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Matrix Theory S Matrix

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The technology required for eikonal scattering amplitude calculations in matrix theory is developed. Using the entire supersymmetric completion of the $v^4/r^2$ matrix theory potential we compute the graviton-graviton scattering amplitude and find agreement with 11 dimensional supergravity at tree level.

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$M$ theory, the 11 dimensional quantum theory underlying perturbative strings, has in recent years headlined dramatic changes in our understanding of string theory. At large distances $M$ theory reduces (by definition) to 11 dimensional supergravity. According to the matrix theory conjecture of [1] the microscopic degrees of freedom of $M$ theory are described by the large $N$ limit of a quantum mechanical supersymmetric U($N$) Yang-Mills model. The model itself arises, on the one hand, as the regulating theory of the 11 dimensional supermembrane [2] and on the other as the short distance description of D0-branes [3,4]. An essential feature of the model is the existence of asymptotic particle states carrying the quantum numbers of the 11 dimensional graviton supermultiplet [1,5].

A principal test of the matrix conjecture is the comparison of scattering amplitudes in the Yang-Mills quantum mechanics with those of 11 dimensional supergravity. To date, typical matrix theory scattering experiments involve the comparison of classical gravity source-probe actions with the background field effective action of super Yang-Mills theory in (1 + 0) dimensions evaluated on straight line configurations (see [6] for an exhaustive list of references). However, a matrix theory computation yielding true $S$-matrix elements, depending on momenta and polarizations of the external particles, has remained elusive. In this Letter we carry out precisely such a computation.

To this end we construct a matrix theory analog of the Lehmann, Symanzik, and Zimmermann (LSZ) reduction formula which relates the Hamiltonian

$H = \frac{1}{2} p_\mu p^\mu + \frac{1}{2} \tilde{p}_\mu \cdot \tilde{p}_\mu + \frac{i}{4} (\tilde{X}_\mu \times \tilde{X}_\nu)^2 + \frac{i}{2} \tilde{X}_\mu \cdot \tilde{\theta} \gamma_\mu \times \tilde{\theta}$

is a sum of an interacting SU(2) part describing relative motions and a free U(1) piece pertaining to the center of mass. We use a vector notation for the adjoint representation of SU(2), $\tilde{X}_\mu = (Y^I_\mu, x_\mu)$ and $\tilde{\theta} = (\theta^I, \theta^3)$ (with $I = 1, 2$ and $\mu = 1, \ldots, 9$) and may choose a gauge in which $Y^I_0 = 0$. The model has a potential with flat directions along a valley floor in the Cartan sector $x_\mu$ and $\theta^3$. The remaining degrees of freedom transverse to the valley are supersymmetric harmonic oscillators in

$\tilde{x}_\mu (\theta) = x_\mu (\theta) + \epsilon \tilde{X}_\mu (\theta)$
the variables $Y_I^\mu$ ($\mu \neq 9$) and $\theta^I$. Upon introducing a large gauge invariant distance $x = \left(\tilde{X}_0 \cdot \tilde{X}_0\right)^{1/2} = x_0$ as the separation of a pair of particles, the Hamiltonian (1) was shown [5] to possess asymptotic two particle states of the form

$$ p_1^\mu, \mathcal{H}^{1}, p_2^\mu, \mathcal{H}^{2} \rangle = \langle 0_B, 0_c \rangle \frac{1}{x_0} e^{i(p_1 - p_2)x} e^{i(p_1 + p_2)X_0} \times |\mathcal{H}^{1}|_{\theta \theta} |\mathcal{H}^{2}|_{\theta \theta}. \quad (2) $$

Here $p_1^\mu$ and $\mathcal{H}^{1}$ are, respectively, the momenta and polarizations of the two particles. The state $\langle 0_B, 0_c \rangle$ is the ground state of the superharmonic oscillators and the polarization states are the 44 $\otimes$ 84 $\otimes$ 128 representation of the $\theta^0 \pm \theta^3$ variables, corresponding to the graviton, three-form tensor, and gravitino, respectively.

For the computation of scattering amplitudes one may now form the $S$ matrix in the usual fashion

$$ S_{fi} = \langle \text{out} | \exp\{-iHT\} | \text{in} \rangle $$

with the desired ingoing and outgoing quantum numbers according to (2). The asymptotic states above are constructed with respect to a large separation in the same direction for both ingoing and outgoing particles, i.e., eikonal kinematics. More general kinematical situations are handled by introducing a rotation operator into the $S$ matrix [12]. The object of interest is then the vacuum to vacuum transition amplitude

$$ e^{\Gamma(v_\mu, b_\mu, \theta^3)} = \langle \text{out} \rangle_{\theta \theta} |\text{in} \rangle \exp\{-iHT\} |\text{out} \rangle_{\theta \theta}. \quad (3) $$

Note that the ground states actually depend on the Cartan variables $x_\mu$ and $x'_\mu$ through the oscillator mass. Also, both the left and right hand sides depend on the operator $\theta^3$.

Our key observation is rather simple. In field theory one is accustomed to expand around a vanishing vacuum expectation value when computing the vacuum to vacuum transition amplitude for some field composed of oscillator modes. In quantum mechanics it is of course exactly the same, and therefore if one is to represent (3) by a path integral one should expand the super oscillators transverse to the valley about a vanishing vacuum expectation value. One may then write the matrix theory $S$ matrix in terms of a path integral with the stated boundary conditions

$$ e^{\Gamma(v_\mu, b_\mu, \theta^3)} = \int_{\tilde{X}_0 = (0, x_\mu'), \tilde{\theta} = (0, \theta^3)} \mathcal{D}(\tilde{X}_0, \tilde{\theta}) \langle \tilde{X}_0, \tilde{\theta} \rangle_{\theta \theta} \exp\left(i \int_{-T/2}^{T/2} L_{\text{SYM}}\right). \quad (4) $$

The Lagrangian $L_{\text{SYM}}$ is that of the supersymmetric Yang-Mills quantum mechanics with appropriate gauge fixing to which end we have introduced ghosts $\tilde{b}, \tilde{c}$ and the Lagrange multiplier gauge field $\tilde{A}$. The effective action $\Gamma(v_\mu, b_\mu, \theta^3)$ is most easily computed via an expansion about classical trajectories $X'^\mu(t) \equiv x'^\mu(t) = b_\mu + v_\mu t$ and constant $\theta^3(t) = \theta^3$ which yields the quoted boundary conditions through the identification $b_\mu = (x'_\mu + x_\mu)/2$ and $v_\mu = (x'_\mu - x_\mu)/T$.

Up to an overall normalization $\mathcal{N}$, our LSZ reduction formula for matrix theory is simply

$$ S_{fi} = \delta^9(k'_\mu - k_\mu)e^{-iE(k'_\mu - k_\mu)T/2} \int d^9x'd^9x \mathcal{N} \exp\{-iw_\mu x'_\mu + iw_\mu x_\mu\} \times \langle \mathcal{H}^{3}|(\mathcal{H}^{4}|e^{i\Gamma(v_\mu, b_\mu, \theta^3)}|\mathcal{H}^{1})|\mathcal{H}^{2}. \quad (5) $$

The leading factor expresses momentum conservation for the center of mass where we have denoted $k_\mu = p_1^\mu + p_2^\mu$ and $k'_\mu = p_1'^\mu + p_2'^\mu$ for the ingoing and outgoing particles, respectively, and similarly for the relative momenta $u_\mu = (p_1^\mu - p_2^\mu)/2$ and $w_\mu = (p_2^\mu - p_2'^\mu)/2$.

In a loopwise expansion of the matrix theory path integral one finds $\Gamma(v_\mu, b_\mu, \theta^3) = v_\mu v_\mu T/2 + \Gamma(1) + \Gamma(2) + \ldots$ of which we consider only the first two terms in order to compare our results with tree-level supergravity. Inserting this expansion into (5) and changing variables $d^9x'd^9x \rightarrow d^9(T^\mu) d^9b$, the integral over $T^\mu$ may be performed via stationary phase. Dropping the normalization and the overall center of mass piece the $S$ matrix then reads

$$ S_{fi} = e^{-i\left((u+\mu)/2\right)T/2} \int d^9b e^{-iu_b \Gamma} (\mathcal{H}^{3}|(\mathcal{H}^{4}|e^{i\Gamma(v_\mu, b_\mu, \theta^3)}|\mathcal{H}^{1})|\mathcal{H}^{2}, \quad (6) $$

where $q_\mu = w_\mu - u_\mu$. It is important to note that in (6) the variables $\theta^3$ are operators $\{\theta^3, \theta^3\} = \delta_{\alpha\beta}$ whose expectation between polarization states $|\mathcal{H}^{3}\rangle$ yields the spin dependence of the scattering amplitude.

The loopwise expansion of the effective action should be valid for the eikonal regime, i.e., large impact parameter $b$ or small momentum transfer $q_\mu$. As we shall see below, this limit is dominated by $t$-channel physics on the supergravity side.

**D0-brane computation of the matrix theory effective potential.**—We must now determine the one-loop effective matrix potential $\Gamma(v, b, \theta^3)$, namely, the $v^4/r^4$ term and its supersymmetric completion. Fortunately the bulk of this computation has already been performed in string theory by [9,10] who applied the Green-Schwarz boundary state formalism of [11] to a one-loop annulus computation for a pair of moving D0-branes. They found that the leading spin interactions are dictated by a simple zero mode analysis and their form is, in particular, scale independent. This observation allows one to extrapolate the results of [9,10] to short distances and suggests a matrix theory description for tree-level supergravity interactions.

Following [9,10], supersymmetric D0-brane interactions are computed from the correlator

$$ \mathcal{V} = \frac{1}{16} \int_0^\infty dt \langle B, \tilde{\gamma} = 0|e^{-2\pi i a' p^+ (P^+ i \hat{\theta}/\hat{x}^+)} \times e^{i\eta^\mu + \bar{\eta}'\bar{\theta}'} e^{V_0}|B, \tilde{\gamma} = \tilde{b}\rangle \quad (7) $$
with $Q^-, \bar{Q}^-$ being the SO(8) supercharges broken by the presence of the D-brane, $|B\rangle$ the boundary state associated with D0-branes, and $V_{B} \oint_{\sigma} d\sigma \left( X^1 \partial_{\sigma} X^1 + \frac{1}{2} S \partial_{\sigma} S \right)$ is the boost operator where the direction 1 has to be identified with the time (see [9,10] for details). Expanding (7) and using the results in section four of [10], one finds the following compact form for the leading one-loop matrix theory potential (normalizing to one the $v^4$ term and setting $\alpha' = 1$)

$$V_{1-loop} = \left[ v^4 + 2i v^2 v_m (\theta \gamma^m \theta) \partial_n - 2v_p v_q (\theta \gamma^m \theta) (\theta \gamma^n \theta) \partial_m \partial_n - \frac{4i}{9} v_q (\theta \gamma^m \theta) (\theta \gamma^n \theta) (\theta \gamma^p \theta) \partial_m \partial_n \partial_p + \frac{2}{63} (\theta \gamma^m \theta) (\theta \gamma^n \theta) (\theta \gamma^p \theta) \partial_m \partial_n \partial_p \right] \frac{1}{r^4},$$

(8)

where $\theta = (\eta^\alpha, \tilde{\alpha}^\alpha)$ should be identified with $\theta^3/2$ of the last section and all indices are nine dimensional. The general structure of this potential was noted in [13] and its first, second, and last terms were calculated in [14], [7], and [8], respectively. (It would be interesting to establish the supersymmetry transformations of this potential. However, acceleration terms, set to zero in our setup, should then be included [15].)

$$\langle \text{in} \rangle = \frac{1}{256} h_m^1 (\lambda^+_m \gamma_m \lambda^+_1) (\lambda^+_1 \gamma_m \lambda^+_1) h^3_p (\lambda^+_2 \gamma_p \lambda^+_2) (\lambda^+_2 \gamma_p \lambda^+_2) | \theta \rangle,$n

$$\langle \text{out} \rangle = \frac{1}{256} (-1) h_m^1 (\lambda^+_m \gamma_m \lambda^+_1) (\lambda^+_1 \gamma_m \lambda^+_1) h^3_p (\lambda^+_2 \gamma_p \lambda^+_2) (\lambda^+_2 \gamma_p \lambda^+_2).$$

(9)

Note that (following [5]) we have complexified the Majorana center of mass and Cartan spinors $\theta^0$ and $\theta^3$ in terms of SO(7) spinors $\lambda^{1,2} = (\theta^3 \pm \theta^0 \pm i \theta^0 \pm i \theta^3)/2$, where $\pm$ denotes projection with respect to $\gamma_9$. Actually the polarizations in (9) are seven dimensional but may be generalized to the nine dimensional case at the end of the calculation. We stress that these maneuvers are purely technical and our final results are SO(9) covariant. The creation and destruction operators $\lambda^+_{1,2}$ and $\lambda_{1,2}$ annihilate the states $| - \rangle$ and $| - \rangle$, respectively.

The resulting one-loop eikonal matrix theory graviton-graviton scattering amplitude is composed of 66 terms and [denoting, e.g., $(q h_1 h_4 v) = q_{\mu} h^1_{\mu} h^4_{\nu} v_{\rho}$ and $(h_1 h_4) = h^1_{\mu} h^4_{\nu} v_{\rho}$] is given by

$$A = \frac{1}{q^2} \left\{ \frac{1}{2} (h_1 h_4) (h_2 h_3) v^4 + 2 (q h_1 h_2 v) (h_1 h_4) - (q h_2 h_3 v) (h_1 h_4) \right\} v^2$$

$$+ (v h_2 v) (q h_3 q) (h_1 h_4) + (v h_3 v) (q h_2 q) (h_1 h_4) - 2 (q h_2 v) (q h_3 v) (h_1 h_4)$$

$$- 2 (q h_1 v) (q h_2 h_3 v) + (q h_1 h_2 v) (q h_2 h_3 v) + (q h_1 h_2 v) (q h_3 h_2 v)$$

$$+ \frac{1}{2} ((q h_1 h_2 h_3 q) - 2 (q h_1 h_2 h_3 q) + (q h_1 h_2 h_3 q) - 2 (q h_2 h_3 q) (h_1 h_4)) v^2$$

$$- (q h_2 v) (q h_3 q) (h_1 h_4) + (q h_2 q) (q h_3 v) (h_1 h_4) - (q h_1 q) (q h_2 h_3 v) + (q h_1 q) (q h_3 h_2 v)$$

$$- (q h_4 q) (q h_2 h_3 v) + (q h_4 q) (q h_3 h_2 v) - (q h_4 q) (q h_3 h_2 v) + (q h_4 q) (q h_3 h_2 v)$$

$$- (q h_4 q) (q h_1 h_2 v) + (q h_4 q) (q h_1 h_2 v) - (q h_1 q) (q h_4 h_3 q) + (q h_1 q) (q h_4 h_3 q)$$

$$- (q h_4 q) (q h_1 h_3 q) + (q h_4 q) (q h_1 h_3 q) + (q h_1 q) (q h_4 h_3 q) - (q h_1 q) (q h_4 h_3 q)$$

$$+ \frac{1}{2} ((q h_1 q) (q h_2 q) (h_1 h_4) + 2 (q h_1 q) (q h_4 q) (h_2 h_3) + 2 (q h_1 q) (q h_3 q) (h_2 h_4)$$

$$+ (q h_3 q) (q h_4 q) (h_1 h_4) + \frac{1}{2} ((q h_1 q) (q h_2 h_3 q) - (q h_1 q) (q h_2 h_3 q) + (q h_1 q) (q h_2 h_3 q)$$

$$- (q h_1 q) (q h_2 h_3 q) - (q h_1 q) (q h_2 h_3 q) - (q h_1 q) (q h_2 h_3 q)$$

$$+ \frac{1}{2} ((q h_1 q) (q h_2 q) (h_1 h_4) + (q h_1 q) (q h_2 q) (h_1 h_4) + (q h_1 q) (q h_2 q) (h_1 h_4)) v^2$$

$$) \right\} .$$

(10)

We have neglected all terms within the curly brackets proportional to $q^2 \equiv q_{\mu} q_{\nu}$, i.e., those that cancel the $1/q^2$ pole. These correspond to contact interactions in the D0-brane computation, whereas this calculation is valid only for noncoincident branes.

$D = 11$ supergravity.—The above leading order result for eikonal scattering in matrix theory is easily shown to agree with the corresponding 11 dimensional field theoretical amplitude. Tree-level graviton-graviton scattering is dimension independent and has been computed in [16]. We have double-checked that work by a type IIA string theory computation.
and will not display the explicit result here which depends on 11 momenta $p_M^i$ (with $i = 1, \ldots, 4$) and polarizations $h_{MN}^i$ subject to the de Donder gauge condition $p_M^i h_{MN}^i - (1/2) p_M^i h_{MN}^i = 0$ (no sum on $i$). Matrix theory, on the other hand, is formulated in terms of on shell degrees of freedom only, namely, transverse physical polarizations and euclidean nine momenta.

Going to light-cone variables for the eleven momenta $p_M$ we take the case of vanishing $p^–$ momentum exchange, i.e., the scenario of our matrix computation,

$$p_M^1 = (–\frac{1}{2}(\nu_\mu - q_\mu/2)^2, 1, \nu_\mu - q_\mu/2),$$
$$p_M^2 = (–\frac{1}{2}(\nu_\mu - q_\mu/2)^2, 1, –\nu_\mu + q_\mu/2),$$
$$p_M^3 = (–\frac{1}{2}(\nu_\mu + q_\mu/2)^2, 1, \nu_\mu + q_\mu/2),$$
$$p_M^4 = (–\frac{1}{2}(\nu_\mu + q_\mu/2)^2, 1, –\nu_\mu - q_\mu/2).$$

[We denote $p_± = p^± = (p^{10} ± p^0)/\sqrt{2}$ and our metric convention is $\eta_{MN} = \text{diag}(−, +, +, +)$.] By transverse Galilean invariance we have set to zero the nine dimensional center of mass momentum. We measure momenta in units of $p^–$ which we set to one. For this kinematical situation conservation of $p^+$ momentum clearly implies $\nu_\mu q_\mu = 0$. Note that the vectors $u_\mu$ and $w_\mu$ of (5) are simply $u_\mu = \nu_\mu - q_\mu/2$ and $w_\mu = \nu_\mu + q_\mu/2$.

We reduce to physical polarizations by using the residual gauge freedom to set $h_{iM}^i = 0$ and solve the de Donder gauge condition in terms of the transverse traceless polarizations $h_{iM}^i$, for which one finds $h_{iM}^i = −p_M^i h_{iM}^i$.

Agreement with the matrix result (10) is then achieved by taking the eikonal limit $\nu_\mu \gg q_\mu$ of the gravity amplitude in which the $t$-pole contributions dominate. (In the above parametrization, the Mandelstam variables are $t = q_\mu^2 = –2p_M^1 p_M^4, s = 4\nu_\mu^2 + q_\mu^2 = –2p_M^1 p_M^2$, and $u = 4\nu_\mu^2 = –2p_M^1 p_M^3 = s - t$.) One then reproduces exactly (10) as long as any pieces canceling the $t$ pole (i.e., the aforementioned $q^2$ terms) are neglected.

Although we have presented here only a matrix scattering amplitude restricted to the eikonal regime, we nevertheless believe the agreement found is rather impressive.

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