Chapter 2

Unobserved Heterogeneity, Solidarity and Experience Rating in Individual Life Insurance

2.1 Introduction

The urn-of-urns model is well known in non-life insurance. It takes into account the fact that not all heterogeneity between separate contracts with respect to the claims distribution can be observed. As time passes, claims experience can be included as an additional, observable risk factor, to update the premiums for each individual contract. In non-life business, each contract can usually make a claim more than once, and hence builds up its own unique claims experience, resulting in a unique individual premium as well. As a consequence, the intensity by which some contracts (the "good risks") in a portfolio subsidize others (the "bad risks") in one and the same portfolio, or the subsidizing solidarity, gradually decreases in time. On the other hand, the basis of insurance, that is that, in a homogeneous risk class, contracts without claims pay for claims incurred by other contracts, or the probabilistic solidarity, remains preserved.

Norberg (1989) showed that experience rating can be applied to group life contracts as well, though the underlying principle is different from the one given above. The claims distribution of a single life depends on the entire contract period, since it depends on the insured’s life history, which is determined throughout the whole time. Hence, experience rating in individual life insurance actually has no real significance: there is no difference between updating premiums to experience and not updating at all. A group life contract, however, usually consists of several of such single lives and thus generates a unique claims experience in time, enabling the insurer to adapt premiums on a regular basis throughout the period of validity of the contract.

In this chapter, we will consider unobserved heterogeneity in individual life insurance, usually characterized by the fact that a premium, determined upon issue, cannot be adjusted to experience later on. Our aim is to derive measures of solidarity, though not only for the entire contract term. Since life insurance treaties usually comprise several years, insurers are interested in quantities, such as profits, allocated to policy years, and therefore also solidarity measures on the level of a year will be derived. In order to be able
to do so, we will apply, at least formally, experience rating in a Bayesian way. Regarding
the urn-of-urns model, it is in that case assumed that all individuals in one and the same
group are affected by the same group specific latent risk factor.

The set-up of this chapter is as follows. In Section 2.2, we will consider a non-life
insurance contract and recall the solidarity measures defined in De Wit & Van Eeghen
(1984). Regarding the non-life case, we will restrict ourselves to insurance treaties allowing
for experience rating after each policy year. Crucial for such contracts is that the premium
is paid only at the beginning of each policy year, covering the loss for that policy year
exclusively. Then individual life contingencies will be discussed. In all subsequent sections,
the solidarity measures derived are based on the approach by De Wit & Van Eeghen
(1984), in the sense that each loss suffered by the insurer on contract level can be split
into two parts. One of these parts, the process effect, is the loss that would have occurred
had the contract paid a premium exactly matching the risk it represents, or, more formally,
had there been equivalence on an individual level. The other component, the parameter
effect is the loss due to the mismatch between premium and risk. All solidarity measures
obtained in this chapter are based on parameter effects.

A life insurance contract will at first, in Section 2.3, be defined in a very general
way, similar to Norberg (1990, 1991, 1992). His approach involves the possibility that
both transition intensities and premium and benefit payments at a certain moment may
depend on the entire life history of the insured. The urn-of-urns model will be intro­
duced and some quantities defined. In Section 2.4, solidarity measures will be defined,
at first, in Subsection 2.4.1, concerning the entire contract period. Compared with non-
life insurance, complications will arise, as premium payment is usually not restricted to
time-upon-issue. In Subsection 2.4.2, a part of the contract period will be considered.
Using Norberg’s definition of partial loss, we will define solidarity measures based on risk
premiums. Comparison between parameter effects concerning the aggregate period and
parts of it, respectively, will be the topic of Section 2.5. In general, the parameter effect
on contract term level is not the same as the sum of the parameter effects allocated to
the respective policy years. An exception arises when premiums are such that reserves at
any moment are equal to zero, irrespective of the state to which they apply. In that case,
the life insurance treaty has much in common with a multi-period non-life agreement.

Norberg’s model has the advantage that it allows for many insurance contracts and
probability spaces, but its drawback is that the quantities derived are difficult to interpret.
This is why, in Section 2.6, the special case of a hierarchical Markov chain will be discussed.
In the given case, any state in the model that has once been left can not be re-entered.
Besides, the loss applying to a certain state at time-upon issue can be clearly separated
into policy years. We will deal first with the general case and then with the fully discrete
life policy. The latter one has the property that benefits and premiums due to transition
are paid either at the beginning or at the end of each policy year. We will restrict
ourselves to the classical model where only the states ”Alive” and ”Dead” apply. This
rather special case is considered, because one of the solidarity measures thus derived will
take a central place in Chapter 3. The case will act as the basis for the so-called frailty
model, where the unobservable random variable acts multiplicatively on the conditional
transition intensity. In Section 2.7, this model will be discussed and also be accompanied
by a numerical example. Section 2.8 concludes.
2.2 The non-life insurance case

Consider a portfolio of $n$ contracts, labelled by $i$ ($i \in \{1, \ldots, n\}$). The risks – to be interpreted as aggregate claim amounts in this non-life context – corresponding to the contracts are denoted by the random variables $X_i$. An urn-of-urns model applies: for each $i \in \{1, \ldots, n\}$, the distribution of $X_i$ depends on the value of the one-dimensional risk parameter corresponding to the contract, being denoted by $\Theta_i$. For each $i \in \{1, \ldots, n\}$ it is assumed that, conditionally given $\Theta_i = \theta_i$, $X_i \sim G_{\theta_i}$, with $G_{\theta}$ some distribution function having parameter $\theta_i$, and that $\Theta_1, \ldots, \Theta_n$ are i.i.d. with common Bayesian prior structure distribution $U$. It is assumed that all contracts are issued at the same time, at which there is no claims experience yet.

For notational convenience, we will drop the subscripts $i$. If the insurer aims for equivalence on a group level (where "group" means the entire portfolio), in this respect an unconditional expected loss of zero, the net premium to be paid by each contract upon issue would be equal to $E[X] = E[E[X|\Theta]]$. Note that this premium is an average premium and that, as a consequence, there is cross-subsidization between contracts, or subsidizing solidarity, a concept which has been quantified by De Wit & Van Eeghen (1984) in the following way.

The given premium does not match the risk of an individual faced with an outcome of the first draw equal to $\theta$. The adequate premium, satisfying equivalence on an individual level or micro-equivalence, would then be equal to $E[X|\Theta = \theta]$. So the amount by which an individual subsidizes the collective, defined by Posthuma (1985) as the ex ante transfer, is equal to

$$E[X] - E[X|\Theta = \theta]. \quad (2.1)$$

If this amount is negative, the individual considered is subsidized by the collective. Now the subsidizing solidarity, denoted by $SS$, has been defined in De Wit & Van Eeghen (1984) as the average (over all individuals) of the squared ex ante transfers, or the variance of the expected loss made by the insurer on the individual contract:

$$SS = \text{Var}_\Theta [E[X|\Theta]] = \int_\theta (E[X] - E[X|\Theta = \theta])^2 dU(\theta). \quad (2.2)$$

Notation 1 If, in the remainder of this chapter, we add a subscript $\Theta$ to an expectation ($E$) or a variance ($\text{Var}$) operator, we mean that this expectation or variance is taken with respect to the random variable $\Theta$ (with common prior distribution $U$). If a subscript is not added to either one of the two operators, this means that the expectation or variance is taken with respect to either the aggregate claim amount $X$ (in this section) or the stochastic process $\{X(t)\}_{t \geq 0}$ to be defined in Subsection 2.3.1.

De Wit & Van Eeghen (1984) indicate that the subsidizing solidarity is just one part of the variability of the insurer’s result on the given individual level. In case of perfect risk classification there would also be fluctuation due to the fact that the actual realization of $X$ differs from the given risk premium $E[X|\Theta = \theta]$. This difference

$$E[X|\Theta = \theta] - X \quad (2.3)$$

is
is called \textit{ex post transfer} by Posthuma (1985). De Wit & Van Eeghen (1984) define the \textit{probabilistic solidarity}, specified by PS, as the average (over all individuals) of the variance of the loss in case of equivalence on an individual level:

\begin{equation}
\text{PS} = E_\Theta [\text{Var}[X|\Theta]].
\end{equation}

Hence

\begin{equation}
\text{Var}[X] = \text{Var}_\Theta [E[X|\Theta]] + E_\Theta [\text{Var}[X|\Theta]] = SS + PS.
\end{equation}

\textbf{Remark 2} De Wit & Van Eeghen (1984) use the name "risk solidarity" for SS. We, however, prefer the term "subsidizing solidarity" instead, since in our view the former name does not indicate the type of solidarity accurately enough: the concepts "probabilistic" and "risk" can easily be confused with one another.

\textbf{Remark 3} There is a rationale for using the term "probabilistic solidarity": within each group consisting of individuals with the same unconditional claims distribution, those who make no claims are solidary with those who make severe claims. Nevertheless, contrary to the explanation of subsidizing solidarity, this interpretation is not entirely in line with what most people understand by the word "solidarity". Wilkie (1997), dealing with current risk classification practices in life insurance, defines two forms of insurance, namely mutuality and solidarity. In case of mutuality, each insured contributes to the insurance pool, according to the risk he brings in, whereas the situation of solidarity is characterized by the lack of such a relationship. Wilkie, however, does not consider any quantitative measures.

In non-life insurance, each individual contract builds up its own claims experience, so premiums can be adjusted based among others on that claims experience, using Bayes' rule. The same applies for a group life insurance contract, as pointed out in Norberg (1989). Both approaches have in common that for each contract, the outcome of the first stage draw applies to all risk units included in the contract. The difference is that, on the one hand, a conditional claims distribution being the same for each policy year does not apply if this risk unit is a single life, as each human life's course of events is unique and depends on the whole lifetime, i.e. it can never be repeated again, but on the other hand, a group life contract usually comprises several of such single lives so a unique claims history will be generated. Hence, as a consequence, the subsidizing solidarity gradually decreases in time.

We will now investigate the unobserved heterogeneity and solidarity forms in individual life insurance. At first a very general life insurance model, derived from Norberg (1990, 1991, 1992), will be considered.

\section{A general individual life insurance treaty}

The individual life insurance treaty according to Norberg will be described in Subsection 3.1, concerning the possible states applying to the insured, and in Subsection 3.2, where the premium and benefit payment functions are dealt with and the urn-of-urns model is introduced.
2.3. A general individual life insurance treaty

2.3.1 The stochastic process of visiting several states

Consider a set \( \zeta = \{0, \ldots, J \} \) of possible states of a general life policy. At time \( t > 0 \), the policy is in one and only one particular state \( j \in \zeta \), while at time \( 0 \), the contract is in state \( 0 \). The policy is represented by a sample path of a stochastic process \( \{X(t)\}_{t \geq 0} \), with \( X(t) \) denoting the state of the policy at time \( t \) with \( t \in [0, \infty) \). The stochastic process is defined on some probability space \( (\Omega, \mathcal{F}, P) \). Hence, \( \{X(t)\}_{t \geq 0} \) is a function from \([0, \infty)\) to \( \zeta \). It is assumed to be right-continuous, with a finite number of jumps from one state to another in each interval. The process generates the filtration \( \mathcal{F} = \{\mathcal{F}(t)\}_{t \geq 0} \), being a family of sub-sigma-algebras of \( \mathcal{F} \), where each \( \mathcal{F}(t) \), being some collection of \( \mathcal{F} \)-events, comprises the events of transition up to and including time \( t \). Hence \( \mathcal{F} \) can be interpreted as a description of the evolution of information about the contract throughout time. It is assumed that the process possesses transition intensities, denoted by \( \lambda_{jk}(t) \), specified as

\[
\lambda_{jk}(t) = I_j(t) \mu_{jk}(t),
\]

where \( I_j(t) \) denotes the indicator of the event that the process is in state \( j \) at time \( t \geq 0 \) and \( \mu_{jk}(t) \) is some \( \mathcal{F} \)-adapted process, meaning that \( X(t) \) is measurable with respect to \( \mathcal{F}(t) \) for each \( t \geq 0 \).

In the next subsection, the benefit and premium payment functions will be introduced, both being predictable with respect to \( \mathcal{F} \), which means that benefits and premiums may at any time depend on the past development of the policy.

2.3.2 The benefit and premium payment functions

**Definition 4** We define for any real valued \( s, t \) and any function \( h \) of a real argument:

\[
h(t-) = \lim_{s \uparrow t} h(s).
\]

The policy is incepted at time \( 0^- \). The aggregate value of all benefits paid by the insurer in \([0, t]\), due to the insurance contract and depending on \( \{\mathcal{F}(t)\}_{t \geq 0} \), is expressed by \( B(t) \), assumed to be non-negative, non-decreasing, finite and right-continuous.

**Definition 5** We define \( \int_u^t \) as integration over \((u, t]\).

It is assumed that \( \delta(t) \), specifying the force of interest at time \( t \in [0, \infty) \), is deterministic and continuous. Then the present value at time \( 0 \) of all benefit payments, denoted by \( V(B) \), equals

\[
V(B) = \int_{[0, \infty)} \nu(t) dB(t),
\]

with \( \nu(t) = e^{-\int_0^t \delta(s)ds} \) and \( \nu(0) = 1 \) by definition.

On the other hand, the structure of premium payments up to and including \( t \) is described by \( C(t) \), which, like \( B(t) \), is supposed to be non-negative, non-decreasing,
finite and right-continuous for \( t \geq 0 \). It is presumed that the present value at time 0 of all premium payments is equal to

\[
\pi^* V(C), \quad (2.9)
\]

with \( \pi^* \geq 0 \) representing the premium level and

\[
V(C) = \int_{[0, \infty)} \nu(t) \, dC(t), \quad (2.10)
\]

with \( \nu(t) \) for \( t \in [0, \infty) \) defined as above. Expression (2.10) can be interpreted as a baseline premium payment function.

Conditionally given any outcome of the stochastic process \( \{X(t)\}_{t \geq 0} \), both \( B(t) \) and \( C(t) \) are determined by the insurance agreement.

The insurer's loss suffered during the entire contract period \([0, \infty)\) is indicated by \( L \). It is equal to the difference between the present value of benefits and the present value of premiums:

\[
L = V(B) - \pi^* V(C). \quad (2.11)
\]

**Remark 6** The term "loss" in the above context is not to be confused with the same word related to decision theory (e.g. "quadratic loss").

We assume that the probability space concerning an individual insured depends on the outcome \( \theta \) of a one-dimensional random variable \( \Theta \) having prior distribution \( U \), similar to the one dealt with in the previous section. Then, recalling that the insurance contract is issued at time \( 0^* \), the premium level satisfying equivalence on an individual level, conditionally given \( \Theta = \theta \), is equal to

\[
\tau^* = \frac{\mathbb{E}[V(B) | \Theta = \theta]}{\mathbb{E}[V(C) | \Theta = \theta]} . \quad (2.12)
\]

In the remainder of this chapter, such premium levels will be specified by \( \tau(\theta) \) and called *individual premium levels*. It is assumed that the outcome of \( \Theta \) is not observable and therefore to each contract an *average premium level* pertains, satisfying equivalence on a group level. Such an average premium level is defined as \( \tau \), so

\[
\tau = \frac{\mathbb{E}[V(B)]}{\mathbb{E}[V(C)]} = \frac{\mathbb{E}_\Theta [\mathbb{E}[V(B) | \Theta]]}{\mathbb{E}_\Theta [\mathbb{E}[V(C) | \Theta]]}. \quad (2.13)
\]

**Remark 7** Note that \( \tau \neq \mathbb{E}_\Theta [\tau(\Theta)]. \)

In the next section, solidarity measures will be defined, based on the quantities defined in the above two equations.
2.4 Solidarity measures

The approach in this section will be as follows. First, in Subsection 2.4.1, the entire period will be studied based on the values of $\pi^*$ and $\pi$ satisfying equivalence on an individual level and equivalence on a group level, respectively, and solidarity will be quantified. Then, in Subsection 2.4.2, the loss related to a certain period will be considered.

2.4.1 Considering the entire period of insurance

With respect to the given urn-of-urns model, the observable second part consists of the realized event history $\{F(\infty)\}$. Suppose that the outcome of the first stage is equal to $\theta$. Then (2.11) can be split up as follows:

$$ L = L^{(1)}(\theta) + (L - L^{(1)}(\theta)), \quad (2.14) $$

with

$$ L^{(1)}(\theta) = V (B) - \pi (\theta) V (C), \quad (2.15) $$

denoting the process effect, or the loss that would be suffered in case of equivalence on an individual level. This quantity is the life-analogue of formula (2.3), though with reversed sign, because $L(\theta)$ is considered from the insurer’s - and not from the insured’s - point of view. The process effect has expectation zero on the level of an individual contract. The remaining part of (2.14), named parameter effect, is given by

$$ L^{(2)}(\theta) = L - L^{(1)}(\theta) = (\pi (\theta) - \pi) V (C). \quad (2.16) $$

Recall from Section 2.2 that in the non-life case the difference between the aggregate loss and the process effect is deterministic, equal to the expected loss for the individual contract. This does, however, not apply for life insurance as can be derived from the above formula, since, in general, $L^{(2)}(\theta)$ is a function of the random variable $V (C)$ and therefore stochastic. The only exception is the case of single premium payment, which will be demonstrated as well.

In order to be able to quantify subsidizing solidarity in an appropriate way, just like we did in Section 2.2, we will, just as in Spreeuw & Wolthuis (1997), divide (2.16) further into its expectation and another remaining part. Conditionally, given $\Theta = \theta$, we finally get:

$$ L = L^{(1)}(\theta) + L^{(2a)}(\theta) + L^{(2b)}(\theta), \quad (2.17) $$

where

$$ L^{(2a)}(\theta) = \mathbb{E} [L - L^{(1)}(\Theta) | \Theta = \theta] = (\pi (\theta) - \pi) \mathbb{E} [V (C) | \Theta = \theta], \quad (2.18) $$

denotes the conditional expected loss for the individual contract, given $\Theta = \theta$, and

$$ L^{(2b)}(\theta) = (\pi (\theta) - \pi) (V (C) - \mathbb{E} [V (C) | \Theta = \theta]) \quad (2.19) $$

the remaining part.
Remark 8 If we change the signs of $L^{(2a)}$, we obtain the ex ante transfer defined in Spreew (1996) (who considers the two states model "Alive" and "Dead", for single and level premium payment). It equals the difference between: i) the actuarial present value, concerning the individual, of all premium payments in case of "solidarity" and "equivalence on a group level" and ii) the actuarial present value of all premiums, again concerning the individual, to be paid when the insurer uses the principle of "equivalence on an individual level".

The following characteristics apply to the three components:

$$E[L^{(1)}(\theta)] = E[L^{(2b)}(\theta)] = 0, \forall \theta \in \Theta.$$  \hspace{1cm} (2.20)

$$E[\theta][E[L^{(i)}(\theta)|\Theta]] = 0, \forall i \in \{1, 2a, 2b\}.$$  \hspace{1cm} (2.21)

As

$$E[L] = E[\theta][E[L|\Theta]] = 0,$$  \hspace{1cm} (2.22)

we have that the unconditional variance of the aggregate loss is equal to

$$\text{Var}[L] = \sum_{i,j \in \{1, 2a, 2b\}} E[\theta][E[L^{(i)}(\theta)L^{(j)}(\theta)|\Theta]].$$  \hspace{1cm} (2.23)

Next the six components of this variance will be analyzed. It turns out that:

$$E[\theta][E[(L^{(1)}(\theta))^2|\Theta]] = E[\theta][\text{Var}[L^{(1)}(\theta)|\Theta]]$$

$$= E[\theta][\text{Var}[V(B) - \pi(\Theta)V(C)|\Theta]];$$  \hspace{1cm} (2.24)

$$E[\theta][E[(L^{(2a)}(\theta))^2|\Theta]] = E[\theta][E[(\pi(\Theta) - \pi)E[V(C)|\Theta]^2|\Theta]]$$

$$= \text{Var}[E[V(B) - \pi V(C)|\Theta]];$$  \hspace{1cm} (2.25)

$$E[\theta][E[(L^{(2b)}(\theta))^2|\Theta]] = E[\theta][\text{Var}[(\pi(\Theta) - \pi)V(C)|\Theta]];$$  \hspace{1cm} (2.26)

$$E[\theta][E[L^{(1)}(\Theta)L^{(2b)}(\Theta)|\Theta]] = E[\theta][\text{Cov}[L^{(1)}(\Theta), L^{(2b)}(\Theta)|\Theta]]$$

$$= E[\theta][\text{Cov}[V(B) - \pi(\Theta)V(C), (\pi(\Theta) - \pi)V(C)|\Theta]];$$  \hspace{1cm} (2.27)

$$E[\theta][E[L^{(1)}(\Theta)L^{(2a)}(\Theta)|\Theta]] = E[\theta][E[L^{(2b)}(\Theta)L^{(2a)}(\Theta)|\Theta]] = 0.$$  \hspace{1cm} (2.28)

This yields:

$$\text{Var}[L]$$

$$= E[\theta][\text{Var}[V(B) - \pi(\Theta)V(C)|\Theta]]$$

$$+ \text{Var}[E[V(B) - \pi V(C)|\Theta]]$$

$$+ E[\theta][\text{Var}[(\pi(\Theta) - \pi)V(C)|\Theta]]$$

$$+ 2E[\theta][\text{Cov}[V(B) - \pi(\Theta)V(C), (\pi(\Theta) - \pi)V(C)|\Theta]].$$  \hspace{1cm} (2.29)

Equation (2.29) shows that the variance of the loss for an arbitrary individual contract consists of:
2.4. Solidarity measures

- the expected loss variance in case of equivalence on an individual level,
- the variance of the expected loss,
- the expected variance, due to the deviation of the actual difference between the aggregate loss and the process effect from the expected difference, and
- two times the expected covariance between, on the one hand, the process effect and, on the other hand, the difference between the aggregate loss and the process effect.

We will, in order to remain as close as possible to the solidarity concepts developed by De Wit & Van Eeghen (1984), define the subsidizing solidarity for the entire period \([0, \infty)\), denoted by \(SS_{[0,\infty)}\), as the second component of (2.29), being the variance of the squared expectations of the parameter effects:

\[
SS_{[0,\infty)} = \text{Var}_\Theta [E[V(B) - \pi V(C)|\Theta]].
\] (2.30)

**Case 9 In case of single premium payment, implying**

\[
V(C) = 1,
\] (2.31)

the last two components of the right hand side of (2.29) vanish and the formula reduces to

\[
\text{Var}[L] = \text{E}_\Theta [\text{Var}[V(B)|\Theta]] + \text{Var}_\Theta [E[V(B)|\Theta]],
\] (2.32)

hence obtaining a formula very similar to the non-life case displayed in (2.5). The above equation demonstrates that the second term of (2.29) can actually be considered as the subsidizing solidarity.

The reason why the aggregate loss variance of a general life contract consists of four parts instead of two lies in the fact that the aggregate value of premium payments is stochastic rather than deterministic. The first component of the right hand side of both (2.29) and (2.32) may be defined as the probabilistic solidarity (since it is the expected value of the squared process effects). One might argue that the third and fourth term of the right hand side of (2.29) may contain elements of solidarity as well but these will not be dealt with in this chapter.

The quantities dealt with up to now concern the entire contract period. In view of the fact that an insurance treaty usually comprises several years, it is desirable to allocate the aggregate loss due to a single contract to several disjoint periods. This is the topic of the next subsection.

### 2.4.2 Considering part of the entire period

When looking at parts of a contract term, such as policy years, one deals with reserves, amounts at risk and risk premiums. All such quantities are to be updated to experience, i.e. the insured's life history. In this respect, single life policies have an essential property, already noticed by Jewell (1978) in his paper on Bayesian prediction: that a single life
cannot gather any information on his distribution of future remaining lifetime. Jewell proved this in a classical two states framework (with states "Alive" and "Dead"). We will show the validity of this statement in our general case for the prospective reserve.

Prospective reserves need to be calculated in order to be able to allocate the aggregate loss to several disjoint periods. They are written as follows for \( t \geq 0 \), following Norberg (1991):

\[
V^+_F(t, B, C, \pi) = E_{\mathcal{F}(t)} V^+ (t, B, C, \pi), \tag{2.33}
\]

where

\[
V^+ (t, B, C, \pi) = \frac{1}{\nu(t)} \int_{(t, \infty)} \nu(\tau) d(B(\tau) - \pi C(\tau)). \tag{2.34}
\]

Norberg (1991) calls the quantity defined in (2.33) the prospective F-reserve, indicating that the reserve depends on \( T(t) \), being the insured’s life history up to and including time \( t \).

**Notation 10** So the expectation in the right hand side of (2.33) is not an expectation with respect to the stochastic process \( \{X(t)\}_{t\geq 0} \) but is instead an expectation, conditionally given the information \( \mathcal{F}(t) \) at time \( t \geq 0 \).

Conditionally given \( \Theta = \theta \), the reserve meeting the future expected benefits minus premiums should be equal to

\[
V^+_{F,\theta}(t, B, C, \pi) = E_{\mathcal{F}(t) \mid \Theta = \theta} \left[ V^+ (t, B, C, \pi) \right]. \tag{2.35}
\]

From the Bayesian point of view, an adequate reserve at time \( t \), conditionally given the information \( \mathcal{F}(t) \), would therefore be

\[
V^+_F(t, B, C, \pi) = \int_\theta V^+_{F,\theta}(t, B, C, \pi) \, dU(\theta \mid \mathcal{F}(t)), \tag{2.36}
\]

where \( dU(\theta \mid \mathcal{F}(t)) \) denotes the posterior distribution of \( \Theta \), conditionally given the life history \( \mathcal{F}(t) \). Define \( I_{\mathcal{F}(t)} \) as the indicator function valued one if the event \( \mathcal{F}(t) \) is realized and zero in other cases. Then, according to Bayes’ rule, this distribution turns out to be

\[
dU(\theta \mid \mathcal{F}(t)) = \frac{E \left[ I_{\mathcal{F}(t)} \mid \Theta = \theta \right] \, dU(\theta)}{\int_\theta E \left[ I_{\mathcal{F}(t)} \mid \Theta = \phi \right] \, dU(\phi)} = \frac{E \left[ I_{\mathcal{F}(t)} \mid \Theta = \theta \right] \, dU(\theta)}{\int_\theta E \left[ I_{\mathcal{F}(t)} \mid \Theta = \phi \right] \, dU(\phi)}; \tag{2.37}
\]

hence

\[
V^+_F(t, B, C, \pi) = \int_\theta V^+_{F,\theta}(t, B, C, \pi) \frac{E \left[ I_{\mathcal{F}(t)} \mid \Theta = \theta \right] \, dU(\theta)}{E \left[ I_{\mathcal{F}(t)} \right]} \tag{2.38}
\]

\[
= \int_\theta E \left[ V^+ (t, B, C, \pi) \mid \Theta = \theta \right] \, dU(\theta) \frac{E \left[ I_{\mathcal{F}(t)} \mid \Theta = \theta \right]}{E \left[ I_{\mathcal{F}(t)} \right]} \tag{2.39}
\]

\[
= \frac{E \left[ V^+ (t, B, C, \pi) \right]}{E \left[ I_{\mathcal{F}(t)} \right]} \tag{2.40}
\]

\[
= E_{\mathcal{F}(t)} V^+ (t, B, C, \pi); \tag{2.41}
\]
which is the reserve, conditionally given the history \( F(t) \). Regarding the single or multiple decrement model including only one non-absorbing state "Alive", assuming that the contract comes to an end once this state is left, the above result implies that there is actually no difference between updating to the experience and not updating at all. In the given case, all the premiums and reserves would be fixed while the insured is alive.

**Remark 11** As mentioned before, Jewell (1978) restricts himself to the classical case of two states. Then \( F(t) \) discloses nothing else than the fact that the individual has survived up to time \( t \).

**Remark 12** Contrary to single life business, in group life insurance the posterior distribution makes real sense, as demonstrated in Norberg (1989). The reason is that collective contracts usually involve more than one individual and hence each contract builds up its own, unique, claims experience. Since it is assumed that the same latent risk factor, being independent of time, applies to all individuals in one and the same group, the claims experience also gives information about the risk profile of the remaining persons. We will, however, not deal with this, since it is beyond the scope of this thesis.

Finally, note that (2.38) remains valid if one replaces the prospective reserve by any function concerning future payments and having \( F(t) \) as argument. Examples are, as mentioned before, amounts at risk and risk premiums.

Although there is no difference between updating quantities such as reserves to the experience and not updating at all, prior distributions can nevertheless help to derive solidarity measures on the level of a partial period and this is what will be done next.

We define (with \( X(t) \) as defined above)

\[
N_{jk}(t) = \# \{ \tau \in (0,t] : X(\tau-) = j, X(\tau) = k \} ; \quad j \neq k, t > 0, \tag{2.39}
\]

as the number of transitions from state \( j \) to state \( k \) up to and including time \( t > 0 \), and \( N_{jk}(0) = 0 \) for all \( j \neq k \).

In the following analyses it is no longer necessary to deal with both a benefit and a premium payment function, because emphasis will be placed on risk premiums, and therefore we will combine \( B \) and \( C \) into one single aggregate payment function \( A(t) \), such that

\[
A(t) = B(t) - \pi C(t), \quad t \in [0,\infty). \tag{2.40}
\]

Next \( A(\cdot) \) itself will be split in quantities \( A_{j}^{+}(\cdot) \) and \( A_{jk}^{-}(\cdot) \), where \( A_{j}^{+}(\cdot) \) is such that for \( t > s \), \( A_{j}^{+}(t) - A_{j}^{+}(s) \) represents the aggregate value of net benefit payments less premium payments during the stay in state \( j \), \( j \in \zeta \), and \( A_{jk}^{-}(\cdot) \) provides the benefit minus premium payments upon a transition from \( j \) to \( k \), \( j, k \in \zeta \). It is assumed that, for any \( j \in \zeta \), \( A_{j}^{+}(\cdot) \) has the same properties as \( A(\cdot) \), hence finite and right-continuous, and that, for any \( j, k \in \zeta \), \( A_{jk}^{-}(\cdot) \) is left-continuous and finite. Both \( A_{j}^{+}(\cdot) \) and \( A_{jk}^{-}(\cdot) \) are supposed to be predictable with respect to \( F \), implying that both benefits and premiums may depend on
the past development of the policy. The total stream of payments $A(t), t \geq 0$, is specified to be of the form

$$A(0-) = 0,$$

(2.41)

and

$$dA(t) = \sum_j I_j(t) dA^*_j(t) + \sum_{j,k,j\neq k} a^*_{jk}(t) dN_{jk}(t).$$

(2.42)

The loss concerning a partial period $(s, t] \subset [0, \infty)$ is, just as in Norberg (1992), defined as

$$L(s, t] = \int_{[s, t]} \nu dA + \nu(t) V^+_F(t, A, \pi) - \nu(s) V^+_F(s, A, \pi),$$

(2.43)

with, for all $s > 0$, $V^+_F(s, A, \pi) = V^+_F(s, B, C, \pi)$, defined in (2.38). Now let

$$V_j(t, \pi) = E_{\mathcal{F}(t)} \left[ V^+(t, A, \pi) | X(t) = i \right]$$

(2.44)

be the conditional reserve given the life history $\mathcal{F}(t)$ and $X(t) = i$. Norberg (1992) derived that the loss expressed in the above formula is

$$L(s, t] = \int_{(s, t]} \sum_{j,k} \nu(\tau) R_{jk}(\tau) (dN_{jk}(\tau) - \lambda_{jk}(\tau) d\tau),$$

(2.45)

with

$$R_{jk}(\tau) = a^*_{jk}(\tau) + V_k(\tau, \pi) - V_j(\tau, \pi),$$

(2.46)

expressing the amount at risk at time $\tau$.

Periodic losses will be considered with a certain state at the beginning of the period given. Let $L^{(s)}_{(s, t]}$ be the loss allocated to period $(s, t]$, conditionally given $X(s) = j$. We first consider this loss quantity with respect to an infinitesimal time interval $(t-, t)$, so $s = t-$. It turns out that:

$$L^{(s)}_{(t-, t]} = \sum_{k \neq j} \nu(t-) R_{jk}(t-) (dN_{jk}(t-) - \mu_{jk}(t-) dt).$$

(2.47)

We denote the intensity of transition from $j$ to $k$ at time $\tau$, conditionally given $X(\tau) = j$ and $\Theta = \theta$, by $\mu_{jk,\theta}(\tau)$. The following relation between marginal and conditional intensities holds:

$$\mu_{jk}(\tau) = \int_\theta \mu_{jk,\theta}(\tau) dU(\theta|\mathcal{F}(\tau), X(\tau) = j).$$

(2.48)
Conditionally given $\Theta = \theta$, $L^{(j)}_{(t_-,.t]}$ can be divided into

\[ L^{(j)}_{(t_-,.t]} = L^{(j)(1)}_{(t_-,.t],\theta} + L^{(j)(2)}_{(t_-,.t],\theta}, \quad (2.49) \]

with

\[ L^{(j)(1)}_{(t_-,.t],\theta} = \sum_{k \neq j} \nu (t_-) R_{jk} (t-) \left( dN_{jk} (t-) - \mu_{jk,\theta} (t-) \right) dt, \quad (2.50) \]

and

\[ L^{(j)(2)}_{(t_-,.t],\theta} = \sum_{k \neq j} \nu (t-) R_{jk} (t-) \left( \mu_{jk,\theta} (t-) - \mu_{jk} (t-) \right) dt, \quad (2.51) \]

representing the process and parameter effect respectively. Note that the latter is deterministic, so the variance of $L^{(j)}_{(t_-,.t]}$ is equal to

\[ \text{Var} \left[ L^{(j)}_{(t_-,.t]} \right] = \text{E}_\theta \left[ \text{Var} \left[ L^{(j)}_{(t_-,.t],\theta} | \Theta, \mathcal{F} (t-) \right] \right] + \text{Var}_\theta \left[ \text{E} \left[ L^{(j)}_{(t_-,.t],\theta} | \Theta, \mathcal{F} (t-) \right] \right], \quad (2.52) \]

where

\[ \text{Var}_\theta \left[ \text{E} \left[ L^{(j)}_{(t_-,.t],\theta} | \mathcal{F} (t-) \right] \right] = \int_\theta \left( \sum_{k \neq j} \nu (t-) R_{jk} (t-) \left( \mu_{jk,\theta} (t-) - \mu_{jk} (t-) \right) dt \right)^2 \cdot dU (\theta | \mathcal{F} (t-), X (t-) = j), \quad (2.53) \]

is a measure for the subsidizing solidarity.

Recall, however, that this quantity is only deterministic from the point of view at time $t-$ when $X (t-)$ is known. Besides, such a division similar to the one in the non-life case is not possible if one considers a period longer than infinitesimal. For instance, if we define for $s < t-$, analogous to (2.50) and (2.51):

\[ L^{(1)}_{(s,.t],\theta} = \int_{(s,.t]} \sum_{j,k \neq j} \nu (\tau) R_{jk} (\tau) \left( dN_{jk} (\tau) - I_j (\tau) \mu_{jk,\theta} (\tau) \right) d\tau, \quad (2.54) \]

and

\[ L^{(2)}_{(s,.t],\theta} = \int_{(s,.t]} \sum_{j,k \neq j} I_j (\tau) \nu (\tau) R_{jk} (\tau) \left( \mu_{jk,\theta} (\tau) - \mu_{jk} (\tau) \right) d\tau, \quad (2.55) \]

respectively, then the latter would be stochastic, even if $X (s)$ were known, due to the fact that more than one transition in the given period may occur. One can, however, similarly to the approach in Subsection 2.4.1, base the subsidizing solidarity on the expected value of (2.55) and thus obtain:

\[ \text{SS}_{(s,.t]} = \text{E}_\theta \left[ \text{E}^2 \left[ L^{(2)}_{(s,.t],\theta} | \Theta \right] \right]. \quad (2.56) \]

In the next section, it will be investigated if the parameter effects for part of the insurance term can be compared with those concerning the entire contract period (derived in the previous subsection).
2.5 Relations between entire and partial periods

Norberg (1992) formulated the partial losses, defined in (2.43), in such a way that, except for lump sum payments at time 0, they together form the loss for the whole contract period, defined in (2.11). This can be shown as follows. For any $0 = t_0 < t_1 < \ldots < t_n = \infty, n \in N \setminus \{0\}$, we have

$$\sum_{i=1}^{n} L(t_{i-1},t_i) = L(0,\infty) = \int_{0,\infty} \nu dA + \nu(\infty)V^+_{\infty}(\infty,A,\pi) - \nu(0)V^+_{0}(0,A,\pi)$$

$$= L - A(0), \quad (2.57)$$

because of the principle of equivalence. In this section, we will, first of all, show that a relationship similar to (2.57) in general does not hold between, on the one hand, the parameter effect for the whole term, defined in (2.16), and, on the other hand, the parameter effect for parts of the term, defined in (2.55). An exception arises under a certain relationship between the benefit functions $A^0_j(\cdot)$ and $a^0_{jk}(\cdot) \ (j,k \in \zeta)$, as will be demonstrated too.

When one compares $L_{(s,t],\pi(\theta)}^{(2)}$, as defined in (2.55) with $L_{(s,t],\theta}^{(2)}$ defined in (2.16), the latter depends both on $\pi(\theta)$ and $\pi$, while the former is not a function of $\pi(\theta)$ (since it is a function of reserves which in turn depend only on $\pi$). To show it in a mathematical sense, the process effect $L_{(s,t],\theta}^{(1)}$, defined in (2.15), can, in similar fashion to (2.45), be allocated to disjoint intervals $(s,t]$:

$$L_{(s,t],\pi(\theta)}^{(1)}(\theta) = \int_{(s,t]} \sum_{j \neq k} \nu(\tau) R_{jk,\theta}(\tau) \left( dN_{jk}(\tau) - \lambda_{jk,\theta}(\tau) \right) d\tau, \quad (2.58)$$

with

$$R_{jk,\theta}(\tau) = a^0_{jk}(\tau) + V_k(\tau,\pi(\theta)) - V_j(\tau,\pi(\theta)). \quad (2.59)$$

In (2.59), $R_{jk,\theta}(\tau)$ is defined as the conditional amount at risk pertaining to the transition from $j$ to $k$, applying in case of equivalence on an individual level, conditionally given $\Theta = \theta$. Similarly, $V_i(\tau,\pi(\theta)), i \in \{j,k\}$, is the conditional reserve pertaining to state $i$, applying in case of equivalence on an individual level, conditionally given $\Theta = \theta$.

However, at least in general

$$L_{(s,t]} - L_{(s,t],\pi(\theta)}^{(1)}(\theta) \neq L_{(s,t],\theta}^{(2)} = L_{(s,t]} - L_{(s,t],\theta}, \quad (2.60)$$

since, again in general,

$$L_{(s,t],\pi(\theta)}^{(1)}(\theta) \neq L_{(s,t],\theta}^{(1)}, \quad (2.61)$$

the reason for the last mentioned being that, in general for any non-absorbing state $i \in \zeta$,

$$V_i(\tau,\pi(\theta)) \neq V_i(\tau,\pi). \quad (2.62)$$
The rationale is that, regarding any life insurance contract, premiums paid in a certain
period are usually not equal to the risk premium for that period. As a consequence,
reserves are in general nonzero.

Next, some sufficient conditions will be derived, under which the inequality-signs in the
above equations can be replaced by equality-signs. We forget about $A$, $\pi$ and $\pi(\theta)$ as
quantities themselves but keep them as an argument of several functions, indicating that
they are based on premiums satisfying equivalence on an individual level or equivalence on
a group level, respectively. The quantities $A_j^\circ$ and $a_{jk}^\circ$ will be redefined. In the remainder
of this section, it is assumed that:

\[ dA_j^\circ (t) \leq 0 \land a_{jk}^\circ (t) \geq 0 \quad \forall j \in \zeta; t \geq 0. \]  

Hence there are premiums to be paid only during sojourn in a certain state while benefits
are due only in case of transition. We specify the average premium due in $(t-, t]$ to be
equal to the expected value of the benefits due:

\[ dA_j^\circ (t) = -\sum_{k \in \zeta} a_{jk}^\circ (t) \mu_{jk} (t) dt, \forall j \in \zeta; t \geq 0; F(t). \]  

Furthermore, we define the individual premium $A_j^{\circ, \theta}(t)$ in the same infinitesimal time
interval to be such that the expected benefits are covered on an individual level:

\[ dA_j^{\circ, \theta} (t) = -\sum_{k \in \zeta} a_{jk}^{\circ, \theta} (t) \mu_{jk, \theta} (t) dt, \forall j \in \zeta; t \geq 0; F(t); \theta. \]  

In other words, each infinitesimal premium paid is always equal to the infinitesimal risk
premium. We will call the premiums as specified in (2.64) and (2.65) natural premiums.

It can be verified recursively that:

\[ V_F^+(s, A, \pi) = 0, \quad \forall s \geq 0; F(t) \]  
\[ V_{F, \theta}^+(s, A, \pi(\theta)) = 0, \quad \forall s \geq 0; F(t); \theta \]

hence:

\[ R_{jk} (t) = R_{jk, \theta} (t) = a_{jk}^\circ (t) \quad \forall t \geq 0; j, k \in \zeta, \]

and

\[ L^{[1]}_{(s, \theta), \pi(\theta)} (\theta) = L^{[1]}_{(s, \theta), \theta}; \quad \forall 0 \leq s < t; \theta, \]

so for any $0 = t_0 < t_1 < \ldots < t_n = \infty, n \in \mathbb{N} \setminus \{0\}$, we have

\[ L^{(2)}_\theta = \sum_{i=1}^{n} L_{(t_{i-1}, t_i), \theta}^{(3)}. \]

**Remark 13** Instead of (2.63), one may also assume:

\[ dA_j^\circ (t) \geq 0 \land a_{jk}^\circ (t) \leq 0, \]

resulting in the same conclusions if one adopts equations (2.64) and (2.65). Regarding
the classical two states model, comprising the states "Alive" and "Dead", this can be used
if one deals with life annuities and hence negative amounts at risk.

In the next section we will consider the hierarchical Markov chain, which enables us to
allocate losses not only to periods, but also to states.
2.6 The hierarchical Markov chain

Assuming the Markov property means that the intensities $\mu_{jk}(\cdot)$ presented in (2.6) are deterministic functions. Furthermore, the benefit functions $A^j \phi(\cdot)$ and $a^j_\phi(\cdot)$ are taken to be deterministic as well. That the Markov chain is hierarchical means that states can only be either strongly transient or absorbing. A state is strongly transient if it cannot be re-entered, once it is left. (A state is transient if the total expected time that will be spend in that state is finite; this does not imply that a state can never be re-entered once it has been left.) A contract entering an absorbing state will remain in that state for ever with certainty.

We will not only deal with the hierarchical Markov chain in general (in Subsection 2.6.1), but also with the special case of fully discrete contracts (in Subsection 2.6.2), since the contract to be considered in Chapter 3, where solidarity is discussed as well, is also based on payments at integer times.

2.6.1 General case

We assume that the contract is in state 0 at initial time 0. Let $T^{(0)}$ be the random variable representing the time-of-departure from state 0. This implies that the contract has remained in state 0 in $[0,T^{(0)}]$. According to Norberg (1992), the loss in state 0 is equal to

$$L^{(0)} = \nu(T^{(0)}) R_{0\zeta}(T^{(0)}) - \int_{[0,T^{(0)}]} \nu(\tau) \sum_{k \in \zeta} R_{0k}(\tau) \mu_{0k}(\tau) d\tau,$$

where $R_{0\zeta}(\cdot) = \sum_{k \in \zeta} R_{0k}(\cdot)$, by definition.

**Definition 14** For any function $f$ having states $0, k \in \zeta$ as an argument, the following notational convention is adopted:

$$f_{0\zeta} = \sum_{k \in \zeta} f_{0,k}.$$

We can then allocate the loss to years $(m, m+1]$, where $m$ is integer valued. Now define

$$d\Pi r_{0k}(\tau) = R_{0k}(\tau) \mu_{0k}(\tau) d\tau$$

as the infinitesimal risk premium based on the marginal transition rate $\mu_{0k}(\tau)$, and

$$pr_{0k}(m,s) = \int_{(m,s]} \nu(\tau) d\Pi r_{0k}(\tau),$$

as the corresponding discounted risk premium concerning the transition from 0 to $k$ over the period $(m,s]$. For convenience, let $pr_{0k}(m) = pr_{0k}(m,m+1)$. Conditionally given $T^{(0)} = t$, we then have, if $\Omega_0(m)$ denotes the loss due to state 0 in the $(m+1)$-th year, or year $(m,m+1]$, $m > 0$ and integer:

$$\Omega_0(m; t) = \begin{cases} -\nu(m) pr_{0\zeta}(m) & m + 1 < t \\ \nu(t) R_{0\zeta}(t) - \nu(m) pr_{0\zeta}(m,t) & m < t \leq m + 1 \\ 0 & m \geq t \end{cases}.$$
Remark 15 Wolthuis (1994), dealing with the hierarchical Markov chain, considers the reduced risk aggregate loss function, in this respect, conditionally given $T^{(0)} = t$, equal to $\nu(t)(A(t) + V_K(t))$, where $K$ is a random variable denoting the next state entered. He shows that this function results in the same loss allocation as above, so regarding hierarchical Markov chains, Norberg's loss function coincides with Wolthuis' reduced risk aggregate loss function.

Just as in the previous section, the above periodic loss can be split in a process effect and a parameter effect, as specified below, conditionally given $\Theta = \theta$, by $\Omega_{0,\theta}^{(1)}(m; t)$ and $\Omega_{0,\theta}^{(2)}(m; t)$, respectively:

\[
\Omega_{0,\theta}^{(1)}(m; t) = \begin{cases} 
-\nu(m) \prod_{0, \Theta}(m) & m + 1 < t \\
\nu(t) R_{0, \Theta}(t) - \nu(m) \prod_{0, \Theta}(m, t) & m < t \leq m + 1 \\
0 & m \geq t 
\end{cases},
\]

and

\[
\Omega_{0,\theta}^{(2)}(m; t) = \begin{cases} 
\nu(m) (\prod_{0, \Theta}(m) - \prod_{0, \Theta}(m)) & m + 1 < t \\
\nu(m) (\prod_{0, \Theta}(m, t) - \prod_{0, \Theta}(m)) & m < t \leq m + 1 \\
0 & m \geq t 
\end{cases}.
\]

In the above specifications we used the following quantity

\[
\prod_{0, \Theta}(m, t) = \int_{(m, t]} \frac{\nu(\tau)}{\nu(m)} d\Pi_{0, \Theta}(\tau),
\]

where

\[
d\Pi_{0, \Theta}(\tau) = R_{0, \Theta}(\tau) \mu_{0, \Theta}(\tau) d\tau
\]

denotes the infinitesimal risk premium, conditionally given $\Theta = \theta$. Note that the quantities in (2.78) are again stochastic. The expected value, equal to

\[
E\left[\Omega_{0,\theta}^{(2)}(m; T^{(0)})\right] = \nu(m) e^{-\int_0^m \mu_{0, \Theta}(\tau) d\tau}
\]

\[
\cdot \left\{ \int_0^{m+1} \frac{\nu(m+s)}{\nu(m)} e^{-\int_m^{m+s} \mu_{0, \Theta}(\tau) d\tau} \mu_{0, \Theta}(m + s) \cdot \left( \prod_{0, \Theta}(m, m + s) - \prod_{0, \Theta}(m, m + s) \right) ds 
\right. 
\]

\[
+ \nu(m+1) e^{\int_m^{m+1} \mu_{0, \Theta}(\tau) d\tau} \cdot \left( \prod_{0, \Theta}(m, m + 1) - \prod_{0, \Theta}(m, m + 1) \right) \}
\]

\[
(2.81)
\]

can again serve as a basis to determine the subsidizing solidarity. The last mentioned quantity is denoted by $SS_{(m, m+1)}$ and displayed below:

\[
SS_{(m, m+1)} = E_{\Theta} \left[ E_{\Theta}^2 \left[ \Omega_{0, \Theta}^{(2)}(m; T^{(0)}) \mid \Theta \right] \right] = \int_0^2 E_{\Theta}^2 \left[ \Omega_{0, \Theta}^{(2)}(m; T^{(0)}) \right] dU(\theta).
\]

(2.82)

One drawback of the above approach may be that the actuarial discounting factors

\[
\nu(m) e^{-\int_0^m \mu_{0, \Theta}(\tau) d\tau}
\]

\[
(2.83)
\]
in equation (2.81) have a diminishing impact on solidarity for later policy years. So the question is whether the quantities (2.82) for subsequent policy years can be compared with one another in a reasonable way.

Therefore, we will also present solidarity measures having the property that actuarial discounting takes place to the beginning of the policy year considered. The contract should at that time still be in state 0 in order to let this approach make sense. Then the basis of the subsidizing solidarity is the expression between large curly brackets in (2.81). This yields the following measure, defined as $SS^*_{(m,m+1)}$:

$$SS^*_{(m,m+1)} = E_{\theta,T(0)>m} \left[ \mathbb{E}^2 \left[ \frac{\Omega_{0,\theta}(m;T(0))}{\nu(m)e^{-\int_0^m \mu_{0,c,\theta}(\tau)d\tau}} \mid \Theta, T(0) > m \right] \right]$$

$$= \int_\theta \mathbb{E}^2 \left[ \frac{\Omega_{0,\theta}(m;T(0))}{\nu(m)e^{-\int_0^m \mu_{0,c,\theta}(\tau)d\tau}} \right] d\Theta \left( \theta \mid T(0) > m \right). \quad (2.84)$$

It will turn out in the following subsection that the formulas (2.82) and (2.84) reduce to simpler expressions in case a discrete life insurance or life annuity in a classical two-states model applies.

## 2.6.2 Fully discrete insurances and annuities in a classical two-states framework

The classical hierarchical Markov model comprises the strongly transient state ”Alive” and the absorbing state ”Dead”. In this subsection, it is assumed that no premiums or benefits are paid due to remaining in the state ”Dead”. As a consequence, the reserve applying to the case of ”Dead” is zero at any moment. We will leave out any subscripts $j$ and $k$ related to states, concerning the intensities $\mu_{jk}(\cdot)$ and $\mu_{jk,\theta}(\cdot)$ (forces of mortality), the benefit-less-premium payment functions $a_{\theta}^j(\cdot)$ (benefit or premium due to dying) and $A_{\theta}^j(\cdot)$ (benefit or premium due to remaining alive), the reserves $V_j(\cdot)$, the amounts at risk $R_{jk}(\cdot)$ and the risk premiums $p_{rjk}(m + \tau)$ and $p_{rjk,\theta}(m + \tau)$, since this all does not give rise to misunderstandings. The same holds for the superscript (0) of the random variable $T^{(0)}$, actually representing the individual’s remaining lifetime.

Fully discrete insurances and annuities have the property that both benefits and premiums, whether they are due to remaining in a state or to transition from one state to another, are paid either at the beginning or at the end of the policy year. In this subsection it is assumed that benefits due upon death in a certain policy year are paid at the beginning of the next year, in order to satisfy, as stated in Subsection 2.4.2, the requirement that $a^\circ(\cdot)$ is left-continuous. Premiums are considered to be negative benefits. So referring to a notation used earlier in this chapter, we have that in policy year $(m, m + 1]$, the benefit paid in case of death is equal to:

$$a^\circ(m + \tau) = \frac{\nu(m + 1)}{\nu(m + \tau)} a^\circ(m + 1), \quad 0 < \tau \leq 1. \quad (2.85)$$

Benefits and premiums due to being alive will be paid at the end of each year:

$$dA^\circ(m + t) = A^\circ(m + 1) d[t], \quad -1 < t < \infty. \quad (2.86)$$
2.6. The hierarchical Markov chain

where \( [t] \) denotes the largest integer smaller than or equal to \( t \). The above specification satisfies the requirement, again as stated in Subsection 2.4.2, that \( A^o (\cdot) \) is right-continuous. The time variable \( t \) has been allowed to fall below zero in order to be able to specify benefit and premium payments at time 0— as well. It is assumed here that, for \( m \in \{0,1,\ldots\} \), premiums paid due to being alive at time \( m - \), i.e. just before \( m \), account for policy year \((m, m + 1]\).

We have, for \( 0 < \tau < 1 \):

\[
V (m + \tau) = \frac{\nu (m + 1)}{\nu (m + \tau)} \left( e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} V ((m + 1) -) + \left( 1 - e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} \right) a^o (m + 1) \right).
\]  

(2.87)

Remark 16 Note that in this case

\[
V (m + 1) = V ((m + 1) -) - A^o (m + 1).
\]  

(2.88)

So \( R(m + \tau) \), being the amount at risk at time \( m + \tau \), with \( 0 < \tau \leq 1 \), satisfies the following equality:

\[
R (m + \tau) = a^o (m + \tau) - V (m + \tau)
\]

\[
= \frac{\nu (m + 1)}{\nu (m + \tau)} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} (a^o (m + 1) - V ((m + 1) -))
\]

\[
= \frac{\nu (m + 1)}{\nu (m + \tau)} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} R ((m + 1) -).
\]  

(2.89)

The risk premium \( pr (m + \tau) \), based on the marginal forces of mortality \( \mu (\cdot) \) for \( 0 < \tau \leq 1 \) thus proves to be

\[
pr (m, m + \tau)
\]

\[
= \int_{[0,\tau]} \frac{\nu (m + s)}{\nu (m)} R (m + s) \mu (m + s) \, ds
\]

\[
= \int_{[0,\tau]} \frac{\nu (m + s) \nu (m + 1)}{\nu (m)} \frac{\nu (m + 1)}{\nu (m + s)} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} R ((m + 1) -) \mu (m + s) \, ds
\]

\[
= \frac{\nu (m + 1)}{\nu (m)} R ((m + 1) -) \int_{[0,\tau]} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} \mu (m + s) \, ds.
\]  

(2.90)

In the same way, the risk premium related to the force of mortality \( \mu_\theta (\cdot) \) is, again for \( 0 < \tau \leq 1 \) equal to

\[
pr_\theta (m, m + \tau)
\]

\[
= \int_{[0,\tau]} \frac{\nu (m + s)}{\nu (m)} R (m + s) \mu_\theta (m + s) \, ds
\]

\[
= \int_{[0,\tau]} \frac{\nu (m + s) \nu (m + 1)}{\nu (m)} \frac{\nu (m + 1)}{\nu (m + s)} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} R ((m + 1) -) \mu_\theta (m + s) \, ds
\]

\[
= \frac{\nu (m + 1)}{\nu (m)} R ((m + 1) -) \int_{[0,\tau]} e^{-\int_{m+\tau}^{m+1} \mu(\tau) d\tau} \mu_\theta (m + s) \, ds.
\]  

(2.91)
Equality (2.81) reduces to

\[ E \left[ \Omega_{m}^{(2)} (m; T) \right] = \nu(m) e^{-\int_{0}^{m} \mu_{0}(r) dr} \frac{\nu(m+1)}{\nu(m)} R((m+1) -) \]

\[ \cdot \int_{[0,1]} e^{-\int_{m}^{m+t} \mu_{0}(r) dr} (\mu_{0} (m+t) - \mu(m+t)) e^{-\int_{m}^{m+t} \mu_{0}(r) dr} dt \]

\[ = \nu(m+1) e^{-\int_{0}^{m} \mu_{0}(r) dr} R((m+1) -) \left( e^{-\int_{m}^{m+1} \mu(r) dr} - e^{-\int_{m}^{m+1} \mu_{0}(r) dr} \right). \]  

(2.92)

Hence, the values for (2.82) and (2.84) are as follows:

\[ SS_{m,m+1} \]

\[ = \left( \frac{\nu(m+1) R((m+1) -)}{\nu(m)} \right)^{2} \int_{\theta} \left( e^{-\int_{0}^{m} \mu_{0}(r) dr} \left( e^{-\int_{m}^{m+1} \mu(r) dr} - e^{-\int_{m}^{m+1} \mu_{0}(r) dr} \right) \right)^{2} dU(\theta); \]  

(2.93)

and

\[ SS_{(m,m+1)}^{(*)} \]

\[ = \left( \frac{\nu(m+1) R((m+1) -)}{\nu(m)} \right)^{2} \int_{\theta} \left( e^{-\int_{m}^{m+1} \mu(r) dr} - e^{-\int_{m}^{m+1} \mu_{0}(r) dr} \right)^{2} dU(\theta \mid T > m). \]  

(2.94)

The latter quantity clearly shows the concept of Subsidizing Solidarity, namely the average, over all individuals, of the squared difference between the average risk premium, equal to:

\[ \frac{\nu(m+1) R((m+1) -)}{\nu(m)} \left( 1 - e^{-\int_{m}^{m+1} \mu(r) dr} \right), \]  

(2.95)

and the risk premium satisfying equivalence on an individual level, equal to

\[ \frac{\nu(m+1) R((m+1) -)}{\nu(m)} \left( 1 - e^{-\int_{m}^{m+1} \mu_{0}(r) dr} \right). \]  

(2.96)

A numerical illustration of both the above quantities will be given in the next section.

### 2.7 Example: frailty models

In a frailty model, the unobservable outcome \( \theta \) acts multiplicatively on the intensity:

\[ \mu_{jk, \theta}(t) = \theta \mu_{jk, 1}(t), \quad \theta \geq 0, \]

(2.97)

where \( \mu_{jk, 1}(t) \) denotes the "baseline" rate of transition from state \( j \) to state \( k \). In this case, \( \theta \) is interpreted as the frailty of the individual: the higher \( \theta \) is, the more likely it
2.7. Example: frailty models

is that the individual will make a transition from \( j \) to \( k \) and hence the "frailer" is the individual considered.

The above specification has given rise to a large literature, taking off at the end of the 1970's. One of the pioneering papers concerning the states "Alive" and "Dead", results of which we will use in this section, is the demographically oriented article Vaupel et al. (1979). For a general treatment of frailty models, the reader is referred to Section IX of Andersen et al. (1993). A clear overview on a more elementary level is given in Aalen (1994), including applications in the field of medical statistics. Lancaster (1990) considers the model from an econometric point of view.

Remark 17 In fact, the frailty model as specified in (2.97) is a special case of the so called mixed proportional hazards model dealt with in Lancaster (1990) and indicated as

\[
\mu_{jk}(z, t) = \theta \phi(z, \beta) \mu_{jk1}(t),
\]  

(2.98)

with \( z \) and \( \beta \) representing a vector of covariates and regression coefficients, respectively, and \( \phi(z, \beta) \) being a real valued function of these vectors, in fact expressing the observed heterogeneity between individuals. The mixed proportional hazards model reduces to (2.97) for \( \phi(z, \beta) = 1 \). The most common specification of \( \phi(z, \beta) \) is

\[
\phi(z, \beta) = e^{z^T \beta}.
\]  

(2.99)

If additionally unobserved heterogeneity is supposed not to be present, the mixed proportional hazards model reduces to the Cox model, which will be discussed in Chapter 6.

The specification (2.97) is also used in Norberg (1989). In his contribution, the unobserved random variable \( \Theta \) is assumed to act on all individuals of a group life contract simultaneously, while we assume \( \Theta \) to be an individual effect.

In this example we will restrict ourselves to the classical two states model comprising "Alive" and "Dead", so, using the same argumentation as in Subsection 2.6.2, the subscripts \( j \) and \( k \) can be left out. The same applies to the random variable \( T^{(0)} \), which is replaced with \( T \), the latter indicating the random variable of remaining lifetime of the individual with respect to time 0, being the time-at-issue of an insurance contract. The consequence of the above specification is that the posterior distribution of \( \Theta \), conditional on survival up to time \( t \), is equal to

\[
dU(\theta \mid \mathcal{F}(t)) = \frac{e^{-\theta H(t)} dU(\theta)}{L_{\Theta}(H(t))},
\]  

(2.100)

as indicated in Hougaard (1984), who instead calls the above quantity the "frailty distribution among survivors at time \( t \)". In (2.100),

\[
H(t) = \int_0^t \mu_1(i) \, dt,
\]  

(2.101)

denotes the baseline hazard rate integrated from 0, representing the time at issue, to \( t \), and \( L_{\Theta}(\cdot) \) specifies the Laplace transform of the random variable \( \Theta \):

\[
L_{\Theta}(y) = \int_0 e^{-\theta y} dU(\theta).
\]  

(2.102)
Hence, \(\mu(t)\), the marginal force of mortality, is obtained as follows by substituting (2.100) into (2.48):

\[
\mu(t) = \tilde{\theta}(t) \mu_1(t),
\]

where \(\tilde{\theta}(t) = E[\Theta \mid T > t]\) specifies the mean of the above posterior distribution. Again according to Hougaard (1984), this quantity proves to be

\[
\tilde{\theta}(t) = \frac{-\Lambda_\Theta'(H(t))}{\Lambda_\Theta(H(t))} = -g'(H(t)),
\]

with \(g(y) = \ln[\Lambda_\Theta(y)]\), \(y > 0\).

Note that

\[
dm = \left(\frac{\Lambda_\Theta'(H(t))}{\Lambda_\Theta(H(t))} - \left(\frac{\Lambda_\Theta'(H(t))}{\Lambda_\Theta(H(t))}\right)^2\right) \mu_1(t)
\]

\[
= -\left(\text{Var} \left[\Theta \mid T > t\right] - \text{Var} \left[\Theta \mid T > t\right]\right) \mu_1(t).
\]

Clearly demonstrating that the mean frailty among survivors decreases as a function of time \(t\). This is what one calls the selection effect: as time passes the more frail individuals are most likely to die. Hence the population remaining consists of gradually more robust individuals than before.

Gamma mixture models have traditionally been a very popular instrument in actuarial applications, not in the last place because of their tractability. Therefore we choose \(U(\cdot)\) to be Gamma(\(\alpha, \beta\)) distributed:

\[
dU(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta} d\theta, \quad \alpha > 0, \beta > 0.
\]

We then have

\[
dU(\theta \mid T > t) = \frac{(\beta + H(t))^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-(\beta + H(t))\theta},
\]

so the posterior distribution, conditionally given survival up to time \(t\), is again Gamma distributed, with parameters \(\alpha\) and \(\beta + H(t)\). It follows that

\[
\tilde{\theta}(t) = \frac{\alpha}{\beta + H(t)},
\]

and that the marginal probability to survive up to time \(t\) is:

\[
e^{-\text{\int}_0^t \mu(s) ds} = \left(\frac{\beta}{\beta + H(t)}\right)^\alpha.
\]
2.7. Example: frailty models

We will consider a discrete endowment insurance with term \( n \) and lump sum \( D \). This implies that, for \( m \in \{0, \ldots, n-1\} \), in case of death somewhere in year \( (m, m+1] \), a benefit \( D \) is paid at time \( (m+1)+ \), where, by definition, for any real valued \( t \):

\[
t^+ = \lim_{s \downarrow t} s.
\]

If the insured is alive at time \( n^- \), he will receive \( D \) from the insurer as well. We take \( \delta(t) \), which was defined to be the force of interest at time \( t \), to be constant and equal to \( \delta \) for all \( t \geq 0 \). Three cases of premium payment are distinguished:

1. Single premium payment.
2. Level premium payment.
3. Natural premium payment.

To quantities pertaining to cases 1, 2 and 3, the superscripts \( SP, LP \) and \( NP \), respectively, will be added. For our next analysis we use the following property which is well known in life actuarial science:

\[
A(n) = 1 - d \cdot \tilde{a}(n),
\]

where \( A(n) \) denotes the single premium for an endowment insurance with duration \( n \) (with the whole life insurance as a special case if \( n = \infty \)), based on some distribution of the remaining lifetime, and \( \tilde{a}(n) \) indicates the single premium for the annuity-due based on the same distribution of remaining lifetime. Finally, \( d = 1 - e^{-\delta} \). Assuming that the individual is aged \( x \) years upon issue of the contract, we will, in order to remain familiar with the field of classical life contingencies, use the following symbol for the single premium of the annuity-due with duration \( n \), based on the marginal distribution of \( T \):

\[
\tilde{a}_{x\cdot n} = E \left[ \int_0^{T_{\wedge n}} e^{-\delta s} ds \right] = \sum_{m=0}^{n-1} e^{-\delta m} \left( \frac{\beta}{\beta + H(m)} \right)^\alpha,
\]

where \( x \wedge y \) denotes the minimum of \( x \) and \( y \). The single premium is paid upon issue, i.e. at time \( 0^- \), and is equal to \( D \left( 1 - d \cdot \tilde{a}_{x\cdot n} \right) \). Then (2.30), actually the second component of (2.32), is equal to

\[
SS_{[0,n]}^{SP} = (dD)^2 \int_\theta \left( \sum_{m=0}^{n-1} e^{-\delta m} \left( e^{-\theta H(m)} - \left( \frac{\beta}{\beta + H(m)} \right)^\alpha \right) \right)^2 dU(\theta).
\]

In case of level premium payment, the premium is paid at time \( m^- \) with \( m = 0, \ldots, n-1 \), while the insured is alive. This level premium is \( D \left( (\tilde{a}_{x\cdot n})^{-1} - d \right) \).

The Subsidizing Solidarity for the entire period in this case proves to be

\[
SS_{[0,n]}^{LP} = D^2 \int_\theta \left( \sum_{m=0}^{n-1} e^{-\delta m} \left( e^{-\theta H(m)} - \left( \frac{\beta}{\beta + H(m)} \right)^\alpha \right) \right)^2 dU(\theta) \left( \tilde{a}_{x\cdot n} \right)^2.
\]
Since
\[ d = \frac{1}{\sum_{m=0}^{\infty} e^{-\delta m}} < \frac{1}{a_{n-1}}, \]  
(2.115)
we have that \( SS_{[0,n]}^{SP} < SS_{[0,n]}^{LP} \) in any case.

The natural premium, also paid at the beginning of each policy year, is such that the reserve at any integer time is equal to zero, implying that this premium is the same as the risk premium (see also Section 2.5). For year \( (m, m+1], \ m \in \{0, \ldots, n-2\} \), the natural premium, paid at time \( m \) while the insured is alive, is therefore equal to
\[
De^{-\delta} E \left[ I_{T \leq m+1} \mid T > m \right] = De^{-\delta} e^{-\int_{m}^{m+1} \mu(s)ds} = De^{-\delta} \left( 1 - \left( \frac{\beta + H(m)}{\beta + H(m+1)} \right)^{\alpha} \right) \]  
(2.116)
In case \( m = n-1 \), however, no natural premium is paid, because the amount at risk at time \( n-1 \) is equal to zero. The reason is that in the final year \( (n-1, n] \) the benefit \( D \) will be paid with certainty at the end of that year. The Subsidizing Solidarity \( SS_{[0,n]}^{NP} \) is hence equal to:
\[
SS_{[0,n]}^{NP} = D^2 \int_{\theta} \left( \sum_{m=0}^{n-2} e^{-\theta(m+1)} e^{-\theta H(m)} X_{[m,m+1]}^{NP} \right)^2 dU(\theta), \]  
(2.117)
with
\[
X_{[m,m+1]}^{NP} = e^{-\theta(H(m+1)-H(m))} - \left( \frac{\beta + H(m)}{\beta + H(m+1)} \right)^{\alpha}. \]  
(2.118)

We will next consider the partial solidarity measures specified in (2.93) and (2.94), and derive, concerning the several policy years, expressions for the formulas (2.93) and (2.94) with respect to the three cases of premium payment. To be able to do so we need to formulate the respective amounts at risk at the end of year \( (m, m+1], \ m \in \{0, \ldots, n-2\} \). (Recall that in the final policy year no amount at risk applies, because the benefit \( D \) related to year \( (n-1, n] \) will be paid with certainty if the insured is alive at time \( n-1 \).)

If the policy is paid by single premium, we have that this quantity is equal to
\[
R^{SP}_{\{m+1\}} = d \cdot D \sum_{t=m+1}^{n-1} e^{-\delta(t-(m+1))} e^{-\int_{m+1}^{t} \mu(s+r)dr} = d \cdot D \cdot \tilde{a}_{x+m+1:n-(m+1)}, \]  
(2.119)
where
\[
\tilde{a}_{x+m+1:n-(m+1)} = \sum_{t=m+1}^{n-1} e^{-\delta(t-(m+1))} e^{-\int_{m+1}^{t} \mu(s+r)dr} \]  
(2.120)
denotes the single premium corresponding to an annuity due for an individual aged $x + t$ years, incepted at time $m + 1$ with duration $n - (m + 1)$ and based on the marginal force of mortality $\mu(\cdot)$.

For level premium payment, the amount at risk at the end of year $(m, m + 1]$, again for $m \in \{0, \ldots, n - 2\}$, equals:

$$R_{LP}^{SP}((m + 1) -) = D \frac{\hat{a}_{x+m+1|n-(m+1)}}{\hat{a}_{x,n}},$$  \hspace{1cm} (2.121)

As argued before, in case of natural premium payment the amount at risk at the end of year $(m, m + 1]$, $m \in \{0, \ldots, n - 2\}$, is equal to the lump sum:

$$R_{NP}^{SP}((m + 1) -) = D,$$ \hspace{1cm} (2.122)

while, similar to the cases of single and level premium payment,

$$R_{NP}^{SP}(n-) = 0.$$ \hspace{1cm} (2.123)

It follows that, for all three methods of premium payment, the Subsidizing Solidarity concerning policy year $(n - 1, n]$ is equal to zero.

Finally, it follows from (2.119), (2.121) and (2.122) that

$$R_{NP}^{SP}((m + 1) -) < R_{LP}^{SP}((m + 1) -) < R_{NP}^{LP}((m + 1) -), \forall m \in \{0, \ldots, n - 2\},$$  \hspace{1cm} (2.124)

hence

$$SS_{(m,m+1]}^{SP} < SS_{(m,m+1]}^{LP} < SS_{(m,m+1]}^{NP},$$ \hspace{1cm} (2.125)

and

$$SS_{(m,m+1]}^{(\ast)SP} < SS_{(m,m+1]}^{(\ast)LP} < SS_{(m,m+1]}^{(\ast)NP}.$$ \hspace{1cm} (2.126)

In our numerical example, we specify the baseline hazard function to be Gompertz with parameters $x = 30$, $b = 2 \cdot 10^{-5}$, and $c = 1.095$; so

$$\mu_{1}(t) = 2 \cdot 10^{-5} \cdot 1.095^{30+t}.$$ \hspace{1cm} (2.127)

Furthermore, we take $D = 1000$, $n = 35$ and $\delta = \ln(1.04)$ (implying a yearly interest rate of 4%). It is assumed that the realization of $T$ has an upper bound of 85. (So if the individual is aged 30, as suggested by the value of the parameter $x$ in specification (2.127), he is supposed to die before his 115-th birthday). The parameters of the Gamma($\alpha, \beta$) mixing distribution, displayed in (2.106) are:

$$\alpha = 1; \hspace{0.5cm} \beta = 0.5.$$ \hspace{1cm} (2.128)

To give an impression of the different possible risk profiles of the insured we will display here the unconditional expectation of remaining lifetime and the variance of the conditionally expected remaining lifetimes:

$$E[T] = E[E[T \mid \Theta]] = 58.80;$$ \hspace{1cm} (2.129)

$$\text{Var}[E[T \mid \Theta]] = 144.43.$$ \hspace{1cm} (2.130)
The solidarity measures comprising the entire contract term are:

\[ SS_{[0,n]}^{SP} = 208.84; \]  
\[ SS_{[0,n]}^{LP} = 381.25; \]  
\[ SS_{[0,n]}^{NP} = 1487.06. \]  

(2.131) \hspace{1cm} (2.132) \hspace{1cm} (2.133)

The solidarity measures concerning the policy years are displayed in Tables 2.1 (discounting to time-at-issue) and 2.2 (discounting to beginning of policy year):
Table 2.1

<p>| Solidarity measures on the level of a policy year $(m, m+1]$ discounted to time-at-issue. |
|---|---|---|
| $m$ | $SS^P_{(m,m+1]}$ | $SS^{dp}<em>{(m,m+1]}$ | $SS^{NP}</em>{(m,m+1]}$ |
| 0  | 0.1473 | 0.2990 | 0.2834 |
| 1  | 0.1565 | 0.2857 | 0.3102 |
| 2  | 0.1659 | 0.3030 | 0.3394 |
| 3  | 0.1756 | 0.3205 | 0.3713 |
| 4  | 0.1854 | 0.3384 | 0.4060 |
| 5  | 0.1953 | 0.3565 | 0.4438 |
| 6  | 0.2051 | 0.3745 | 0.4849 |
| 7  | 0.2149 | 0.3924 | 0.5295 |
| 8  | 0.2245 | 0.4099 | 0.5780 |
| 9  | 0.2338 | 0.4268 | 0.6305 |
| 10 | 0.2426 | 0.4429 | 0.6874 |
| 11 | 0.2508 | 0.4578 | 0.7490 |
| 12 | 0.2581 | 0.4713 | 0.8155 |
| 13 | 0.2645 | 0.4829 | 0.8872 |
| 14 | 0.2697 | 0.4924 | 0.9644 |
| 15 | 0.2735 | 0.4994 | 1.0474 |
| 16 | 0.2757 | 0.5033 | 1.1365 |
| 17 | 0.2760 | 0.5039 | 1.2318 |
| 18 | 0.2742 | 0.5006 | 1.3337 |
| 19 | 0.2701 | 0.4931 | 1.4422 |
| 20 | 0.2635 | 0.4810 | 1.5575 |
| 21 | 0.2542 | 0.4641 | 1.6797 |
| 22 | 0.2422 | 0.4421 | 1.8087 |
| 23 | 0.2273 | 0.4149 | 1.9445 |
| 24 | 0.2095 | 0.3825 | 2.0869 |
| 25 | 0.1891 | 0.3453 | 2.2355 |
| 26 | 0.1663 | 0.3036 | 2.3900 |
| 27 | 0.1415 | 0.2584 | 2.5498 |
| 28 | 0.1154 | 0.2107 | 2.7141 |
| 29 | 0.0888 | 0.1622 | 2.8821 |
| 30 | 0.0629 | 0.1149 | 3.0527 |
| 31 | 0.0391 | 0.0714 | 3.2247 |
| 32 | 0.0192 | 0.0350 | 3.3967 |
| 33 | 0.0053 | 0.0096 | 3.5672 |
| 34 | 0.0000 | 0.0000 | 0.0000 |</p>
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2.8 Conclusions and final comments

This chapter concerns the application of the urn-of-urns model to an individual life insurance contract. The aim has been to derive subsidizing solidarity measures, both on the level of the entire contract term and on the level of parts of that term. Starting point of all analysis has been the general life insurance model developed by Norberg (1990, 1991, 1992).

For non-life contracts, the subsidizing solidarity has been based in De Wit & Van Eeghen (1984) on the parameter effect, equal to the actual aggregate loss less the aggregate loss suffered in case of equivalence on an individual level. Regarding the entire period a treaty is valid, developing solidarity measures is in general not as easy as it is in the non-life insurance case. The reason lies in the fact that the parameter effect is stochastic rather than deterministic, except when the contract is paid by single premium. This implies that the unconditional loss variance in general consists of four components instead of two. We have taken the expected value of the above mentioned parameter effect as the basis of the subsidizing solidarity.

Losses allocated to a policy year depend on risk premiums and amounts at risk applying to that year. These quantities are in their turn dependent on the reserves pertaining to the same year. It would be in the spirit of the urn-of-urns model to update all these quantities to the individual’s claims’ experience. However, contrary to non-life and group life business, there is actually no difference between updating the relevant quantities, such as risk premiums, to the experience and not updating at all. The reason is that an individual cannot learn in time about the distribution of his remaining lifetime. This was already proved by Jewell (1978) for the classical two states model, i.e. a model where only the states ”Alive” and ”Dead” apply, and in this chapter we have showed it in general.

We have based the definition of the parameter effect corresponding to a partial period (such as a policy year) on the difference between the average risk premium and the risk premium paid in case of equivalence on an individual level. With regard to those risk premiums, the amount at risk is in both cases a function of the marginal transition intensities. The parameter effect is, similar to the one related to the whole contract period, not deterministic. However, again just as in the entire-period case, one can generate a solidarity measure as a function of the expected values of the several possible parameter effects.

The parameter effect for the entire contract term is in general not equal to the sum of the parameter effects for the several disjoint partial periods. This is due to the fact that reserves pertaining to intermediate points of time are usually nonzero. There is, however, an exception if, at any point of time, the premiums or benefits due to remaining in a certain state exactly match the benefits or premiums resulting from leaving that state. The reason is that all the reserves vanish in that case.

The complications described do not disappear if Norberg’s model reduces to the hierarchical Markov chain, where states, once left, can never be re-entered. The latter model has, however, the advantage that solidarity measures can be derived, which not only refer to periods, but also to states. These measures are even easier to handle in case of a discrete time policy in a classical two states framework.

All measures of the subsidizing solidarity depend on the specification of the a priori
mixing distribution, which is, like the case of experience rating in the Bayesian sense, an arbitrary issue to at least some extent. But once this prior has been chosen, the several solidarity measures help the insurer to assess how the intensity of the mutual cross-subsidization between different risks in the same portfolio develops as time passes. Such analyses are important in order to forecast the threat of adverse selection. This may exist not only at issue but also thereafter if individuals have the opportunity to withdraw their policies.