Dynamics of Price Formation in Financial Markets

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Market Failure in the Presence of Technical Analysts

7.1 Introduction

The preceding chapter showed how investors that do not invest in information acquisition or neglect any form of news regarding fundamentals, can still gain from trade by means of simple trading rules. Their common prior is not determined by an exogenous information arrival, but arises endogenously through the observation of price realizations. These investors are thus pure technical analysts whose demands are determined by a sequence of prices only. The striking implication is that these agents owe their existence entirely to the information-based trading activities of other investors in the economy. A natural question that arises is whether this parasitical type of behavior, that freely increases agents’ utilities, imposes a threat to those that do collect information. This question is the focus of this chapter.

We show that if too many technical analysts are present the market may break down. Market breakdowns have been motivated in other work. For instance, Bhattacharya and Spiegel[1991] show that if an agent is too uninformed relative to other market participants he will not engage in any trading activity, leading to a failure of the market. In our model quite the opposite motivates a market breakdown. Here, if the market is too uninformed on average, informed traders refuse to engage in trading activities. This market breakdown is not the result of information friction, but is due to the impossibility of informed agents to collect a sufficiently large risk premium. The risk premium is needed as compensation for the higher price volatility that occurs in a dominant presence of technical analysts. This illustrates that information asymmetry in our model has a different role. Commonly, information asymmetry gives rise to an adverse selection component. Uninformed investors in our model, however, owe their existence entirely to the partial revelation of information: they free-ride on the information collection of informed investors.

Another focus of this chapter is how market statistics are affected by the presence of technical analysts. It is shown that even though they are contrarians, buying and selling against the mean reverting liquidity supply, an increase in their presence increases volatility,
magnifies auto-correlations and decreases market depth. Indeed, their rational trading strategy does not have the dampening effect on prices one perhaps would expect. Rather, their presence decreases the average competitiveness of the market, leading to a higher sensitivity of price to liquidity shocks.

Additionally, we consider how the degree of information leakage through prices impacts the ex ante utilities of informed and uninformed investors. The (lack of) price informativeness determines the relative advantage of informed investors compared to technical analysts. Obviously, the comparative advantage of informed agents increases with diminishing informativeness. Notwithstanding this however, the ex ante utility of technical analysts is decreasing in the degree of information revelation, even though they derivate their existence entirely from the informativeness of prices. The reason for this -perhaps counter intuitive- result is the reduced competition among informed agents if informativeness is low. Prices are less responsive to information, and hence less predictable. As a result, risk premia are higher. In fact, because Bertrand competition is absent, these risk premia more than offset the increase in the variance of future wealth. Ultimately this leads to higher expected utilities for both informed agents and technical analysts.

We derive these results within a noisy rational expectations framework similar to Hellwig[1980] and Wang[1993]. In our multi-period economy agents can trade a risky asset that is infinitely long lived and does not generate any dividends. This structure captures the idea that dividends are not paid out in a continuous fashion, and that stocks are not liquidated regularly. Instead, investors try to benefit from the realization of capital gains through delta positions conditional on their information. Moreover, within rational expectations models, cash flow distributions such as dividends, also have an informational role (as in Wang[1993]). We, however, want to focus exclusively on the information contained in prices. The fundamental value of the asset varies stochastically through time, and is observed by the informed investors only. Technical analysts are present who only observe price realizations. They use their knowledge of the statistical properties of the price process to mimic the actions of informed traders as closely as possible. Last but not least, there are liquidity traders who cause the per capita excess supply to vary randomly, thereby motivating the existence of trade.

Other work that considers technical analysis can be found in Brown and Jennings[1989] and Grundy and McNichols[1989]. They provide a rationale for the existence of technical analysis. As such, our work is a direct extension of their work in that we consider how the presence of pure technical analysts impacts a financial market. Wang[1993] also considers trading behavior of relatively uninformed investors, and the impact of the structure of a market on market statistics. In his model, uninformed investors do observe public information, and
as such his model differs fundamentally from ours.

This paper is organized as follows. Section 2 introduces our model. Section 3 studies the stationary equilibrium. The impact of technical analysts on market statistics and utilities is considered in section 4. Section 5 derives the market failure result. Additional properties of our economy are presented in section 6. Section 7 concludes.

7.2 The Model

In this section we introduce the assumptions underlying our model.

A. The Assets

We assume that two assets are traded, a risky and a riskless asset. The riskless asset yields a fixed return of zero in each period. The risky asset does not yield any distributions, and is not liquidated in any nearby future period. With this assumption we deviate from Wang[1993,1994], and Campbell, Grossman and Wang[1993] who study a multi-period model in which the risky asset in each period yields dividends, and Brennan and Cao[1996], Brown and Jennings[1989] and Grundy and McNichols[1989] who allow for liquidation of the asset. The type of asset we model resembles a pure growth stock. Our model reflects a market in which the time horizons of investors are small compared to the lifetime of the asset traded\(^1\).

The fundamental value of the asset is denoted by \( V_t \), corresponding to the expected liquidation value of the asset at a liquidation date infinitely far in the future. This fundamental value experiences a shock in each period, due to the arrival of new information. Explicitly, the fundamental value follows a random walk process, i.e.

\[
V_{t+1} = V_t + \delta_{t+1}
\]

where \( \delta_{t+1} \sim N[0, \sigma^2] \), and the asset is liquidated at a point \( T \) at \( V_T \), with \( T = \infty \).

B. The Investors

Investors are assumed to behave myopically, and maximize their expected utility one period ahead. As stated, they have two investment opportunities, the risky and the riskless asset.

\(^1\)That agents re-trade the asset is similar to the ideas of De Long et.al. [1990] on their work on noise traders. Indeed, as in their models, the additional uncertainty through future noise when re-trading the asset, leads to an increase in risk premia and volatility of prices. However, we do not model the behavior of noise traders as they do, but focus on the actions of price taking rational uninformed investors, and the informativeness of prices.
Formally, investor $i$ at time $t$ maximizes

$$\max_{d_{it}} \mathbb{E}_t[U_{t+1}(W_t(d_{it}))[I_t^i, P_t]] = \max_{d_{it}} \mathbb{E}_t[-\exp[-\rho_t W_{t+1}(d_{it})]][I_t^i, P_t]$$

where $W_t$ is his wealth, $I_t^i$ his information set and $\rho_t$ his risk aversion coefficient. The price of the risky asset at time $t$ is denoted by $P_t$. We assume that economy consists of a continuum of investors, which we index by $i \in [0,1]$. For tractability reasons, the average risk tolerance of the informed and uninformed are assumed to be identical, i.e. $A = \frac{1}{w} \int_{\text{inun}} \frac{1}{\rho_t} \text{d}t^2$.

We assume myopia of investors mainly for tractability of the model. If agents’ time horizons extend over multiple periods, they generally trade more aggressively: agents then exploit correlations between subsequent price changes to reduce the variance of their ultimate payoff. We refer to the analysis presented in chapter 5 for elaboration on the impact of longer time horizons.

C. Liquidity Demand

The per capita excess supply $\hat{Z}_t$ of the risky asset varies over time, due to the presence of noise or liquidity traders. This quantity is normally distributed in each period with standard deviation $\Delta$ and mean zero. Observe that this implies that two sources of uncertainty occur in each period, the shock to the fundamental value of the asset $\hat{d}_t$, and the shock to the supply level of the economy, $\hat{Z}_t$.

D. Information Structure

Investors are of two types: informed traders, who perfectly know the current state $\hat{\varphi}_t = (\hat{V}_t, \hat{Z}_t)$ of the economy, and uninformed traders (the technical analysts) who only observe the price realization in each period. The uninformed share a common prior about the current

\footnote{The integral $\int \text{d}i$ is heuristically defined as the limiting economy. That is $\int \text{d}i = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \text{d}i$. Note that $w = \int_{\text{inun}} \text{d}i$.}

\footnote{A variety of possibilities can be invoked to model the extra source of noise. We can assume that each trader is endowed with a random portion of the risky asset (such as in Brennan and Cao[1996]). We can assume that the asset is in fixed supply, and let the risk premia vary randomly (such as in Campbell, Grossman and Wang[1993]).}

\footnote{The normality assumptions, though commonly encountered (see for instance Wang[1993,1994], Vives[1995], Brennan and Cao[1996] among others) may seem somewhat unrealistic since the value of the asset can become negative. We cannot circumvent this without adding more complexity to the model. However, as noted for instance by Campbell, Grossman and Wang[1993], if we assume that our investors have mean-variance utility functions, and restrain the fundamental value from becoming negative, the results reported would be the same.}
state, which we denote by $\tilde{q}_t \equiv E[\tilde{q}_t | P_t, P_{t-1}, \ldots]$. The uncertainty of the current state after observing the price is denoted by the variance-covariance matrix $O = \text{var}[\tilde{q}_t | P_t, P_{t-1}, \ldots]$. We denote the uncertainty regarding the fundamental value by $o \equiv O_{11} = \text{var}[V_t | P_t, P_{t-1}, \ldots]$. The informed are assumed to make up a constant fraction $w$ of the total number of investors.

**E. Equilibrium**

We study the steady state equilibrium for this economy, with the pricing functional linear in the common prior of the market and the current state of the economy. Explicitly, we study the equilibrium in which

$$P_t = p_c V_t^c + p_f V_t - \gamma_c Z_t^c - \gamma_f Z_{t-1}^c$$

where $p_c = (\pi_c, -\gamma_c)'$ and $p_f = (\pi_f, -\gamma_f)'$. These coefficient vectors capture the impact of uninformed and informed investors information on the market-clearing price respectively. In the steady state equilibrium we demand that (a) the coefficients of the pricing functional remain constant over time, (b) the market clears in each period, and (c) $V_t$ represents the discounted liquidation value at infinity. Observe that requirement (c) can be interpreted as a transversality condition.

Initially, we impose the fraction of informed exogenously, and as such the equilibrium we study is a partial equilibrium. In section 6 we consider an economy where information can be acquired at a certain cost.

**F. Additional Definitions**

For expositional purposes we define the informativeness of the price system, denoted by $\theta$, as

$$\theta \equiv \frac{\sigma^2}{o}$$

Theoretically $\theta$ is greater than $0$, and unbounded on the positive side. Furthermore, we introduce the quantity $\kappa$, defined as

$$\kappa = 2 \frac{\Delta \sigma}{A}.$$ 

$\kappa$ plays a central role in our analysis and measures the turnover variance relative to the average risk tolerance. It can be seen as a measure for the relative risk aversion level in the market.

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\[5\] In fact, $\kappa$ is inversely related to the parameter definition of $\zeta$ in chapter 4, which we depicted as a relative risk tolerance parameter. Again, we may relate this parameter to the development of a market as well (see chapter 4).
7.3 A Stationary Equilibrium

A. Introduction

The model is a special case of the economy described in the previous chapter. As such, we can directly use the results therein to determine the steady state equilibrium in this setup. We briefly recall some of the difficulties involved in solving for this equilibrium. Agents maximize over a next period payoff that is solely determined by the prevailing market clearing price at that time. The result is that the coefficients of the pricing functionals are related between subsequent periods, leading to a complex pricing problem. The uninformed estimate of the fundamental value is determined through the complete history of price realizations, and is updated in each period. These updates are performed using a Kalman filter, given the Gaussian structure of the model. Additionally, their uncertainty is recursively determined. If we demand a stationary equilibrium, necessarily the uncertainty of the uninformed should be constant. Hence, both the common prior of the market and its precision are endogenously determined. This feature contrasts with most price formation models where the common prior or its uncertainty arises (at least) partially through exogenous information signals.

B. Equilibrium

We first solve for the equilibrium pricing coefficients. Using the results from the previous chapter, in the appendix it is shown that the following theorem applies.

**Theorem 7.1** If \( \kappa \pi_f < w \), for any \( \pi_f \in [0, 1] \) there exists a stationary equilibrium, satisfying the conditions stated in section 7.2.E, given by

\[
\tilde{P}_t = \pi_c \tilde{V}_t + \pi_f \tilde{V}_t - \gamma_f \tilde{Z}_t
\]

(7.2)

with coefficients

\[
\pi_c = \frac{(1-w)(\theta + 1 + \theta^2)}{w\theta + \theta^2 + 1 + \theta} \left( \frac{\theta}{\theta} \right)
\]

(7.3)

\[
\gamma_f = \frac{w \sigma}{\Delta \kappa} \left( 1 - \frac{\pi_f^2 \sigma^2}{\kappa^2} / w^2 \right)
\]

The prior expected value of the uninformed with respect to the true value of the firm, \( \tilde{V}_t \), is updated in each period using

\[
\tilde{V}_t = w \frac{1 + \theta}{w\theta + \theta^2 + 1 + \theta} \tilde{V}_{t-1} + \frac{1}{\pi_f} \frac{\theta}{1 + \theta} \tilde{P}_t
\]

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6In these models, this is either a public signal or a disguised public signal in the form of dividends.
and the fraction $\gamma_f^2/\pi_f^2$ can be written as

$$\frac{\gamma_f^2}{\pi_f^2} = \frac{\sigma^2}{\Delta^2} \frac{1 + \theta}{\theta^2}$$

Note that the prior of the uninformed regarding the liquidity shock, $Z_t$, does not enter the pricing function. This is a natural result of the type of process we have imposed on this variable: conditional on any information set prior to $t$, its expectation is zero. Moreover, observe (from equation (7.3)) how the relative impact of the uninformed, given by $\pi_c/\pi_f$, is determined partially by their fraction in the economy $(1 - \omega)$, and by the informativeness $\theta$. A peculiar feature of this equilibrium is that $\pi_f$ cannot be determined in this general setting.

C. Liquidation

That $\pi_f$, the weight attached to the informed (perfect) knowledge of the true value of the asset, is undetermined, owes the models' overlapping generations feature. In fact, the true value $\bar{V}_t$ need not to be related to the true value of the firm at all. Also observe that we have an infinity of possible pricing functionals: including a constant term in the relation (7.2) again yields a viable equilibrium, since agents know that they can re-trade the asset with the same constant bias. This freedom for bubbles is a well-known feature of overlapping generations models (Tirole[1982,1989]). However, we can tie the equilibrium down uniquely by assuming that agents anticipate a future liquidation date. Consistency with our stationary equilibrium then demands that this liquidation date is an infinite number of periods further. In the following we adopt this requirement, through the assumption that the asset is liquidated at $\bar{V}_T$, with $T$ infinite. This leads to the following theorem.

**Theorem 7.2** If at a future point in time the asset is liquidated, the equilibrium pricing functional is characterized by

$$\pi_f + \pi_c = 1$$

**(7.4)**

**Proof.** See theorem (6.4). $\blacksquare$

The liquidation restriction leads to an appealing result. At all points in time, the informational component of price realizations is given by a weighted average of the beliefs of the uninformed and the informed investors.

With this restriction, in the stationary equilibrium the pricing functional reduces to

$$\bar{P}_t = \bar{V}_t + \pi_f(\bar{V}_t - \bar{V}_{t-1}) - \gamma_f \bar{Z}_t$$
This implies that the ex ante expected value of the price given the information set of the uninformed investors, always equals the expected future liquidation value, i.e.

$$E_t [\tilde{P}_{t+n} | I_{t-1}^e] = V_t$$

This result appeals, as it indicates that the ex ante market clearing price reflects information in an unbiased fashion. Moreover, although the liquidation point lies an infinite periods away in the future, this financial market accommodates the shocks to liquidity in the economy, even though investors act myopically.

We can now solve for the unique equilibrium given the additional requirement imposed by relation (7.4). Using (7.3), we see that $\pi_f$ should solve

$$\pi_f + \frac{(1 - w) (\theta + 1 + \theta^2)}{\theta + \theta^2 + 1 + \theta} \delta \pi_f = 1.$$ 

This condition can be stated more explicitly. Introduce a new variable $m$ for ease of exposition, defined by

$$m = \frac{\gamma^2 \Delta^2}{\pi_f^2 \sigma^2} = 1 + \theta.$$ 

This quantity effectively measures the relative variability of prices due to liquidity versus information shocks. The solution to the price problem can now be depicted as in the following theorem.

**Theorem 7.3** If the asset is liquidated at $V_T$ at $T = \infty$, the equilibrium is given by theorem 7.1 with $\pi_f = \pi_f^*$ the solution to

$$\pi_f = \frac{\left(\frac{1}{2} w (-1 + \sqrt{1 + 4m}) + 1 + m \right) \left(1 + \frac{1}{2} (-1 + \sqrt{1 + 4m})\right)}{\left(\frac{1}{2} w (-1 + \sqrt{1 + 4m}) + 1 + m (1 - w)\right) \left(1 + \frac{1}{2} (-1 + \sqrt{1 + 4m}) + m\right)}$$ 

(7.5)

with $m$ given by

$$m = \frac{w^2}{\kappa^2 \pi_f} \left(1 - \sqrt{\left(1 - \frac{\kappa^2 \pi_f^2}{w^2}\right)^2}\right).$$

The above equation specifying $\pi_f^*$ is impossible to solve explicitly. However, comparative statics regarding the solution can be derived and are summarized in the theorem below.

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7Recall that $\theta$ also depends on $\pi_f$.  
8In fact, this equality is of $8^{th}$ order in $\pi_f$.  

7.3. A Stationary Equilibrium

The response rate \( \pi_f^* \) as a function of \( \kappa \) for \( w = 0.5 \) and \( w = 0.75 \). The sensitivity to information shocks decreases with the relative risk aversion \( \kappa \).

**Figure 7.1.**

**Theorem 7.4**

A. \( \pi_f^* \) is a function of \( w \) and \( \kappa \) only. B. \( \pi_f^* \) is strictly increasing in the fraction of informed, \( w \). C. The fraction \( \pi_f^*/w \) is strictly decreasing in the fraction of informed, \( w \). D. \( \pi_f^* \) is strictly decreasing in \( \kappa \) for \( w < 1 \).

**Proof.** See appendix. ■

The pricing functional is thus characterized by

\[
P_t = V_t + \pi_f^*(w, \kappa)(V_t - \tilde{V}_t) - \pi_f^*(w, \kappa)\sqrt{\sigma^2(w, \kappa)}\frac{\sigma}{\Delta}\tilde{Z}_t
\]

In the figures (7.1) and (7.2), some of the dependencies of \( \pi_f^* \) are displayed. The quantity \( \pi_f^* \) measures the impact of the information of the informed on the pricing functional relative to the impact \( (1 - \pi_f^*) \) of the uninformed' estimate of the fundamental value. Hence, a change in \( \pi_f^* \) can be given a direct economic interpretation. Decreasing \( \pi_f^* \) makes the pricing functional less sensitive with respect to new information. In other words, the stickiness or staleness of prices is increased. Note that this typically occurs if \( \kappa \) increases. That is, the weight attached to the informed’s knowledge of the fundamental value diminishes if the economy on average is more risk averse or the size of the shocks to the economy increases.
An elegant feature of the model is that $\pi^{\gamma}$ is determined by two parameters only: $w$, the fraction of informed investors, and the fraction $\kappa (=2\Delta^2/\hat{A})$, i.e. $\pi^{\gamma} = \pi^{\gamma}(w, \kappa)$. This implies that multiplying the standard deviation of liquidity shocks while reducing the standard deviation of information shocks with the same factor leaves $\pi^{\gamma}$ unaffected. It should be noted, however, that the risk premium coefficient $\gamma^{\gamma}$ does change under such transformation.

The following theorem regarding the quantity $m$, is useful.

**Theorem 7.5** The relative sensitivity of the price to liquidity shocks and information shocks, quantified by $m \equiv \frac{\gamma^{\gamma}_{\Delta^2}}{\pi^{\gamma}_{\sigma^2}}$ is: A. a function of $w$ and $\kappa$ only. B. strictly decreasing in $w$. C. $m_{\text{max}} = 1$, $m_{\text{min}} = 0$. D. strictly increasing in $\kappa$.

**Proof.** See appendix.

Observe that since $m$ decreases with $w$, the sensitivity of the price with respect to liquidity shocks as compared to its sensitivity to information shocks increases if the fraction of uninformed grows. This can be understood as follows. On the one hand, the impact of the informed decreases simply because they are less in number. However, the informed also cannot as precisely estimate the future pricing functional: their current knowledge has less
impact on the future pricing functional. Hence, they increase the sensitivity with respect to the liquidity shocks \((\gamma_f)\) which magnifies risk premia, compensating them for the reduction in certainty.

Another feature of this equilibrium deserves some attention. Note that the right hand side of (7.5) is strictly decreasing in \(w\) (see also the appendix). A lower bound for \(p_0\) is therefore implied by substituting the value \(w = 0\) into (7.5). If we rewrite the resultant in terms of \(m\), this bound is given by

\[
\pi_f \geq \frac{1 + \sqrt{(1 + 4m)}}{1 + \sqrt{(1 + 4m)} + 2m}
\]

Given the maximum value for \(m\), \(\inf \{\pi_f\} = -\frac{1}{2} + \frac{1}{2}\sqrt{5} (\sim .61803)\).

D. Informativeness

There is a close connection between \(m\), measuring the relative sensitivity of price to liquidity shocks, and \(\theta\), the informativeness of the price system. The latter can be expressed in terms of \(m\) as

\[
\theta = \frac{1}{2m} \left(1 + \sqrt{(1 + 4m)}\right)
\]

Given theorem 7.5 above, the following corollary can be derived.

**Corollary 7.1** The relative informativeness of the price system, quantified by \(\theta = \frac{\sigma^2}{\sigma}\), is:

A. a function of \(w\) and \(\kappa\) only. B. strictly increasing in the fraction of informed, \(w\). C. \(\theta_{\min} = \frac{1}{2}(1 + \sqrt{5})\). D. strictly decreasing in \(\kappa\).

This corollary shows that the informativeness decreases if the fraction of uninformed grows. Furthermore, it decreases if (1) the standard deviation of the liquidity shocks increases, (2) the standard deviation with respect to news increases, and (3) the average risk tolerance level decreases. These dependencies are all quite intuitive. If more informed agents are present, the stronger their impact on price realizations and hence the more informative the price system. That the minimal informativeness equals the golden ratio, the most irrational number, cannot readily be assigned any economic intuition. It is the point where the degree of information asymmetry equals the degree of informativeness.

7.4 On the Impact of Technical Analysts

In this section we consider how the presence of technical analysts affects the equilibrium. First, we consider trading activities of technical analysts in more detail.
7.4.1 TRADING RULES

Each price realization contains an imperfect signal about the current fundamental value. The uninformed combine this signal with their previous estimate about the value of the asset, by means of a recursive update rule (a Kalman filter). This dynamic updating of beliefs causes the uninformed to adopt a dynamic trading strategy, which is characterized by the following theorem.

**Theorem 7.6** For the economy described above, the uninformed investors are contrarians and use an exponential moving average rule to determine their trades. That is, their demand is proportional to the difference between the price and an exponential moving average of the price $M_t$, i.e.

$$d^c_t \propto M_t - \hat{P}_t$$

with $M_t$ updated in each period using the following recursion relation:

$$M_t = \lambda \hat{P}_{t-1} + (1 - \lambda)M_{t-1}$$

with $\lambda = w(\theta^2 + w\theta + 1 + \theta)^{-1}(1 + \theta) > 0$.

Hence, the demand of the uninformed investor is determined by the difference between the price of the asset and an exponential moving average of past price realizations. Given their non-zero demand, one can directly assess that the uninformed increase their utility by participating in the trading process. Indeed, the uninformed provide immediacy for liquidity traders, and earn a risk premium for this transfer of risk. The reason we have contrarian trading rules is the following. In our model, uninformed agents rationally explain price changes by a liquidity and an informational component. Hence, their update regarding the fundamental value of the asset will be smaller than the price change itself. Since the expected future price equals their estimate of the fundamental value, investors will tend (i.e. conditional on their prior of course) to go short when the price goes up and long if the price goes down.

The moving average exponent $\lambda$ decreases as the fraction of informed increases, i.e. the weight attached to each price realization decreases with $w$. This dependency is shown for several values of $\kappa$ in figure (7.3). Though it may seem counter-intuitive, note that as the informativeness of each price realization increases, the informativeness of past prices is increased as well. Put differently, if the fraction of informed increases, the precision of the common prior increases relative to the precision of the information signal in each price.
7.4.2 Market Statistics

A Autocorrelation and Variance

In this subsection we turn our attention to empirically measurable quantities. The unconditional variance of price changes (see the appendix for its functional form) is plotted as a function of the fraction of informed in Figure (7.4). Indeed, instead of what one may expect, the price variance is increasing in the fraction of technical analysts. Although they act as contrarians, their trading strategy does not have a dampening impact on price evolution. Instead, an increase in their presence increases risk premia, and as a result magnifies price changes.

The first order autocorrelation as a function of the fraction of informed is plotted in figure (7.5). This quantity is always negative, due to the transitory nature of the liquidity component. If we allow for persistence of supply levels, positive autocorrelations may be found (see for instance chapter 6). We also remark that in absence of a presence of chartists, higher order autocorrelations disappear. Indeed, only the impact of the trading activities of technical analysts leads to these higher order patterns.

Observe that the significance of the autocorrelation increases in the fraction of uninformed.
FIGURE 7.4. Relative variance of price changes as a function of the fraction of informed investors for $\kappa = 0.25$ and $\kappa = 0.5$. Volatility increases with the fraction of chartists and the relative risk aversion $\kappa$.

This dependency is directly related to the fact that the sensitivity to liquidity shocks is larger when the fraction of technical analysts increases. Higher negative correlation between subsequent price changes is indicative of higher risk premia. These risk premia are either the result of higher volatility of shocks to the economy, or lower informedness of the economy.

B. Market Depth

We define the market depth as the inverse of $\gamma_f$, the liquidity cost parameter. We have

$$\gamma_f^{-1} = \frac{\Delta}{\sigma \pi_f} \sqrt{\theta} \left( \frac{\theta^2}{\theta + 1} \right) = \frac{\Delta}{\sigma} \left( 1 + \frac{w}{w \theta^2 + 1 + \theta} \right) \sqrt{\theta + 1}$$

(7.6)

Inspection of this quantity leads to the following theorem.

Theorem 7.7 Market depth is strictly decreasing in the fraction of technical analysts.

Proof. Direct differentiation shows that the right hand side of (7.6) strictly increases with both $w$ and $\theta$. Since $\theta$ increases with $w$, the theorem follows.

Indeed, the competitiveness of the market is reduced when a larger fraction of rational investors apply technical analysis, leading to a decrease in market depth. The financial market
imposes higher liquidity costs with a larger fraction of chartists.

We emphasize that all of above results are derived under the assumption that the total number of traders relative to the total supply variance is assumed constant. Technical analysts, due to their greater uncertainty, can less effectively compete. If their fraction increases, the average competitiveness of the market diminishes. The result is a lower market depth, higher price variance, and an increase in the magnitude of autocorrelations between price shocks.

7.4.3 UTILITIES
One of the questions that arises when considering the impact of technical analysts concerns the ex ante expected utilities of agents. We denote the ex ante utility, conditional on past prices, but prior to the receipt of information signals and the price realization, by $V_f$ for the informed and by $V_c$ for the chartists. Formally, these quantities are defined according to

$$V_f = \mathbb{E}[U_t^{\text{informed}}|I_t^f], \quad \text{and} \quad V_c = \mathbb{E}[U_t^{\text{uninformed}}|I_t^c]$$

The next theorem concerns the dependency of $V_f$ and $V_c$ on $w$ and $\theta$. 
Theorem 7.8 The ex ante expected utilities of the uninformed and informed, conditional on past prices, are given by

\[ V_c = -\sqrt{\frac{1}{(\omega(1+\theta+\phi^2))^2 + 1}} \]

\[ V_f = -\sqrt{\frac{1 + \theta + \phi^2}{(\omega(1+\theta+\phi^2))^2 + (1+\theta)^2}} \]

Proof. See appendix. 

Inspection of the above expected utilities shows that the ex ante utility of the uninformed is decreasing in the informativeness of the price system. Hence informed and uninformed agents objectives are aligned in the dependency on the informativeness of the price system: both benefit most when the informativeness is as small as possible. As the informativeness decreases with increasing fraction of uninformed, the following corollary readily follows.

Corollary 7.2 Both informed investors and technical analysts benefit from an increasing fraction of technical analysts.

The interesting implication is that although technical analysts free-ride on the activities of informed investors, the latter benefit if the fraction of chartists increases. Also the uninformed benefit from an increasing fraction of their kind.

7.5 Market Failure in the Presence of Too Many Technical Analysts

The foregoing analysis was performed assuming that there exists a solution to the equality (7.5) specifying \( p_0 \). However, as was already indicated in section 7.3.D, the price system needs a minimal informativeness for an equilibrium to be viable. Indeed, this informativeness can only be guaranteed if there is a sufficient number of informed investors in the economy. The following theorem formalizes this notion.

Theorem 7.9 A necessary and sufficient condition for a stationary equilibrium to exist, is given by the inequality

\[ \frac{1 + \sqrt{3} + \omega}{3 + \sqrt{3} - \omega} \frac{1}{w} \leq \frac{A}{2\Delta_i} \]

Note that the left hand side of (7.7) decreases in the fraction of informed, \( w \). This implies that if the fraction of informed investors is below some critical value, the stationary equilibrium cannot exist. Solving the above equation for \( w \), we obtain for the minimal fraction of informed
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A plot of $w_{\text{min}}$ showing the dependency on $\kappa$ is shown in figure (7.6).

Hence, the equilibrium breaks down for (a) Low risk tolerance (b) Large variance of liquidity shocks (c) Large variance of informational shocks, and (d) A small fraction of informed investors. Interestingly, the liquidity shocks, which are necessary to make this equilibrium work with costly information (Grossman and Stiglitz[1980]), can also cause the equilibrium to break down. Observe that, given the maximum value of unity for $w$, a minimal requirement is that $\kappa \leq 1$, or $2\sigma \Delta \leq A$. This implies that the variance of market turnover relative to the average risk tolerance should not exceed one-half.

Here, we see the failure of financial markets under some circumstances. If the fraction of uninformed is too large, the equilibrium will break down. Observe that increasing the necessary noise in the economy $\Delta$ only worsens things, as $\kappa$ increases with $\Delta$ and so does the minimal fraction of informed $w_{\text{min}}$. Why does the market break down in the presence of (relatively) too many uninformed traders? The answer lies in the combination of the increased uncertainty in their presence and the dual role of the sensitivity of price to liquidity shocks.
With an increase in the fraction of technical analysts, the uncertainty of the market increases. To compensate, higher risk premia are necessary, which can be established by increasing the sensitivity of price to liquidity shocks\(^9\). However, increasing this sensitivity also increases the variability of price changes and hence uncertainty of investors. Indeed, this gives rise to an iterative mechanism. A convergence is established as long as uncertainty is not too high. If however, uncertainty is too large, the adverse effect of liquidity sensitivity dominates.

In the literature it has been extensively discussed how information asymmetry may cause markets to break down. However, usually a presence of too many informed investors (insiders) is the cause of such a market failure (see for instance Bhattacharya and Spiegel[1991]). In our model exactly the opposite motivates a breakdown: a relatively too significant presence of (totally) uninformed investors may cause prices to have too little information content and exhibit too much uncertainty. The overlapping generations aspect plays an important role here. Information then only has value if it is reflected in (future) prices. A similar reasoning can be found in Dow and Gorton[1994]. In their model, informed investors can only perform short-term arbitrage if their private information is impounded in future short term price movements, leading to an arbitrage chain that breaks down if transaction costs are too high. As an analogy, in our stationary economy this arbitrage chain is infinitely long, given that information regarding fundamentals is not realized before time infinity. As such, transaction costs can also in our model lead to a market breakdown.

### 7.6 Costly Information

The previous analysis assumed that information is an endowment. In this section, we relax this assumption and assume that information can be acquired at a certain cost. Hence, agents can observe the true value of the firm in each period at a cost \(c\). The decision of each agent to acquire information or become a technical analyst is determined by the comparison of the expected utility of informed with the expected utility of staying uninformed. We adopt a similar analysis as in Grossman and Stiglitz[1980], and consider the fraction of informed investors which would arise endogenously in a competitive market given this cost to information acquisition.

Given a cost to the acquisition of information, \(c\), the decision to become informed depends

\(^9\)Observe that this is similar to the noise trader risk described in DeLong et.al.[1990]. The overlapping generations nature of their model also induces noise traders' activities to be priced.
on the ratio of ex ante expected utilities. This ratio, which we denote by $q$, follows immediately from theorem (7.8)

$$q = \frac{\mathcal{V}_f(c)}{\mathcal{V}_c} = e^{pc} \frac{\theta^2 + 1 + \theta}{(1 + \theta)^2}.$$ 

Utilities are negative, hence, a higher level of $q$ is associated with a smaller difference between uninformed and informed ex ante expected utilities. Observe that -as in Verrecchia[1982]- higher risk tolerance leads to lesser impact of costs on utility. This reveals that if and only if all risk aversion levels are equal across traders, the average risk tolerance of the informed traders will equal the average risk tolerance of the uninformed traders. A necessary assumption is therefore that the market is homogenous in risk tolerance levels. Let us therefore adopt this requirement and study the interior equilibrium that arises.

Given the comparative statics derived for $\theta$, the following theorem follows.

**Theorem 7.10** The relative utility of the informed compared to the uninformed is increasing in $\kappa$ and decreasing in the fraction of informed, $w$.

**Proof.** Use corollary 7.1. ■
In a general equilibrium, informed as well as uninformed should be indifferent between switching their types. Hence, an interior equilibrium is characterized by \( q = 1 \). Using the above expression for \( q \), such an equilibrium corresponds to a value of \( \theta \) of

\[
\theta^* = \frac{1}{2} \left( 2 - e^{2pc} + e^{pc} \sqrt{4 - 3e^{2pc}} \right) \frac{e^{2pc}}{e^{2pc} - 1}.
\]

The dependency of \( \theta^* \) as a function of \( pc \) is given in figure (7.7). In an overall equilibrium therefore, \( \theta^* \) is determined by the cost of information acquisition and the risk aversion level only.

The following implications can be derived: (1) If \( c \) increases, \( \theta^* \) decreases. Therefore the informativeness of the price system is diminishing in the cost to information acquisition. This intuitive result has been noted by others (e.g. Grossman and Stiglitz[1980] and Verrecchia[1982]), (2) The informativeness of the price system is independent of the standard deviation of the shocks to the fundamental value, and the standard deviation of supply shocks. This result can be ascribed to the endogenous determination of the common prior in our model. In Grossman and Stiglitz[1980], price informativeness also depends on the precision of the information of insiders. (3) The fraction of technical analysts increases with information costs. This indicates that the ex ante expected utility of the informed relative to the uninformed reaches its maximum when their number is as small as possible, i.e. if \( w = w_{\text{min}}(\kappa) \), or \( \theta = \frac{1}{2} (1 + \sqrt{5}) \). The maximum cost to becoming informed, \( c_{\text{max}} \), is therefore given by the solution to \( q(pc_{\text{max}}, \theta = \frac{1}{2} (1 + \sqrt{5})) = 1 \).

7.7 Conclusion

In this chapter, we have considered how the informativeness of prices gives rise to the existence of a group of investors that can be depicted as pure technical analysts. The main objective of the analysis was to examine how the presence of these rational agents impacts the utilities of investors and market statistics. It is shown that though technical analysts are contrarians, they do not have a smoothing effect on price evolution. Instead, their presence leads to an increase in market volatility and a reduction in market depth. Both information collectors and technical analysts benefit, however, when the fraction of technical analysts is as large as possible. This occurs to the detriment of liquidity investors that pay a high risk premium for the accommodation of risk.

It was also shown that the market may fail if the fraction of technical analysts is too large. This type of market breakdown is quite unique. The informed traders refuse to trade
in a too uninformed market, rather than the opposite type of market breakdown investigated for instance by Bhattacharya and Spiegel[1991]. To resolve this problem, regulation should induce agents to collect information, and make pure technical analysis unprofitable. A trading cost could warrant this.

Our analysis can be extended along several dimensions. One is the exogenously assumed presence of liquidity traders. They play an important role in the model. It is likely that even liquidity traders are sensitive to the size of risk premia, and as such implementing a more natural relation between the competitiveness of a market and its liquidity would accord better with reality. In our model, such a relation is absent, which, combined with the limited competitiveness, leads to the result that non-liquidity traders prefer volatile markets with large risk premia over markets with opposite characteristics.

Another point that deserves more attention concerns the myopia of our investors. If agents have a time horizon that extends beyond the next period, noise trader risk will have a less profound impact on the equilibrium. An extension, including multi-horizon investors, in the spirit of chapters 5 and 6 would therefore be of interest. It is likely that our market is more robust under this extension. However, it is our contention that the comparative statics of the equilibrium we derived, will stand the test of this exercise.
7. Appendix

A Proof of Theorem 1

We derive the equilibrium using the results from chapter 5. Adopting the same notation, in this specification, we have that \( \tilde{\varphi}_t = (\tilde{V}_t, \tilde{Z}_t) \) follows the AR(1) process

\[ \tilde{\varphi}_{t+1} = G \tilde{\varphi}_t + \eta_{t+1} \]

where \( G = \text{diag}(1, 0) \) and \( \Xi = \text{diag}(\sigma^2, \Delta^2) \). Uninformed agents update their beliefs of \( \tilde{\varphi}_{t+1} \) conditional on observing the price at time \( t \) according to

\[ \tilde{\varphi}_{t+1} = G((1 - kp^e_J)\tilde{\varphi}_t + kp^e_J\tilde{\varphi}_t) \]

with \( k = \Omega^c p_f(p_f'\Omega^c p_f)^{-1} \), and \( \Omega^c \) the solution to \( \Omega^c = \Xi + G(I - \Omega^e p_f(p_f'\Omega^c p_f)^{-1}p_f')\Omega^c G' \). The relation between the conditional uncertainty regarding the current fundamental state \( O \) and \( \Omega^c \) is thus given by \( \Omega^c = \Xi + GOG' \). Note that we have that \( kp^e_JO = 0 \), since \( kp^e_J\tilde{\varphi}_t \) is perfectly observable. This property implies that \( O_{xz} = O_{xu} = \pi_f \gamma_f^{-1}O_e \), and that \( O_{zz} = \pi_f^2 \gamma_f^{-2}O_{ee} = \pi_f^2 \gamma_f^{-2}o \). It then follows immediately that \( \Omega^c = \text{diag}(\sigma^2 + o, \Delta^2) \), and that

\[ k = (\pi_f^2(\sigma^2 + o) + \gamma_f^2 \Delta^2)^{-1}(\pi_f(o^2 + o) - \gamma_f \Delta^2) \]

The static Ricatti equation is given explicitly for \( o \) by

\[ o = \frac{\Delta^2(o^2 + o)}{\pi_f^2 \sigma^2 + \pi_f^2 o + \gamma_f^2 \Delta^2} \]

which is solved by the positive root of this equation

\[ o = \frac{1}{2} \sigma^2 \left(1 - \sqrt{1 + 4 \gamma_f^2 \Delta^2 / \pi_f^2 \sigma^2}\right) \]

Lemma 6.2 shows that we can write

\[ p'_f = -(1 - \alpha)p'^e_JG + (1 - \alpha)k_v^{-1}k_e \]

which implies that \( \pi_c = (1 - \alpha)k_v^{-1}(1 - \pi_f k_e) \) and \( \gamma_e = 0 \). The quantities \( \alpha \) and \( S \) are given by

\[ \alpha = \frac{wA}{\pi_f^2 \sigma^2 + \gamma_f^2 \Delta^2} \quad \text{and} \quad S = \frac{wA}{\pi_f^2 \sigma^2 + \gamma_f^2 \Delta^2 + \frac{1}{\alpha} \pi_f^2 \sigma^2 + \gamma_f^2 \Delta^2} \]

The pricing coefficients \( \pi_f \) and \( \gamma_f \) follow from theorem 6.3, which states that

\[ p'_f(G - I) = \frac{1}{\alpha S'} \]

\[ p'_c = p'^e_c \]

\[ p'_c = p'^e_c \]
This equation leaves \( \pi_f \) unspecified, and gives for \( \gamma_f \) the solution, if \( \kappa \pi_f = 2 \Delta \sigma A^{-1} \pi_f < w \)

\[
\gamma_f = \frac{wA}{2\Delta^2} \left( 1 - \sqrt{1 - \frac{\Delta^2 \pi_f^2}{w^2 A^2}} \right)
\]

Using the definition of \( \theta = \sigma^2 \sigma^{-1} \) and some additional algebra, we obtain the theorem. ■

B. Proof of Theorem 7.4

(A) Follows immediately from the expression for \( m \) and \( f \).

(B) First we prove the following lemmas.

Lemma 7.1 \( \frac{\partial}{\partial w} f(w, m) \geq 0 \).

Proof. Differentiation leads to this result immediately. ■

Lemma 7.2 \( \frac{\partial}{\partial m} f(w, m) \leq 0 \).

Proof. Define \( g(m) \equiv \frac{1}{2} (-1 + \sqrt{1 + 4m}) \), then \( m(g(w, \kappa, p)) = \left(-\frac{1}{4} + \frac{1}{2} (2g + 1)\right)^2 \). Note that \( m \) is strictly increasing in \( g \), and vice versa. In terms of \( g \), \( f(w, m) \) becomes

\[
f(w, m(g)) = 4 \frac{wg + 1 + 2g^2 + g^3 + g}{(4 + 4g^2 - 4wg^2 + 4g + w)(1 + g)^2}
\]

Taking the derivative with respect to \( g \), we have that

\[
\frac{\partial f(w, m(g))}{\partial g} = 4 \left( -4 - 4g^2 + 4wg^2 - 4g - w \right)^2 \left( 1 + g \right)^3 \times \\
\left( -8 + 3w - 8g + w^2 - 8g^2 + 11wg^2 + 7wg + wg^3 \\
+ 4w^2 g^2 - w^2 g + 12w^2 g^3 - (1 - w)(4g^5 + 12g^4) - 8g^3 \right)
\]

The sign of the derivative is thus satisfies, using that \( w \in [0,1] \)

\[
\text{sign} \left( \frac{\partial f(w, m(g))}{\partial g} \right) \propto -8 + 3w - 8g + w^2 - 8g^2 + 11wg^2 + 7wg + wg^3 \\
+ 4w^2 g^2 - w^2 g + 12w^2 g^3 - (1 - w)(4g^5 + 12g^4) - 8g^3 \leq -4 - g + 7g^2 + 5g^3 \leq 0
\]

for \( g \leq \frac{(-1 + \sqrt{5})}{2} \) or \( m(g(w, \kappa, p)) \leq 1 \). Since \( f \) is monotonically decreasing in \( g \), \( f \) is monotonically decreasing in \( m \) for \( m \leq 1 \). ■
Lemma 7.3 \( \frac{\partial}{\partial w} m(w, p, \kappa) \leq 0, \frac{\partial}{\partial p} m(w, p, \kappa) \geq 0 \) and \( \frac{\partial}{\partial \kappa} m(w, p, \kappa) \geq 0. \)

Proof. Follows directly from taking derivatives. ■

Lemma 7.4 \( \frac{df(w, m)}{dw} \geq 0. \)

Proof. Taking the derivative:
\[
\frac{df(w, m)}{dw} = \frac{\partial f(w, m)}{\partial w} + \frac{\partial f(w, m)}{\partial m} \frac{\partial m(w, p, \kappa)}{\partial w} \geq 0, \text{ lemma 7.1} < 0, \text{ lemma 7.2} < 0, \text{ lemma 7.3}
\]
the lemma follows. ■

Lemma 7.5 \( f(w, m(w, p, \kappa)) \) is monotonically decreasing in \( p. \)

Proof. Taking the derivative, we have
\[
\frac{df(w, m)}{dp} = \frac{\partial f(w, m)}{\partial m} \frac{\partial m(w, p, \kappa)}{\partial p} \leq 0, \text{ lemma 7.1} < 0, \text{ lemma 7.2} > 0, \text{ lemma 7.3}
\]
Now define \( p' \) and \( p'' \), the solutions to \( p' = f(w', \kappa, p') \) and \( p'' = f(w'', \kappa, p'') \) with \( w' < w'' \).
Assume that \( p' > p'' \), then we have that
\[
p' = f(w', \kappa, p') \leq f(w'', \kappa, p') \leq f(w'', \kappa, p'') = p''
\]
which contradicts with \( p' > p'' \). Hence \( p' \leq p'' \) for \( w' < w'' \). The solution \( p \) is monotonically increasing in \( w \). This concludes the proof of 7.4.B. ■

(C) Define \( z \equiv p/w \) then \( z \) solves
\[
z = h(w, n(z, \kappa))
\]
where
\[
h(w, n) = \frac{1}{w \left( \frac{1}{2} w (-1 + \sqrt{1 + 4n}) + 1 + n \right) \left( 1 + \frac{1}{2} (-1 + \sqrt{1 + 4n}) \right)} \left( \frac{1}{2} w (-1 + \sqrt{1 + 4n}) + 1 + n (1 - w) \right) \left( 1 + \frac{1}{2} (-1 + \sqrt{1 + 4n}) + n \right)
\]
and \( n \) is given by
\[
n(z, \kappa) = \frac{1}{\kappa^2 z^2} \left( 1 - \sqrt{1 - \kappa^2 z^2} \right)^2
\]

Lemma 7.6 \( \frac{\partial}{\partial z} n(z, \kappa) \geq 0. \)

Proof. Follows directly from taking the derivative. ■

Lemma 7.7 \( \frac{\partial}{\partial n} h(w, n) \leq 0. \)
Lemma 7.8 $\frac{\partial}{\partial w} h(w, n) \leq 0$.

Proof. The sign of the derivative is determined by
\[
\text{sign}\left(\frac{\partial}{\partial w} h(w, n)\right) = \frac{-2 - 4n + 2w - 2w\sqrt{(1 + 4n) + 6nw - w^2 + w^2\sqrt{(1 + 4n)}}}{-3nw^2 - 2n^2 + 4n^2w + w^2\sqrt{(1 + 4n)n} - 2w\sqrt{(1 + 4n)n}}
\]
\[
\leq -2 - 2n^2 \leq 0
\]
where we used that $w \in [0, 1]$ and $n \in [0, 1]$.

Lemma 7.9 $\frac{\partial}{\partial z} h(w, n(z, \kappa)) \leq 0$.

Proof. Taking the derivative
\[
\frac{d}{dz} h(w, n(z, \kappa)) = \frac{\partial}{\partial n} h(w, n) \frac{\partial}{\partial z} n(z, \kappa) \leq 0
\]
\[
\leq 0, \text{ lemma 7.6} \geq 0, \text{ lemma 7.7}
\]

We now prove that $z$ is decreasing in $w$ by proving its double negation. Define $z'$ and $z''$, to be the solutions to $z' = h(w', n(\kappa, z'))$ and $z'' = h(w'', n(\kappa, z''))$ where $w' < w''$. Assume that $z' < z''$
\[
z' = h(w', n(\kappa, z')) \geq h(w'', n(\kappa, z')) \geq h(w'', n(\kappa, z'')) = z''
\]
which contradicts the assumption $z'(w') > z''(w'')$. Hence for $z$ is monotonically decreasing in $w$.

This proves theorem 7.4.C.

(D) Proof. Note that $f$ is monotonically decreasing in $k$, which can be easily shown using lemma 7.2 and lemma 7.3. Again we use the method of reductio ad absurdum. Define $p'$ and $p''$, the solutions to $p' = f(w, \kappa', p')$ and $p'' = f(w, \kappa'', p'')$ with $\kappa' < \kappa''$. Assume that $p' < p''$, then we have that
\[
p' = f(w, \kappa', p') \geq f(w, \kappa'', p') \geq f(w, \kappa'', p'') = p''
\]
which contradicts with $p' < p''$. Hence $p' \geq p''$ for $\kappa' < \kappa''$.

C. Proof of theorem 7.5

Proof. Note that the function $k(w, \kappa, p) = f(w, \kappa, p) - p$ is strictly decreasing in $p$. Furthermore, note that $m(w, \kappa, p)$ is only defined if $p \leq \frac{\mu}{\kappa}$. Hence $k(\cdot)$ is only defined on $[0, \frac{\mu}{\kappa}]$. Because it is
continuous, it immediately follows that there exists a unique root of $k(w, \kappa, p)$ if and only if $k(0) > 0$, and $k(1) < 0$. Or, iff

$$k(0) = 1 > 0, \text{ and } k(1) = \frac{1 + \sqrt{5} + w}{3 + \sqrt{5} - w} < 1$$

The theorem follows. •

D. Proof of theorem 7.6

From theorem (6.5), we know that chartists have a demand schedule that has the form

$$d_t \propto (1 - q'Gk)(M_t - \hat{P}_t)$$

where $q' = p'_f - p'_c$, and

$$M_t = \frac{q'G(1 - kq')}{1 - q'Gk} \left[ \sum_{n=0}^{\infty} (G - Gkq')^n Gk \hat{P}_{t-n} \right]$$

(7.8)

Next, imposing our economy, we obtain the following property ($n > 2$)

$$(G - Gkq')^n = \lambda(G - Gkq')^{n-1}$$

with $\lambda = \left( \pi_f^2(\sigma^2 + \omega) + \gamma_f^2 \Delta^2 \right)^{-1}((\pi_f^2 - \pi_f^2)(\sigma^2 + \omega) + \gamma_f^2 \Delta^2)$. Moreover,

$$\frac{q'G(1 - kq')}{1 - q'Gk}Gk = 1 - \lambda$$

Rewriting in terms of $\theta$ yields the theorem. •

E. Market Statistics

From chapter 5, section 6.5 we know that the variance of price changes is given by

$$\text{var}(\hat{Q}_{t+1}) = S^{-2} \epsilon_t' \Sigma \epsilon_t + p'_f \Xi p_f + S^{-2}(\alpha^2 - 1) \epsilon_t' \left( \frac{2\alpha - 1}{\alpha} (1 - kp'_f)\Omega - \Omega (1 - kp'_f)^2 \right) \epsilon_t$$

and the covariance between price innovations is given by ($n > 0$)

$$\text{cov}(\hat{Q}_{t+1}, \hat{Q}_{t+1-n}) = S^{-2} \epsilon_t' G^n \Sigma \epsilon_t$$

$$+ S^{-1} \epsilon_t' \left( G^{-1} - (\alpha^2 - 1)(1 - kp'_f)(G - Gkp'_f)^{n-1} \right) \Xi p_f$$

$$- S^{-2} \epsilon_t' (1 - \alpha) \left( \frac{2\alpha - 1}{\alpha} (1 - kp'_f)(G - Gkp'_f)^n - G^n \right) \Omega (1 - kp'_f) \epsilon_t$$

where $\Sigma \equiv \sum_{j=0}^{\infty} G^j \Xi G^j$ is the unconditional variance of state of the economy. Here we have that $\Sigma \equiv \sum_{j=0}^{\infty} G^j \Xi G^j = \text{diag}(0, \Delta^2)$. Rewriting the above formula, one obtains for the variance of price changes:

$$\text{var}(\hat{Q}_{t+1}) = S^{-2} \Delta^2 + \pi_f^2 \sigma^2 + \gamma_f^2 \Delta^2 - \frac{(1 - \alpha)^2 \Delta^2 \pi_f^2 (\sigma^2 + \omega)}{\alpha^2 S^2 (\pi_f^2 (\sigma^2 + \omega) + \gamma_f^2 \Delta^2)}$$
and the first order autocorrelation is given by
\[
\text{cov}(\tilde{Q}_{t+1}, \tilde{Q}_t) = -\gamma_f \Delta^2 \alpha^2 \alpha^2 + 2\pi_j \Delta^2 \alpha - \gamma_f \Delta^2 \alpha - \pi_j \Delta^2
\]
\[
\frac{\alpha}{\pi_j (\sigma^2 + \alpha)} \cdot (-1 + \alpha) (2\alpha - 1) \frac{\gamma_f^2 \Delta^2 (\sigma^2 + \alpha) + \gamma_f^2 \gamma_f \Delta^2}{\alpha S^2 \left( \pi_j^2 (\sigma^2 + \alpha) + \gamma_f^2 \Delta^2 \right)}.
\]
and higher order autocorrelation are given by
\[
\text{cov}(\tilde{Q}_{t+1}, \tilde{Q}_{t+n}) = (-1 + \alpha) \gamma_f \Delta^2 \alpha^n \pi_f \left( \pi_f (\sigma^2 + \alpha) \gamma_f \right)^n
\]
\[
\frac{\alpha}{\pi_j (\sigma^2 + \alpha)} \cdot (-1 + \alpha) (2\alpha - 1) \frac{\gamma_f^2 \Delta^2 (\sigma^2 + \alpha) + \gamma_f^2 \gamma_f \Delta^2}{\alpha S^2 \left( \pi_j^2 (\sigma^2 + \alpha) + \gamma_f^2 \Delta^2 \right)}
\]

F. Proof of Theorem 7.9

First we proof the following lemma:

Lemma 7.10 The ex ante expected utility of being informed, \( V_f \), conditional on the information in past prices can be written as
\[
E[V_f \mid \tilde{I}_t, \tilde{P}_t] = \frac{\text{var}(\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t)}{\text{var}(\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t)} V_f
\]
where \( V_f \) is the ex ante utility of staying uninformed.

Proof. Note that we have
\[
E[U_{t+1} \mid \tilde{I}_t, \tilde{P}_t] = -\exp \left[ \frac{1}{2} \frac{(E[\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t] - \tilde{P}_t)^2}{\text{var}(\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t)} \right]
\]
\[
E[U_t \mid \tilde{I}_t, \tilde{P}_t] = -\exp \left[ \frac{1}{2} \frac{(E[\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t] - \tilde{P}_t)^2}{\text{var}(\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t)} \right]
\]
Because \( \tilde{I}_t \) is a sub-tribe of \( \tilde{I}_t \), distributions are normal and the linearity of the price function, it follows that \( E[\tilde{P}_{t+1} \mid \tilde{I}_t, P_t] - \tilde{P}_t \) is normally distributed. Now define
\[
\varphi = \text{var}(E[\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t] \mid \tilde{I}_t, \tilde{P}_t)
\]
\[
\omega = \frac{E[\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t] - \tilde{P}_t}{\sqrt{\varphi}}
\]
Then the expected utility of the informed can be written as
\[
E[U_{t+1} \mid \tilde{I}_t, \tilde{P}_t] = -\exp \left[ \frac{1}{2} \frac{(E[\tilde{P}_{t+1} \mid \tilde{I}_t, \tilde{P}_t] - \tilde{P}_t)^2}{\varphi} \right]
\]
Conditional on $I_t^f$, this quantity is given by

$$
E[U_{t+1}^f|I_t^f, \tilde{P_t}, \tilde{P_t}] = \frac{1}{\sqrt{1 + \frac{\sigma^2}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})}}} \exp \left[ \frac{1 - (E[P_{t+1}^f|I_t^f, \tilde{P_t}])^2}{\frac{1}{2} \cdot \frac{\sigma^2}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})}} \right] 
$$

where we used that $E[U_{t+1}^f|I_t^f, \tilde{P_t}, \tilde{P_t}] = E[U_{t+1}^f|I_t^f, \tilde{P_t}] - \tilde{P_t}$.

Since the following equality holds:

$$\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t}) = \text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t}) + \text{var}(E[P_{t+1}^f|I_t^f, \tilde{P_t}]) - \tilde{P_t},$$

we have that

$$E[U_{t+1}^f|I_t^f, \tilde{P_t}, \tilde{P_t}] = \frac{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t}) + \sigma^2} \exp \left[ \frac{1 - (E[P_{t+1}^f|I_t^f, \tilde{P_t}] - \tilde{P_t})^2}{\frac{1}{2} \cdot \frac{\sigma^2}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})}} \right]$$

and the lemma is proven. 

Armed with this expression, we now turn to the computation of the ex ante expected utilities. Because of lemma 7.10 we only have to compute the ex ante expected utility of the uninformed, which is given by

$$\mathcal{V}^c = E[U_{t+1}^f|I_t^f, \tilde{P_t}] = -E[\exp \left( \frac{1}{2} \cdot \frac{\sigma^2}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})} \right) | I_t^f]$$

Conditional on $I_t^f$, the quantity

$$\theta = E[\tilde{P}_{t+1}|I_t^f, \tilde{P_t}] - \tilde{P_t} = (q(I - kp)G - \tilde{P_t})X_{t-1} + (qk - 1)pX_t$$

has mean zero, and variance

$$\sigma^2_{\theta} = \text{var}(\theta|I_t^f) = (qk - 1)p(GG' + \Sigma)p'(qk - 1)$$

Hence $\mathcal{V}^c$ is given by

$$\mathcal{V}^c = -E[\exp \left( \frac{1}{2} \cdot \frac{\sigma^2_{\theta}}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})} \right) | I_t^f] =$$

$$= \frac{1}{\sqrt{\sigma^2_{\theta} + \text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})}} \exp \left( \frac{1}{2} \cdot \frac{\sigma^2_{\theta}}{\text{var}(\tilde{P}_{t+1}|I_t^f, \tilde{P_t})} \right) \cdot \left( 1 - \frac{1}{(1 - \pi^2 I\theta(1 + \theta))} \right)^2 + 1$$
It follows that

\[ \psi' = \sqrt{\frac{\theta^2 + 1 + \theta}{(1 + \theta)^2}} \nu c = -\sqrt{\frac{\theta^2 + 1 + \theta}{\left(1 + \theta - \pi^{-1} \nu \right)^2 + (1 + \theta)^2}} \]
whence it follows that $\mathbb{E}(X) = \mathbb{E}(X)$, and

$$
\mathbb{E}(X) = \mathbb{E}(X)
$$

For the following identity holds:

$$
\mathbb{E}(X) = \mathbb{E}(X)
$$

we prove that

$$
\mathbb{E}(X) = \mathbb{E}(X)
$$

and the lemma is proven.

Agreed with this expression, we have now the approximation of the expected value.

Denoting by $\mathcal{L}$ the set of linear functions, we only have to choose the discrete approximated value of the approximation, which is given by

$$
\mathcal{L} = \mathbb{E}(X)
$$

Conditionally on $\mathcal{L}$, the quantity

$$
\mathbb{E}(X) = \mathbb{E}(X)
$$

has value 50%, i.e., we consider

$$
\mathbb{E}(X) = \mathbb{E}(X)
$$

Hence $\mathcal{L}$ is given by

$$
\mathcal{L} = \mathbb{E}(X)
$$