Dynamics of Price Formation in Financial Markets

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On Technical Analysis and Time Series Properties of Asset Returns

6.1 Introduction

In the preceding chapters an information structure is imposed in which agents in each period observe a public signal that perfectly reveals the asset's fundamental value of previous periods. This assumption is commonly encountered. In fact, most rational expectations models assume the existence of some exogenous common prior from which agents derive their subsequent beliefs updates. This prior arises as a result of a public signal or signals hidden in for instance dividend pay-outs. In absence of such exogenous form of public information however, prices are the only transmitters that provide public signals. A commonly known prior is then determined through the observation of prices only, and arises endogenously due to trading. If prices contain noisy signals, as dictated by the Grossman-Stiglitz paradox, also past prices may contribute in the estimation of fundamentals. The usage of past market statistics to predict fundamentals falls under the denominator of technical analysis. In this chapter, we focus on the rationale for this activity in detail. We study an economy in which public information is absent, and where, consequently, investors that do not invest in direct information gathering, rely on technical analysis alone. Although their inference of fundamental values through charting obviously is less precise than by direct observation of information, their activity still has added value. In our model, chartists profit from the occurrence of systematic, yet transient, shocks to liquidity that need to be accommodated by a risk averse investor community composed of finitely lived generations. This applies, in spite of their rational awareness of their aggregate price impact, and the existence of investors that perfectly known the assets’ fundamental value. As such, (pure) technical analysis is an integral part of the rational expectations equilibrium we consider. Given the CARA-Gaussian framework of our model, technical analysts apply a simple, exponential moving average type of trading rule to adjust their positions dynamically over time. Depending on the magnitude of the persistence of the shocks to liquidity, these uninformed agents are trend-followers or contrarians.

We also consider the time series properties of stock returns. The type of process that the
supply level in the economy follows plays a profound role in this area. Rationality demands that the fundamental value of stock should necessarily be void of any auto-correlations patterns. Hence, all patterns in returns are ultimately derived from the supply process. In our model, we allow the supply level to follow an autoregressive process of arbitrary dimensions. As such, we can for instance incorporate dependencies on past supply levels and include correlations between subsequent supply shocks. Within the context of charting this generosity is especially important. Ultimately, chartists’ efforts are focused on estimating the markets’ current excess supply level. The result is that the supply process eventually determines the type of trading rule agents use.

Although common to the rational expectations literature is a supply level dynamics limited to a simple AR(1) specification (see for instance Wang[1993,1994], Brennan and Cao[1996] among others), we think that in reality the dynamics of liquidity supply may indeed have a much richer structure. For instance, it is likely that the supply level is mean-reverting, but on a short time scale highly persistent. Liquidity investors may ultimately step into (or leave) the market again, but they will tend to hold their positions for at least a number of trading periods. It is also likely that subsequent innovations in supply are correlated: if a fundamental shock establishes a non-informational need to trade among liquidity investors, the absorption of this supply shock will most likely not happen instantaneously. Some agents may be able to postpone their liquidity trade. However, because a single fundamental shock is the cause for innovations in liquidity supply, a correlation pattern arises. In an application of our framework, we consider the properties of such an economy in detail. We show that if the liquidity level is highly persistent, autocorrelations will be positive. If liquidity shocks are correlated however, short-term correlations will be positive while long-term autocorrelation may be negative. We additionally consider how the presence of technical analysts affects the time series properties of prices. We demonstrate that, perhaps contrary to what one may expect, an increase in the fraction of technical analysts, magnifies correlation patterns.

We undertake this study by using the competitive rational expectations approach as developed by Grossman and Stiglitz[1980] and Hellwig[1980]. We apply an infinite period model that incorporates a continuum of traders who actively exchange a risky asset. The asset is not liquidated in any nearby period, and is void of cash distributions such as dividends. It has a fundamental value associated that varies stochastically through time. Three types of traders are present. Informed investors who perfectly observe the true value of the asset in each period. The second group of investors consists of chartists. Their demand schedules are based on the history of prices alone. Both types of investors, informed and chartists, derive their profitability of trading from the presence of the third group, the liquidity traders. These
traders cause the per capita excess supply to vary randomly over time. As stated previously, we allow this supply process to be an autoregressive process of arbitrary dimensions.

We consider two types of equilibria. First, we derive the equilibrium conditions if agents have consumption horizons that lie beyond a number of periods. A novel feature of this equilibrium is that, at all points in time, the trading community consists of agents with different time horizons. Although a closed form solution is not feasible, our analysis allows for a relatively easy numerical computation of the equilibrium. The other type of equilibrium includes only myopic investors. This allows us to more explicitly derive some of its properties, such as the trading rules chartists apply and time series properties of asset returns. We continue with the consideration of an application in which we assume that the stochastic supply level shocks are highly persistent and correlated between subsequent periods.

Our model has a strong connection with Wang[1993]. He also studies an economy in which informed and uninformed investors trade an infinitely long lived asset. Wang[1993], however, allows for a continuous stream of dividends. These dividends represent a public signal, which leads uninformed investors to not only focus on prices alone. As a result, the common prior is not completely endogenous, but depends on the informativeness of dividends as well. Additionally, the supply process is limited to an Ornstein-Uhlenbeck process. Other recent work on multi-period rational expectations model can be found in Vives[1995] and Brennan and Cao[1996, 1997]. In these papers, investors are modeled that have time horizons that extend over a number of trading periods, and coincide with the liquidation date of the asset. An important point to note is that in these models, shocks to liquidity are persistent. When considering a stationary model as we do, the liquidity level should always revert to some mean, for else bubbles arise. This restriction makes the inclusion of investors with longer-term horizons more difficult.

A salient feature of our model is its overlapping generations aspect. In this respect, our model has connections with the work of De Long et.al.[1990], Palomino[1996]. As in these models, liquidity traders introduce an additional risk premium through the price-risk agents face, when re-trading the asset. However, we model a large market, and as such strategic price manipulation is not present. A recent contribution by Spiegel[1998] also relies on the features of an overlapping generations model. The so-called negative root equilibrium he studies, which is associated with extremely volatile prices, can also be found in our model. His model, however, focuses on a homogeneously informed economy, and as such, technical analysis is absent.

1We do not however explore the properties of this equilibrium.
The risky asset we model is infinitely long lived, and void of cash distributions. Its fundamental value is, therefore, an abstract quantity. In our model, it is a discounted expectation of a liquidation value at an infinite liquidation date. As such, short-lived information, which is modeled by Holden and Subrahmanyam[1996], is not present. Information is persistent, which in fact is the only feature that distinguishes information from noise. In this paper, we also do not try to exploit a possible feature of long-term assets which has been noted by Vishny and Schleiffer[1990]. They argue that long-term assets may exhibit more persistent mis-pricing than short-term assets, due to the fact that arbitrage on a short time scale is less risky and thus less costly. Indeed, the arbitrage found in our model is reminiscent of the Dow and Gorton[1994] arbitrage chains. As in their model, today's optimal actions depend on the actions of future agents. They show that short-term arbitrage may not occur, if transaction costs are too high. Though we use a quite different model, were we to include such trading frictions, a market breakdown may result.

Technical analysis has been motivated from a rational expectations perspective by a number of authors. Brown and Jennings[1989] and Grundy and McNichols[1989] show that uninformed investors can extract more information by combining the noisy signals displayed by subsequent prices. Blume, Easley and O'Hara[1994] also consider how volume may provide additional information. The contribution of our paper in this respect is that we explicitly derive the trading rules technical analysts should use, given an arbitrary stochastic process for the liquidity supply. This allows us to hint at the structure noise may have, given the popularity of trading rules observed in practice, and the empirically found time series properties of asset returns.

From a bounded rationality perspective, the usage of technical analysis has been motivated by De Long et.al. [1990], and Brock and Hommes[1998]. They show that heterogeneous beliefs about the correct prediction model may be persistently biased. This contrasts the contention of Friedman[1953] that technical analysts will be driven out of the market, given their suboptimal trading decisions, and that beliefs will ultimately converge. Brock and Hommes[1998] additionally show how the financial market may get stuck in equilibria with very rich dynamics. In fact, the aggregate impact of investors may lead to chaotic behavior of price realizations. Our approach, however, shows that even under the strict rational expectations requirement, technical analysis may be an integral part of a financial market equilibrium.

This paper is organized as follows. Section 2 introduces the model. In section 3, we derive the equilibrium conditions if we allow investors to have consumption horizons that extend over multiple periods. In section 4, we simplify our model by assuming that agents act myopically.
This allows us to derive the equilibrium conditions more explicitly. We derive the trading rules of uninformed investors and the time series properties of excess returns. Section 5 applies the framework to an economy in which the liquidity supply level is stochastic, yet highly persistent. We extend this application with the allowance of correlated liquidity shocks. Section 6 discusses the implications of our model. We conclude in section 7 with a discussion of possible extensions.

6.2 The Model

A. The Assets

We assume that two assets are traded, a risky and a riskless asset. The riskless asset yields a fixed gross return $R$ in each period. The risky asset does not yield any distributions, and is not liquidated in any nearby future period. With this assumption we deviate from Wang[1993, 1994], and Campbell et.al.[1993] who study a multi-period model in which the risky asset in each period yields dividends, and Brennan and Cao[1996], Brown and Jennings[1989] and Grundy and McNichols[1989] who allow for liquidation of the asset. In this model, the risky asset is infinitely long-lived without any distributions. It can be thought of as a pure growth stock\(^2\).

The fundamental value of the asset is denoted by $V_t$. It represents the discounted future liquidation value conditional on the aggregate information in the market, and is defined recursively as follows:

$$V_t = R^{-1}E_t[V_{t+1} | I_t]$$

where $I_t$ represents the complete information set at time $t$.

In each period, (informed) agents receive new information regarding this quantity. As such, the fundamental value is constantly subject to shocks. We explicitly assume that the

\(^2\)One may also view this model as describing a market on a short time scale compared to the interval in which dividends are paid out.
fundamental value follows the process\(^3\)

\[
V_{t+1} = R V_t + \tilde{\eta}_{t+1}
\]

(6.1)

where \( \tilde{\eta}_{t+1} \sim N[0, \sigma^2] \). We have not included a drift for exposition purposes. The model is easily extended however if we include a constant drift.

**B. The Investors**

The economy consists of an infinity of investors, indexed by \( i \in \mathcal{N} = \{1, 2, \ldots \} \). Investors have a certain lifetime, which may vary, and we depict by \( \{T_i\}_{i \in \mathcal{N}} \). In each period a new generation is ‘born’ that replaces the agents that have arrived at their consumption horizon. As stated, investors have two investment opportunities, the risky and the riskless asset. The price of the risky asset at time \( t \) we denote by \( P_t \). Because of its properties, investors can only make profits from a position in the risky asset through the realization of capital gains\(^4\).

Denoting the wealth of investor \( i \) at time \( t \) by \( W_t^i \), his maximization problem can be written as

\[
V^i(\Psi; t) = \max_{d_t} E[U_t^i | T_t] = \max_{d_t} -E[\exp \left(-\rho_t \left(W_t^i + \sum_{n=t+1}^{T_t} \Delta W_{t+n}^i \right)\right) | T_t] \quad (6.2)
\]

where \( \Delta W_{t+1}^i = (P_{t+1} - R P_t) d_t^i \), with \( d_t^i \) the demand of the investor at time \( t \), \( T_t \) his information set, and, \( \rho_t \) his risk aversion coefficient.

Given the infinity of investors, we need to consider the aggregation over investors. The integration over investors is defined as usual by introducing a triple \( \mathcal{N}^i, \mathcal{P}, \mu \), where

\text{subject to:}

\[
V_t = \sum_{n=0}^{\infty} R^{-n} E_t D_{t+n} + S_t
\]

then we have that

\[
V_{t+1} = \sum_{n=0}^{\infty} R^{-n} E_{t+1} D_{t+n+1} + S_{t+1} = \sum_{n=0}^{\infty} R^{-n} E_{t+1} D_{t+n+1} + R S_t + R D_t = RV_t + \delta_{t+1}
\]

which matches precisely our state process.

\text{According to some references in the literature (e.g. Hirshleifer and Riley [1992]), this feature would relegate our investors to being pure speculators. However, since they do have a clear function in this economy in the form of the accommodation of risk, within our model this characterization is not complete.}
6.2. The Model

\( g(N) \) denotes the set of all possible partitions of \( N \), and \( \mu \) is a Jordan measure defined by \( \forall A \in g(N), \mu(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} 1_{\{i \in A\}} \). We denote the integration of a random variable \( x_i \) over investors by \( \int x_i \, d\mu = \sum_{i \in N} x_i \mu(i) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} x_i \).

C. Liquidity Demand and Hidden Variables

We assume that the per capita excess supply \( \tilde{z}_t \) of the risky asset varies stochastically over time, due to the presence of noise or liquidity traders\(^5\). We allow this stochastic process to be quite generic, but do demand that there is no correlation between the fundamental value and the shocks to liquidity. The process may depend on hidden state variables (these may include lagged excess supplies), but is restricted to an AR specification. We denote the vector of hidden variables and the excess supply by \( \Theta_t \), with the convention that the first element of \( \Theta_t \) is the excess supply. The number of hidden variables is denoted by \( N_h \). Given a set of hidden variables by \( \{\tilde{h}_i, t\}_{1 \leq i < N_h} \), we have \( \Theta_t = (\tilde{z}_t, \tilde{h}_{1,t}, ..., \tilde{h}_{N_h})' \). The stochastic process followed by \( \Theta_t \) is represented by

\[
\Theta_{t+1} = G_\Theta \Theta_t + \tilde{\eta}_{\Theta, t+1}
\]

with \( G_\Theta \) a \((N_h + 1) \times (N_h + 1)\) matrix, and \( \tilde{\eta}_{\Theta, t+1} \) a \((N_h + 1)\) dimensional, normally distributed vector with mean 0 and variance-covariance matrix \( \Lambda \).

For a proper solution, additional requirements are needed. We further demand that:

1. the supply process is transient, i.e. \( \lim_{n \to \infty} (G_\Theta)^n \to 0 \). This requirement is necessary, for else bubbles arise.
2. the matrix \( RI - G_\Theta \) is invertible\(^6\).
3. innovations in the supply process and the fundamental value process are not correlated, i.e. \( \text{cov}(\tilde{\eta}_{\Theta, t}, \tilde{\eta}_{h, t}) = 0 \) \( \forall t \).

In the applications section, we study two explicit structures for the supply process. In the first model, we assume that \( N_h = 0 \), but allow the innovations in the supply level to have

\(^5\)A variety of possibilities can be invoked to model the extra source of noise. We can assume that each trader is endowed with a random portion of the risky asset (such as in Brennan and Cao[1996]). We can assume that the asset is in fixed supply, and let the risk premia vary randomly (such as in Campbell, Grossman and Wang[1993]).

\(^6\)The identity-matrix is depicted by \( I \). For notational simplicity, we do not explicitly give its dimension. It is implied by the context it appears in. Hence, in this case it is of dimension \( N_h + 1 \).
some persistence. Specifically, we assume that 

$$\tilde{Z}_{t+1} = a\tilde{Z}_t + \eta_{t+1},$$

which is captured by

$$G_\Theta = (a), \text{ with } a \in [0, 1)$$

In the second model, we additionally incorporate a correlation between subsequent supply shocks. A specification that matches such description is

$$\tilde{Z}_{t+1} = a\tilde{Z}_t + b(\tilde{Z}_t - \tilde{Z}_{t-1}) + \eta_{t+1}.$$ Introducing one hidden variable \(\tilde{h}_{1,t}\) that equals the lagged supply, the process can be represented as

$$\begin{pmatrix} \tilde{Z}_t \\ \tilde{h}_{1,t} \end{pmatrix} = \begin{pmatrix} a + b & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Z}_{t-1} \\ \tilde{h}_{1,t-1} \end{pmatrix} + \eta_{t+1}.$$ 

### D. Notation and the State Space

The state of the economy at time \(t\) is given by an \((N_h + 2)\) dimensional vector, which we denote by \(\varphi_t\), i.e. \(\varphi_t = (\tilde{V}_t, \tilde{\Theta}_t)'\). Given the properties of \(V_t\) and \(\tilde{\Theta}_t\), this state variable develops according to

$$\hat{\varphi}_{t+1} = G\hat{\varphi}_t + \tilde{\eta}_{t+1}, \quad (6.3)$$

with \(G\), a square matrix of dimension \(N_h + 2\), given by

$$G = \begin{pmatrix} R & 0 \\ 0 & G_\Theta \end{pmatrix}, \quad (6.4)$$

and \(\tilde{\eta}_{t+1}\) normal with mean zero and variance-covariance matrix

$$\Xi = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \Lambda \end{pmatrix}.$$ 

### E. Information Structure and Preferences

Investors are of two types: uninformed traders, or chartists, who only observe the price realization in each period due to the absence of public revelation of information, and, informed traders (the fundamentalists) who perfectly know the current state of the economy, \(\varphi_t\). We depict the information sets of chartists and fundamentalists by \(I_t^c\) and \(I_t^f\) respectively. Denoting the history of a random variable \(A_t\) up to and including time \(t\) by \(A_t\), we have the following representations for the information sets:

$$I_t^c = \{A_t\}, \text{ and } I_t^f = \{\hat{P}_t, \hat{\varphi}_t\}.$$
Chartists hold identical beliefs regarding the distribution of the state of economy. We denote their expectation of $\tilde{\varphi}_t$, prior to the trading session at $t$, by $\varphi_t^e = (\tilde{V}_t^e, \tilde{\Theta}_t)$, and the variance of their estimation error by $\Omega_{t-1}$. Hence, $\varphi_t^e = \mathbb{E}[\tilde{\varphi}_t | \{P_{t-1}\}]$, and $\Omega_{t-1} = \text{var}[\tilde{\varphi}_t | \{P_{t-1}\}]$. Because $\tilde{\varphi}_t$ is $\mathcal{I}_t^f$-measurable, the informed estimation error is zero. We assume that the informed make up a constant fraction $w$ of the investor community. Furthermore, we denote the average risk tolerance of the informed and uninformed by $A_f$ and $A_c$ respectively, where $A_f = \frac{1}{1-w} \int_{t \in \mathfrak{S}} \frac{1}{\mu} dt$ and $A_c = \frac{1}{w} \int_{t \in \mathfrak{S}} \frac{1}{\mu} dt$.

F. Equilibrium

Equilibrium is defined as the existence of a price function $\hat{P}_t$ that maps information and supplies into prices. We assume that this pricing functional is linear in the augmented state variable $\Psi_t \equiv (\tilde{\varphi}_t, \tilde{\varphi}_t^e)'$. Explicitly, we assume that

$$\hat{P}_t = p_t^i \Psi_t = p_t^{c,t} \tilde{\varphi}_t + p_t^{f,t} \tilde{\varphi}_t$$

where $p_t = (p_{c,t}, p_{f,t})'$, and $p_{c,t}, p_{f,t}$ are $(N_h + 2)$ dimensional vectors that measure the impact of the informed and uninformed information respectively. We partition these vectors as $p_{c,t} = (\pi_{c,t}, -\gamma_{c,t})'$, and $p_{f,t} = (\pi_{f,t}, -\gamma_{f,t})'$. $\pi_{c,t}$ and $\pi_{f,t}$ have dimension unity, and measure the impact of the fundamental, persistent part of the economy on prices, $(\tilde{V}_t, \tilde{V}_t^e)$. $\gamma_{c,t}$ and $\gamma_{f,t}$ have dimension $N_h + 1$ and measure the impact of the transient part of the state on prices, $(\tilde{\Theta}_t, \tilde{\Theta}_t^e)$. The pricing function is thus equivalently represented by

$$\hat{P}_t = \pi_{c,t} \tilde{V}_t^e - \gamma_{c,t} \tilde{\Theta}_{t-1} + \pi_{f,t} \tilde{V}_t - \gamma_{f,t} \tilde{\Theta}_t$$

In equilibrium we demand that all agents act optimally and that the market clears, i.e.

$$\int d_t^i(P_t; \mathcal{I}_t) dt = \tilde{Z}_{t,i}$$

with $d_t^i$ the solution to (6.2). In order to tie down the equilibrium uniquely, a boundary condition is needed. We therefore assume that agents foresee that the asset is liquidated at a liquidation period $T$, with $T = \infty$. Ultimately, we focus on the steady state equilibrium in which the pricing coefficients are independent of time.

6.3 The Multi-Horizon Equilibrium

Of fundamental importance is the way chartists update their common prior. Given the Gaussian-Markov property of $\tilde{\varphi}_t$, the common beliefs of these agents are constructed from the combination of the price realization and the previous common prior by means of a Kalman filter.
Specifically, agents use the signal in prices represented by \( p_{t+1}' \), and update their beliefs according to

\[
\tilde{\phi}_{t+1} = G((1 - k_t p_{t+1}') \bar{\phi}_t + k_t p_{t+1} \bar{\phi}_t).
\]

The vector \( k_t \) (of dimension \( N_h + 2 \)) represents a vector of regression coefficients. It depends on the uncertainty of chartists, \( \Omega^c_t \), and is given by \( k_t = \Omega^c_t p_{t+1} (p_{t+1}' \Omega^c_t p_{t+1})^{-1} \). The estimation error of our technical analysts, \( \eta^c_{t+1} = \tilde{\phi}_{t+1} - \phi^c_{t+1} \), is updated as

\[
\Omega^c_{t+1} = \text{var}(\tilde{\phi}_{t+1}) = \Xi + G(I - k_t p_{t+1}') \Omega^c_t G'.
\]

For notational convenience\(^7\), we define two basis vector \( \tilde{e}_f = (1, 0)' \) and \( \tilde{e}_c = (0, 1)' \). The augmented state variable variables can now be represented as \( \Psi_t = (\phi_t, \phi^c_t)' = \phi_t \otimes \tilde{e}_f + \phi^c_t \otimes \tilde{e}_c \). This vector follows an AR (1) process, which we depict by

\[
\Psi_{t+1} = H_{t+1} \Psi_t + \eta_{t+1} \otimes \tilde{e}_f
\]

with

\[
H_{t+1} = \begin{pmatrix}
G & 0 \\
Gk_t p_{t+1}' & G((1 - k_t p_{t+1}')
\end{pmatrix}
\]

(6.5)

Conditional on observing the price at time \( t \), the prediction of chartists of the future state, \( \Psi'_{t+1} \), is given by \( \Psi'_{t+1} = \phi_{t+1} \otimes (\tilde{e}_f + \tilde{e}_c) \). Informed investors can use their knowledge of \( \Psi_t \) to arrive at \( \Psi'_{t+1} = G \phi_t \otimes \tilde{e}_f + \phi^c_{t+1} \otimes \tilde{e}_c \). The following lemma concerning these quantities is useful.

**Lemma 6.1** For each investor \( i \in N \), there is matrix \( B_i \) such that the prediction of investor \( i \) regarding the future state conditional on the price realization, can be written as

\[
\tilde{\Psi}^i_{t+1} = B_i H_{t+1} \Psi_t
\]

and the error \( \tilde{\Psi}^i_{t+1} - \Psi_{t+1} \) can be written as \( \tilde{\tilde{\Psi}}^i_{t+1} = \tilde{\eta}_{t+1} \otimes \tilde{e}_f \), with \( \tilde{\eta}_{t+1} \sim N(0, \Omega^i_{t+1}) \).

**Proof.** The first part of the lemma follows immediately by noting that for informed investors \( B_f \) equals the identity matrix, and for uninformed investors it is given by

\[
B_c = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
\]

(6.6)

\(^7\)We can use this type of notation because the dimension of the noisy state variables equals the dimension of the (measurable) priors. We need to split up the augmented state space into noisy and measurable variables, because otherwise variance-covariance matrices become ill-defined.
Regarding the way in which we can represent the prediction error, we note that for chartists, this quantity is given by
\[ \tilde{Y}_{t+1}^c = \bar{\Psi}_{t+1} - \tilde{\Psi}_{t+1}^c = \bar{y}_{t+1}^c \otimes \hat{e}_f. \]
For informed investors, the error in their estimate is given by
\[ \tilde{Y}_{t+1}^f = \bar{\Psi}_{t+1} - \tilde{\Psi}_{t+1}^f = \bar{y}_{t+1}^f \otimes \hat{e}_f, \]
with \( \bar{y}_{t+1} = \bar{\eta}_{t+1} \).

We now turn to the maximization problem of investors. We have denoted the value function at time \( t \) as \( V^i(W_t^i; \Psi_t; t) \) (see (6.2)). This function should solve the Bellman equation, i.e. it is the solution to
\[
0 = \max_{d_t^i} \left[ \mathbb{E}[V(W_{t+1}^i; \Psi_{t+1}; t + 1); T_t^i] - V(W_t^i; \Psi_t; t) \right]
\]
under the constraints that
\[ W_{t+1}^i = W_t^i + (p_t^i + \Psi_t) d_t^i, \quad \text{and} \quad V(W_t^i; \Psi_t; T_t^i) = -e^{-\rho_i W_t^i}, \]
By backward induction it can be shown that the following theorem applies to the solution.

**Theorem 6.1** The value function of investor \( i \) at time \( t \) can be written as an exponent of a quadratic form in \( \Psi_t \), i.e.
\[
V^i(W_t^i; \Psi_t; t) = -C_t^i \exp \left[ -\rho_i W_t^i - \frac{1}{2} \Psi_t \Gamma_t^i \Psi_t \right]
\]
with \( C_t^i \) a constant, independent of \( \{\Psi_t\}_{t \leq T} \), under the optimal investment policy represented by a demand schedule \( d_t^i \) that is measurable with respect to \( T_t^i \), and given by
\[
d_t^i = \frac{p_t^i (I - K^i_{t+1} \Gamma^i_{t+1}) B_t H_{t+1} - R p_t^i \Psi_t}{p_t^i K_t^i + p_t^i + 1}
\]
with
\[ K_{t+1}^i \equiv \hat{m}_f \otimes \left( (\Omega^i_{t+1})^{-1} + (\Gamma^i_{t+1})_{ff} \right)^{-1}, \]
where \( (\Gamma^i_{t+1})_{ff} \) is the upper left \( (N_h + 2) \times (N_h + 2) \) square of the symmetrical matrix \( \Gamma^i_{t+1} \) which is recursively determined through
\[
\Gamma_t^i = (B_t H_{t+1})^T \Gamma_{t+1}^i \left( I - K_{t+1}^i \Gamma_{t+1}^i \right) B_t H_{t+1}
\]
\[
\quad + \frac{p_t^i (I - K_{t+1}^i \Gamma_{t+1}^i) B_t H_{t+1} - R p_t^i}{p_t^i K_t^i + p_t^i + 1}
\]
and \( \Gamma_{T_t^i} = 0. \)

**Proof.** Assume that the theorem holds for \( t + 1 \). At \( t \), the investor is faced with the optimization problem
\[
d_t^i = \arg \max_{d_t^i} \left\{ -C_t^i \mathbb{E}_t^i \left[ \exp \left[ -\rho_i W_t^i - \rho_i (p_{t+1}^i \Psi_{t+1} - R p_{t+1}^i \Psi_t) d_t^i - \frac{1}{2} \Psi_{t+1} \Gamma^i_{t+1} \Psi_{t+1} \right] \right] \right\}
\]
\[ := -\rho_i \phi_{t+1}^i \]

\[ 6.3. \text{The Multi-Horizon Equilibrium} \]
This expectation can be calculated readily. Using the definition of $\tilde{Y}_t$ and lemma (6.1) we can rewrite $\phi_t^{i+1}$ according to

$$
\phi_t^{i+1} = \left(p_{t+1}^{i}\tilde{\psi}_{t+1}^{i} - R p_{t+1}^{i}\tilde{\psi}_{t+1}^{i}\right) d_t' + \frac{1}{2\rho_i} \tilde{\psi}_{t+1}^{i+1} \Gamma_{t+1}^{i+1} \tilde{\psi}_{t+1}^{i+1} + \left(d_t' p_{t+1} + \rho_i^{-1} \Gamma_{t+1}^{i+1} \tilde{\psi}_{t+1}^{i+1}\right) \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right) + \frac{1}{2\rho_i} \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right) \Gamma_{t+1}^{i+1} \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right),
$$

where we used the symmetrical property of $\tilde{g}_{t+1}$. Using the definition for $K_{t+1}^{i}$, the expectation of $-\exp\left[-\rho_i \phi_t^{i+1}\right]$ can be found using a standard formula (see appendix 2.A),

$$
E[-\exp\left[-\rho_i \phi_t^{i+1}\right] | I_t] = -C_t^i \exp\left[-\rho_i \left(\left(p_{t+1}^{i}\tilde{\psi}_{t+1}^{i} - R p_{t+1}^{i}\tilde{\psi}_{t+1}^{i}\right) d_t' + \frac{1}{2\rho_i} \tilde{\psi}_{t+1}^{i+1} \Gamma_{t+1}^{i+1} \tilde{\psi}_{t+1}^{i+1} + \left(d_t' p_{t+1} + \rho_i^{-1} \Gamma_{t+1}^{i+1} \tilde{\psi}_{t+1}^{i+1}\right) \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right) + \frac{1}{2\rho_i} \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right) \Gamma_{t+1}^{i+1} \left(\tilde{g}_{t+1} \otimes \tilde{e}_{f}\right)\right]}
$$

(6.8)

with

$$
C_t^i = |\Omega_{t+1}^i|^{-\frac{1}{2}} \langle X_{t+1}^i \rangle^{-1} + \langle X_{t+1}^i \rangle J_{\infty}^{\frac{1}{2}} C_t^i.
$$

The maximum value is attained for the demand schedule given by

$$
d_t' = \frac{p_{t+1}' \left(I - K_{t+1}^{i+1} \Gamma_{t+1}^{i+1}\right) \tilde{\psi}_{t+1}^{i+1} - R p_{t+1}^{i}\tilde{\psi}_{t+1}^{i}}{R p_{t+1}^{i} K_{t+1}^{i+1} p_{t+1}}
$$

which upon applying lemma (6.1) reduces to the expression in the theorem. Substituting this optimal policy in (6.8), we obtain the form for the value function as in the theorem with $\Gamma_{t+1}^{i}$ given by (6.7).

Armed with the optimal demand schedule of each agent, we are ready to consider market clearing. The market clearing condition is given by

$$
\int d_t' d_t = \int p_{t+1}' \left(I - K_{t+1}^{i+1} \Gamma_{t+1}^{i+1}\right) B_t H_{t+1} - R p_{t+1}^{i}\tilde{\psi}_{t+1}^{i} d_t = \tilde{Z}_t
$$

Hence, the pricing coefficients are recursively determined through

$$
p_t' = R^{-1} \left[ \int \left(\rho_i p_{t+1}' K_{t+1}^{i+1} p_{t+1}\right)^{-1} d_t\right]^{-1} \left[p_{t+1}' \int \left(I - K_{t+1}^{i+1} \Gamma_{t+1}^{i+1}\right) B_t d_t H_{t+1} - c_t'\right]
$$

(6.9)

Note that $\rho_t'$ depends on $p_{t+1}$, and vice versa. The reason is that $p_{t+1}'$ is determined partially by the uncertainty of chartists at $t+1$. This uncertainty in turn depends on the informativeness of past prices, and hence $p_t'$. Therefore, in equilibrium all price coefficients $\{p_t'\}_{t=1}^\infty$ need to be solved simultaneously.

The state process is stationary. This implies that if the liquidation date is an infinite periods in the future, the economy may converge to a steady state. For consistency, however, the
dispersion of time horizons in the economy needs to be constant as well. Let us assume that this is indeed the case, and that additionally, the distribution of consumption horizons is independent of risk aversion and informedness. Denote the fraction of investors that have $\tau$ periods until their consumption horizon by $\beta_\tau$. Using the above expressions, it is easy to show that the following theorem applies to suchs markets.

**Theorem 6.2** If a stationary equilibrium exists, the equilibrium price function is given by

$$\tilde{P}_t = p'\Psi_t = p'_c\varphi^c_t + p'_f\varphi^f_t$$

with $p'$ the solution to

$$p' = (RS)^{-1} \left[ p' \left[ \sum_{\tau=1}^\infty \beta_\tau \left( wA_f \left( I - K^f_{\tau} \Gamma^f_{\tau} \right) p'_{\tau} K^f_{\tau} + (1 - w)A_r \left( I - K^c_{\tau} \Gamma^c_{\tau} \right) B_r \right) \right] H - I \right]$$

with $S = wA_f \sum_{\tau=0}^\infty \beta_\tau (p' K^f_{\tau} p')^{-1} + (1 - w)A_r \sum_{\tau=0}^\infty \beta_\tau (p' K^c_{\tau} p')^{-1}$, a proxy for the average trading aggressiveness, and where $K^c_{\tau}$ represents the uncertainty matrix for investor $i$ with time to consumption $\tau$. This quantity is given by $K^c_{\tau} \equiv \hat{\mu}_f \otimes (\Omega^{-1} + (\Gamma^c_{\tau})_{eff})^{-1}$ for informed investors and by $K^c_{\tau} \equiv \hat{\mu}_f \otimes (\Omega^{-1} + (\Gamma^c_{\tau})_{eff})^{-1}$ for chartists, where $\Omega_c$ is the solution to the static Ricatti equation

$$\Omega^c = \Xi + G(I - \Omega^c p_f (p'_f \Omega^c p_f)^{-1} p'_f)\Omega^c G'$$

The matrix $\Gamma^c_{\tau}$ is computed recursively, with $\Gamma^c_0 = 0$. The recursion relation for informed investors is given by

$$\Gamma^c_{\tau} = H\Gamma^c_{\tau-1} \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) H$$

$$+ \left( \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) H - R \right)^t p' \left( \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) H - R \right)$$

$$p' K^c_{\tau-1} p$$

and for uninformed by

$$\Gamma^c_{\tau} = (B_c H) \Gamma^c_{\tau-1} \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) B_c H$$

$$+ \left( \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) B_c H - R \right)^t p' \left( \left( I - K^c_{\tau-1} \Gamma^c_{\tau-1} \right) B_c H - R \right)$$

$$p' K^c_{\tau-1} p$$

with $H$ given by the static version of (6.5), and $B^c$ given by (6.6).

The complexity of this setup prohibits any explicit derivation of pricing coefficients or comparative statics. This theorem however does tell us explicitly how to solve this equilibrium numerically. In the next sections we focus however on the markets in which agents are myopic.
6.4 The Myopic Equilibrium

The expressions derived for the multi-horizon case allow for a numerical solution, but do not lend to an easy derivation of comparative statics. We proceed therefore by assuming that agents act myopically. This simplifies the model significantly and allows for a more explicit characterization of the properties of the equilibrium.

In the first subsection, the steady state of the economy is considered. It is a degenerate equilibrium, due to the overlapping generations feature of the economy. We fix it by imposing the boundary condition that agents foresee a liquidation date an infinity of periods hence. Armed with these results, we next derive the chartists trading rules, and the variances and auto-covariances of excess returns.

6.4.1 The Steady State

Consider theorem (6.2). Given the myopia of agents, the matrix $\Gamma^t$ equals the null matrix. Consequently, the uncertainty matrices are given by $K^i_t = n^i_t \otimes \Xi$ for the informed, and $K^c_t = n^c_t \otimes \Omega^c_t$ for the chartists. For ease of exposition, we define $S$, a proxy for the average trading aggressiveness, as

$$S = w A_f (p'_f \Xi p'_f)^{-1} + (1 - w) A_c (p'_c \Omega_c p'_c)^{-1}, \quad (6.10)$$

and $\alpha$ as the relative contribution of the informed to this quantity, i.e. $\alpha = S^{-1} w A_f (p'_f \Xi p'_f)^{-1}$. Using these definitions, from theorem (6.2) it immediately follows that, in the steady state, pricing coefficients solve

$$p' = R^{-1} p' (\alpha + (1 - \alpha) B_c) H - R^{-1} S^{-1} v'_c. \quad (6.11)$$

Using that $p = (p'_f, p'_c)'$, we can split up this equation. For $p'_c$ we obtain the identity

$$p'_c = (1 - \alpha) p'_f (R (R - G (I - k p'_f)))^{-1} - I)$$

Hence, the impact of chartists on the price function, is solved explicitly in terms of $p'_f$. In the appendix, it is shown that this relation can be rewritten as stated in the following lemma.

**Lemma 6.2** The impact of the uninformed on the market clearing price is given by

$$p'_c = -(1 - \alpha) p'_f + (1 - \alpha) (k_v)^{-1} v_c. \quad (6.12)$$

Indeed, the impact of uninformed to price realizations is proportional to their contribution to the average trading aggressiveness in the market $S$, measured by $(1 - \alpha)$. 
6.4. The Myopic Equilibrium

The other relation which can be derived from (6.11), gives us the identity
\[ S^{-1} \gamma'_c = p'_c G k p'_f - R p'_f + \alpha p'_f G + (1 - \alpha) p'_f G k p'_f \]
Using lemma (6.2), it readily follows that
\[ p'_f (G - R) = (\alpha S)^{-1} \gamma'_c \]
Decomposing this relation and using the definitions for \( \alpha \) and \( S \), we finally obtain the following theorem characterizing the equilibrium under myopia of agents.

**Theorem 6.3** In the myopic, steady state equilibrium, the pricing function is given by
\[ P_t = \pi_f (V_t - (1 - \alpha)(1 - k'_c \pi_f^{-1}) V'_t) - \gamma'_f (\Theta_t - (1 - \alpha) \Theta'_t) \]
with \( \pi_f \) unspecified, and \( \gamma'_f \) the solution to
\[ \gamma'_f (RI - G) = w^{-1} A^{-1}_f \gamma'_c \]
Observe that the equation (6.14) consists of \( N_h + 1 \) equations, of which \( N_h \) are linear in \( \gamma'_j \), and one is quadratic. Two implications follows from this. First, it implies that the pricing coefficients that measure the impact of hidden variables are linear in the vector of coefficients \( \gamma'_j \) that measures the impact of the supply level on prices. Second, it implies that we can solve this set of equations explicitly. However, a solution need not exist. In fact, as shown in the following chapter, in an application of this framework, the exogenous parameters may need to satisfy certain restrictions for equilibrium existence.

That \( \pi_f \) is undetermined owes to the overlapping nature of the model. Indeed, an infinity of equilibria exists, and as such this particular equilibrium allows for bubbles.

In the derivation of theorem (6.3), we used \( \gamma'_c = -(1 - \alpha) \gamma'_f \). This gives us that, conditional on past prices, the expected discount on price is given by \( D_t = E[\gamma'_f (\Theta_t - (1 - \alpha) \Theta'_t) | \{ P_{t-1} \}] = \alpha \gamma'_f \Theta'_t \). Observe that relation (6.14) implies that \( \alpha \gamma'_f = S^{-1} \gamma'_c (R - G) \). Therefore, the discount is also represented by
\[ D_t = S^{-1} \gamma'_c (R - G) \Theta'_t \]
Going back to the definition of \( S \) (6.10), it is evident that if \( w \), the fraction of informed decreases, \( S \) diminishes as well. Hence, with an increasing fraction of chartists, the average trading aggressiveness decreases. This implies that the discount on price, \( D_t \), increases in the fraction of chartists. We therefore expect that in markets where many technical analysts are present, risk premia will be high and market depth will be low. Indeed, in the next chapter it is shown that a too high fraction of uninformed investors can lead to a market breakdown.
6.4.2 TYING DOWN THE EQUILIBRIUM

The many equilibria are a direct consequence of the overlapping generations aspect of the model. In fact, as shown by Tirole[1982,1989], a bubble can exist if it grows at the risk free rate. Indeed, if we do not demand \( \tilde{V}_t \) to be related to some liquidation value, any \( \tilde{V}_t \) will suffice if agents believe it will. However, true rationality would demand that there is a consistency between the true value and the liquidation value of the firm. In this subsection, we impose this requirement: we assume that \( \tilde{V}_t \) represents the (discounted) liquidation value at the liquidation date. In this case, the following theorem applies.

**Theorem 6.4** If the asset is liquidated at its true value at an infinite date, the steady state equilibrium is determined uniquely by the requirement that

\[
\pi_c + \pi_f = 1
\]  

(6.16)

**Proof.** Decomposing relation (6.9) leads to two equalities. The first equation gives us that

\[
p'_{c,t} = R^{-1} \left( p'_{c,t+1} + (1 - \alpha_{t+1}) p'_{f,t+1} \right) G (I - k_f p'_{f,t})
\]

(6.17)

where we defined \( \alpha_{t+1} \equiv w A_f (S_{t+1})^{-1} \left( p'_{f,t+1} - \pi_f p'_{f,t+1} \right) \) and \( S_{t+1} \equiv w A_f (p'_{f,t+1} - \pi_f p'_{f,t+1})^{-1} + (1 - w) A_c \left( p'_{f,t+1} \Omega_{t+1} p'_{f,t+1} \right) \). Using this expression for \( p'_{c,t} \), the second equality can be written as

\[
(p'_{c,t} + p'_{f,t}) = R^{-1} \left( p'_{c,t+1} + p'_{f,t+1} \right) G - R^{-1} S_{t+1}^{-1} I
\]

For the individual components we thus obtain

\[
\pi_{c,t} + \pi_{f,t} = \pi_{c,t+1} + \pi_{f,t+1}, \text{ and}
\]

\[
(\pi'_{c,t} + \pi'_{f,t}) = R^{-1} \left( \pi_{c,t+1} + \pi_{f,t+1} \right) G (1 - \alpha) - R^{-1} S_{t+1}^{-1} I
\]

(6.18)

Imposing the boundary conditions at \( T, \pi_{c,T} = 0, \) and \( \pi_{f,T} = 1 \) the theorem follows. ■

This implies that the equilibrium is given by the following pricing functional:

\[
\tilde{P}_t = \tilde{V}_t^c + \pi_f (\tilde{V}_t - \tilde{V}_t^c) - \pi_f (\tilde{V}_t - (1 - \alpha) \tilde{V}_t^c)
\]

with \( \pi_f \) the solution to \( \alpha \pi_f + (1 - \alpha) k_v^{-1} = 1 \). The equation is highly complex through the non-linear dependency of both \( \alpha \) and \( k_v \) on \( \pi_f \).

Condition (6.16) is in fact a transversality condition. It can be shown that under this very condition we have

\[
\lim_{n \to -\infty} R^{-n} E[P_{t+n} | Z_t] = E[\tilde{V}_t | Z_t]
\]
Hence, the discounted expected price at infinity equals the expected true value of the firm, for any information set $I_t$. From this observation we can immediately derive the following corollary.

**Corollary 6.1** If agents foresee that the asset will ultimately be liquidated at its fundamental value, then, unconditionally, the equilibrium price is always an unbiased estimator of the fundamental value.

A similar result can be found in Tirole[1982,1985] where it is shown that if agents hold rational expectations, bubbles cannot exist.

6.4.3 CHARTISTS’ TRADING RULES

**A The Trading Rule**

The uninformed investors (chartists) dynamically adjust their beliefs about the current state using a Kalman filter to extract information from the market clearing price. This in turn causes the uninformed investors to adopt a dynamic trading strategy characterized in the following theorem.

**Theorem 6.5** Chartists submit a demand proportional to the difference between the market clearing price and an exponential moving average of past prices, i.e.

$$d_c t \propto (R - q'Gk)(M_t - \hat{P}_t) \tag{6.19}$$

where $q \equiv p_I + p_c$, and

$$M_t = \frac{q'G(1 - kq')}{R - q'Gk} \left[ \sum_{n=0}^{\infty} (G - Gkq')^n Gk\hat{P}_{t-n} \right] \tag{6.20}$$

**Proof.** The demand of uninformed investors is proportional to their expectation of the excess return. The latter is given by, using the notation of section 6.3, $(p' B^c H - R p') \Psi t$. Using the expression for $H$ and $B^c$, and writing the signal in price as $\hat{P}_t - p'_i \varphi^c_{t-1}$, we obtain the following expression for the chartists’ demand

$$d_c t \propto (q'Gk - R) \hat{P}_t + q'G(1 - kq')\varphi^c_t$$

Next, define

$$M_t = \frac{q'G(1 - kq')}{R - q'Gk} \varphi^c_t \tag{6.21}$$

and the above expression for the demand follows. Expanding the update rule of the uninformed as

$$\varphi^c_t = G(I - kp'_I)\varphi^c_{t-1} + Gkp'_I\hat{P}_{t-1} = \ldots = \sum_{n=0}^{\infty} (G - Gkq')^n Gk\hat{P}_{t-n}$$
we obtain expression (6.20) for $M_t$. 

In this case all investors have a zero unconditional holding in the asset. This is due to our assumption that the mean level of the excess supply is zero. In case of a non-zero fixed supply, agents have a non-zero conditional holding that exactly offsets this bias in supply (see for instance Slezak[1994]).

**B. Trend-following versus Contrarian Trading Rules**

Of special interest are the characteristics of the technical trading rule uninformed investors use. Specifically, we want to know whether this trading rule is trend-following or contrarian. From relation (6.19) it follows that the coefficient $(R - q'Gk)$ determines this classification. Hence, if this coefficient is negative (positive) the uninformed investor is trend-following (contrarian). The following theorem concerns this observation.

**Theorem 6.6** Uninformed investors are trend-followers if $k_z > 0$, and contrarians if $k_z < 0$.

**Proof.** Note that since $p'_c = -(1 - a)p'_f + (1 - a)(k_v)^{-1}v_i$ we can write $q' = ap'_f + (1 - a)(k_v)^{-1}v_i$.

The coefficient can then be expressed as follows:

$$q'Gk - R = ap'_fGk + (1 - a)R - R = ap'_fGk - aR$$

$$= (S^{-1}v'_z - aRp'_f)k - aR = S^{-1}k_z$$

where we used the equality (6.13), and the fact that $p'_f k = 1$. 

Hence if $k_z > 0$, the investors trading behavior is trend-following. Note that $k_z = v'_z \Omega_c p_f (p'_f \Omega_c p_f)^{-1}$ represents the conditional correlation between price and the current liquidity supply level in the market. A positive value of this coefficient may seem an unusual condition. It implies that if price goes up, unconditionally, liquidity demand has most likely decreased. Is the asset a Giffen good then? No, it is not. The argument is subtle, and has an 'is the glass half-full or half-empty' flavor. The explanation is that chartists try to estimate the absolute level of the supply, and not its latest innovation. If $k_z$ is positive, an increasing price is most likely explained by a decrease in the supply level, going from a high supply level to a low supply level. Still, however, the liquidity supply level is positive (i.e. demand negative), and a further decrease in the supply level is expected. Chartists do best by going long in the asset, taking a counter position in the form of a positive demand. If $k_z$ is negative, an increasing price is most probably due to an increase in the demand of liquidity traders, moving for instance from a null level to positive demand level. In that case, chartists act as contrarians, and go short against the long position of the liquidity traders.
6.4.4 Market Statistics

In this subsection, we briefly consider the time series properties of excess returns. Define the excess return as \( \tilde{Q}_{t+1} = \tilde{P}_{t+1} - R\tilde{P}_t \), and the estimation error of chartists as \( \tilde{y}_t = \tilde{\varphi}_t - \varphi_t \).

The next result, which is derived in the appendix, shows how these estimation errors affect excess returns.

**Lemma 6.3** The excess return can be written as

\[
\tilde{Q}_{t+1} = S^{-1}\tilde{Z}_t + p'_f\tilde{y}_{t+1} - (1 - \alpha)(\alpha S^{-1})v' \left(1 - kp'_f\right)\tilde{y}_t
\]

Observe that the average trading aggressiveness \( S \) is crucial to the magnitude of the excess returns. Conform intuition, a more aggressive or risk tolerant trader community decreases risk premia. Note also, that if \( \alpha \) equals 1, corresponding to a fully informed economy, the estimation error of the uninformed does not affect the excess return, as logic would demand.

This follows from rewriting \( \tilde{Q}_{t+1} \) upon implementing \( \alpha = 1 \):

\[
\tilde{Q}_{t+1} = S^{-1}\tilde{Z}_t + p'_f\tilde{y}_{t+1}
\]

If chartists are absent, the excess return is only determined by the shock to the economy \( \tilde{y}_{t+1} \), and the current supply level, \( \tilde{Z}_t \). All other variables that do play a role in the generation of \( \tilde{Z}_t \) are not priced, and have no risk premium associated. With the presence of chartists, also these variables play a role generally to the construction of risk premia or excess returns, through the distorting effect they have on the estimate of the supply shock.

The second moments are characterized in the following theorem.

**Theorem 6.7** The unconditional variance of price innovations is given by

\[
\text{var}(\tilde{Q}_{t+1}) = S^{-2}\nu^T\Sigma_\nu + p'_f\Xi p_f
\]

\[
+ S^{-2} (\alpha^{-1} - 1)v'_\nu \left(\frac{2\alpha - 1}{\alpha}(1 - kp'_f)\Omega^c - \Omega^c (1 - kp'_f)'v'\right) v_\nu
\]

and the autocovariance of excess returns is given by \( (n > 0) \)

\[
\text{cov}(\tilde{Q}_{t+1}, \tilde{Q}_{t+1-n}) = S^{-2}\nu^T G^n \Sigma_\nu
\]

\[
+ S^{-2-n} (G^{m-1} - (\alpha^{-1} - 1)(1 - kp'_f)(G - Gkp'_f)^{n-1}) \Xi p_f
\]

\[
- S^{-2} (1 - \alpha) \left(\frac{2\alpha - 1}{\alpha}(1 - kp'_f)(G - Gkp'_f)^n - G^n\right) \Omega^c (1 - kp'_f)'v' \nu.
\]
The expressions derived do not allow for an easy characterization of the correlogram. In the next section, however, we exemplify our findings by means of an application of our framework. However, do note that the $i_z$ already hints at the fact that changes in the fundamental value do not contribute to higher order correlations.

### 6.5 An Application

The preceding analysis does not specify the way in which the liquidity component evolves through time. Given this generality, we can impose various structures and use the theorems to derive the characteristics of the equilibrium. In this section, we consider a simple economy in order to extract additional implications from the model. In the following chapter we elaborate on an implementation of the model when the liquidity component is pure noise.

The motivation for the structure of the application we consider is as follows. The rational investors in our model should be associated with professional traders that maintain the financial market and accommodate the transfer of supply. When we consider what happens in practice, such traders maintain large positions but only for a short period of time, trying to exploit discrepancies in price levels on a short time scale. The liquidity traders should be associated with institutional or individual investors with a long time horizon. They are assumed to be insensitive to the additional risk premium they pay if their demand accidentally coincides with the aggregate demand in the market, as it is small relative to the length of time they will hold the asset. Their time horizons are long compared to the time horizon of our professional traders, which leads us to model the aggregate supply in the market as being highly persistent on the time scale of the latter group of traders.

#### 6.5.1 Persistence of Liquidity Supply

**A. The Model**

The structure we impose on the liquidity supply $\tilde{Z}_t$ is simple: we assume it follows a mean reversion process given by

$$ \tilde{Z}_t = a \tilde{Z}_{t-1} + \eta_{z,t}. $$

The parameter $a$ effectively measures the persistence of the liquidity supply level. The state
process can thus be written as

\[ \begin{pmatrix} \tilde{V}_t \\ \tilde{Z}_t \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \tilde{V}_{t-1} \\ \tilde{Z}_{t-1} \end{pmatrix} + \eta_t, \]

where \( \eta_t \) has mean zero, and variance covariance matrix \( \Xi \), given by

\[ \Xi = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \Delta^2 \end{pmatrix}. \]

Furthermore, assume that the average risk tolerance of informed and uninformed is equal i.e. \( A_f = A_e \equiv A \).

**B. The Steady State Equilibrium**

Though we ultimately have to calculate the equilibrium numerically, first we solve the equilibrium stepwise, to illustrate the results derived in the previous section. Theorem (6.3) establishes that the pricing function under this structure is given by the form

\[ \hat{P}_t = \pi_f \tilde{V}_{t-1} - \gamma_c \tilde{Z}_{t-1} + \pi_f \tilde{V}_t - \gamma_f \tilde{Z}_t. \]

The equilibrium condition as given in (6.14), reduces to

\[ \gamma_c (R - a) = \frac{1}{wA} (\sigma^2 + \gamma_f^2 \Delta^2) \]

which yields two solutions

\[ \gamma_f^{\pm} = -\frac{wA(R - a)}{2\Delta^2} \left( 1 + \sqrt{1 - \left( \frac{2\pi_f \Delta \sigma}{wA(R - a)^2} \right)^2} \right). \]

The intuitive solution is \( \gamma_f^{+} \), corresponding to an economy in which absence of supply shocks implies that \( \gamma_f = 0 \). This is the type of solution which is commonly used in rational expectations models. In a recent paper by Spiegel[1998], however, the so-called negative root equilibrium (corresponding in our model to \( \gamma_f^{-} \)) is explored. In this equilibrium, the corresponding value of \( \gamma_f \) can be very large. In Figure (6.1) the two solutions are shown as a function of \( a \). Indeed, such high values (of a magnitude 10 times larger) correspond to extremely volatile price movements. We will, however, not consider this type of equilibrium. One reason follows from inspection of relation (6.18) that determines the pricing coefficients recursively. In dynamical systems' vocabulary, the two roots (critical points) are classified into a positive and a negative attractor. The negative attractor corresponds to the negative
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FIGURE 6.1. The two solutions to the equilibrium specification.

...root equilibrium, which means that if the pricing coefficients only deviate slightly from the equilibrium values, they will diverge from the equilibrium point, whereas the other solution is a positive attractor to which solutions will converge when taking the limit of $T$ to infinity. In short, though of interest, within our model the negative root equilibria are quite unstable and likely to explode, ultimately resulting in a market crash.

Observe that the above relation puts constraints on the values we may choose in equilibrium. Indeed, we should have that $\pi_f \leq \frac{\mu \Lambda (R - \alpha)}{2 \Delta \sigma}$. This tells us that the market does not function if the risk tolerance level is too small compared to the variance of shocks to the economy. The factor $(R - \alpha)$ is also a determinant in this condition. Indeed, due to the fact that agents act myopically, and can only profit from trading with liquidity traders, an extremely high persistence of liquidity supply ($\alpha$ near 1) restricts them from being able to resell the asset in the next period. This ultimately leads to a market breakdown. Furthermore, observe that if the fraction of uninformed investors $(1 - w)$ becomes too large this also happens.

Note that if we allow the economy to have bubbles, our calculation of the equilibrium is finished at this point, for we can then consider $\pi_f$ as an exogenous parameter in the model. We, however, also impose the transversality condition.
C. Tying the Equilibrium Down

The remaining unknown quantity at this point is the pricing coefficient \( \pi_f \). This quantity can be found using the transversality condition, which demands that \( \pi_c + \pi_f = 1 \).

First, however, consider the uncertainty of the uninformed. Given that the signal in price is perfectly observable, it can be shown that, defining \( \Omega_{\pi} = \pi_f \),

\[
\Omega_{\pi}^e = \pi_f^2 \gamma_f - 2 \alpha, \quad \text{and} \quad \Omega_{\pi} = \pi_f - \pi_f \gamma_f^{-1} \alpha
\]

Hence, to find \( \Omega_{\pi}^e \), we only need to consider the solution for \( \alpha \). The Ricatti equation tells us that

\[
\alpha = \frac{\sigma^2 \Delta^2 \gamma_f^2 + \sigma^2 \sigma^2 \pi_f^2 + \sigma R^2 \Delta^2 \gamma_f^2}{\pi_f^2 \sigma^2 + \pi_f^2 \alpha (R - \alpha)^2 + \Delta^2 \gamma_f^2}
\]

This quadratic equation in \( \alpha \) is readily solved. The solution can be represented as \( \alpha(\mathcal{E}; \pi_f) \), where \( \mathcal{E} \) represents the set of exogenous parameters of the economy, i.e. \( \mathcal{E} = \{ \sigma, \Delta, R, \alpha \} \).

Next consider \( \pi_c \): the transversality condition (6.16) can thus be written as

\[
\pi_c(\mathcal{E}; \pi_f) + \pi_f = 1
\]

Although this equation cannot be solved explicitly for \( \pi_f \), numerically, it is an easy task. The following discussion builds on the numerical results. Our parameter of focus is the persistence parameter \( \alpha \). Unless stated otherwise, we choose the other parameters of our economy as \( \sigma^2 = 0.1, \Delta^2 = 0.1, \sigma = 16, R = 1.03, \text{and} w = 0.4 \).

C. Trend-following Trading Rules

As stated previously, the sign of \( k_z \), the regression coefficient between price and liquidity supply, determines the type of trading behavior technical analysts exhibit. A positive value corresponding to trend-following behavior. Because chartists attempt to trade with liquidity traders, they need to estimate the current liquidity supply level. This estimate combines two effects. The first effect is the measurement of the direct impact of the current liquidity shock on price, which causes agents to be contrarians. The second effect is caused by uncertainty with regard to the previous supply shock. A previously high magnitude of the supply level may erode with a similar impact on price, but implying the opposite trading strategy. To notice this, consider \( k_z \)

\[
k_z \propto \alpha \sigma \pi_f^2 (R - \alpha) - \Delta^2 \gamma_f^2
\]

where we used that \( \gamma_f \) is always positive. Observe how the two effects determine the ultimate sign of \( k_z \). If past uncertainty \( \alpha \) is very large and if supply shocks are highly correlated, an increasing (decreasing) price is more likely to be due to the erosion of a large negative
(positive) liquidity demand. If instead supply shocks are more transient, an increasing (decreasing) price is most likely caused by an increase in the magnitude of liquidity demand (supply). In other words, if prices rise, this can be due to liquidity traders buying today, but it can also be due to liquidity traders buying back the assets they sold in the previous period. In the latter case, agents assume that liquidity demand is negative. If the variance of prices increases dramatically, and liquidity supply is highly persistent, this effect dominates. Figure (6.2) acknowledges this switch in trading behavior.

Trend-following trading behavior has also been motivated by Wang [1993] and Brennan and Cao [1996, 1997] who show that depending on the relative informedness of agents they may exhibit trend-following or contrarian trading characteristics. This behavior is driven by the occurrence of public information shocks. If fundamental information is publicly revealed, it impacts the price realization. However, informed investors also impact the price with their superior information. The result is that the public signal can only partially explain the change in price. The uninformed, on the other hand, place a heavy weight on the public signal, and even more so than on the change in price. Hence, a price increase will lead uninformed investors to go long. The more positive the public signal, the higher the price, but also the higher the absolute difference between the impact on the uninformed investors beliefs and the impact
6.5. An Application

The variance of price shocks, depicted in Figure (6.3), is seen to increase quite dramatically as $\gamma$ nears unity. It should be noted though, that this increase in variance is not as dramatic as in the negative root equilibrium studied by Spiegel[1998], but is more in line with the increase in variance as has been reported by Wang[1993].

E. Auto-correlations

The first order autocorrelation of excess returns is given in figure (6.4), we observe that it switches sign as $\gamma$ enters the high persistence region. This effect is similar to what we observe for the type of trading rule technical analysts use. Of course, this switch is the very cause of the switch in the type of trading rule and occurs for all higher order auto-correlations. In the critical region, the correlogram of excess returns has a typical form which is depicted in
FIGURE 6.4. First order autocorrelation as a function of $a$. Conform the trading strategy of chartists, it switches from negative to positive in the high persistence regime.

Figure (6.5). That the autocorrelation is negative for low persistence of liquidity and positive for higher persistence is a feature of the equilibrium that coincides with the findings of Wang[1993].

6.5.2 CORRELATED SHOCKS TO LIQUIDITY

The above analysis captures the notion that liquidity supply may be persistent on the time scale of our professional traders. However, it is not likely that, if liquidity pressure in the economy changes, it is immediately absorbed by the market. Instead, it is more likely that it takes time to digest the shift in the supply level. This implies that, apart from a persistence in liquidity supply, a correlation between subsequent liquidity shocks can be expected. In this subsection, we include this feature by an extension of the previous model.

A The Model

We model the correlation between supply shocks, by adding another term, i.e., we assume that liquidity supply follows:

$$Z_{t+1} = aZ_t + b(Z_t - Z_{t-1}) + \eta_{t+1}$$
The correlation between shocks to liquidity supply is measured by $b$. Note that if we put $b = 0$, we obtain the same model as in the previous section.

We capture this process by means of the following presentation. Introducing an additional variable $\tilde{h}_{t,t} \equiv \tilde{z}_{t-1}$, the lagged supply level, we can write

\[
\begin{pmatrix}
\tilde{V}_t \\
\tilde{Z}_t \\
\tilde{h}_t
\end{pmatrix} =
\begin{pmatrix}
R & 0 & 0 \\
0 & a + b & -b \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{V}_{t-1} \\
\tilde{Z}_{t-1} \\
\tilde{h}_{t-1}
\end{pmatrix} + \eta_t
\]

where $\eta_t$ has mean zero, and (pseudo\textsuperscript{5}) variance-covariance matrix $\Xi$, given by

\[
\Xi =
\begin{pmatrix}
\sigma^2 & 0 & 0 \\
0 & \Delta^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\textit{B. Equilibrium}

\textsuperscript{5}This is not a proper variance-covariance matrix due to its singularity. However, for our analysis this specification is without consequences.
FIGURE 6.6. Correlogram of excess returns in the high persistence region for two values of the momentum parameter $b$. A higher momentum leads to a more complex correlation pattern.

In this case, two equations specify the steady state:

$-(a + b - R) \gamma_{f,z} - \gamma_{f,h} = \frac{1}{wA} \left( p_v^2 \sigma^2 + \gamma_{f,z}^2 \Delta^2 \right)$, and

$b \gamma_{f,z} + R \gamma_{f,h} = 0$

Hence, it follows that $\gamma_{f,h} = -R^{-1}b \gamma_{f,z}$, and that

$\gamma_{f,z} = -\frac{\kappa}{2\Delta^2} \left(-1 + \sqrt{1 - 4\kappa^{-2} \Delta^2 p_v^2 \sigma^2} \right)$

where $\kappa = wAR^{-1} \left(R(a + b) - R^2 - b \right)$. Again, $\pi_f$ is determined numerically by additionally imposing the transversality condition. We adopt the values of the previous section, and consider the autocorrelations that arise in the high persistence region. Hence, we choose $a = 0.98$.

C. Autocorrelations

The pattern of correlations that arises is given in Figure (6.6). The graph shows how the momentum term enhances the magnitude of the positive short-term autocorrelation. However, if the momentum term is large (in the figure $\beta = 0.5$), longer-term correlations become negative.

Another interesting phenomenon concerns the way in which the presence of chartists impacts correlation patterns. In Figure (6.7), for high momentum and high persistence, the
FIGURE 6.7. Correlogram of excess returns for \( w = 0.4 \) and \( w = 1 \). Uninformed investors make the correlation pattern more profound.

correlogram is shown for \( w = 0.4 \) and \( w = 1 \). Indeed, perhaps contrary to what one might expect, an increase in the fraction of technical analysts increases the significance of the correlation patterns. This is in line with our discussion in section 6.4.1, where it was deduced that an increase in the fraction of technical analysts increases risk premia. The result is that a more profound correlation pattern arises (recall that changes in the fundamental value do not contribute to a correlation pattern).

6.6 Implications

A The Efficient Market Hypothesis

The use of an exponential moving average rule seems in direct conflict with the weak form efficient market hypothesis. The latter states that one cannot make extranormal profits using past market statistics. The argument is that competition drives prices to their efficient levels. Our model, though competitive in the sense that all investors are price takers, is not competitive however in the Bertrand sense. Therefore, this notion of efficient markets cannot be applied directly to our setting.

In our model, the uninformed are allowed to predict future returns to some extent since they
need to be compensated for the risk they induce when taking a portion of the risky asset out of the hands of those that experience a liquidity shock. From this point of view, the ability to predict is in our economy not more than the usual risk premia that are associated with risky assets. Since the sign and size of the liquidity shocks change over time, also the sign and the size of risk premia vary over time. This in turn causes chartists to adopt a dynamic trading strategy. However, they do increase their expected utility through trading, regardless of their personal preferences or initial endowments. This implies that the risk premia chartists receive for holding the risky asset, are more than they need to offset their decrease in expected utility from the risk it induces. In other words, this financial market offers a free lunch. Therefore, in order to justify this equilibrium, we need to impose a trading cost for each agent. This cost arises naturally however if we assume that there is a cost associated with the maintenance or the setup of a financial market.

As noted by Brown and Jennings[1989], whether this type of technical analysis violates the weak form efficient market hypothesis, depends on the definition used. They examine different definitions under the specific assumption of their model, that at a date 0, there is some exogenous common prior. In our model, however, a common prior can only be estimated using technical analysis, and that being the case, most definitions cannot be examined within our model. An exception is the definition proposed by Malkiel[1992]: a market is efficient with respect to some information set, if prices do not change if this information is revealed to all market participants. Indeed, by construction our model is weak form efficient in this sense. However, technical analysis is necessary within our model: not using past prices would definitely lead to a decrease in expected utility.

Last but not least, we remark that also informed agents need to apply technical analysis in our model, in order to calculate the estimates of rational uninformed investors. Hence, our point is somewhat stronger than just providing a rationale for technical analysis. Our model indicates that all agents should use technical analysis.

B. Technical Trading Rules

An integral part of the rational expectations equilibrium derived in this paper, is the use of an exponential moving average type of trading rule by uninformed investors. Moreover, in the application we studied, we have shown that if liquidity shocks are highly persistent, it results in trend-following trading behavior of these agents. Our model implies that it may be perfectly rational for uninformed investors to adopt an exponential moving average type of trading rule as is observed in practice (Pring[1991]). That moving average rules are used extensively can be derived from a survey by Taylor and Allen[1992] that revealed that about
64% of professional traders use some form of trend-following technical trading rule that influences their trading decisions. Moreover, studies by Brock et al. [1990] and Kho [1994] show that these trading rules can indeed be considered profitable. The explanation that our model implies is as follows: agents simply use a Kalman filter to estimate the true value of the asset, which results in the consideration of a moving average of prices. Trend-following behavior is optimal when chartists are highly uncertain about past supply levels, which makes price changes more likely to be due to an erosion of the absolute magnitude of liquidity than to a temporary increase in the latter.

C. Time Series Properties

In the application we showed that a high persistence of liquidity supply can lead to positive autocorrelation of excess returns. If additionally, liquidity shocks are positively correlated, this leads to a rapidly decaying structure of serial correlations, which can even amount in a negative long-term autocorrelation. Conrad [1989] found such a positive correlation between price changes on a short-time scale of daily to weekly returns. Long-term correlations are not found significant in this study. Within our model this can be explained from the fact that the sign of long-term autocorrelations is more sensitive to the structure of the market. If liquidity shocks are highly correlated, a negative autocorrelation is expected, while a mild correlation between liquidity shocks leads to a positive long-term correlation. Jegadeesh [1990] finds negative first order autocorrelation on the scale of monthly returns. He also finds positive autocorrelations for longer time intervals. The second order supply process we studied, cannot capture this effect. We would need an additional dimension in the supply process to establish such a pattern.

An interesting result is that though technical analysts exploit correlation patterns in returns, an increase in the fraction of technical analysts, only increases the significance of these patterns. Hence, the idea that chartists will arbitrage away these correlation patterns does not hold in our model. However, it should be noted that this result only holds if we keep the total number of investors constant. Were we to allow free entry into trading, two effects accumulate. On the one hand, the variance of liquidity shocks per capita would decrease, which diminishes the sensitivity to supply shocks, while on the other hand the above effect shows up. Which effect dominates may be an interesting topic for further analysis of the model.

The empirical literature is however not conclusive regarding patterns in time series (see for instance Boudoukh et al. [1994] or Conrad and Kaul [1989, 1998]).
6.7 Concluding Remarks

In this chapter, we proposed a stationary noisy rational expectations economy in which the aggregation of noise and information in prices motivates the presence of pure technical analysts. A closed form solution was derived for a class of economies where liquidity supply is allowed to follow any type of AR(n) process. For the special case where investors trade myopically, we additionally derived the characteristics of chartists’ trading rules and the unconditional moments of price changes. It was shown that technical trading rules may exhibit trend-following characteristics. In the process, we argued how a presence of technical analysts may increase risk premia and decrease market depth. Moreover, we showed that if agents believe that the asset is ultimately liquidated at its true value, prices are an unbiased estimate of the fundamental value.

In an application of our framework, the impact was studied of a persistence of liquidity supply and a correlation between liquidity shocks, on the trading behavior of technical analysts and the time series properties of returns. As we demonstrated, contrary to what one may expect, an increase in the fraction of technical analysts, leads to an increase in the significance of correlation patterns.

It is tempting to try to compare our economy directly with some empirical studies. In particular, we would like to try to calibrate the values of the parameters in our model. Also, it is of interest to elaborate more on the way empirically measurable quantities from time series data, such as autocorrelations, are dependent on the exogenous parameters in our model. In a next version, we will try to extend our story along these lines.

Many authors have considered the possibility that bubbles may exist and that the overlapping generations aspect of stock trading may provide a rationale for this feature of financial markets. As we have shown, such bubbles will not exist if investors anticipate that the stock will be liquidated at its intrinsic value, even if this liquidation date is infinitely far in the future. However, it may be not too realistic to assume that agents can indeed find out what this intrinsic value actually is. As such, when we allow agents to herd on a proxy for this true value in the form of $V_t$, then the equilibrium is not necessarily tied down. Related to this issue is the work of Dow and Gorton[1994] who show that prices may be limited in their information content if the realization of this information is too distant. This result is due to transaction costs present in the market, that make short-term arbitrage unprofitable. In our model, such transaction costs would lead to a market breakdown. This also hints that more elaboration on the multi-horizon economy we derived is important.
6. Appendix

A. Proof of lemma 6.2

The proof relies on the fact that $G$ is a block diagonal matrix. Partition the vector $k$ and $p_f$, as follows

$$p_f = \left( \pi_f, -\gamma_f' \right)'$$

$$k = \left( k_v, k_e \right)'$$

We can write the following

$$R - G(I - kp'f) = \begin{pmatrix} Rk_v \pi_f & Rk_v \gamma'_f \\ G_e k_e \pi_f & R - G_e + G_e k_e \gamma'_f \end{pmatrix}$$

Using the inverse rule for partitioned matrices (see appendix 2.A), we obtain

$$(R - G(I - kp'f))^{-1} = \begin{pmatrix} \frac{1}{Rk_v \pi_f} (1 + \gamma'_f CC_e k_e) & -\frac{1}{R} \gamma'_f C \\ -\frac{1}{Rk_v \pi_f} C & C \end{pmatrix}$$

where $C = (R - G_e)^{-1}$. Straightforward matrix algebra then yields

$$Rs'_f (R - G(I - kp'f))^{-1} = (k_v)^{-1} k_v$$

Substituting this expression in the equation for $p'_c$ gives the lemma. ■

B. Proof of theorem 6.3

We can write the return $\hat{Q}_{t+1}$ as

$$\hat{Q}_{t+1} = \hat{p}'_f \hat{\varphi}_{t+1} + \hat{p}'_e \hat{\varphi}_{t+1} - R \hat{p}'_f \hat{\varphi}_t - R \hat{p}'_e \hat{\varphi}_t$$

$$= \hat{p}'_c G_k p'_f \hat{\varphi}_t + \hat{p}'_f (G - R) \hat{\varphi}_t + \hat{p}'_c (G - R) \hat{\varphi}_t + \hat{p}'_f \hat{\varphi}_{t+1}$$

Using the equality $p'_c = -(1 - \alpha)p'_f + (1 - \alpha)(k_v)^{-1} k_v$, and $p'_f (G - R) = (\alpha S)^{-1} \varphi_z$, we have that

$$p'_c (G - R) = -(1 - \alpha)p'_f (G - R) + (1 - \alpha)(k_v)^{-1} k_v (G - R)$$

$$= -(1 - \alpha)(\alpha S)^{-1} \varphi_z$$

and

$$p'_f (G - R) \varphi_t + p'_c (G - R) \varphi_t = (\alpha S^{-1}) \varphi_t - (1 - \alpha)(\alpha S)^{-1} \varphi_z' \varphi_t$$

$$= (\alpha S)^{-1} \varphi_t + S^{-1} \varphi_z' \varphi_t$$

and $p'_c G k_p = -(1 - \alpha)(\alpha S)^{-1} \varphi_z' \varphi_t$. Using these results, we can rewrite $\hat{Q}_{t+1}$ as

$$\hat{Q}_{t+1} = (\alpha S)^{-1} \varphi_t (1 - k p_f') \hat{\varphi}_t + S^{-1} \varphi_t (k p_f' \hat{\varphi}_t + \varphi_t') + p'_f \hat{\varphi}_{t+1}$$

$$= S^{-1} \hat{Z}_t + p'_f \hat{\varphi}_{t+1} - (1 - \alpha)(\alpha S)^{-1} \varphi_z (1 - k p_f') \hat{\varphi}_t$$
where we used \( \varphi_t = \sum_{n=1}^{\infty} C^n k_p f_j \bar{y}_{t-n} \), and \( \bar{y}_t = \bar{y}_t + \sum_{n=1}^{\infty} C^n k_p f_j \tilde{y}_{t-n} \).

C. Variances and Covariances

To calculate the variances and covariances, we rewrite the excess return in terms of the innovation process \( \tilde{y}_t \). Since,

\[
\bar{y}_t = G(I - k_p f_j) \bar{y}_{t-1} + \eta_t = \sum_{i=0}^{\infty} (G - G k_p f_j)^i \tilde{y}_{t-i},
\]

we have

\[
\tilde{y}_{t+1} = S^{-1} e' \sum_{i=0}^{\infty} G^n \tilde{y}_{t-i} + p'_f \tilde{y}_{t+1} - (1 - \alpha)(\alpha S)^{-1} e' (1 - k_p f_j) \sum_{i=0}^{\infty} (G - G k_p f_j)^i \tilde{y}_{t-i}
\]

\[
= S^{-1} e' \sum_{i=0}^{\infty} W(i) \tilde{y}_{t-i} + p'_f \tilde{y}_{t+1},
\]

where we defined \( W(i) = G^{-1} (\alpha - 1)(1 - k_p f_j) (G - G k_p f_j)^i \).

The covariance between price changes can be written as

\[
\text{cov}(\tilde{y}_{t+1}, \tilde{y}_{t+1-n}) = \text{cov}(S^{-1} e' \sum_{i=0}^{\infty} W(i) \tilde{y}_{t-i} + p'_f \tilde{y}_{t+1}, S^{-1} e' \sum_{j=0}^{\infty} W(j) \tilde{y}_{t-n-j} + p'_f \tilde{y}_{t+1-n})
\]

\[
= \sum_{j=0}^{\infty} S^{-2} e' W(j+n) \Xi W(j) e' + S^{-1} e' W(n-1) \Xi p'_f I_{\{n>0\}} + p'_f \Xi p'_f I_{\{n=0\}}
\]

We can simplify this expression. Note that we have that

\[
\Omega^c = \Xi + (G - G k_p f_j) \Omega^c G = \sum_{i=0}^{\infty} (G - G k_p f_j)^i \Xi G^i
\]

Also observe that

\[
(1 - k_p f_j) \Omega^c (1 - k_p f_j)' = (1 - k_p f_j)(\Omega^c - \Omega^c k_p f_j) (1 - k_p f_j)(\Omega^c - k G f_j (p'_f \Omega^c p_f))
\]

\[
= (1 - k_p f_j) \Omega^c
\]

where we used \( k = \Omega^c p_f (p'_f \Omega^c p_f) \) and \( (1 - k_p f_j)k = 0 \). This equality implies that \( \Omega^c \) is also represented by

\[
\Omega^c = \Xi + (G - G k_p f_j) \Omega^c (G - G k_p f_j)' = \sum_{i=0}^{\infty} (G - G k_p f_j)^i \Xi (G - G k_p f_j)^i
\]

Using these expressions, after some algebra, it follows that

\[
\sum_{j=0}^{\infty} W(j+n) \Xi W(j)' = G^n \Xi - (1 - \alpha) \left( \frac{2\alpha - 1}{\alpha} (1 - k_p f_j) G k_p f_j)^n \Omega^c - G^n \Omega^c (1 - k_p f_j)^n \right)
\]
where $\Sigma = \sum_{j=0}^{\infty} G^j \Xi G^j$ is the unconditional variance of the state of the economy.

Hence, we arrive at the following expression for the variance of price changes

$$\text{var}(\tilde{Q}_{t+1}) = S^{-2}t_s^2 \Sigma t_z + \nu_f^2 \Xi p_f^2$$

$$+ S^{-2}(\alpha^{-1} - 1)t_s^2 \left( \frac{2\alpha - 1}{\alpha} (1 - kp_f') \Omega - \Omega'(1 - kp_f') \right) t_z$$

and for $n > 0$

$$\text{cov}(\tilde{Q}_{t+1}, \tilde{Q}_{t+1-n}) = S^{-2}t_s^2 G^n \Sigma t_z$$

$$+ S^{-1}t_s^2 (G^n - (\alpha^{-1} - 1)(1 - kp_f')(G - Gkp_f')^{n-1}) \Xi p_f$$

$$- S^{-2}t_s^2 (1 - \alpha) \left( \frac{2\alpha - 1}{\alpha} (1 - kp_f')(G - Gkp_f')^n - G^n \right) \Omega(1 - kp_f') t_z.$$
where we define $W(g) \in \mathcal{C}^r(x^2 - (1 - ky)^2 = \frac{x^2}{a^2})$.

The covariance between power changes can be shown to be:

$$
\text{cov}(Q_i, i_j) = \left( x^2 - \sum y_i \right) W_{i_i} W_{j_j} x^2 - \sum y_i \left( \sum k_i \right) W_{i_i} W_{j_j} x^2 - \sum y_i \left( \sum k_i \right) W_{i_i} W_{j_j} x^2
$$

We can simplify this expression. Notice that we have that:

$$
\Gamma = H < \sum (G - O_h) y > = \sum_{i=1}^{N} (G - O_h) y_i
$$

Also observe that:

$$
\text{cov}(Q_i, i_j) = \left( x^2 - \sum y_i \right) W_{i_i} W_{j_j} x^2 - \sum y_i \left( \sum k_i \right) W_{i_i} W_{j_j} x^2 - \sum y_i \left( \sum k_i \right) W_{i_i} W_{j_j} x^2
$$

This quantity represents the $\Gamma$ in the expression:

$$
\text{cov}(Q_i, i_j) = \sum (G - O_h) y_i
$$

Using the expression above, we generalize it follows that:

$$
\sum_{i=1}^{N} (G - O_h) y_i
$$