UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

The Nori-Hilbert scheme is not smooth for 2-Calabi-Yau algebras

Bocklandt, Raf; Galluzzi, Federica; Vaccarino, Francesco

DOI
10.4171/JNCG/247

Publication date
2016

Document Version
Final published version

Published in
Journal of Noncommutative Geometry

License
CC BY

Link to publication

Citation for published version (APA):
Bocklandt, R., Galluzzi, F., & Vaccarino, F. (2016). The Nori-Hilbert scheme is not smooth for 2-Calabi-Yau algebras. *Journal of Noncommutative Geometry*, *10*(2), 745-774. https://doi.org/10.4171/JNCG/247

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, P.O. Box 19185, 1000 GD Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:13 Nov 2026

J. Noncommut. Geom. 10 (2016), 745–774
DOI 10.4171/JNCG/247

Journal of Noncommutative Geometry
© European Mathematical Society

The Nori–Hilbert scheme is not smooth for 2-Calabi–Yau algebras

Raf Bocklandt, Federica Galluzzi* and Francesco Vaccarino**

Abstract. Let k be an algebraically closed field of characteristic zero and let A be a finitely generated k-algebra. The Nori–Hilbert scheme of A, Hilb^n_A, parameterizes left ideals of codimension n in A. It is well known that Hilb^n_A is smooth when A is formally smooth.

In this paper we will study Hilb^n_A for 2-Calabi–Yau algebras. Important examples include the group algebra of the fundamental group of a compact orientable surface of genus g, and preprojective algebras. For the former, we show that the Nori–Hilbert scheme is smooth only for $n = 1$, while for the latter we show that a component of Hilb^n_A containing a simple representation is smooth if and only if it only contains simple representations. Under certain conditions, we generalize this last statement to arbitrary 2-Calabi–Yau algebras.

Mathematics Subject Classification (2010). 14C05, 14A22, 16G20, 16E40.

Keywords. Representation theory, Calabi–Yau algebras, Nori–Hilbert scheme.

1. Introduction

Let A be a finitely generated associative k-algebra with k an algebraically closed field of characteristic zero. In this paper we study the Nori–Hilbert scheme Hilb^n_A whose k-points parameterize left ideals of A with codimension n.

When A is commutative, this is nothing but the classical Hilbert scheme Hilb^n_X of n points on $X = \mathrm{Spec}\, A$. It is well known that Hilb^n_X is smooth when X is a quasi-projective irreducible and smooth curve or surface. The scheme Hilb^n_A is smooth when A is formally smooth, hence of global dimension one, proved by L. Le Bruyn (see [30, Prop. 6.3.]). The same holds when A is finitely unobstructed [3].

The main result of this paper is to show that the above results do not extend to dimension two in the non-commutative case.

*Supported by the framework PRIN 2010/11 "Geometria delle Varietà Algebriche", cofinanced by MIUR. Member of GNSAGA.

**Partially supported by the TOPDRIM project funded by the Future and Emerging Technologies program of the European Commission under Contract IST-318121.

The smoothness results on Hilb^n_X are heavily based on the use of Serre Duality, so it seems natural to investigate the geometry of Hilb^n_A when A is a Calabi–Yau algebra of global dimension two. These are algebras for which $\mathrm{Ext}^\bullet_{A^e}(A, A) \cong A[2]$, which implies that the double shift is a Serre functor for their derived category.

Important examples of 2-dimensional Calabi–Yau algebras are tame and wild preprojective algebras (see Bocklandt [6]) and group algebras of fundamental groups of compact orientable surfaces with nonzero genus (a result of Kontsevich [20, Corollary 6.1.4.]).

In this paper we will investigate the smoothness of the Nori–Hilbert scheme for these two types of algebras. The main results are the following:

Theorem 1.1. *Let $A_g = k[\pi_1(S)]$ be the group algebra of the fundamental group of a compact orientable surface S of genus $g > 1$. The scheme $\mathrm{Hilb}^n_{A_g}$ is irreducible of dimension $(2g-2)n^2 + n + 1$ and it is smooth if and only if $n = 1$.*

Theorem 1.2. *Let $\Pi(Q)$ be the preprojective algebra attached to a non-Dynkin quiver Q and let α be a dimension vector for which there exist simple representations. The component of $\mathrm{Hilb}^n_{\Pi(Q)}$ containing the α-dimensional representations is irreducible of dimension*

$$1 + 2\sum_{a \in Q_1} \alpha_{h(a)}\alpha_{t(a)} + \sum_{v \in Q_0} (\alpha_v - 2\alpha_v^2)$$

and it is smooth if and only if Q has one vertex and $\alpha = (1)$ (or equivalently all α-dimensional representations are simple).

After these two results we look into the case of more general 2-CY algebras. Using results by Van den Bergh [42], we show that locally the representation space of any finitely generated 2-CY algebra can be seen as the representation space of a preprojective algebra. This fact will allow us to generalize the main result to all finitely generated 2-CY algebras.

Theorem 1.3. *Let A be a finitely generated 2-CY algebra and let ρ be a simple representation such that the dimension of its component in $\mathrm{Rep}^n_A /\!\!/ \mathrm{GL}_n$ is bigger than 2. The component of Hilb^n_A containing ρ is smooth if and only if all representations in this component are simple.*

The paper goes as follows. In Section 3 we recall the definition and the principal known results on the smoothness of Hilb^n_A. We also introduce quivers and generalize the definition to Hilb^α_A for arbitrary dimension vectors α. We consider the representation scheme Rep^α_A of an associative algebra A and the open subscheme U^α_A whose points correspond to α-dimensional cyclic A-modules. The general linear group GL_α acts naturally on U^α_A. We show that $\mathrm{U}^\alpha_A/\mathrm{GL}_\alpha$ represents Hilb^α_A and that $\mathrm{U}^\alpha_A \to \mathrm{Hilb}^\alpha_A$ is a universal categorical quotient and a GL_α-principal bundle.

After introducing Calabi–Yau algebras in Section 4, we carefully analyze the tangent space of Rep^α_A and of Rep^n_A, the representation scheme of the n-dimensional representations of A, in Section 5. If A is a 2-CY algebra having a suitable resolution,

we find a sharp upper bound for the dimension of the tangent space of a point in U_A^α corresponding to an A-module M. In Theorem 5.2 we prove that this dimension is completely controlled by $\dim_k(\mathrm{End}_A(M))$. This is achieved by using Hochschild cohomology and the equality $\dim_k(\mathrm{End}_A(M)) = \dim_k(\mathrm{Ext}_A^2(M,M))$ given by the Calabi–Yau condition. This method was inspired by a similar one used by Geiss and de la Peña (see [18]), which works for finite-dimensional k-algebras only.

We then prove the first two main theorems, 1.1 and 1.2, by combining our results on the tangent spaces of $\mathrm{Rep}_{A_g}^n$ and $\mathrm{U}_{A_g}^n$ with the description of $\mathrm{Rep}_{A_g}^n$ and $\mathrm{Rep}_{\Pi(Q)}^\alpha$ given in [6,7,14,36] and in [12].

In Section 6, we show that locally the representation space of a 2-CY algebra is the representation space of a preprojective algebra and we deduce from this that, for simple dimension vectors, the smooth semisimple locus equals the simple locus. Finally, we combine the results from Sections 6 and 5.2 to prove Theorem 1.3 and we give a couple of examples that illustrate it.

2. Notations

Unless otherwise stated, we adopt the following notations:

- k is an algebraically closed field of characteristic zero.
- $F = k\{x_1, \ldots, x_m\}$ denotes the associative free k-algebra on m letters.
- $A \cong F/J$ is a finitely generated associative k-algebra.
- $\mathcal{N}_R$, $\mathcal{C}_R$ and Set denote the categories of R-algebras, R-commutative algebras and sets, respectively, where R is a given commutative ring.
- The term "A-module" indicates a left A-module.
- A-mod denotes the category of left A-modules.
- We write $\mathrm{Hom}_{\mathcal{A}}(B,C)$ in a category $\mathcal{A}$ with B, C objects in $\mathcal{A}$. If $\mathcal{A} = A$-mod, then we will write $\mathrm{End}_A(B)$, for $B \in A$-mod.
- A^{op} denotes the opposite algebra of A and $A^e := A \otimes A^{\mathrm{op}}$ denotes the envelope of A. It is an A-bimodule and a k-algebra. One can identify the category of the A-bimodules with A^e-mod and we will do this thoroughly this paper.
- $A_g = k[\pi_1(S)]$ is the group algebra of the fundamental group of a compact orientable surface S of genus $g > 1$.
- Ext_A^i denotes the Ext groups on the category A-mod.
- Q will denote a quiver, Q_0 its vertices and Q_1 its arrows. The maps $h, t : Q_0 \to Q_1$ assign to each arrow its head and tail.
- kQ will be the path algebra of Q.
- $\alpha : Q_0 \to \mathbb{N}$ will denote a dimension vector and its size is $n = |\alpha| = \sum_{v \in Q_0} \alpha_v$.

- If R is a ring, $\mathrm{Mat}_n(R)$ denotes the ring of $n \times n$ matrices with elements in R.
- $\mathrm{Mat}_\alpha(R) := \prod_{v \in Q_0} \mathrm{Mat}_{\alpha_v}(R)$ and its group of invertible elements is GL_α.
- The standard module over $\mathrm{Mat}_\alpha(R)$ will be denoted by $R^\alpha = \oplus_{v \in Q_0} R^{\alpha_v}$ and $\mathrm{Mat}_\alpha(R)$ sits inside $\mathrm{Mat}_n(R) = \mathrm{End}_R(R^\alpha)$ with $n = |\alpha|$.

3. Nori–Hilbert schemes

3.1. Definitions. Let $A \in \mathcal{N}_k$ be fixed. Consider the functor of points

$$\mathcal{H}ilb^n_A : \mathcal{C}_k \to \mathsf{Set},$$

given by

$$\mathcal{H}ilb^n_A(B) := \{\text{left ideals } I \subset A \otimes_k B \text{ such that } M = (A \otimes_k B)/I \text{ is a projective } B\text{-module of rank } n\} \quad (3.1)$$

for all $B \in \mathcal{C}_k$.

It is a closed subfunctor of the Grassmannian functor, so it is representable by a scheme Hilb^n_A (see [41, Proposition 2]) and we call it the *Nori–Hilbert scheme*. Its k-points are the left ideals of A of codimension n.

Nori introduced it for $A = \mathbb{Z}\{x_1, \ldots, x_m\}$ in [34]. It was then defined in a more general setting in [41] and in [37]. Van den Bergh showed that for $A = F$ the scheme Hilb^n_F is smooth of dimension $n^2(m-1)+n$, (see [41]).

It is also called *the non-commutative Hilbert scheme* (see [17,37]) or the *Brauer–Severi scheme* of A (see [29, 30, 41]), in analogy with the classical Brauer–Severi varieties parameterizing left ideals of codimension n of central simple algebras (see [4]).

Let now A be commutative and $X = \mathrm{Spec}\, A$. The k-points of Hilb^n_A parameterize zero-dimensional subschemes $Y \subset X$ of length n. It is the simplest case of Hilbert scheme parameterizing closed subschemes of X with fixed Hilbert polynomial P. In this case P is the constant polynomial n. The scheme Hilb^n_A is usually called the *Hilbert scheme of n points on X* (see for example Chapter 7 in [10,21] and Chapter 1 in [33]).

There is the following fundamental result.

Theorem 3.1 (See [15,16,22]). *If X is an irreducible smooth quasi projective variety of dimension d (with $d = 1, 2$), then the Hilbert scheme of n points over X is a smooth irreducible scheme of dimension dn.*

This theorem can be partially extended to the Nori–Hilbert scheme. The scheme Hilb^n_A is smooth if A is finitely unobstructed i.e. if $\mathrm{Ext}^2_A(M, M) \cong 0$ for all finite dimensional A-modules M. This follows by [3, Corollary 4.2.] and Theorem 3.13.

Remark 3.2. If A is hereditary then it is finitely unobstructed and it was well known that Hilb^n_A is smooth for hereditary algebras which are finite dimensional (see [9, Proposition 1]).

If $A = kQ/J$ is the path algebra of a quiver with relations, then to every left ideal $I \in \mathcal{H}ilb^n_A(B)$ we can assign a dimension vector

$$\alpha : v \mapsto \mathrm{rank}(vA \otimes_k B)/I.$$

So we can define the subset $\mathcal{H}ilb^\alpha_A(B) \subset \mathcal{H}ilb^n_A(B)$ containing all ideals with dimension vector α. We denote its representing scheme by Hilb^α_A. Hilb^n_A decomposes as a disjoint union of all Hilb^α_A with $|\alpha| = n$.

3.2. Representation schemes. Let $A \in \mathcal{N}_k$ be fixed. The covariant functor

$$\mathcal{R}ep^n_A : \mathcal{C}_k \longrightarrow \mathsf{Set}$$

given by

$$\mathcal{R}ep^n_A(B) := \mathrm{Hom}_{\mathcal{N}_k}(A, \mathrm{Mat}_n(B)) \tag{3.2}$$

for all $B \in \mathcal{C}_k$, is represented by a commutative algebra $V_n(A)$ (see [35, Ch.4, §1]). We write Rep^n_A to denote $\mathrm{Spec}\, V_n(A)$. It is called the *scheme of the n-dimensional representations of A*. It is considered as a k-scheme.

Let now $A = kQ/J$ be a path algebra of a quiver with relations. The vertices in Q will correspond to orthogonal idempotents in kQ, which will generate a subalgebra $\ell = \oplus_{v \in Q_0} kv \cong k^{Q_0}$ and both A and kQ can be seen as ℓ-algebras. We can choose a generating set of relations $\{r_i | i \in J\}$ such that each r_i sits in $vkQw$ for some idempotents $v, w \in Q_0$. We will denote these idempotents by $h(r_i), t(r_i)$.

Fix a dimension vector $\alpha : Q_0 \to \mathbb{N}$ with $|\alpha| = n$. Let k^α be the ℓ-module consisting of the direct sum of α_v copies of the simple module corresponding to each vertex v. The space $\mathrm{Mat}_n(k)$ can be given the structure of a ℓ-bimodule/ℓ-algebra by identifying it with $\mathrm{Hom}_k(k^\alpha, k^\alpha)$. An α-dimensional representation ρ is a ℓ-algebra homomorphism from A to $\mathrm{Mat}_n(k)$, this homomorphism extends the ℓ-module structure on k^α to an A-module structure.

For any commutative k-algebra B we set $B^\alpha = k^\alpha \otimes B$ and $\mathrm{Mat}_n(B) = \mathrm{Mat}_n(k) \otimes B$.

Definition 3.3. Let $A = kQ/J$ and $B \in \mathcal{C}_k$. By an *α-dimensional representation of A* over B we mean a homomorphism of ℓ-algebras $\rho : A \to \mathrm{Mat}_n(B)$.

It is clear that this is equivalent to give an A-module structure on B^α. The assignment $B \to \mathrm{Hom}_{\mathcal{N}_\ell}(A, \mathrm{Mat}_n(B))$ defines a covariant functor

$$\mathcal{C}_k \longrightarrow \mathsf{Set}.$$

This functor is represented by a commutative k-algebra. More precisely, there is the following lemma.

Lemma 3.4 ([35, Ch. 4, §1 extended to quivers]). *For all $A \in \mathcal{N}_\ell$ and each dimension vector α, there exist a commutative k-algebra $V_\alpha(A)$ and a representation $\pi_A : A \to \mathrm{Mat}_n(V_\alpha(A))$ such that $\rho \mapsto \mathrm{Mat}_n(\rho) \cdot \pi_A$ gives an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_k}(V_n(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{N}_\ell}(A, \mathrm{Mat}_n(B)) \tag{3.3}$$

for all $B \in \mathcal{C}_k$.

Definition 3.5. We denote $\mathrm{Rep}_A^\alpha := \mathrm{Spec}\, V_\alpha(A)$. It is considered as a k-scheme.

The scheme Rep_A^α is also known as *the scheme of α-dimensional A-modules*.

Remark 3.6. Any path algebra with relations $A = kQ/I$ can also be seen as the quotient of a free algebra: $A \cong F/J$, so it makes sense to define both Rep_A^n and Rep_A^α. It is known that there is the following relation between the two

$$\mathrm{Rep}_A^n = \coprod_{|\alpha|=n} \mathrm{Rep}_A^\alpha \times_{\mathrm{GL}_\alpha} \mathrm{GL}_n$$

where the action of GL_α on GL_n is by multiplication.

Examples 3.7.

1. If $A = F$, then $\mathrm{Rep}_F^n(k) \cong \mathrm{Mat}_n(k)^m$ (because a free algebra is the path algebra of a quiver with one vertex, a dimension vector in this case is just a number n).
2. If $A = F/J$, the B-points of Rep_A^n can be described in the following way:

   $$\mathrm{Rep}_A^n(B) = \{(X_1, \ldots, X_m) \in \mathrm{Mat}_n(B)^m \ : \ f(X_1, \ldots, X_m) = 0 \text{ for all } f \in J\}.$$

   The scheme Rep_A^n is a closed subscheme of Rep_F^n.
3. If $A = \mathbb{C}[x, y]$, then

   $$\mathrm{Rep}_A^n(\mathbb{C}) \ = \ \{(M_1, M_2) \ : \ M_1, M_2 \in \mathrm{Mat}_n(\mathbb{C}) \text{ and } M_1 M_2 = M_2 M_1\}$$

   is the *commuting scheme*, see [39].
4. If $A = kQ$, then $\mathrm{Rep}_A^\alpha(k) \cong \bigoplus_{a \in Q_0} \mathrm{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(k)$. For each arrow a, $\rho(a)$ is an $n \times n$ matrix with zeros everywhere except on a block of size $\alpha_{h(a)} \times \alpha_{t(a)}$.

Because A is finitely generated, Rep_A^n is of finite type. Note that Rep_A^n may be quite complicated. It is not reduced in general and it seems to be hopeless to describe the coordinate ring of its reduced structure.

3.3. Principal bundles over Nori–Hilbert schemes.

Fix $A \in \mathcal{N}_k$.

Definition 3.8. For each $B \in \mathcal{C}_k$, consider the set

$$\mathcal{U}_A^n(B) \ = \ \{(\rho, v) \in \mathrm{Rep}_A^n(B) \times \mathbb{A}_k^n(B) \ : \ \rho(A)(Bv) = B^n\}.$$

The assignment $B \mapsto \mathcal{U}_A^n(B)$ is functorial in B and the corresponding functor is representable by a scheme U_A^n which is an open subscheme in $\mathrm{Rep}_A^n \times \mathbb{A}_k^n$.

Remark 3.9. Note that points $\rho \in \mathrm{Rep}_A^n$ such that there is a $v \in \mathbb{A}_k^n$ with $(\rho, v) \in \mathrm{U}_A^n$ correspond to n-dimensional cyclic A-modules.

Let GL_n be the general linear group scheme over k whose B-points form the group $\mathrm{GL}_n(B)$ of invertible matrices in $\mathrm{Mat}_n(B)$.

Definition 3.10. Given $B \in \mathcal{C}_k$, $\mathrm{GL}_n(B)$ acts on $\mathrm{Rep}_A^n(B)$:

$$\begin{array}{ccc} \mathrm{GL}_n(B) \times \mathrm{Rep}_A^n(B) & \longrightarrow & \mathrm{Rep}_A^n(B) \\ (g, \rho) & \longrightarrow & \rho^g \ : \ \rho^g(a) = g\rho(a)g^{-1}, \end{array}$$

and on $\mathrm{Rep}_A^n(B) \times \mathbb{A}_k^n(B)$:

$$\begin{array}{ccc} \mathrm{GL}_\alpha(B) \times \mathrm{Rep}_A^n(B) \times \mathbb{A}_k^n(B) & \longrightarrow & \mathrm{Rep}_A^n(B) \times_k \mathbb{A}_k^n(B) \\ (g, \rho, v) & \longrightarrow & (\rho^g, gv). \end{array}$$

The open subscheme U_A^n is clearly closed under the action above.

Remark 3.11. The A-module structures induced on B^n by two representations ρ and ρ' are isomorphic if and only if there exists $g \in \mathrm{GL}_n(B)$ such that $\rho' = \rho^g$.

Definition 3.12. We denote by $\mathrm{Rep}_A^n//\mathrm{GL}_n = \mathrm{Spec}\, V_n(A)^{\mathrm{GL}_n(k)}$ the categorical quotient (in the category of k-schemes) of Rep_A^n by GL_n. It is the *(coarse) moduli space of n-dimensional representations* of A.

There is the following

Theorem 3.13 ([40, Theorem 5.5.]). *The scheme $\mathrm{U}_A^n/\mathrm{GL}_n$ represents Hilb_A^n and $\mathrm{U}_A^n \to \mathrm{Hilb}_A^n$ is an universal categorical quotient and a GL_n-principal bundle. Therefore the scheme Hilb_A^n is smooth iff U_A^n is smooth.*

Let now $A = kQ/J$ and α any dimension vector with $|\alpha| = n$. Identify the B-points of the n-dimensional affine scheme $\mathbb{A}_k^n$ with the elements of the module B^α. Denote GL_α the group scheme over k whose B-points form the group $\mathrm{GL}_\alpha(B)$ of invertible matrices in $\mathrm{Mat}_\alpha(B) = \mathrm{End}_\ell(B^\alpha)$.

We can define in the same way as before an open subscheme U_A^α in $\mathrm{Rep}_A^\alpha \times \mathbb{A}_k^\alpha$ and actions of GL_α on Rep_A^α and on $\mathrm{Rep}_A^\alpha \times \mathbb{A}_k^\alpha$.

Theorem 3.13 can be easily generalized as follows.

Corollary 3.14. *The scheme $\mathrm{U}_A^\alpha/\mathrm{GL}_\alpha$ represents Hilb_A^α and $\mathrm{U}_A^\alpha \to \mathrm{Hilb}_A^\alpha$ is an universal categorical quotient and a GL_α-principal bundle. Therefore the scheme Hilb_A^α is smooth iff U_A^α is smooth.*

Proof. By Remark 3.6 it follows

$$\mathrm{U}_A^n = \coprod_{|\alpha|=n} \mathrm{U}_A^\alpha \times_{\mathrm{GL}_\alpha} \mathrm{GL}_n$$

and hence

$$\mathrm{Hilb}_A^n \cong \mathrm{U}_A^n/\mathrm{GL}_n = \coprod_{|\alpha|=n} \mathrm{U}_A^\alpha/\mathrm{GL}_\alpha. \qquad \square$$

Remark 3.15. Consider the forgetful map $\mathrm{Rep}_A^n \times \mathbb{A}_k^n \longrightarrow \mathrm{Rep}_A^n$, which sends (ρ, v) to ρ. The existence of a cyclic vector is an open condition, so the image of U_A^n is an open subset of Rep_A^n and the preimage of $\rho \in \mathrm{Rep}_A^n$ is an open subset of $\{\rho\} \times \mathbb{A}_k^n$. This implies that

$$\dim T_{(\rho,v)}\mathrm{U}_A^n = \dim T_\rho \mathrm{Rep}_{\mathrm{A}}^{\mathrm{n}} + \dim \mathbb{A}_k^n \tag{3.4}$$

and U_A^n is smooth if and only if its image in Rep_A^n is smooth. Analogously, U_A^α is smooth if and only if so is its image through the forgetful map $\mathrm{Rep}_A^\alpha \times \mathbb{A}_k^\alpha \longrightarrow \mathrm{Rep}_A^\alpha$.

These results lead us to study the local geometry of Rep_A^n and Rep_A^α. For general algebras this study is quite hard, but we are interested in a special class of algebras: 2-Calabi–Yau algebras.

4. Calabi–Yau algebras

Calabi–Yau algebras have been defined by V. Ginzburg in [20] and R. Bocklandt in [6] following the notion of Calabi–Yau triangulated category introduced by Kontsevich. For alternative approaches and further reading see [1,23,24] and [26]. We first recall the following.

Definition 4.1 ([19, Definition 20.6.1]). An algebra A is called *homologically smooth* if A has a finite resolution by finitely-generated projective (left) A^e-modules.

Definition 4.2 ([20, Definition 3.2.3]). A homologically smooth algebra A is *$d-$Calabi–Yau* ($d-$CY for short) if there are A^e-module isomorphisms

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} A & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

We note some properties of Calabi–Yau algebras.

Proposition 4.3. *If A is $d-$CY, then*

(1) *The global dimension of A is $\leq d$.*

(2) *If there exists a nonzero finite-dimensional A-module M, then the global dimension of A is exactly d.*

(3) *If $M, N \in A$-mod are finite-dimensional, then*

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^{d-i}_A(N, M)^*.$$

(4) *For every finite dimensional A-module M there is a trace map $\mathrm{Tr}_M : \mathrm{Ext}^d(M, M) \to k$, compatible with the product of* Ext*'s:*

$$\mathrm{Tr}_N(fg) = (-1)^{i(d-i)}\mathrm{Tr}_M(gf)$$

for $f \in \mathrm{Ext}^i(M, N)$ and $g \in \mathrm{Ext}^{d-i}(N, M)$.

Proof. These are standard results, see for example [2, Proposition 2.4], [5, Section 2] or [6, Prop. 2.2] for proofs. □

Examples 4.4.

1. The polynomial algebra $k[x_1, \ldots, x_n]$ is $n-$CY.
2. Let X be an affine smooth Calabi–Yau variety (i.e. the canonical sheaf is trivial) of dimension n. Then $\mathbb{C}[X]$ is $n-$CY.
3. If Q is a quiver, denote by $\overline{Q}$ the *double* quiver of Q obtained by adjoining an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. The *preprojective* algebra is the associative algebra

$$\Pi(Q) := k(\overline{Q})/ < \sum_{a \in Q_1} [a, a^*] >$$

   where $[x, y] = xy - yx$ denotes the commutator. If A is any path algebra with homogeneous relations for the path length grading, then A is 2-CY if and only if A is the preprojective algebra of a non-Dynkin quiver (see [6, Theorem 3.2.]).
4. Let $k[\pi_1(M)]$ be the group algebra of the fundamental group of a compact aspherical orientable manifold M of dimension n. Kontsevich proves that $k[\pi_1(M)]$ is $n-$CY (see [20, Corollary 6.1.4.]). This algebra is not positively graded. Thus, if S is a surface of genus $g \geq 1$, the algebra $A_g := k[\pi_1(S)]$ is 2-Calabi–Yau. The fundamental group $\pi_1(S)$ has presentation

$$< X_1, Y_1, \ldots, X_g, Y_g | X_1 Y_1 X_1^{-1} Y_1^{-1} \cdots X_g Y_g X_g^{-1} Y_g^{-1} = 1 > . \qquad (4.1)$$

5. Local geometry

Let $A = kQ/J$ be the path algebra of a quiver with relations and α a fixed dimension vector with $|\alpha| = n$. A point $x \in \mathrm{Rep}_A^\alpha(k)$ corresponds to a pair (M, μ) where $M \cong k^\alpha$ has an A-module structure given by the ℓ-algebra homomorphism

$$\mu : A \to \mathrm{Mat}_n(k) \cong \mathrm{End}_k(M).$$

The linear representation μ makes $\mathrm{End}_k(M)$ an A^e-module.

We write M for a point x in $\mathrm{Rep}_A^\alpha(k)$ and $T_M \mathrm{Rep}_A^\alpha$ to denote the tangent space to Rep_A^α at x and to stress the dependence on M.

Proposition 5.1 ([19, 12.4]). *For $M \in \mathrm{Rep}_A^\alpha(k)$*

$$T_M \mathrm{Rep}_A^\alpha \cong \mathrm{Der}_\ell(A, \mathrm{End}_k(M)).$$

Proof. An element $p \in T_M \mathrm{Rep}_A^\alpha$ corresponds to a morphism of ℓ-algebras $q : A \to \mathrm{Mat}_n(k[\epsilon])$ such that $q(a) = \theta(a)\epsilon + \mu(a)$ for all $a \in A$, where $\mu : A \to \mathrm{End}_k(M)$ is the ℓ-algebra morphism associated to M. By using $q(ab) = q(a)q(b)$ one can easily see that $\theta \in \mathrm{Der}_\ell(A, \mathrm{End}_k(M))$ and $\theta(\ell) = 0$. On the other hand, for all $\theta \in \mathrm{Der}_\ell(A, \mathrm{End}_k(M))$, the pair (θ, μ) gives a point of $T_M \mathrm{Rep}_A^\alpha$ in the obvious way. □

Let now $M \in \mathrm{Rep}_A^\alpha(k)$. It is easy to check (see [19, 5.4]) that we have the following exact sequence

$$0 \to \mathrm{Ext}_A^0(M,M) \to \mathrm{End}_\ell(M) \to \mathrm{Der}_\ell(A, \mathrm{End}_k(M)) \to \mathrm{Ext}_A^1(M,M) \to 0$$

and therefore

$$\begin{aligned} \dim_k T_M \mathrm{Rep}_A^\alpha &= \dim_k \mathrm{Der}_\ell(A, \mathrm{End}_k(M)) \\ &= \alpha^2 + \dim_k(\mathrm{Ext}_A^1(M,M)) - \dim_k(\mathrm{Ext}_A^0(M,M)) \end{aligned}$$

where α^2 stands for the inner product of α with itself:

$$\alpha^2 = \sum_{v \in Q_0} \alpha_v^2 = \dim_k \mathrm{End}_\ell(M) = \dim_k(\mathrm{Mat}_\alpha(k)).$$

The above computation specializes to

$$\dim_k T_M \mathrm{Rep}_A^n = n^2 + \dim_k(\mathrm{Ext}_A^1(M,M)) - \dim_k(\mathrm{Ext}_A^0(M,M)) \tag{5.1}$$

and, therefore, the local dimension of the representation spaces Rep_A^α and Rep_A^n is controlled by the dimensions of Ext_A^0 and Ext_A^1.

If A is 2-CY admitting a suitable resolution, one can actually say more.

Theorem 5.2. *Let A be a 2-CY and $F_\bullet$ be a resolution by finitely-generated projective (left) A^e-modules. Suppose that the functions*

$$c^i : \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, \quad M \longmapsto c_M^i := \dim_k(\mathrm{Hom}_{A^e}(F_i, \mathrm{End}_k(M))$$

are locally constant for $i = 0, 1, 2$. Then the dimension of the tangent space $T_M \mathrm{Rep}_A^n$ is an increasing function of $\dim_k(\mathrm{End}_A(M))$ on the irreducible components of $\mathrm{Rep}_A^n(k)$.

Proof. To compute the dimension of the tangent space $T_M \mathrm{Rep}_A^n$ at $M \in \mathrm{Rep}_A^n(k)$ we need to compute the groups $\mathrm{Ext}_A^i(M,M)$, $i = 0, 1$. We use the isomorphisms

$$\mathrm{Ext}_A^i(M,M) \cong \mathrm{H}^i(A, \mathrm{End}_k(M)) \tag{5.2}$$

(see [11, Corollary 4.4]) where $\mathrm{H}^i(A, \mathrm{End}_k(M))$ denotes the Hochschild cohomology with coefficients in $\mathrm{End}_k(M)$. Note also that

$$\mathrm{End}_A(M) \cong \mathrm{Ext}_A^0(M,M) \cong \mathrm{Hom}_{A^e}(A, \mathrm{End}_k(M)).$$

Take the resolution $F_\bullet$ of A and consider the associate complex

$$0 \longrightarrow \mathrm{Hom}_{A^e}(A, \mathrm{End}_k(M)) \longrightarrow \mathrm{Hom}_{A^e}(F_0, \mathrm{End}_k(M)) \xrightarrow{d_M^0}$$

$$\longrightarrow \mathrm{Hom}_{A^e}(F_1, \mathrm{End}_k(M)) \xrightarrow{d_M^1} \mathrm{Hom}_{A^e}(F_2, \mathrm{End}_k(M)) \xrightarrow{d_M^2} \cdots$$

Set

$$\begin{aligned} k^i &: \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, \qquad M \longmapsto k_M^i := \dim_k \ker d_M^i, \\ h^i &: \mathrm{Rep}_A^n(k) \longrightarrow \mathbb{N}, \qquad M \longmapsto h_M^i := \dim_k \mathrm{Ext}_A^i(M, M). \end{aligned} \tag{5.3}$$

The following relations hold by the rank-nullity theorem

$$\begin{cases} h_M^i = k_M^i + k_M^{i-1} - c_M^{i-1} \\ h_M^0 = k_M^0 \end{cases} \tag{5.4}$$

Recall that $\dim T_M \mathrm{Rep}_A^n = n^2 + h_M^1 - h_M^0$ (see (5.1)), but, since A is 2-CY, we have $h_M^0 = h_M^2$, so that

$$\begin{aligned} \dim T_M \mathrm{Rep}_A^n &= n^2 + h_M^1 - h_M^2 \\ &= n^2 + (k_M^1 + k_M^0 - c_M^0) - (k_M^2 + k_M^1 - c_M^1) \\ &= n^2 - c_M^0 + c_M^1 - k_M^2 + h_M^0. \end{aligned} \tag{5.5}$$

The algebra A has global dimension 2, therefore $h^3 \equiv 0$ on $\mathrm{Rep}_A^n(k)$. From (5.4) it follows then that $k_M^3 + k_M^2 = c_M^2$. The function c^2 is locally constant, so by observing that the functions k^i are (locally) upper semicontinuous, it follows that the functions k^3 and k^2 are locally constant as well. Therefore by (5.5) one has that $\dim T_M \mathrm{Rep}_A^n = N + h_M^0$ where N is locally constant. □

The two main examples of Calabi–Yau algebras under consideration fit into this picture.

Proposition 5.3. *Let S be a compact orientable surface S of genus g. The algebra $A_g = k[\pi_1(S)]$ admits a finite free resolution.*

Proof. A resolution $F_\bullet$ is provided by Davison in the proof of [14, Theorem 5.2.2.]:

$$0 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A_g \longrightarrow 0$$

where $F_i = A_g \otimes k^{d_i} \otimes A_g$ and d_i is the number of non-degenerated i-th dimensional simplices in a simplicial complex Δ homeomorphic to S. □

Since the F_i's are finitely generated and free, the functions $c^i = n^2 \dim_k F_i$ are constant.

Proposition 5.4. *The preprojective algebra of a non-Dynkin quiver $A = \Pi(Q)$ admits a resolution by finitely-generated projective A^e-modules $F_\bullet$ such that the functions c^i are constant.*

Proof. Here we follow [6]. Consider the standard projective resolution given in [6, Remark 4.5]

$$\bigoplus_{i\in Q_0} F_{ii} \longrightarrow \bigoplus_{(a,a^*)} F_{t(a)h(a)} \oplus F_{t(a^*)h(a^*)} \longrightarrow \bigoplus_{i\in Q_0} F_{ii} \xrightarrow{m} A$$

where $F_{ij} := Ai \otimes jA$ and $i, j \in Q_0$. The crucial observation now is that, if $M \in \mathrm{Rep}^n_A(k)$, then

$$\dim_k(\mathrm{Hom}_{A^e}(F_{ij}, \mathrm{End}_k(M))) = \dim_k(i\,\mathrm{End}_k(M)\,j) = \alpha_i\alpha_j.$$

This means that the dimensions $\dim_k(\mathrm{Hom}_{A^e}(F_{ij}, \mathrm{End}_k(M))$ are constant. □

5.1. Proof of Theorem 1.1. We start with two preliminary lemmas.

Let $A \in \mathcal{N}_k$. We say that a (left) ideal I of A is of codimension n if $\dim_k(A/I) = n$.

Recall (see (5.3)) that $h^i_M = \dim_k \mathrm{Ext}^i_A(M, M)$ for $M \in \mathrm{Rep}^n_A(k)$.

Lemma 5.5. *Let A be an associative k-algebra. A codimension n ideal $I \subset A$ is two-sided if and only if $h^0_{A/I} = n$.*

Proof. If I is two-sided, then $h^0_{A/I} = n$. Let now I be such that $h^0_{A/I} = n$. We have $\mathrm{End}_A(A/I) = \mathcal{I}/I$ where $\mathcal{I}$ is the idealizer of I, that is the subalgebra of A which is maximal among those algebras where I is two-sided. Therefore, $I \subset \mathcal{I} \subset A$ and $\mathcal{I}/I \cong A/I$. This implies $\mathcal{I} = A$ and I two-sided. □

Lemma 5.6. *If $A = A_g$ and $g > 1$, for all $n \geq 1$ there is $I \in \mathrm{Hilb}^n_{A_g}(\mathbb{C})$ which is a two-sided ideal.*

Proof. Recall that (see (4.1))

$$A_g = \mathbb{C}[< X_1, Y_1, \ldots, X_g, Y_g | X_1Y_1X_1^{-1}Y_1^{-1} \cdots X_gY_gX_g^{-1}Y_g^{-1} = 1 >]$$

so that $A_1 \cong \mathbb{C}[x, y]$. Let J be a $\mathbb{C}$-point in $\mathrm{Hilb}^n_{A_1}$. Consider the following composition

$$A_g \xrightarrow{\alpha} A_1 \xrightarrow{\pi} A_1/J$$

where α maps X_1 to x, Y_1 to y and all the others X_i and Y_i to 1. The map π is the quotient map. Let I be the kernel of the composition $\pi\alpha$, which is onto, then $A_g/I \cong A_1/J \cong \mathbb{C}^n$, since $J \in \mathrm{Hilb}^n_{A_1}(\mathbb{C})$. Therefore I is a two-sided ideal in $\mathrm{Hilb}^n_{A_g}(\mathbb{C})$. □

Proof of Theorem 1.1. In [36] it is shown that $\mathrm{Rep}^n_{A_g}$ is irreducible for every n of dimension

$$\dim \mathrm{Rep}^n_{A_g} = \begin{cases} (2g-1)n^2+1 & \text{if } g>1 \\ n^2+n & \text{if } g=1 \end{cases}$$

We know that $\mathrm{U}^n_{A_g}$ is open in $\mathrm{Rep}^n_{A_g} \times \mathbb{A}^n_k$ and hence also irreducible with dimension

$$(2g-1)n^2+1+n.$$

By Theorem 3.13, $\mathrm{Hilb}^n_{A_g}$ must also be irreducible with dimension

$$(2g-1)n^2+1+n-n^2 = (2g-2)n^2+n+1,$$

if $g>1$. The argument in [36, p. 25] shows that there exist simple representations of A_g for any dimension when $g>1$. Since simple modules are cyclic, to a simple n-dimensional representation of A_g corresponds a point in the image of $\mathrm{U}^n_{A_g}$ by the forgetful map $\mathrm{Rep}_{A_g} \times \mathbb{A}^n_k \longrightarrow \mathrm{Rep}_{A_g}$. The module M corresponding to such a point has $\dim_k(\mathrm{End}_{A_g}(M))$ minimal. We also know by Lemma 5.5 that $\mathrm{Hilb}^n_{A_g}$ contains k-points corresponding to two-sided ideals, so these points correspond to modules with $\dim_k(\mathrm{End}_{A_g}(M)) = n$. Thus, we can say that the image of $\mathrm{U}^n_{A_g}$ by the forgetful map contains k-points where the dimension of the tangent space is different, since A_g verifies hypotheses of Theorem 5.2. We conclude that $\mathrm{U}^n_{A_g}$ is not smooth, or equivalently, $\mathrm{Hilb}^n_{A_g}$ is not smooth. □

5.2. Proof of Theorem 1.2. Through this section $A = \Pi(Q)$, the preprojective algebra of a connected quiver Q, see Example 4.4.3.

The situation for $\mathrm{Hilb}^\alpha_{\Pi(Q)}$ is a bit more complicated.

First of all $\mathrm{Hilb}^\alpha_{\Pi(Q)}$ might not be irreducible. Take for instance $Q = \circ \to \circ$ with dimension vector $(1,1)$. In this case $\mathrm{Rep}^\alpha_{\Pi(Q)}$ is the union of 2 intersecting lines and all representations are cyclic, so $\mathrm{Hilb}^\alpha_{\Pi(Q)}$ is not smooth. If we take the dimension vector $(1,2)$ then $\mathrm{Rep}^\alpha_{\Pi(Q)}$ is the union of two planes intersecting in the zero representation, so it is still not irreducible. The zero representation is not cyclic so $U^\alpha_{\Pi(Q)}$ is smooth and hence so is $\mathrm{Hilb}^\alpha_{\Pi(Q)}$. If we take the dimension vector $(1,3)$, then $\mathrm{Rep}^\alpha_{\Pi(Q)}$ is the union of two 3-dimensional spaces intersecting in the zero representation, but now no representation is cyclic so $\mathrm{Hilb}^\alpha_{\Pi(Q)}$ is empty.

To avoid these pathologies, we will restrict to the case where $\mathrm{Rep}^\alpha_{\Pi(Q)}$ contains simple representations. The quivers and dimension vectors for which there exist simple representations have been characterized by Crawley-Boevey in [12]. The main ingredient to state the results is a quadratic form on the space of dimension vectors:

$$p(\alpha) = 1 - \alpha \cdot \alpha + \sum_{a \in Q_1} \alpha_{h(a)}\alpha_{t(a)}.$$

We also need the notion of a positive root. This is a dimension vector α for which $\mathrm{Rep}^{\alpha}_{Q}$ has indecomponible representations (we use the shorthand $\mathrm{Rep}^{\alpha}_{Q}$ for $\mathrm{Rep}^{\alpha}_{kQ}$). If α is a positive root then $p(\alpha)$ equals the dimension of $\mathrm{Rep}^{\alpha}_{Q}/\!\!/\mathrm{GL}_{\alpha}$. We call a positive root real if $p(\alpha) = 0$ and imaginary if $p(\alpha) > 0$. In particular the elementary dimension vector ϵ_v, which is one in vertex v and zero in the other vertices, is a real positive root if v has no loops and an imaginary positive root if it has loops. Dynkin quivers have no imaginary roots, extended Dynkin quivers have precisely one imaginary root δ, with $p(\delta) = 1$. If Q is not Dynkin or extended Dynkin we will call it wild.

Theorem 5.7 (Crawley-Boevey [12, Theorem 1.2]).

- $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *contains simple representations if and only if* α *is a positive root and* $p(\alpha) > \sum_1^r p(\beta^i)$ *for each decomposition of* $\alpha = \beta^1 + \cdots + \beta^r$ *into* $r \geq 2$ *positive roots.*
- *If* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *contains simple representations, then* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *is an irreducible variety of dimension* $2p(\alpha) + \alpha \cdot \alpha - 1$ *and the quotient variety* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}/\!\!/\mathrm{GL}_{\alpha}$ *has dimension* $2p(\alpha)$.

We say that $\alpha \geq \beta$, if $\alpha_v \geq \beta_v \ \forall v \in Q_0$. In [31] Le Bruyn observes the following interesting property of dimension vectors of simples.

Lemma 5.8. *If* α *is the dimension vector of a simple representation of* $\Pi(Q)$ *then there is an extended Dynkin subquiver of* Q *with imaginary root* δ *such that* $\alpha \geq \delta$.

Remark 5.9. Note that combined with Crawley-Boevey's result, this implies that the quotient variety has dimension at least 4 unless Q is extended Dynkin. Indeed if $\alpha \neq \delta$, we can use the elementary dimension vectors ϵ_v to make a decomposition in positive roots $\alpha = \delta + \sum n_v \epsilon_v$ with $n_v = \alpha_v - \delta_v$ and we get

$$p(\alpha) > p(\delta) + \sum n_v p(\epsilon_v) \geq 1.$$

We will also need a local description of the quotient space of representations.

Theorem 5.10 (Crawley-Boevey [13]). *If* ξ *is a point in* $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *corresponding to a semisimple representation with decomposition* $S_1^{e_1} \oplus \cdots \oplus S_k^{e_k}$, *then there is a quiver* Q_L *and a* $\mathrm{Stab}_{\xi} = \mathrm{GL}_{\beta}$*-equivariant morphism* $\kappa : \mathrm{Rep}^{\beta}_{\Pi(Q_L)} \to \mathrm{Rep}^{\alpha}_{\Pi(Q)}$ *which maps* 0 *to* ξ. *The corresponding quotient map*

$$\mathrm{Rep}^{\beta}_{\Pi(Q_L)}/\!\!/\mathrm{GL}_{\beta} \to \mathrm{Rep}^{\alpha}_{\Pi(Q)}/\!\!/\mathrm{GL}_{\alpha}$$

is étale at 0. *The vertices of* Q_L *correspond to the simple factors in* ξ *and the dimension vector* β *assigns to each vertex the multiplicity of the corresponding simple.*

Remark 5.11. Theorem 5.10 means that if ζ' is a β-dimensional semisimple representation of $\Pi(Q')$ that is "close enough" to the 0, the corresponding

representation $\zeta' = \kappa(\zeta) \in \mathrm{Rep}^{\alpha}_{\Pi(Q)}$ is semisimple. The stabilizers of these two points are the same so the decomposition in simples has the same structure. To determine the dimensions of the simples of ζ one can look at the centralizer of Stab_ζ in GL_α:

$$C_{\mathrm{GL}_\alpha}\mathrm{Stab}_\zeta = \prod_i \mathrm{GL}_{\dim S_i}.$$

The dimension of each simple in ζ must be at least the dimension of the corresponding simple in ζ', because $\mathrm{Stab}_{\zeta'} = \mathrm{Stab}_\zeta$ and $\mathrm{GL}_\beta \subset \mathrm{GL}_\alpha$ so $C_{\mathrm{GL}_\beta}\mathrm{Stab}_{\zeta'} \subset C_{\mathrm{GL}_\alpha}\mathrm{Stab}_\zeta$.

Lemma 5.12. *If M is a semisimple representation of $\Pi(Q)$ with decomposition $S_1^{e_1} \oplus \cdots \oplus S_k^{e_k}$ then M is cyclic if and only if $e_i \leq \dim S_i$, $\forall i$.*

Proof. Because M is semisimple, the map $\rho_M : A \to \mathrm{End}_k(M)$ factorizes through the surjective map $A \to \oplus_i \mathrm{Mat}_{\dim S_i}(k)$. Using the idempotents $1\!\!1_{e_i}$ we can split up every cyclic vector v into cyclic vectors $1\!\!1_{e_i} v$ for $1\!\!1_{e_i} M$ and vice versa. Looking at the summands separately, this reduces the problem to showing that the $\mathrm{Mat}_d(k)$-representation $(k^d)^{\oplus e}$ is cyclic if and only if $e \leq d$. This condition is clearly necessary as otherwise $\dim_k \mathrm{Mat}_d(k) < \dim_k (k^d)^{\oplus e}$. It is also sufficient because we can take $b_1 \oplus \cdots \oplus b_e$ where $(b_i)_{1 \leq i \leq d}$ is the standard basis of k^d. □

Lemma 5.13. *If $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ contains simple representations and $\alpha \neq (1)$, then there exists a cyclic α-dimensional representation M with $\mathrm{End}_A(M) \neq k$.*

If, after deleting the zero vertices, Q is not extended Dynkin of type D_n or E_n then one can choose this representation to be semisimple.

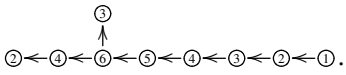
Proof. We assume that α is sincere in the sense that for all $v \in Q_0 : \alpha_v \neq 0$, otherwise we can delete the vertices with $\alpha_v = 0$.

We first do the one vertex case. If Q has 0 or 1 loop then the only dimension vector with simples is (1). If Q has more than 1 loop, then Crawley-Boevey's criterion implies that there are simple representations in every dimension vector. For $\alpha = (n)$ we can take the direct sum of n different 1-dimensional simple representations, which is cyclic by Lemma 5.12.

If Q has more than one vertex we have to distinguish between three cases.

(1) Q is Dynkin. In this case there are no dimension vectors with simple representations except the elementary ones.

(2) Q is extended Dynkin. In this case the only sincere dimension vector which has simple representations is the imaginary root. In each case we can find a cyclic representation which is not indecomposable. If $Q = \tilde{A}_n$ then the zero representation is cyclic because the dimension vector only contains ones. In the other cases assume that Q is oriented with arrows that move away from a chosen vertex with dimension 1 as illustrated below in the case of $\tilde{E}_8$:



We pick a representation which assigns to each arrow in Q a map of maximal rank except for terminal arrows, for which we take a matrix with rank equal to the terminal dimension -1. To the starred arrows we assign zero maps. This is a cyclic representation of $\Pi(Q)$: If we look at the dimension of the tail of the incoming nonstarred arrow in any given vertex, we see that it is at least one less than the dimension of this vertex. Therefore we can find vectors $\vec{w}_v \in vk^\alpha$ such that in each vertex v, the vectors $\vec{w}_v$ and $v\Pi(Q)_{\geq 1}v'\vec{w}_{v'}, v' \in Q_0$ generate vk^α. The sum $\sum_{v\in Q_0} w_v$ is a cyclic vector for this representation, but endomorphism ring of this representation contains $k \oplus k$ because there is a direct summand in the terminal vertex.

(3) Q is wild, so by Remark 5.9 $\dim \mathrm{Rep}^\alpha_{\Pi(Q)}/\!\!/\mathrm{GL}_\alpha \geq 4$. We work by induction on $|\alpha| = \sum_v \alpha_v$. If α only consists of ones then by Lemma 5.12 the zero representation is semisimple and cyclic so we are done.

If $\alpha_v > 1$ for some v, Lemma 5.8 shows that we can always find a subquiver of extended Dynkin type (or a 1 vertex 1 loop quiver, which is essentially $\tilde{A}_0$) such that α is bigger than the imaginary root δ. We can find a semisimple representation in $\rho \in \mathrm{Rep}^\alpha_Q$ which is the direct sum of a simple nonzero representation of the extended Dynkin subquiver, together with elementary simples with multiplicity $\alpha_v - \delta_v$ for each vertex v. By Theorem 5.10 there is a GL_β-equivariant morphism $\mathrm{Rep}^\beta_{\Pi(Q_L)} \to \mathrm{Rep}^\alpha_{\Pi(Q)}$ that maps 0 to ρ, which induces a morphism $\mathrm{Rep}^\beta_{\Pi(Q_L)}/\!\!/\mathrm{GL}_\beta \to \mathrm{Rep}^\alpha_{\Pi(Q)}/\!\!/\mathrm{GL}_\alpha$ that is étale at zero.

This implies that the dimension of $\mathrm{Rep}^\beta_{\Pi(Q_L)}/\!\!/\mathrm{GL}_\beta$ is the same as the dimension of $\dim \mathrm{Rep}^\alpha_{\Pi(Q)}/\!\!/\mathrm{GL}_\alpha$ and also that $\mathrm{Rep}^\beta_{\Pi(Q_L)}$ contains simples: just lift a simple "close enough" to ρ. So (Q_L, β) is again wild and $|\beta| < |\alpha|$.

By induction there is a semisimple cyclic representation $\xi \in \mathrm{Rep}_\beta \Pi(Q_L)$, which we can choose in the appropriate neighborhood of the zero representation because $\Pi(Q_L)$ is graded and hence $\mathrm{Rep}^\beta_{\Pi(Q_L)}$ has a k^*-action by scaling. Under the étale morphism, ξ corresponds to a semisimple point $\rho' \in \mathrm{Rep}^\alpha_{\Pi(Q)}$ which has the same stabilizer.

By Remark 5.11, the dimensions of the simples in the decomposition of ρ' are not smaller than those in the decomposition of ξ. Lemma 5.12 now implies that the representation ρ is also cyclic. □

Theorem 5.14. *Let $\Pi(Q)$ be the preprojective algebra of a quiver Q and let α be a dimension vector for which there exist simple representations. Then* $\mathrm{Hilb}^n_{\Pi(Q)}$ *is irreducible of dimension* $1 + 2\sum_{a\in Q_1} \alpha_{h(a)}\alpha_{t(a)} + \sum_{v\in Q_0}(\alpha_v - 2\alpha_v^2)$ *and it is smooth if and only if Q has one vertex and $\alpha = (1)$ (or equivalently all α-dimensional representations are simple).*

Proof. From Theorem 5.7 we know that in this case $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ is an irreducible variety with dimension $2p(\alpha)+\alpha\cdot\alpha-1$, where $p(\alpha)$ is the quadratic form we defined before. Using the fact that $\mathrm{Hilb}^{n}_{\Pi(Q)}$ is a quotient of an open subset of $\mathrm{Rep}^{\alpha}_{\Pi(Q)}\times k^{\alpha}$ with fibers of dimension $\alpha\cdot\alpha$, we arrive at the desired formula for the dimension. Unless $\alpha=(1)$ the previous lemma shows that we can always find a cyclic representation ρ with $\mathrm{End}(\rho)\neq k$. By Theorem 5.2 this representation will correspond to a singularity in the Hilbert scheme. □

The crucial element in the proof for preprojective algebras rests on the fact that one can describe the representation space around any semisimple point again as the representation space of a preprojective algebra. If we want to generalize our result to other Calabi–Yau algebras, we need to find a similar description. This will be done in the final part of the paper.

6. The local structure of representations spaces of 2-Calabi–Yau algebras

In this section we explain how the local structure of the representation space of a 2-Calabi–Yau algebra can be seen as the representation space of a preprojective algebra. This result enables us to show that the semisimple representations that correspond to smooth points in the representation space are precisely the simple points. Moreover we show that if a neighborhood of a semisimple contains simples and the dimension of the quotient space is bigger than 2, then we can also find non-simple semisimple cyclic representations. This implies that there is a singularity in the corresponding component of the Hilbert scheme.

The results described here follow from a combination of results by many authors. First we will explain the A_∞-perspective on deformation theory as developed by Kontsevich and Soibelman [27, 28] and apply it to representation theory. This point of view is also studied by Segal [38]. Then we will use results by Van den Bergh on complete Calabi–Yau algebras in [42] to show that locally 2-CY algebras can be seen as completed preprojective algebras. This observation is a generalization of a result by Crawley-Boevey in [13]. It also allows us to classify the semisimple representations that correspond to smooth points in the representation space of a Calabi–Yau algebra.

6.1. Deformation theory. We are going to reformulate some concepts in geometric representation theory to the setting of deformation theory. To do this we need to recall some basics about A_∞-algebras from [25] and [27].

Let ℓ be a finite dimensional semisimple algebra over k. An A_∞-algebra is a graded ℓ-bimodule B equipped with a collection of products $(\mu_i)_{i\geq 1}$, which are

ℓ-bimodule morphisms of degree $2-i$

$$\mu_i : \underbrace{B \otimes_\ell \cdots \otimes_\ell B}_{i \text{ factors}} \to B$$

subject to the relations[1]

$$[M_n] \sum_{u+v+j=n} \pm \mu_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) = 0.$$

Note that μ_1 has degree 1 and $[M_1]$ implies $\mu_1^2 = 0$, so B has the structure of a complex. Moreover if $\mu_i = 0$ for $i > 2$ we get a dg-algebra, so A_∞-algebras can be seen as generalizations of dg-algebras. If it is clear which product we are talking about we drop the index i.

Morphisms between two A_∞-algebras B and C are defined as collections of ℓ-bimodule morphisms $(F_i)_{i\geq 1}$ of degree $1-i$

$$F_i : B \otimes_\ell \cdots \otimes_\ell B \to C$$

subject to the relations

$$\sum_{u+v+j=n} \pm F_{u+v+1}(1^{\otimes u} \otimes \mu_j \otimes 1^{\otimes v}) + \sum_{i_1+\cdots+i_l=n} \pm \mu_l(F_{i_1} \otimes \cdots \otimes F_{i_l}) = 0.$$

The power of A_∞-structures lies in the fact that they can be transported over quasi-isomorphisms between two complexes. If B is an A_∞-algebra, C a complex of ℓ-bimodules and $\phi : B \to C$ a quasi-isomophism then we can find an A_∞-structure on C and a quasi-A_∞-isomorphism $F_\bullet : B \to C$ with $F_1 = \phi$.

An important result in the theory of A_∞-algebras is the minimal model theorem [25, 27, 28]:

Theorem 6.1. *Every A_∞-algebra is A_∞-isomorphic to the product of a minimal one (i.e. $\mu_1 = 0$) and a contractible one (i.e. $\mu_{>1} = 0$ and zero homology). Two A_∞-algebras are quasi-isomorphic if they have isomorphic minimal factors.*

Given an A_∞-algebra B we can define the Maurer–Cartan equation

$$\mu(x) + \mu(x,x) + \mu(x,x,x) + \cdots = 0$$

The standard way to make sense of this equation is to demand that $x \in B_1 \otimes \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal in $R = k[t]/(t^n)$ (or some other local artinian commutative ring $R = k \oplus \mathfrak{m}$) and to let R commute with the μ_i. The set of solutions will be denoted by $\mathrm{MC}(B)_\mathfrak{m}$ and as such $\mathrm{MC}(B)$ can be considered as a functor from local artinian rings to sets.[2]

[1] For the specific sign convention we refer to [25].

[2] In fact it is a functor to groupoids, because one can integrate the $B_0 \otimes \mathfrak{m}$-action on $\mathrm{MC}(B)_\mathfrak{m}$.

If B_0 and B_1 are finite dimensional we can also make sense of this by looking at the local ring

$$\widehat{\mathrm{MC}}(B) := k[\![B_1^*]\!]/\langle \xi\mu^1 + \xi\mu^2 + \xi\mu^3 + \cdots | \xi \in B_1^* \rangle$$

where $\xi\mu^k$ is interpreted as the homogeneous polynomial function that maps $x \in B_1$ to $\xi(\mu_k(x, \ldots, x))$. This ring can be seen as the complete local ring corresponding to the zero solution in the formal scheme of solutions to the Maurer–Cartan equation.

B_0 has an infinitesimal action on $\widehat{\mathrm{MC}}$

$$b \cdot \xi := \sum_{i=0}^{\infty} \pm \xi\mu_b^i$$

where $\xi\mu_b^k$ is interpreted as the element in $(B_1^*)^{\otimes k-1}$ that maps x to

$$\xi(\mu_k(b, \ldots, x) \pm \cdots \pm \mu_k(x, \ldots, b)).$$

We denote the ring of invariants of this action by

$$\widehat{\mathrm{MC}}^{\mathrm{inv}}(B) := \{ f \in \widehat{\mathrm{MC}}(B) | \forall b \in B_0 : b \cdot f = 0 \}.$$

If $F_\bullet : B \to C$ is an A_∞-isomorphism then the map

$$\phi_F : \widehat{\mathrm{MC}}(C) \to \widehat{\mathrm{MC}}(B) : \xi \mapsto \sum_{i=0}^{\infty} \xi F^i$$

is an isomophism which maps $\widehat{\mathrm{MC}}^{\mathrm{inv}}(C)$ to $\widehat{\mathrm{MC}}^{\mathrm{inv}}(B)$.

If an A_∞-algebra is a product of two subalgebras, the set of solutions to the Maurer–Cartan equations for an A_∞-algebra is the product of the set of solutions to the Maurer–Cartan equations of its two factors. Likewise, the corresponding local ring is the completed tensor product of the local rings of the two factors and the invariant ring is the completed tensor product of the two invariant rings.

If B is contractible then as vector spaces $B_0 \cong \ker \mu_1|_{B_1}$. As the higher products vanish $\widehat{\mathrm{MC}}(B) \cong k[\![B_0^*]\!]$ and the invariant ring is $\widehat{\mathrm{MC}}^{\mathrm{inv}}(B) = k$. Combined with the minimal model theorem this implies that quasi-isomorphic A_∞-algebras have isomorphic invariant rings.

6.2. Representation spaces. For $A = kQ/J$ a path algebra of a quiver with relations, we can describe the space Rep_A^α as a deformation problem. Fix an α-dimensional representation ρ and construct the following complex $R^\bullet$:

$$R^i = \mathrm{Hom}_{\ell e}(A \otimes_\ell \cdots \otimes_\ell A, \mathrm{Mat}_n(k))$$

with the following products

$$\begin{aligned}\mu_1 f(a_1,\dots,a_{i+1}) &= \rho(a_1)f(a_2,\dots,a_{i+1}) - f(a_1a_2,\dots,a_{i+1}) + \cdots \\ &\quad \cdots \pm f(a_1,\dots,a_ia_{i+1}) \mp f(a_1,\dots,a_i)\rho(a_{i+1}) \\ \mu_2(f,g)(a_1,\dots,a_{i+j}) &= f(a_1,\dots,a_i)g(a_{i+1},\dots,a_{i+j})\end{aligned}$$

The Maurer–Cartan equation for this algebra reduces to finding ℓ-linear maps $f : A \to \mathrm{Mat}_n(k) \otimes \mathfrak{m}$ for which

$$\rho(a)f(b) - f(ab) + f(a)\rho(b) + f(a)f(b) = 0,$$

which is precisely the condition that $\rho + f$ is a α-dimensional representation. So the map $f \mapsto (\rho(a) + f(a))_{a\in Q_1}$ maps $MC(R)_{\mathfrak{m}}$ bijectively to the $k \oplus \mathfrak{m}$-points that lie over the point $\rho \in \mathrm{Rep}_\alpha A$. In this way $R^\bullet$ captures the local information of the representation scheme Rep_A^α around ρ.

In fact, we can interpret the complex $R^\bullet$ as

$$\mathrm{Hom}_{A^e}(\mathcal{A}_\bullet, M \otimes M^\vee)$$

where $\mathcal{A}_\bullet$ is the bar resolution of A and M is the A-module corresponding to the representation ρ. Therefore $R^\bullet$ is quasi-isomorphic to the complex $\mathrm{Ext}_A^\bullet(\rho,\rho)$ with a corresponding A_∞-structure. For more information on this we refer to [32].

6.3. Koszul Duality. In general if $A = kQ/J$ and none of the relations r_i contains paths of length ≤ 1, we can consider the zero representation corresponding to the module $\underline{\ell} := A/Q_1A$.

Definition 6.2. The Koszul dual of A is

$$A^! := \mathrm{Ext}_A^\bullet(\underline{\ell},\underline{\ell}).$$

As explained in the previous section this space has the structure of an A_∞-algebra over ℓ coming from the isomorphism

$$\mathrm{Ext}_A^\bullet(\underline{\ell},\underline{\ell}) \cong H(\mathrm{Hom}_{A^e}(\mathcal{A}_\bullet, \underline{\ell} \otimes \underline{\ell}^\vee).$$

The ordinary product in this A_∞-structure is equal to the standard Yoneda product [32].

Note that $\mathrm{Ext}_A^0(\underline{\ell},\underline{\ell}) = \ell$ and as a ℓ-bimodule $\mathrm{Ext}_A^1(\underline{\ell},\underline{\ell})$ is spanned by elements $[a]$ corresponding to the arrows while $\mathrm{Ext}^2(\underline{\ell},\underline{\ell})$ is spanned by elements $[r_i]$ corresponding to a minimal set of relations. The complete structure of the A_∞-products can become very complicated but one has the following identity [38]

$$\mu([a_1],\dots,[a_s]) = \sum_i c_i[r_i] \tag{6.1}$$

where c_i is the coefficient of the path $a_1\cdots a_k$ in r_i.

For every dimension vector α we also have a zero representation $\rho_0 = \underline{\ell} \otimes_\ell k^\alpha$ and in that case

$$\mathrm{Ext}_A^\bullet(\rho_0, \rho_0) = \mathrm{Ext}_A^\bullet(\underline{\ell} \otimes_\ell k^\alpha, \underline{\ell} \otimes_\ell k^\alpha) = k^\alpha \otimes_\ell \mathrm{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}) \otimes_\ell k^\alpha.$$

If $\{b_i | i \in \mathcal{I}\}$ is a graded ℓ-basis for $A^!$, then elements in $\mathrm{Ext}_A^\bullet(\rho_0, \rho_0)$ can be seen as linear combinations $\sum B_i b_i$ where B_i is an $\alpha_{h(b_i)} \times \alpha_{t(b_i)}$-matrix. The higher multiplications are matrix-versions of the original ones:

$$\mu(B_1 b_1, \ldots, B_i b_i) = B_1 \cdots B_i \mu(b_1, \ldots, b_i).$$

In combination with (6.1) it is easy to see that, just as expected, $\sum A_i[a_i] \in \mathrm{Ext}_A^1(\rho_0, \rho_0) \otimes \mathfrak{m}$ is a solution to the Maurer–Cartan equation if and only if the matrices A_i satisfy the relations. From the point of view of local rings we see that

$$\widehat{\mathrm{MC}}(\mathrm{Ext}_A^\bullet(\rho_0, \rho_0)) \cong \widehat{k[\mathrm{Rep}_A^\alpha]}_{\rho_0}.$$

It can also easily be checked that

$$\widehat{\mathrm{MC}}^{\mathrm{inv}}(\mathrm{Ext}_A^\bullet(\rho_0, \rho_0)) \cong \widehat{k[\mathrm{Rep}_A^\alpha]}_{\rho_0}^{\mathrm{GL}_\alpha}.$$

Now we return to the general situation and look at a semisimple α-dimensional representation ρ with decomposition into simple representations $\rho = \sigma_1^{\oplus e_1} \oplus \cdots \oplus \sigma_m^{\oplus e_m}$. We can rewrite

$$\mathrm{Ext}_A(\rho, \rho) = \bigoplus_{i,j=1}^{m} \bigoplus_{r=1}^{e_i} \bigoplus_{s=1}^{e_j} \mathrm{Ext}(\sigma_i, \sigma_j) = k^\epsilon \otimes_l \mathrm{Ext}(\underline{\rho}, \underline{\rho}) \otimes_l k^\epsilon.$$

In this notation $\underline{\rho}$ is the representation that contains one copy of each simple σ_i, $l = k^m$ is the semisimple algebra $\mathrm{Ext}_A^0(\underline{\rho}, \underline{\rho})$ and k^ϵ is the module over this algebra with dimension vector $\epsilon = (e_1, \ldots, e_m)$. If we can find an l-algebra B such that $B^! = \mathrm{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$, then we can say that locally (up to a product with an affine space) the space of α-dimensional representations of A around ρ looks like the space of ϵ-dimensional representations of B around the zero representation.

How do we find B? Because $B^! = \mathrm{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$, the algebra B should be the Koszul dual of $E := \mathrm{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$, so we need to take a look at the construction of the Koszul dual of an A_∞-algebra. We restrict to the relevant case where $E = l \oplus V$ is a finite dimensional augmented l-algebra with an A_∞-structure on E such that $\mu_1(l) = 0$, μ_2 is the ordinary multiplication and $\mu_n(\ldots, l, \ldots) = 0$ for all $n > 2$. For this cases we will follow the construction in [42, Appendix A].

As is explained in [42] the Koszul dual of a finite dimensional algebra should be a complete algebra. First we construct the completed tensor-algebra $\widehat{T_l V^*}$ with $V^* := \mathrm{Hom}(V, k)$. Using a graded l-basis $\mathcal{B}$ for V, this algebra consist of all formal

sums of words $b_1^* \otimes \cdots \otimes b_s^*$ with $b_1, \ldots, b_s \in \mathcal{B}$. We give this algebra a grading by setting $\deg b^* = 1 - \deg b$ for all $b \in \mathcal{B}$.

We turn this completed tensor-algebra into a dg-algebra by adding a differential. Using the Leibniz rule, linearity and completion, the differential is completely defined if we give expressions for db^* with $b \in \mathcal{B}$. We set

$$db^* = \sum_{s=1}^{\infty} \sum_{b_1,\ldots,b_s \in \mathcal{B}} c_{b_1,\ldots,b_s} b_1^* \otimes \cdots \otimes b_s^*,$$

where $c_{b_1,\ldots,b_s}$ is the coefficient of b in $\mu(b_1, \ldots, b_s)$. We will call the dg-algebra $(\widehat{T_l V^*}, d)$ the Koszul dual of (E, μ) and denote it by $E^!$. Note that if (E, μ) is an ordinary algebra, $(\widehat{T_l V^*}, d)$ can be seen as the Koszul complex used to calculate $\mathrm{Ext}_E(\underline{l}, \underline{l})$, so $E^!$ is quasi-isomorphic to the classical Koszul dual of E. If $E = \mathrm{Ext}_A^\bullet(\underline{\ell}, \underline{\ell}) = A^!$ then $E^!$ is formal and its homology is the completion of A by path-length concentrated in degree 0. In other words $\hat{A}$ is the minimal model of $E^!$ (see [42, Proposition A.5.4]).

In general $E^!$ is not formal, but the degree zero part of its homology, $H_0(E^!)$, is enough to construct the Maurer–Cartan equation for E. Indeed, the Maurer–Cartan equation for E only depends on $\mu_i|_{E_1^{\otimes i}}$. In $E^!$ these are encoded in the map $d : E^!_{-1} \to E^!_0$.

Note that because all degrees in $E = \mathrm{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$ are nonnegative, the degrees in $E^!$ are nonpositive. The degree zero part of $E^!$ is the completed tensor algebra $\widehat{T_l E_1^*}$, which can be seen as a completed path algebra of a quiver Q_{loc} with m vertices and $\dim i E_1^* j = \dim \mathrm{Ext}_A^1(\sigma_i, \sigma_j)$ arrows from i to j. This quiver is called the *local quiver of* ρ. $E^!_{-1} = \widehat{k Q_{\mathrm{loc}}} \otimes_l E_2^* \otimes_l \widehat{k Q_{\mathrm{loc}}}$ and the image of $d|_{E^!_{-1}}$ is the $\widehat{k Q_{\mathrm{loc}}}$-ideal generated by the ds_i where the s_i form an l-basis for E_2^*. Hence, $H_0(E^!)$ is the completed path algebra of the quiver Q_{loc} with relations ds_i and

$$\mathrm{Ext}^i_{H_0(E^!)}(l, l) = E_i \text{ for } i \leq 2 \text{ and } \mu_n\big|_{\left(\mathrm{Ext}^1_{H_0(E^!)}(l,l)\right)^{\otimes n}} = \mu_n|_{E_1^{\otimes n}}.$$

This allows us to conclude

Theorem 6.3. *Let A be a finitely presented algebra. If ρ is an α-dimensional semisimple representation of A with decomposition $\rho = \sigma_1^{\oplus e_1} \oplus \cdots \oplus \sigma_m^{\oplus e_m}$ then the local structure of the representation space around ρ is the same (up to a product with an affine space) as the local structure of the representation space around the ϵ-dimensional zero representation of $H_0(E^!)$ with $E = \mathrm{Ext}_A^\bullet(\underline{\rho}, \underline{\rho})$ and $\epsilon = (e_1, \ldots, e_n)$.*

If $H_0(E^!)$ is the completion of a path algebra with relations L we can write

$$\widehat{k[\mathrm{Rep}_A^\alpha]}_\rho \cong \widehat{k[\mathrm{Rep}_L^\epsilon]}_{\rho_0} \otimes k[\![X_1, \ldots, X_s]\!] \text{ and } (\widehat{k[\mathrm{Rep}_A^\alpha]^{\mathrm{GL}_\alpha}})_\rho \cong (\widehat{k[\mathrm{Rep}_L^\epsilon]^{\mathrm{GL}_\epsilon}})_{\rho_0}.$$

Remark 6.4. The number s equals the difference $\dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_\epsilon = \alpha\cdot\alpha - \epsilon\cdot\epsilon$ and we can also identify $\widehat{k[\mathrm{Rep}_L^\epsilon]}_{\rho_0} \otimes k[\![X_1,\ldots,X_s]\!]$ with

$$\widehat{k[\mathrm{Rep}_L^\epsilon \times_{\mathrm{GL}_\epsilon} \mathrm{GL}_\alpha]}_{(\rho_0,1)}.$$

Remark 6.5. If A is hereditary then $\mathrm{Ext}_A^{\geq 2}(\rho,\rho) = 0$ and the Maurer–Cartan equation becomes trivial. The algebra $H_0(E^!)$ is equal to $E^!$ and is just the completed path algebra of the local quiver without any relations. Hence, locally the representation space of an hereditary algebra looks like the representation space of a quiver without relations. This result is an analogue of the local quiver theorem by Le Bruyn in [30]. We will now have a look at generalizations of this result to the 2-CY case.

6.4. Generalizations to 2-Calabi–Yau algebras. Suppose for now that A is 2-CY and M is a semisimple A-module with $\mathrm{End}_A(M) = l = k^m$. In this case $\mathrm{Ext}_A^1(M,M)$ has a nondegenerate antisymmetric l-bilinear form $\langle f,g\rangle := \mathrm{Tr}_M(fg)$ and hence we can find a symplectic l-basis of the form $\{[a_i],[a_i]^*|i\in\mathcal{I}\}$ such that $\langle[a_i],[a_j]\rangle = 0$, $\langle[a_i]^*,[a_j]^*\rangle = 0$ and $\langle[a_i],[a_j]^*\rangle = \delta_{ij}$. Similarly $\mathrm{Ext}^0(M,M)$ is dual to $\mathrm{Ext}^2(M,M)$ so each "vertex" $[v]\in l$ has a dual element $[v^*]$ and we have $[a_i][a_i^*] = [v^*]$ and $[a_i^*][a_i] = -[w^*]$ for some v and w which we can consider as the head and tail of a_i in the local quiver.

If we take the Koszul dual of $\mathrm{Ext}_A^\bullet(M,M)$, it is the completed path algebra of the local quiver Q_{loc} with an extra loop v^* in every vertex v. If we put $z = \sum_{v\in Q_0} v^*$ then we get

$$dz = \sum_{a\in Q_1} aa^* - a^*a + h.o.t.$$

Following the same reasoning as in the proof of Theorem 11.2.1 in [42] one can show that, up to a change of variables, these higher order terms vanish. This implies that

$$H_0(\mathrm{Ext}_A^\bullet(M,M)^!) \cong \widehat{kQ_{\mathrm{loc}}}/\langle dz\rangle \cong \widehat{kQ_{\mathrm{loc}}}/\langle \sum_{a\in Q_1} aa^* - a^*a\rangle.$$

This last algebra is the completed preprojective algebra, so in this case $L = \Pi(Q_L)$ for some quiver Q_L, for which Q_{loc} is the double.

To summarize

Theorem 6.6. *If A is a 2-CY and M is a semisimple A-module, then the algebra $H_0(\mathrm{Ext}_A^\bullet(M,M)^!)$ is isomorphic to a preprojective algebra.*

This means that locally representation spaces of 2-CY algebras look like preprojective algebras around the zero representation. This result can be seen as a generalization of Theorem 5.10.

To solve the question which semisimple representations are smooth, we need to classify the local quivers and dimension vectors for which the zero representation

of the preprojective algebra is smooth. Note that by construction such a dimension vector is sincere, i.e. $\forall v \in Q_0 : \alpha_v \neq 0$.

Theorem 6.7. *The only quivers and sincere dimension vectors for which* $\mathrm{Rep}^{\alpha}_{\Pi}$ *is smooth at the zero representation are disjoint unions of quivers with one vertex and an arbitrary number of loops and dimension vector* 1*, or quivers with one vertex and no loops and arbitrary dimension vector.*

Proof. First note that if the quiver is a disjoint union of two subquivers, the preprojective algebra is the direct sum of two smaller preprojective algebras and the representation space is the product of the corresponding representation spaces of these smaller algebras. So we can assume that Q is connected.

The tangent space to the zero ρ_0 representation in $\mathrm{Rep}^{\alpha}_{\Pi(Q)}$ is equal to $\mathrm{Rep}^{\alpha}_{Q}$ because the derivative

$$\sum [\rho(a), \rho_0(a^*)] + [\rho_0(a), \rho(a^*)] = \sum [\rho(a), 0] + [0, \rho(a)]$$

is identical to zero. Therefore the zero representation is smooth if and only if $\mathrm{Rep}^{\alpha}_{\Pi(Q)} = \mathrm{Rep}^{\alpha}_{Q}$. This means that the relation $\sum [\rho(a), \rho(a^*)] = 0$ must be identical to zero. This only happens when all arrows are loops and the dimension in the vertex is 1 or there are no arrows. □

Corollary 6.8. *Let* A *be a* 2*-CY algebra and let* $\rho \in \mathrm{Rep}^{\alpha}_{A}$ *be semisimple. Then* ρ *is smooth in* $\mathrm{Rep}^{\alpha}_{A}$ *if it is a direct sum of simples without extensions between them, where a simple can occur with higher multiplicity if it has no self-extensions.*

If ρ *has simple representations in its neighborhood then* ρ *itself must be simple.*

Finally we need to look at cyclic representations.

Lemma 6.9. *Let* A *be a* 2*-CY algebra and let* ρ *be a non-simple semisimple representation. If* X *is a component of* $\mathrm{Rep}^{\alpha}_{A}$ *containing* ρ *such that*

- *there are simples in* X*, and*
- $\dim X /\!\!/ \mathrm{GL}_{\alpha} > 2$*,*

then this component contains a cyclic non-simple semisimple representation.

Proof. We will look at the representation space $\mathrm{Rep}^{\epsilon}_{L} /\!\!/ \mathrm{GL}_{\epsilon}$, corresponding to the representation ρ. Artin's approximation theorem applied to the isomorphism

$$\widehat{k[\mathrm{Rep}^{\alpha}_{A}]^{\mathrm{GL}_{\alpha}}}_{\rho} \cong \widehat{k[\mathrm{Rep}^{\epsilon}_{L}]^{\mathrm{GL}_{\epsilon}}}_{\rho_0}$$

implies that there is a diagram of étale covers

$$\mathrm{Rep}^{\alpha}_{A} /\!\!/ \mathrm{GL}_{\alpha} \leftarrow U \rightarrow \mathrm{Rep}^{\epsilon}_{L} /\!\!/ \mathrm{GL}_{\epsilon}.$$

Pulling back and pushing forward we can find a semisimple representation $\tilde{\sigma}$ of A for every semisimple representation σ that is close enough to the zero representation $\rho_0 \in \mathrm{Rep}^{\epsilon}_{L} /\!\!/ \mathrm{GL}_{\epsilon}$. The representation $\tilde{\sigma}$ will be simple if and only if σ is simple.

Because A is 2-CY the local algebra L is a preprojective algebra and $\epsilon \neq 1$ because ρ is semisimple but not simple. Furthermore $\mathrm{Rep}^{\epsilon}_{L}/\!\!/\mathrm{GL}_{\epsilon}$ contains simples because X does and L is not the preprojective algebra of an extended Dynkin because $\mathrm{Rep}^{\epsilon}_{L}/\!\!/\mathrm{GL}_{\epsilon} = \dim X/\!\!/\mathrm{GL}_{\alpha} > 2$. By Lemma 5.13 we can find a semisimple cyclic representation σ, corresponding to a non-smooth point in $\mathrm{Rep}^{\epsilon}_{L}/\!\!/\mathrm{GL}_{\epsilon}$. We can choose σ in any neighborhood of the zero representation by rescaling. By Remark 5.11, the dimensions of the simple factors of its counterpart $\tilde{\sigma} \in \mathrm{Rep}^{\alpha}_{A}/\!\!/\mathrm{GL}_{\alpha}$ are at least those of σ, so Lemma 5.12 implies that $\tilde{\sigma}$ is cyclic. □

Proof of Theorem 1.3. If all representations in the component are simple, all local algebras are preprojective algebras over quivers with one vertex and with dimension vector (1). This implies that $\mathrm{Rep}^{\epsilon}_{L}/\!\!/\mathrm{GL}_{\epsilon}$ is smooth for all representations in the component and hence both the component and the Hilbert scheme are smooth.

If the component contains a non-simple semisimple representation, Lemma 6.9 implies that we can find a cyclic semisimple non-simple representation. By Theorem 6.3 this representation corresponds to a non-smooth point in $\mathrm{Rep}^{\alpha}_{A}$ and because it is cyclic also to a non-smooth point in $\mathrm{Hilb}^{\alpha}_{A}$. □

To illustrate this theorem we end with 3 examples.

Example 6.10. Let A_g be the fundamental group algebra of a compact orientable surface with genus $g > 1$. On this algebra we have an action of the group $G = \mathbb{Z}_2^{2g}$ which maps each generator X_i, Y_j to $\pm X_i, \pm Y_i$. Because these transformations leave the relation $\prod_i X_i Y_i X_i^{-1} Y_i^{-1} - 1$ invariant, the skew group ring $A_g \rtimes G$ will be 2-CY.

It can be seen as the quotient of the path algebra of a quiver with $2g$ vertices. The vertices are connected to each other with arrows coming from the X_i, Y_i. Because these are invertible, any representation of $A_g \rtimes G$ will have a dimension vector which assigns the same dimension to every vertex.

If $n = 2g$ all vertices have dimension 1 and all arrows must be represented by invertible numbers. This implies that the space $\mathrm{Rep}_n A_g \rtimes G/\!\!/\mathrm{GL}_n$ only has simple representations and its dimension is $2g$, so its Hilbert scheme is smooth. If $n = 2gm$ with $m > 1$, there are nonsimple representations which are direct sums of simples with dimension $2g$, so these Hilbert schemes are not smooth.

Example 6.11. Let $\mathcal{K}$ be an affine part of a 2-CY variety, such as the product of 2 elliptic curves, an abelian or a K3 surface. The coordinate ring $R = k[\mathcal{K}]$ is a 2-CY-algebra and by Corollary 3.6.6 of [20] it can be written as $D/\langle w \rangle$ where D is a formally smooth algebra. Furthermore there is a noncommutative symplectic form $\omega \in (\Omega^2 D)_{cyc}$ such that $dw = i_\delta \omega$ (where δ is the standard derivation $\delta(a) = 1 \otimes a - a \otimes 1$).

Now consider the free product of m copies of D: $\tilde{D} = D_1 * \cdots * D_m$ and let $\tilde{w} = w_1 + \cdots + w_m$ be the sum of the correponding m copies of w. Again we have a noncommutative symplectic form $\tilde{\omega} = \omega_1 + \cdots + \omega_m$ and $d\tilde{w} = i_\delta \tilde{\omega}$. Theorem 3.6.4

of [20] implies that $A = \tilde{D}/\langle\tilde{w}\rangle$ is 2-CY. The dimension of $\mathrm{Rep}_n A/\!\!/\mathrm{GL}_n$ is at least $2kn$ because for each k-tuple of points in $\mathcal{K}$ we can make a 1-dimensional representation of $\tilde{D}$ that factors through A. All the 1-dimensional representations are clearly simple so the Hilbert scheme for $n = 1$ is smooth. If $n > 1$ the space $\mathrm{Rep}_n A$ contains nonsimple representations, so the Hilbert scheme of A is only smooth for $n = 1$.

Example 6.12. The main ingredient in the proof of Theorem 1.3 is that all local quivers are wild and hence they have cyclic semisimple nonsimple representations. Because of Lemma 5.13, the proof also works in the case that some of the local quivers are extended Dynkin of type A_n. As this is the only extended Dynkin quiver with root $(1, \ldots, 1)$, this means that if the component of $\rho \in \mathrm{Rep}_n A/\!\!/\mathrm{GL}_n$ is two-dimensional and has a nonsimple semisimple multiplicity-free representation, then it is also not smooth.

We can illustrate this with a variation on the McKay correspondence. Consider the elliptic curve $\mathcal{C}$ with coordinate ring $k[X, Y]/(Y^2 - X^3 - 1)$. The product $\mathcal{C} \times \mathcal{C}$ has coordinate ring

$$R = k[X_+, Y_+, X_-, Y_-]/(Y_+^2 - X_+^3 - 1, Y_-^2 - X_-^3 - 1)$$

and on this ring we have an action of the group $\mathbb{Z}_6$ where the generator acts by

$$X_\pm \mapsto e^{\pm 2\pi i/3} X_\pm \quad \text{and} \quad Y_\pm \mapsto -Y_\pm.$$

Because this action preserves the volume form, the skew group ring $R \rtimes G$ is a 2-CY algebra. The quotient $\mathrm{Rep}_n R \rtimes G/\!\!/\mathrm{GL}_n$ for $n = 6$ will contain a component that is isomorphic to $\mathcal{C} \times \mathcal{C}/\!\!/G$. Some points in $\mathcal{C} \times \mathcal{C}$ have a nontrivial stabilizer (e.g. $(0, -1, 0, -1)$) and therefore there are representations of $R \rtimes G$ that are not simple. If $\mathfrak{p} \lhd R$ is a point with a nontrivial stabilizer G_p then the corresponding representation $\rho \in \mathrm{Rep}_n R \rtimes G/\!\!/\mathrm{GL}_n$ is $R/\mathfrak{p} \rtimes kG$ and splits in G_p different components parametrized by the characters of G_p. This implies that ρ is multiplicity-free. From the discussion above we can conclude that the corresponding component of the Hilbert scheme is not smooth.

Acknowledgements. We would like to thank Corrado De Concini for sharing his ideas with us and for proposing us this interesting question. We would also like to thank Claudio Procesi for many interesting discussions.

References

[1] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier*, **59** (2009), no. 6, 2525–2590. Zbl 1239.16011 MR 2640929

[2] C. Amiot, O. Iyama and I. Reiten, Stable categories of Cohen–Macaulay modules and cluster categories, *Amer. J. Math.*, **137** (2015), no. 3, 813–857. Zbl 06458529 MR 3357123

[3] A. Ardizzoni, F. Galluzzi and F. Vaccarino, A new family of algebras whose representation schemes are smooth, *Ann. Inst. Fourier (Grenoble)*, **66** (2016), no. 3, 1261–1277.

[4] M. Artin, Brauer–Severi Varieties, in *Brauer Groups in Ring Theory and Algebraic Geometry, (Wilrijk, 1981)*, 194–210, Lect. Notes Math., 917, Springer, Berlin-New York, 1982. Zbl 0536.14006 MR 0657430

[5] R. Berger and R. Taillefer, Poincaré–Birkhoff–Witt deformations of Calabi–Yau algebra, *J. Noncommut. Geom.*, **1** (2007), 241Ű–270. Zbl 1161.16022 MR 2308306

[6] R. Bocklandt, Graded Calabi Yau algebras of dimension 3, *Journal of Pure and Applied Algebra*, **212** (2008), no. 1, 14–32. Zbl 1132.16017 MR 2355031

[7] R. Bocklandt, *Noncommutative Tangent Cones and Calabi Yau Algebras*, preprint, 2007. arXiv:0711.0179

[8] R. Bocklandt and L. Le Bruyn, Necklace Lie algebras and noncommutative symplectic geometry, *Math. Z.*, **240** (2002), no. 1, 141–167. Zbl 1113.16019 MR 1906711

[9] K. Bongartz, A geometric version of the Morita equivalence, *J. Algebra*, **139** (1991), 159–171. Zbl 0787.16011 MR 1106345

[10] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, 231, Birkhäuser, Boston, 2005. Zbl 1072.14066 MR 2107324

[11] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956. Zbl 0933.18001 MR 0077480

[12] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, *Compositio Math.*, **126** (2001), no. 3, 257Ű–293. Zbl 1037.16007 MR 1834739

[13] W. Crawley-Boevey, Normality of Marsden–Weinstein reductions for representations of quivers, *Math. Ann.*, **325** (2003), 55–79. Zbl 1063.16016 MR 1957264

[14] B. Davison, Superpotential Algebras and Manifolds, *Adv. Math.*, **231** (2012), no. 1, 879–912. Zbl 1278.53083 MR 2955196

[15] P. Deligne, Cohomologie á support propres, in *Théorie des Topos et Cohomologie Étale des Schemas*, Tome 3, 250–480, Séminaire de Géometrie Algébrique du Bois Marie (1963/64), SGA 4, Lecture Notes in Mathematics, 305, Springer-Verlag, Berlin-New York, 1973. Zbl 0255.14011 MR 0354654

[16] J. Fogarty, Algebraic families on an algebraic surface, *Amer. Jour. of Math.*, **90** (1968), 511–521. Zbl 0176.18401 MR 0237496

[17] F. Galluzzi and F. Vaccarino, Hilbert–Chow Morphism for Non Commutative Hilbert Schemes and Moduli Spaces of Linear Representations, *Algebras and Representation Theory*, **13** (2010), 491–509. Zbl 1203.14004 MR 2660858

[18] C. Geiss and J. A. de la Pena, On the Deformation Theory of Finite Dimensional Modules and Algebras, *Manuscripta Math.*, **88** (1995), 191–208. Zbl 0851.16011 MR 1354106

[19] V. Ginzburg, *Lectures on Non Commutative Geometry*, preprint, 2005 arXiv:math/0506603

[20] V. Ginzburg, *Calabi–Yau algebras*, preprint, 2007. arXiv:math.AG/0612139

[21] T. Iarrobino, Hilbert scheme of points: Overviews of the last ten years, in *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, 297–320, Proc. Sympos. Pure Math., 46, Part 2, Amer. Math. Soc., Providence, RI, 1987. Zbl 0646.14002 MR 0927986

[22] B. Iversen, *Linear determinants with applications to the Picard Scheme of a family of algebraic curves*, Lecture Notes in Mathematics, 174, Springer, 1970. Zbl 0205.50802 MR 0292835

[23] O. Iyama and I. Reiten, Fomin–Zelevinsky mutation and tilting modules over Calabi–Yau algebras, *Amer. J. Math.*, **130** (2008), no. 4, 1087–1149. Zbl 1162.16007 MR 2427009

[24] B. Keller, Deformed Calabi–Yau Completions. With an appendix by Michel Van den Bergh, *Journal für die Reine und Angewandte Mathematik (Crelle's Journal)*, **654** (2011), 125–180. Zbl 1220.18012 MR 2795754

[25] B. Keller, Introduction to A-infinity algebras and modules, *Homology Homotopy Appl.*, **3** (2001), no. 1, 1–35. Zbl 0989.18009 MR 1854636

[26] B. Keller and I. Reiten, *Acyclic Calabi–Yau Categories*. With an appendix by Michel Van den Bergh, *Compos. Math.*, **144** (2008), no. 5, 1332–1348. Zbl 1171.18008 MR 2457529

[27] M. Kontsevich and Y. Soibelman, Notes on A_∞-algebras, A_∞-categories and non-commutative geometry, in *Homological mirror symmetry*, 153–219, Lecture Notes in Phys., 757, Springer, Berlin, 2009. Zbl 1202.81120 MR 2596638

[28] M. Kontsevich and Y. Soibelman, *Deformation theory*. Available at: https://www.math.ksu.edu/~soibel/Book-vol1.ps

[29] L. Le Bruyn and G. Seelinger, Fibers of generic Brauer–Severi schemes, *J. Algebra*, **214** (1999), 222–234. Zbl 0932.16025 MR 1684876

[30] L. Le Bruyn, *Noncommutative Geometry and Cayley-smooth Orders*, Pure and Applied Mathematics, 290, Chapman and Hall, 2008. Zbl 1131.14006 MR 2356702

[31] L. Le Bruyn, *Simple roots of deformed preprojective algebras*, preprint, 2001. arXiv:math.RA/0107027

[32] D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, A-infinity structure on Ext-algebras, *Journal of Pure and Applied Algebra*, **213** (2009), no. 11, 2017–2037. Zbl 1231.16008 MR 2533303

[33] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, 18, Amer. Math. Soc., Providence, 1999. Zbl 0949.14001 MR 1711344

[34] M. V. Nori, Appendix to the paper by C. S. Seshadri: Desingularisation of the moduli varieties of vector bundles over curves, in *Proceedings of the International Symposium on Algebraic Geometry, (Kyoto, 1977)*, 155–184, Kinokuniya Book Store, Tokyo, 1977. Zbl 0412.14005 MR 578858

[35] C. Procesi, *Rings with Polynomial Identities*, Lecture Notes in Pure and Appl. Math., 17, Dekker, New York, 1973. Zbl 0262.16018 MR 0366968

[36] A. S. Rapinchuk, V. V. Benyash-Krivetz and V. I. Chernousov, Representation varieties of the fundamental groups of compact orientable surfaces, *Israel J. Math.*, **93** (1996), 29–71. Zbl 0857.14012 MR 1380633

[37] M. Reineke, Cohomology of non-commutative Hilbert schemes, *Algebr. Represent. Theory*, **8** (2005), 541–561. Zbl 1125.14006 MR 2199209

[38] E. Segal, The A_∞ deformation theory of a point and the derived categories of local Calabi-Yaus, *J. Algebra*, **320** (2008), no. 8, 3232–3268. Zbl 1168.18005 MR 2450725

[39] F. Vaccarino, Linear representations, symmetric products and the commuting scheme, *J. Algebra*, **317** (2007), 634–641. Zbl 1155.13007 MR 2362934

[40] F. Vaccarino, Moduli of Linear Representations, Symmetric Products and the non Commutative Hilbert scheme, in *Geometric Methods in Representation Theory*, II, 435–456, Séminaires et Congress (Grenoble), 24, Soc. Math. France, Paris, 2012. Zbl 1314.14004 MR 3203038

[41] M. Van den Bergh, The Brauer–Severi scheme of the trace ring of generic matrices, in *Perspectives in Ring Theory, (Antwerp, 1987)*, 333–338, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 233, Kluwer Acad. Publ., Dordrecht, 1988. Zbl 0761.13003 MR 1048420

[42] M. Van den Bergh, Calabi–Yau algebras and superpotentials, *Selecta Math. (N.S.)*, **21** (2015), no. 2, 555–603. Zbl 06441727 MR 3338683

Received 31 October, 2014; revised 29 February, 2016

R. Bocklandt, Korteweg de Vries institute, University of Amsterdam (UvA),
Science Park 904, 1098 XH Amsterdam, The Netherlands
E-mail: raf.bocklandt@gmail.com

F. Galluzzi, Dipartimento di Matematica, Universitá di Torino,
Via Carlo Alberto n. 10, Torino 10123, Italy
E-mail: federica.galluzzi@unito.it

F. Vaccarino, Dipartimento di Scienze Matematiche, Politecnico di Torino,
C. so Duca degli Abruzzi n. 24, Torino 10129, Italy;
and ISI Foundation, Via Alassio 11/c, 10126 Torino, Italy
E-mail: francesco.vaccarino@polito.it; francesco.vaccarino@isi.it