks to the economy imply more dominance of both $\pi_{1,t}$ and $\pi_{2,t}$. Although, the above expressions for the pricing coefficients $\pi_{1,t}$ and $\pi_{2,t}$, in terms of $Q_t$ and $R_t$ are identical to the expressions found by Admati [1985], the quantities $Q_t$ and $R_t$ are of an intrinsically more complex nature. Their meaning is however unchanged. $Q_t$ can be interpreted as a proxy for the average information precision (the quality of information) of the market. To illustrate this, note that we can rewrite the price function as

$$\tilde{P}_t = \tilde{F}_{t-1} + \pi_{1,t}(\delta_t - Q_t^{-1}R_t^{-1}\tilde{Z}_t)$$

Hence, the signal revealed by the price, $\tilde{\xi}_t$, can be written as

$$\tilde{\xi}_t = \delta_t - \pi_{1,t}\pi_{2,t}\tilde{Z}_t = \delta_t - Q_t^{-1}R_t^{-1}\tilde{Z}_t$$

The second term on the right hand side is the error of this signal. Therefore, its precision is given by

$$\text{var}^{-1}(\delta_t|\tilde{\xi}_t) = Q_tR_tU_t^{-1}R_tQ_t$$

(3.12)

The implication is that the quality of information conveyed by the price increases with $Q_t$. Hence, $Q_t$ proxies the information precision of the price system.

The quantity $R_t$ can be given more meaning as well. It is directly related to the effective risk tolerance of the market. If all variances are increased by a factor $x$, the quantity $R_t$ is unchanged. However, the multiplication of all risk tolerances by a factor $x$ would magnify $R_t$ by a factor $x$ as well. Note that this implies that if the average risk tolerance increases, also the informativeness of prices, represented by (3.12) increases. This is in accordance with the standard results found in the rational expectations literature. If agents are more risk tolerant, they tend to exploit their information more aggressively, resulting in a higher information precision of price realizations.
C. Investors’ Beliefs

It is readily derived that for investor $i$, the expectation of the future price is given by

$$E_i[P_{t+1}] = E_i[F_t] = \hat{F}_{t-1} + \Gamma_{i,t}^{-1} \left(S_{i,t}^{-1}(\tilde{F}_{t} - \hat{F}_{t-1}) + Q_t R_t U_t^{-1} R_t Q_t \pi_{i,t}^{-1}(\hat{F}_{t} - \hat{F}_{t-1}) \right)$$  (3.13)

where

$$\Gamma_{i,t} = (V_t^{-1} + S_{i,t}^{-1} + Q_t R_t U_t^{-1} R_t Q_t).$$  (3.14)

The quantity $\Gamma_{i,t}$ represents the posterior information precision of agent $i$ regarding the innovation in the fundamental value $\delta_t$. The estimate of this innovation combines the observation of the private signal $\tilde{Y}_{t,i}$ and the signal displayed through the price, $\pi_{i,t}^{-1}(\hat{F}_t - \hat{F}_{t-1})$. Both signals are weighted with their information precisions, $S_{i,t}^{-1}$ and $Q_t R_t U_t^{-1} R_t Q_t$, respectively, as can be observed from (3.13).

In this economy, agents are only concerned with future price realizations. The uncertainty of investor $i$ regarding the future price change is given by

$$\text{var}_t[\hat{P}_{t+1}] = \Gamma_{i,t}^{-1} + \pi_{i,t+1}(V_{t+1} + Q_{t+1} R_{t+1} U_{t+1}^{-1} R_t Q_t),$$

Note that the second term on the right hand side is equal across investors. It represents the residual uncertainty of the economy regarding the future price realization. Even combining all knowledge available cannot resolve this uncertainty.

D. Investors’ Utilities

The value function of investor $i$ at time $t$, $V_{i,t}$, is defined as his expected utility under the optimal demand trading strategy depicted in lemma (3.1), and formally represented by (3.1). The following theorem reports the functional form of this value function.

**Theorem 3.2** The value function of agent $i$ at time $t$, prior to the receiving his private signal, given by

$$V_{i,t}(W_{i,t}; t) = - \left( \frac{1}{T-1} \prod_{t=0}^{T-1} \frac{|G_{i,t}|}{|G_{i,t} + c_T|} \right) \times \exp \left[ -r_{-1} W_{i,t} - \frac{1}{2} \tilde{\Pi}_{i,t+1}^{-1} \Omega_{i,t+1}^{-1} \tilde{\Pi}_{i,t+1}^{-1} \right],$$

with $G_{i,t}$ given by (3.5), and $\tilde{\Pi}_{i,t+1}^{-1} = E[P_{t+1} - P_t | I_t^i]$, and $c_T = \tilde{m} \otimes \Omega_{i,t+1}^{-1} \tilde{m}$ with $\tilde{m}$ a 2x2 matrix defined as $\tilde{m} \equiv \text{diag}(0,1)$.

**Proof.** See appendix.
This value function allows us to derive the ex ante expected utility of agents. Denote the initial date of agent \( i \) by \( t_0 \), and his initial wealth by \( W^i_0 \). Straightforward algebra yields that the ex ante expected utility is given by

\[
E[-\exp(W^i_{T_i})] = -\frac{1}{\sqrt{\text{var}_{t_0}^{\text{un}}(\Pi^{t_0}_{t_0})/\text{var}_{t_0}^{\text{un}}(\Pi^{t_0}_{t_0})^{-1} + \Omega_i^{-1}}}
\times (\prod_{r=t_0}^{T_i-1} \left| G_{i,r} \right| / \left| G_{i,r} + c_i \right|) \exp[-r_i^{-1}W^i_0]
\]

where \( \text{var}_{t_0}^{\text{un}}(\Pi^{t_0}_{t_0}) \) is the unconditional variance of the conditional expectation of \( \Pi^{t_0}_{t_0} \).

### 3.4 Special Cases

To illustrate the generality of the model, and to point out explicitly some of its features, this section discusses various special cases.

#### A. A Multi-Asset Model with Biased Aggregate Knowledge

Assume that we have a two-period economy where in the first period agents are allowed to trade, and in the second period the asset is liquidated at the fundamental value. In-between the two periods, the fundamental value does not change. Allow the information signals agents receive, to have a common noise term with variance-covariance matrix \( \Sigma \). Given the one-period optimization problem, the uncertainty matrix \( \Omega_i \) has a simple form, given by

\[
\Omega_i = \text{var}^i(\hat{P}_T) = \text{var}^i(\hat{F}_T) = \Sigma + I^{-1} = \Sigma + (V^{-1} + S^{-1} + Q'R'R^{-1}Q)^{-1}
\]

Elementary algebra shows that this special case is solved by the pricing function in theorem 3.1, with \( \Phi \) and \( R \) given by

\[
\Phi = \int_{i \in N} r_i ((V^{-1} + S^{-1} + \Phi U^{-1} \Phi) \Sigma + I)^{-1} S^{-1} di \tag{3.15}
\]

and

\[
R = \int_{i \in N} r_i ((V^{-1} + S^{-1} + \Phi U^{-1} \Phi) \Sigma + I)^{-1} di \tag{3.16}
\]

Observe that through the inclusion of a signal error, represented by \( \Sigma \), the implicit relation for \( \Phi \) remains. We can simplify the model further by assuming a single asset economy, in which agents have identical characteristics, i.e. \( r_i = r \), \( S_i = S \) \( \forall i \). In that case, we obtain a cubic equation for \( \Phi \), i.e.

\[
\Phi = r((V^{-1} + S^{-1} + \Phi U^{-1} \Phi) \Sigma + 1)^{-1} S^{-1}
\]
Although this equation can be solved explicitly, it is a complicated relation indeed. It illustrates the complexity that arises if the future payoff of the asset is not measurable with respect to the aggregate knowledge in the market. However, if we desire to study a stationary economy, complete resolution of uncertainty should be absent. Allowing the aggregate knowledge to predicted the whole future of prices, would violate this requirement.

B. Admati’s Multi-asset Market

If we do allow the signal bias to be zero, in the single period economy, we obtain Admati’s[1985] multi-asset market. Inserting $\Sigma = 0$ into (3.15) and (3.16), we readily obtain

$$Q = R_t^{-1}\Phi_t = R_t^{-1} \int_{i \in \mathcal{N}} \frac{r_i S_i}{v_i} di$$
$$R = \int_{i \in \mathcal{N}} r_i di$$

which is Admati’s solution. The explicit expressions exemplify the relation between $Q$ and the "average" information precision, and the connection between $R$ and the average risk tolerance.

C. A Two Type Investor Economy

Due to the aggregation over a continuum of differentially informed agents, explicit expressions for the pricing coefficients are impossible to derive. An often encountered simplification which captures the notion of information asymmetry is the assumption of two types of investors, informed and uninformed. Denoting the fraction of informed investors by $w$, the integrals can be replaced by relatively simple expressions. We assume that a fraction $w$ is perfectly informed, i.e. $S_{i,t} = 0$, while the other part $(1 - w)$ is uninformed, i.e. $S_{i,t} = \infty$.

First consider the expression for $\Phi_t$, this quantity is now solved through

$$\Phi_t = \int_{i \in \mathcal{N}_I} r_i \Omega_{i,t}^{-1} di = r w \Omega_{I,t}^{-1}$$

where we defined $\Omega_{I,t}^{-1} = w^{-1} \int_{i \in \mathcal{N}_I} \Omega_{i,t}^{-1} di$ as the average pseudo-variance of informed investors. The expression $R_t$ is given by

$$R_t = r (1 - w) \Omega_{U,t}^{-1} (V_t^{-1} + r^2 w^2 \Omega_{U,t}^{-1} U_t^{-1} \Omega_{I,t}^{-1})^{-1}$$

where we similarly defined $\Omega_{U,t}^{-1}$. The information precision $Q_t$ becomes

$$Q_t = w (1 - w)^{-1} (V_t^{-1} + r^2 w^2 \Omega_{U,t}^{-1} U_t^{-1} \Omega_{I,t}^{-1}) \Omega_{U,t} \Omega_{I,t}^{-1}$$

Note however that we have a different definition for $Q$. In her model $Q$ is given by the quantity we call $\Phi$. 
Note that if an economy is perfectly informed, we have that $R'_t = 0$, $Q'_t = \infty$, and $R'_t Q'_t = r \Omega_{t,t}^{-1}$. Substituting this into the pricing functional yields:

$$
P_t = \bar{F}_t - r^{-1} \Omega_{t,t} \tilde{Z}_t
$$

Similarly we can derive for an economy in which investors only observe public signals, that $R'_t = r \Omega_{t,t}^{-1} V_t$, and $Q'_t = 0$, which results in a pricing functional

$$
P_t = \bar{F}_{t-1} - r^{-1} \Omega_{t,t} \tilde{Z}_t
$$

The similarity in the two expressions already hints at the equivalence between the two types of markets. It can be shown that for stationary economies one has that $\Omega_t = \Omega_{t,t}$. This is a rather intuitive result. Information signals in this model are exclusively realized through price changes. Although the informed economy has a continuous information advantage over the uninformed economy, econometric analysis of market statistics cannot distinguish between the two. Only the dispersion of information quality across a financial market affects the properties of the price process. In the next chapter we elaborate on this feature of re-trade economies.

**D. Myopic Investors**

The model is greatly simplified if we assume that investors act myopically. In that case, the matrix $\Psi_{i,t}$ that determines the trading aggressiveness of investors, reduces to the uncertainty of investor $i$ with regard to the future price realization, i.e. $\Omega_{i,t} = \text{var}_i(\tilde{P}_{t+1})$. This quantity can be written as

$$
\Omega_{i,t} = \Sigma_{t+1} + \Gamma_{i,t}^{-1}
$$

where $\Sigma_{t+1} = \pi_{1,t+1} V_{t+1} \pi_{1,t+1} + \pi_{2,t+1} U_{t+1} \pi_{2,t+1}$, and $\Gamma_{i,t}$ is given by (3.14). The variance matrix $\Sigma_{t+1}$ represents the unanticipated future price change, i.e. the change in price that cannot be predicted even if given all available information at time $t$. Indeed, if an agent is infinitely well informed, characterized by $\Gamma_{i,t} = \infty$, the common residual uncertainty $\Sigma_{t+1}$ remains. This demonstrates one key feature of this model, namely that prices can never be perfectly predicted through the arrival of new information and the presence of noise traders.

Again, the myopic equilibrium is too general to be solved explicitly. Let us therefore study the two-type equilibrium in this setting. In the following theorem this two-type economy is characterized.

**Theorem 3.3** In the myopic two-type economy the pricing functional is given by

$$
P_t = \bar{F}_{t-1} + \pi_{1,t} \tilde{X}_t - \pi_{2,t} \tilde{Z}_t
$$
3. A Multi-Period Multi-Asset Noisy Rational Expectations Equilibrium

with its coefficients determined by the recursion relations

\[ \pi_{1,t} = w \left( 1 - w \right) \left( I + V_t \Sigma_{t+1}^{-1} + w r^2 V_t^{-1} \Sigma_{t+1}^{-1} U_t^{-1} \Sigma_{t+1}^{-1} \right)^{-1} + w \]

and \[ \pi_{2,t} = (wr)^{-1} \pi_{1,t} \Sigma_{t+1}. \]

Proof. Follows directly upon substituting \( S_i = 0 \) for the informed investors, \( S_i = \infty \) for the uninformed investors, and (3.17) into (3.11) and (3.10), and using theorem 3.1. ■

The above theorem gives an explicit means derive the pricing functions in each period. This allows us to numerically derive the properties of the equilibrium as a function of time, given a certain term structure for the exogenous parameters such as \( V_t \) and \( U_t \). Hence, given a time series \( \{ V_{t+1}, \ldots, V_T; U_{t+1}, \ldots, U_T \} \), the equilibrium pricing functionals are uniquely defined.

E. On the Equivalence between Informed and Uninformed Stationary Economies

The above recursion relations may lead to a stationary equilibrium if we assume a regularity in the series \( \{ V_{t+1}, \ldots, V_T; U_{t+1}, \ldots, U_T \} \), and take the limit of \( T \) to infinity. For instance if we assume that the variance-covariance matrices of the shocks to the economy are constant through time. In that case, taking the limit to infinity may yield a steady state in which the coefficients of the pricing functional are constant. This steady state equilibrium exists if there is a solution to

\[ \pi_1 = w \left( 1 - w \right) \left( I + V \Sigma^{-1} + w r^2 V \Sigma^{-1} U^{-1} \Sigma^{-1} \right)^{-1} + w \]

with \( \Sigma = \pi' V \pi_1 + (wr)^{-2} \Sigma' \pi_1 U \pi_1 \Sigma \).

Again consider the perfectly informed economy. In that case we have that \( w = 1 \), and the pricing functional reduces to

\[ \tilde{P}_t = \tilde{F}_t - \frac{1}{r} \Sigma_I \tilde{Z}_t, \]

with \( \Sigma_I \) the solution to \( \Sigma_I = V + r^{-2} \Sigma' \Sigma \). In case of a perfectly uninformed economy, we have that

\[ \tilde{P}_t = \tilde{F}_{t-1} - \frac{1}{r} \Sigma_U \tilde{Z}_t, \]

with \( \Sigma_U \) the solution to \( \Sigma_U = r^{-2}(\Sigma_U + V)U(\Sigma_U + V)' \). Though at first glance, these two economics seem to price risk in a different way, closer inspection reveals that the following theorem holds.

Theorem 3.4 Liquidity costs are identical in the uninformed and informed economy, i.e., we have that \( \Sigma_U = \Sigma_I \).
Proof: Note that $X \equiv \Sigma_I - V$ is the solution to $X + V = V + \frac{1}{\Gamma}(X + V)U(X + V)'$, or $X = \frac{1}{\Gamma}(X + V)U(X + V)'$. Comparison with the equation for $\Sigma_I$, leads to the deduction that $\Sigma_I = X = \Sigma_I - V$. □

The theorem implies that the price functions in the two economies are nearly identical. The difference is the informational component, which in the informed case leads the informational component of the uninformed economy by one period. That the two equilibria only differ in this aspect is to be expected: the investors’ payoffs are only determined through future prices, hence, their expected utility and thus their demand is only determined through their relative information advantage. This relative information advantage is the same for each investor in both cases, leading to the same equilibria. This illustrates an important feature of the re-trade economy. The absolute degree of informativeness is irrelevant for the pricing of risk, it is the degree of information dispersion that matters.

3.5 Concluding Remarks

In this chapter we derived the equilibrium conditions for a multi-asset, multi-period market in which the assets are re-traded and are devoid of dividend distributions. The primary purpose of this exercise was in contributing to the existing rational expectations literature by providing a more generic extension of many of the models found in the literature. Additionally, it is a basis for the simplified frameworks that are studied in the next two chapters.

The extraction of explicit quantitative statements is particularly cumbersome when studying a multi-asset market. We therefore have not engaged in such activity in this chapter. However, studying multi-asset markets does have a clear added value. In particular it allows one to study the correlation between price changes, and its dependency on aspects such as information asymmetry and the presence of noise traders. This is an area where a model such as we have derived can be of use. In particular in light of risk management techniques that make extensive use of properties of correlation matrices and their assumed stability through time, more understanding with respect to the intertemporal features of these quantities may be of help.

On a technical level, our model would benefit from including a drift in the process for the fundamental value, and including a fixed non-zero mean in the per capita excess supply. This would allow for an explicit specification of the cost of capital in these markets. This extension is relatively easy to perform, and is likely candidate for future extensions of this model.
3. A Multi-Period Multi-Asset Noisy Rational Expectations Equilibrium

3.A Appendix

A. Proof of Lemma 3.1

We denote the wealth of investor \( i \) by \( W_{i,t} \), and the vector of price changes between subsequent periods by \( \Pi_t = P_t - P_{t-1} \). Agent \( i \)'s wealth is recursively determined through his investment decisions characterized by the demand vector \( \{d_{i,t}\}_{T}^{T} \). Explicitly, we have \( W_{i,t} = W_{i,t-1} + d'_{i,t-1} \Pi_t \). The optimization problem of investor \( i \), given his time horizon at \( T_i \), is given by

\[
V_{i,t} = \max_{d_{i,t}} E \left[ -\exp \left( -r^{-1} W_{i,t+1} \right) | I_{t,t} \right]
\]

where we defined the value function \( V_{i,t} \). Additionally, define \( \hat{\Pi}^{t+1}_{t+2} = E_{t+1}[P_{t+2} - P_{t+1}] \), which represents the expected future price change following the next period.

First we show that the following lemma holds.

Lemma 3.3 The maximization problem,

\[
\max_{d_{i,t}} E \left[ -\exp \left( -r^{-1} W_{i,t+1} \right) - \frac{1}{2} \hat{\Pi}^{t+1}_{t+2} \Omega^{-1} \hat{\Pi}^{t+1}_{t+2} \right] | I_{t,t} \]

has the solution

\[
d_{i,t} = r^{-1} \Omega^{-1} E[\hat{\Pi}^{t+1}_{t+2} | I_{t,t}]
\]

if \( E_t[\hat{\Pi}^{t+1}_{t+2}] = 0 \).

Proof. The maximization problem can be written as

\[
V_{i,t} = \max_{d_{i,t}} E \left[ -\exp \left( -r^{-1} W_{i,t} - r^{-1} d'_{i,t} \Pi_{t+1} - \frac{1}{2} \hat{\Pi}^{t+1}_{t+2} \Omega^{-1} \hat{\Pi}^{t+1}_{t+2} \right) | I_{t,t} \right]
\]

Next define the vector \( Z_{t+1} = (\Pi_{t+1} - \hat{\Pi}^{t+1}_{t+2})' \), in terms of which we have

\[
V_{i,t} = \max_{d_{i,t}} E \left[ -\exp \left( -a_t - b'_{t} Z_{t+1} - \frac{1}{2} Z'_{t+1} c_{t} Z_{t+1} \right) | I_{t,t} \right]
\]

where we defined

\[
a_t = r^{-1} W_{i,t} + r^{-1} d'_{i,t} \hat{\Pi}^{t}_{i,t+1}
\]

\[
b_t = r^{-1} (d_{i,t}, 0)'
\]

\[
c_t = \begin{pmatrix}
0 & 0 \\
0 & \Omega^{-1}_{t,t+1}
\end{pmatrix}
\]

The expectation (3.19) can be calculated using the standard formula, and is given by
\[ \mathcal{V}_{i,t} = -\max_{d_{i,t}} \sqrt{\frac{|G_{i,t}|}{|G_{i,t} + c_t|}} \exp[-\phi_{i,t}(d_{i,t})] \]

\[ = -\max_{d_{i,t}} \sqrt{\frac{|G_{i,t}|}{|G_{i,t} + c_t|}} \exp[-(a_t - \frac{1}{2} d_{i,t} (G_{i,t} + c_t)^{-1} b_t)] \]

where \( G_{i,t} \) is given by

\[ G_{i,t} = \left( \frac{\text{var}_{i}^2(\Pi_{t+1})}{\text{cov}_{i}^2(\Pi_{t+1}, \Pi_{t+2}^{t+1})} \right)^{-1} \]

The exponent \( \phi_{i,t} \) in this expression can be written as

\[ \phi_{i,t}(d_{i,t}) = r_t^{-1} W_{i,t} + \frac{1}{2} r_t^{-2} d_{i,t} (G_{i,t} + c_t)^{-1} d_t \]

Taking the derivative to \( d_{i,t} \), we obtain that the first order condition of the maximization problem is given by

\[ r_t^{-1} \Pi_{t+1} - r_t^{-2} ((G_{i,t} + c_t)^{-1})_{11} d_t = 0 \]

or

\[ d_{i,t} = r_t ((G_{i,t} + c_t)^{-1})_{11}^{-1} \Pi_{t+1} \]

Defining \( \Omega_{i,t} = ((G_{i,t} + c_t)^{-1})_{11} \), we have

\[ d_{i,t} = r_t^{-1} \Omega_{i,t}^{-1} (E[P_{t+1}] | T_i, t] - P_t] \]

the value function is thus given by

\[ \mathcal{V}_{i,t} = -\sqrt{\frac{|G_{i,t}|}{|G_{i,t} + c_t|}} \exp[-r_t^{-1} W_{i,t} - \frac{1}{2} \Omega_{i,t}^{-1} \Pi_{i,t+1}] \]

and hence satisfies the form given by relation (3.18).

It now immediately follows that the lemma holds, since at time \( T_i = 1 \) the maximization problem of investor \( i \) satisfies (3.18) with \( \Omega_{i,T_i}^{-1} = 0 \) as a boundary condition.

Note that we have used that \( E_t^i [E_{t+1}^i (P_{t+2} - P_{t+1})] = 0 \). Given the form of the pricing function, this requirement demands that for each \( i \)

\[ E_t^i [E_{t+1}^i (P_{t+2} - P_{t+1})] = E_t^i (P_{t+2} - P_{t+1}) = [\pi_{0,t+2} + \pi_{1,t+2} - \pi_{0,t+1} - \pi_{1,t+1}] E_t^i (P_t) \]

In other words, if the equality

\[ \pi_{0,t+2} + \pi_{1,t+2} = \pi_{0,t+1} + \pi_{1,t+1} \]

is satisfied for all \( t \), the lemma holds for all \( t \). Note that this requirement is in fact a transversality condition.
The recursion relation for $\Omega_{t,t}$ then follows by using the inverse rule for partitioned matrices. Note that the value function is given by

$$V_{t,t} = -\left(\prod_{n=t}^{T-1} \frac{|G_{i,n}|}{|G_{i,n} + c_n|}\right) \exp\left[-r_t^{-1} W_{t,t} - \frac{1}{2} \Pi_{t+1} \Omega_{t,t} \Pi_{t+1}^{-1}\right]$$

B. Proof of Lemma 3.2

The sum of $\pi_{it}$ and $\pi_{0t}$ can be written as

$$\pi_{0t} + \pi_{1t} = \pi_{2t} \int_0^1 r_s \Omega_{s,t}^{-1} (C_{0s,t} + C_{1s,t}) ds$$

For notational simplicity, we will drop the time index in this section. Part of this derivation resembles the work of Admati [1985]. The shortcut she uses to simplify the derivation cannot be used in our model, leading to some additional algebra to derive the equilibrium conditions.

We will first derive the quantities $C_{ji}$. Note that the triple $(F, Y_1, P)$ has the following variance covariance matrix

$$\begin{pmatrix}
V & V & V \pi_1 \\
V & V + S_1 & V \pi_1 \\
\pi_1 V & \pi_1 V & \pi_1 V + \pi_2 U \pi_2'
\end{pmatrix}$$

The projection theorem then implies that

$$(C_{1i}, C_{2i}) \begin{pmatrix} V + S_1 & V \pi_1 \\
\pi_1 V & \pi_1 V + \pi_2 U \pi_2'
\end{pmatrix} = (V, V \pi_1')$$ (3.20)
and

\[ V_i = V - C_{1i} V - C_{2i} A_1 V, \]  

(3.21)

Consider first the inverse of the matrix on the left hand side of (3.20),

\[
\begin{pmatrix}
(V + S_i)^{-1} + (V + S_i)^{-1} V \pi_i L_i V (V + S_i)^{-1} & -(V + S_i)^{-1} V \pi_i L_i \\
-L_i \pi_i V (V + S_i)^{-1} & L_i 
\end{pmatrix}
\]

where

\[ L_i = (\pi_1 (V - V(V + S_i)^{-1} V) \pi_1^T + \pi_2 U \pi_2^{-1})^{-1} \]

Define

\[ K_i = V - V(V + S_i)^{-1} V \]  

and

\[ H_i = \pi_1^{-1} L_i^{-1} (\pi_1^{-1})' = K_i + \Phi^{-1} U \Phi^{-1} \]

(3.22) \hspace{1cm} (3.23)

Then

\[ C_{1i} = (I - K_i H_i^{-1}) V (V + S_i)^{-1} \]  

and

\[ C_{2i} = K_i H_i^{-1} \pi_1^{-1} = K_i H_i^{-1} \Phi^{-1} \pi_2^{-1}. \]

(3.24) \hspace{1cm} (3.25)

where we defined \( \Phi \equiv \pi_2^{-1} \pi_1 \). Define \( \Gamma_i \equiv (V^{-1} + S_i^{-1} + \Phi U^{-1} \Phi) = (K_i - K_i H_i^{-1} K_i)^{-1} \), representing the posterior precision of investor \( i \) regarding the fundamental value of the asset of investor \( i \). We can rewrite \( C_{1i} \) and \( C_{2i} \) as

\[ C_{1i} = \Gamma_i^{-1} S_i^{-1} \]

\[ C_{2i} = \Gamma_i^{-1} \Phi U^{-1} \pi_2^{-1} \]

The matrix \( \Phi = \pi_2^{-1} \pi_1 \) can be written as

\[ \Phi = \int_0^1 r_i \Omega_i^{-1} C_{1i} di = \int_0^1 r_i \Omega_i^{-1} \Gamma_i^{-1} S_i^{-1} di \]

Moreover,

\[ \pi_2^{-1} = \int_0^1 r_i \Omega_i^{-1} (I - \Gamma_i^{-1} \Phi U^{-1} \pi_2^{-1}) di \]

Rearranging yields

\[ \pi_2 = \left( \int_0^1 r_i \Omega_i^{-1} di \right)^{-1} \left( I + \int_0^1 r_i \Omega_i^{-1} \Gamma_i^{-1} \Phi U^{-1} di \right) \]

Now define the following integral :

\[ R \equiv \int_0^1 r_i \Omega_i^{-1} \Gamma_i^{-1} di \]

Then :

\[ \int_0^1 r_i \Omega_i^{-1} di = \int_0^1 r_i \Omega_i^{-1} \Gamma_i^{-1} (V^{-1} + S_i^{-1} + \Phi U^{-1} \Phi) di \]

\[ = R V^{-1} + \Phi + R \Phi U^{-1} \Phi \]
We then obtain for the pricing coefficients, (inserting time indices again)

\[
\pi_2 = \left( R_t (\Phi_t U_t^{-1}\Phi_t + V_t^{-1}) + \Phi_t \right)^{-1} \left( I + R_t \Phi_t U_t^{-1} \right)
\]

\[
\pi_1 = \left( R_t (\Phi_t U_t^{-1}\Phi_t + V_t^{-1}) + \Phi_t \right)^{-1} \left( I + R_t \Phi_t U_t^{-1} \right) \Phi_t
\]

\[
\pi_0 = I - \left( R_t (\Phi_t U_t^{-1}\Phi_t + V_t^{-1}) + \Phi_t \right)^{-1} \left( I + R_t \Phi_t U_t^{-1} \right) \Phi_t
\]

with \( \Phi_t \) the solution to

\[
\Phi_t = \int_0^T r_i \Omega_{i,t}^{-1}(V_i^{-1} + S_{i,t}^{-1} + \Phi_t U_t^{-1}\Phi_t)^{-1} S_{i,t}^{-1} dt
\]

and \( R_t \) defined as

\[
R_t = \int_0^T r_i \Omega_{i,t}^{-1}(V_i^{-1} + S_{i,t}^{-1} + \Phi_t U_t^{-1}\Phi_t)^{-1} dt
\]

Using the definition for \( Q_t = R_t^{-1}\Phi_t \), we obtain the expressions in terms of \( Q_t \) and \( R_t \) presented in the theorem.