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A method for forward and inverse solutions of a three-dimensional model of the cochlea

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In this Letter a method is described for computing the response of a linear three-dimensional classical model of the cochlea. The method can be applied when the geometry of the model lends itself to separation of variables, for instance, when the model has the shape of a homogeneous rectangular block or box. The method is an improvement over previously published computation methods, because it handles forward and retrograde traveling waves symmetrically, and it can be used for forward as well as inverse solutions. Furthermore, the method can be extended to a time-domain solution. © 1998 Acoustical Society of America. [S0001-4966(98)06606-5]

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INTRODUCTION

In principle, there are two strategies that can be followed in modeling the mechanism of cochlear functioning. In one, the structure and the parameters of the model are specified to the largest degree of precision and detail, and the model equations are solved by a laborious procedure (a recent example: Kolston and Ashmore, 1996). In the other strategy, geometry and structure of the model are simplified as much as possible (cf. de Boer, 1981, 1991, 1996) so that the fluid equations lead to separation of the variables, which allows the problem to be reduced to a not-too-complicated equation in terms of one variable (the longitudinal place coordinate, x). On modern PC-type computers, the solution for a linear model of this form can generally be obtained in a few seconds.

In the present report a solution method is described that belongs to the second category. We will restrict ourselves to linear models because it is felt that the response of the cochlea with low levels of stimulation (for instance, below 30 dB SPL) is very close to linear. The reader is referred to de Boer (1997a) to find to which extent *nonlinear* effects in the cochlea can be considered in terms of *linear* systems and models. The solution method to be treated here is based upon the same simple three-dimensional structure that was used in recent reports by the author (de Boer, 1995a,b, 1996). However, a new and potentially more useful method is given for the solution. The main advantage is that reflection problems are treated faithfully because all expressions are symmetrical in x (on a micro-scale, wave propagation is symmetrical in x). This implies that the method can be extrapolated to a time-domain solution in a straightforward way. Furthermore, problems associated with evoked oto-acoustical emissions can be handled with ease and confidence. The present report describes the basic theory and the solution methods for the forward as well as the inverse problem in the frequency domain.

I. THEORY

The model we are going to use has a rectangular cross section which is the same over the full length in the longitudinal (x)-direction. The basilar membrane (BM) occupies a fraction, ϵ , of the width of the channel, and that fraction is constant over the full length too (see Fig. 1 of de Boer, 1995a, or Fig. 5.1 of de Boer, 1996). The function $Q(k)$ plays the central part in the general theory of waves in this type of model (de Boer, 1991, Section 2.3; de Boer, 1996, Sections 5.1 to 5.3). This function describes the hydrodynamics of the fluid contained in the cochlear channels, the real variable k is the wave number (2π divided by the wavelength), $i\omega\rho Q(k)$ is the impedance the fluid in one channel presents to the BM when a single wave with wave number k and radian frequency ω travels in the fluid (ρ is the fluid density). This concept is only valid if the model's geometry is homogeneous and the fluid is ideal (i.e., incompressible, inviscid and linear). The function $Q(k)$ takes into account *three-dimensional* effects, i.e., the fact that the BM is moving over only a fraction of its width, that the fluid can move in three dimensions, and that long as well as short waves and the transition between these may be present. We quote the basic equation for a linear model (cf. de Boer, 1991, Eq. 2.3.f) that results from the analysis:

$$\frac{2i\omega\rho}{Z_{\text{BM}}(x)} \int_{-\infty}^{\infty} Q(k) V_{\text{BM}}(k) e^{-ikx} dk = -v_{\text{BM}}(x). \quad (1)$$

The meaning of the variables is as follows: x is the longitudinal coordinate along the BM, $Z_{\text{BM}}(x)$ is the BM impedance, $v_{\text{BM}}(x)$ is the velocity of the BM and $V_{\text{BM}}(k)$ its Fourier transform (only real values of k are involved). In all variables and parameters the independent variable ω has been omitted for simplicity. For the derivation of Eq. (1) and the form of $Q(k)$ for the rectangular-block model we refer to de Boer (1981).

An alternative, more general, solution method is based on the concept of the Green's function (see for an early application Allen, 1977, and for a more recent one Mammano and Nobili, 1993). For the simple regular type of model geometry we are considering here, the Green's function is pro-

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portional to the inverse Fourier transform $q(x)$ of $Q(k)$. Sondhi (1978) has studied the way to circumvent problems with the logarithmic singularity which $q(x)$ displays at $x = 0$. Conversely, the function $Q(k)$ has a singularity at $k = \infty$, where it behaves as $(1/|k|)$ —which is a nonanalytic function of k —to describe the character of short waves.

In de Boer and van Bienema (1982), the function $Q(k)$ for the three-dimensional model was approximated by the quotient of two polynomials in k , one of the second and one of the third degree in k , and a solution method for three-dimensional models was based on it. The same approximation has been used for the “inverse solution” described in de Boer (1995a,b). However, the so-obtained $Q(k)$ function is not symmetrical in k , it behaves in the right manner for $k \rightarrow +\infty$ but not for $k \rightarrow -\infty$; therefore, the way reflected waves are treated is unpredictable. Furthermore, the approximate $Q(k)$ function cannot be extended to a realizable spatial operator as would be needed for a time-domain solution. To improve upon these points, approximate $Q(k)$ by the *symmetrical* function

$$Q(k) = \frac{1}{h_{\text{eff}} k^2} \frac{1 + a_2 k^2 h^2 + a_4 k^4 h^4}{1 + b_2 k^2 h^2 + b_4 k^4 h^4}, \quad (2)$$

where h is the height of each of the cochlear channels, ϵ is the fraction of the width of the cochlear partition occupied by the (flexible) BM, and the “effective height” h_{eff} equals h/ϵ . That we have to go from the relatively simple third-order expression in de Boer and van Bienema (1982) to the sixth order in Eq. (2) is due to the problem of properly representing $Q(k)$ for large values of $|k|$ by a quotient of *even* polynomials in k . Appendix A gives numerical values for the coefficients a_2 to b_4 for various values of ϵ which produce a good approximation of the “true” $Q(k)$ function by Eq. (2). For example, in the case $\epsilon = 0.3$ it is found that Eq. (2), with the coefficients filled in according to Appendix A, provides a satisfactory approximation to the “true” $Q(k)$ function for $|kh| < 50$. In fact, larger values of $|kh|$ than 50 should not play a noticeable part in cochlear modeling. When a_4 and b_4 are zero, the approximation is good to $|kh| = 10$.

II. FORWARD SOLUTION

Substitute Eq. (2) in the standard equation for a “classical” model [Eq. (1)]:

$$\frac{2i\omega\rho}{Z_{\text{BM}}(x)} \int_{-\infty}^{\infty} \frac{1}{h_{\text{eff}} k^2} \frac{1 + a_2 k^2 h^2 + a_4 k^4 h^4}{1 + b_2 k^2 h^2 + b_4 k^4 h^4} \times V_{\text{BM}}(k) e^{-ikx} dk = -v_{\text{BM}}(x). \quad (3)$$

Introduce the new variable $u(x)$ by defining its Fourier transform $U(k)$ in the following way:

$$\frac{1}{h_{\text{eff}} k^2} \frac{1}{1 + b_2 k^2 h^2 + b_4 k^4 h^4} V_{\text{BM}}(k) = U(k). \quad (4)$$

Because in the k domain each multiplication with $(-ik)$ is equivalent to differentiation with respect to x , the x domain counterparts $v_{\text{BM}}(x)$ and $u(x)$ are related by

$$v_{\text{BM}}(x) = h_{\text{eff}} \left[-\frac{d^2}{dx^2} + b_2 h^2 \frac{d^4}{dx^4} - b_4 h^4 \frac{d^6}{dx^6} \right] u(x). \quad (5)$$

When $u(x)$ is known, $v_{\text{BM}}(x)$ can be computed from Eq. (5). To solve for $u(x)$, substitute Eq. (5) in model Eq. (3):

$$\frac{2i\omega\rho}{h_{\text{eff}} Z_{\text{BM}}(x)} \int_{-\infty}^{\infty} (1 + a_2 k^2 h^2 + a_4 k^4 h^4) U(k) e^{-ikx} dk = \left[\frac{d^2}{dx^2} - b_2 h^2 \frac{d^4}{dx^4} + b_4 h^4 \frac{d^6}{dx^6} \right] u(x), \quad (6)$$

and reduce this relation to an equation in x :

$$\frac{2i\omega\rho}{h_{\text{eff}} Z_{\text{BM}}(x)} \left[1 - a_2 h^2 \frac{d^2}{dx^2} + a_4 h^4 \frac{d^4}{dx^4} \right] u(x) = \left[\frac{d^2}{dx^2} - b_2 h^2 \frac{d^4}{dx^4} + b_4 h^4 \frac{d^6}{dx^6} \right] u(x). \quad (7)$$

Equation (7) is a differential equation of the sixth order in $u(x)$, and can be solved by standard methods. Appendix B defines the matrix elements for solving Eq. (7) digitally. Note that, when all coefficients (a_2 to b_4) are zero, Eq. (7) reduces to the standard long-wave equation with h_{eff} as the channel height. In that case, the *long-wave case*, the channel pressure $p(x)$ is proportional to $u(x)$:

$$p(x) = i\omega\rho u(x). \quad (8)$$

This relation can be used to formulate the boundary conditions at and near the stapes.

From the function $u(x)$ the BM velocity $v_{\text{BM}}(x)$ can be obtained by using Eq. (5). Or, alternatively, Eqs. (5) and (7) can be combined to express the BM velocity in terms of lower-order derivatives of $u(x)$:

$$v_{\text{BM}}(x) = \frac{-2i\omega\rho}{Z_{\text{BM}}(x)} \left[1 - a_2 h^2 \frac{d^2}{dx^2} + a_4 h^4 \frac{d^4}{dx^4} \right] u(x). \quad (9)$$

III. INVERSE SOLUTION

The inverse solution can be obtained by direct substitution of $v_{\text{BM}}(x)$ into Eq. (1) and using the exact expression for $Q(k)$. There may arise problems with convergence for large values of $|kh|$, however. It is often better to use the approximation of Eq. (2). Then, $u(x)$ can be obtained directly by solving Eq. (5). The effective BM impedance $Z_{\text{BM}}(x)$ is found from:

$$Z_{\text{BM}}(x) = \frac{-2i\omega\rho}{v_{\text{BM}}(x)} \left[1 - a_2 h^2 \frac{d^2}{dx^2} + a_4 h^4 \frac{d^4}{dx^4} \right] u(x), \quad (10)$$

which is a simple variation of Eq. (9).

IV. APPLICATION NOTES

It has been our general experience that the forward-computation method as described in this paper is very robust. In testing locally active models of the cochlea, and trying out variations of the parameters or introduction of irregularities, it is often found that the region of the response peak becomes the source of strong reflections. The solution method described in this paper can cope well with reflected waves. This stands in contrast to the de Boer–van Bienema (1982)

method which often fails in such cases (see, for instance, the resynthesis results in de Boer, 1995b). We may conclude that the method developed is a good and useful instrument for the further development of cochlear models, especially when it is required that the models be three-dimensional in nature and should include and properly represent retrograde waves or reflections.

On a tentative basis, we applied the method to locally active models like that of Neely and Kim (1986) and that of Geisler and Sang (1995). The model of Neely and Kim is a ‘‘classical’’ model in the sense that the reaction of the cochlear partition at location x is completely determined by the BM movement at the same location. The model of Geisler and Sang (1995) is not a ‘‘classical’’ model because the dynamics of the cochlear partition at location x is influenced by events taking place elsewhere. The response of that model is, however, identical to the response of a ‘‘classical’’ model when the two models have the same effective BM impedance (de Boer, 1997b). Both of these models have been conceived as long-wave models. We found that their parameters need to be modified considerably before these models produce an acceptable response in a three-dimensional setting. This is undoubtedly due to the fact that in the response-peak region, the waves in those models are not long waves but are closer to being short waves, and the parameters are clearly not right for these.

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APPENDIX A: POLYNOMIAL COEFFICIENTS FOR EQ. (1)

Suitable polynomial coefficients for the three-dimensional model are

ϵ	Numerator		Denominator	
	a_2	a_4	b_2	b_4
0.1	0.9	5.0×10^{-3}	1.2×10^{-2}	8.0×10^{-6}
0.15	0.75	5.0×10^{-3}	2.0×10^{-2}	8.0×10^{-6}
0.2	0.85	6.0×10^{-3}	3.0×10^{-2}	1.0×10^{-5}
0.3	0.55	1.0×10^{-3}	1.0×10^{-2}	3.0×10^{-6}
0.4	0.4	1.0×10^{-3}	1.4×10^{-2}	3.0×10^{-6}
0.5	0.4	1.0×10^{-3}	1.6×10^{-2}	4.0×10^{-6}
0.6	0.4	1.0×10^{-3}	2.0×10^{-2}	5.0×10^{-6}
0.7	0.38	1.0×10^{-3}	2.5×10^{-2}	7.0×10^{-6}
0.8	0.35	1.0×10^{-3}	2.5×10^{-2}	8.0×10^{-6}
0.9	0.3	1.0×10^{-3}	2.5×10^{-2}	1.0×10^{-5}

APPENDIX B: FORWARD SOLUTION, MATRIX COMPOSITION

We consider the solution of Eq. (7) by way of discretization of the x axis. We have to express the second, fourth and sixth derivatives to an accuracy of the order $(\delta x)^7$, where δx is the length of one segment in the x direction.

Consider the sequence y_i (with i an integer) as a function of x_i with $x_i = x_0 + i \delta x$; both sequences are samples of the continuous variables $y(x)$ and x . A Taylor’s series expansion of y around its value $y_0 = y(x_0)$ can be made in the range from -3 to $+3$ of i , and leads to

$$\begin{aligned}
 y_{-3} - y_0 &= -3D^{(1)} + 9D^{(2)} - 27D^{(3)} \\
 &\quad + 81D^{(4)} - 243D^{(5)} + 729D^{(6)}, \\
 y_{-2} - y_0 &= -2D^{(1)} + 4D^{(2)} - 8D^{(3)} + 16D^{(4)} \\
 &\quad - 32D^{(5)} + 64D^{(6)}, \\
 y_{-1} - y_0 &= -D^{(1)} + D^{(2)} - D^{(3)} + D^{(4)} \\
 &\quad - D^{(5)} + D^{(6)}, \\
 y_{+1} - y_0 &= +D^{(1)} + D^{(2)} + D^{(3)} + D^{(4)} \\
 &\quad + D^{(5)} + D^{(6)}, \\
 y_{+2} - y_0 &= +2D^{(1)} + 4D^{(2)} + 8D^{(3)} + 16D^{(4)} \\
 &\quad + 32D^{(5)} + 64D^{(6)}, \\
 y_{+3} - y_0 &= +3D^{(1)} + 9D^{(2)} + 27D^{(3)} \\
 &\quad + 81D^{(4)} + 243D^{(5)} + 729D^{(6)},
 \end{aligned} \tag{B1}$$

where the symbol $D^{(j)}$ denotes

$$D^{(j)} = \frac{(\delta x)^j}{j!} \frac{d^j}{dx^j} y(x_0). \tag{B2}$$

Inversion of the coefficient matrix in (B1) (for instance, via MATLAB®) yields:

$$\begin{aligned}
 D^{(2)} &= (4y_{-3} - 54y_{-2} + 540y_{-1} + 540y_{+1} \\
 &\quad - 54y_{+2} + 4y_{+3} - 980y_0)/720, \\
 D^{(4)} &= (-5y_{-3} + 60y_{-2} - 195y_{-1} - 195y_{+1} \\
 &\quad + 60y_{+2} - 5y_{+3} + 280y_0)/720, \\
 D^{(6)} &= (y_{-3} - 6y_{-2} + 15y_{-1} + 15y_{+1} \\
 &\quad - 6y_{+2} + y_{+3} - 20y_0)/720.
 \end{aligned} \tag{B3}$$

From Eqs. (B2) and (B3) the elements of the matrix to be used in solving Eq. (7) are easily found. Obviously, the matrix is a band matrix, with three elements on both sides of the main diagonal.

The equation set needs six boundary conditions, to be formulated in the matrix rows that are incomplete. For stimuli of not too low frequency, the conditions at the end of the model are easily formulated: the value of $u(x)$ as well as of its first and second derivative are set to zero. For stimuli that do not have their short-wave region near the stapes, the boundary condition at $x = 0$ can be formulated by using Eq. (8) to relate $u(x)$ to the stapes velocity. We further prescribe that $u(x)$ obeys the long-wave equation at the two next points.

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