Heterogeneity of Hazard Rates in Insurance.
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Chapter 3

Proportional Mortality Result Sharing: Loss Variance and Solidarity Aspects

3.1 Introduction

For any type of insurance business, each private insurance company is confronted with the problem of risk classification. This is the process of separating potential insured into homogeneous subgroups (subgroups consisting of contracts representing the same risk), and, as a result, premium differentiation. If the classification scheme of a private company fails in any way, adverse selection will be the result. The answer to the question how refined a classification system should be, depends among others on the type of insurance business, the insurer's strategy, societal and ethical issues and the danger of adverse selection.

In practice, irrespective of the type of insurance, the subgroups mentioned in the previous paragraph are never completely homogeneous, however refined the classification system might be. So there will always be mutual cross-subsidization (or subsidizing solidarity, as defined in Chapter 2) between contracts. In order to decrease this effect, at least over time, a group life or non-life insurer can apply experience rating, as discussed in Chapter 2. It was also shown in that chapter that this makes no sense in individual life business, as there is actually no difference between updating premiums to the individual experience and not updating at all. Moreover, concerning an individual life portfolio where only the two states "Alive" and "Dead" apply, individuals with the same observable risk characteristics who survive a certain period all have the same claims experience, i.e. not having died yet. Hence, they all pay the same premium in the next year, so unobserved heterogeneity on an individual level can never be disclosed. Besides, individuals only die once and contracts are then removed from the portfolio. So an individual life insurer aiming to let premiums and risks be more matched with one another will have to work out other methods and this chapter is an attempt to do so. For the sake of simplicity, we will restrict ourselves to elementary life contingencies, i.e. contracts where only the states "Alive" and "Dead" apply.

In this chapter, an individual life insurance system is introduced, based on reimbursing
part of the aggregate mortality result, realized in a certain period and for a certain group, to either the survivors or the deaths' heirs. A fixed, predetermined proportion of the mortality result will, under certain conditions concerning the number of deaths, be distributed equally at the end of the period. The mortality result may be positive or negative. Then in the former case the members of the group considered receive an extra benefit while in the latter case they are charged an extra premium. For simplicity we assume that the amount at risk is the same for each insured in the group. To each individual contract, the same average risk premium, to be paid at the beginning of the period applies. It is calculated on the basis of equivalence on the level of the group. The main purpose of this chapter is to investigate if the given sharing system will diminish cross-subsidization among the individuals in a heterogeneous group.

The mechanism resulting in this decreased subsidizing solidarity works as follows. Consider, for instance, a positive amount at risk. If no sharing takes place, those with a low mortality rate (the low risks) subsidize those with a high mortality rate (the high risks), since the risk premium is average. Assume that the agreement states that distribution to the survivors will take place if the numbers of deaths is such low that the mortality result to be distributed is positive. The low risks are more likely to share in the mortality profit than the high risks and so they may be compensated for subsidizing the high risks.

The outline of this chapter is as follows. In Section 3.2, a formula for the average risk premium, satisfying equivalence on a group level, is derived. It will also be shown that systems exist which, common to real practice, allow only for sharing of positive profits. The average risk premium applies to both the variants "distribution to the survivors" and "distribution to the deaths' heirs", since it is calculated from the insurer's point of view. The same pertains to the aggregate loss variance considered in Section 3.3. It will be proved that in any case this variance is minimized when the intensity of distribution is equal to one, implying that the entire mortality result will be refunded. This would imply that, if the conditions regarding the distribution were satisfied, the whole mortality result would be shared. So even if the mortality result does not lower the mutual cross-subsidization, the sharing system at least contributes to a lower aggregate loss variance. Section 3.4 considers solidarity aspects, being the main topic of this chapter, as mentioned previously. Matters are then considered from the insured's perspective; hence, the cases "distribution to the survivors" and "distribution to the deaths' heirs" will (only in that section) be treated separately. As in Chapter 2, the Subsidizing Solidarity (SS) will act as the solidarity measure in this section. This quantity is derived by calculating, for each individual, the risk premium satisfying equivalence on an individual level. By squaring the difference between this risk premium and the average risk premium, and then averaging the result over all individuals in the portfolio, the SS is obtained. It follows that the definition of the SS used here is exactly in line with the Subsidizing Solidarity defined in Chapter 2, Section 2.7.

It will turn out that there are possibilities to decrease the SS, but that this may be at the cost of the original intention of insurance, that is, to cover against contingent claims. This is the reason why we will discuss an alternative measure, namely the Relative Subsidizing Solidarity (RSS), a concept which explicitly relates the SS to a quantity representing the level of insurance coverage. We will restrict ourselves to some particular cases as it will turn out to be hard to draw conclusions in general.
3.2. The model and its basic assumptions

Conclusions and recommendations for further research will be given in Section 3.5.

This chapter relies on the assumption that the insurer knows the mortality rate of each insured in the portfolio. This assumption seems strange as the company can charge each individual a risk premium corresponding to the individual’s mortality rate (an individual risk premium), thus removing our reason for using a mortality result sharing system (that is, to decrease the mutual cross-subsidization). That reason would be valid in case of imperfect information, i.e. when the insurer does not know the mortality rate per individual. The ultimate aim of this research is to develop a model of proportional mortality result sharing under the assumption that an urn-of-urns model applies to each individual in the portfolio. It will turn out that the formulas in this chapter for the given case of complete and perfect information are already complicated. Without any doubt, they would increase in complexity in case of an urn-of-urns model (because calculation of the relevant quantities such as the subsidizing solidarity would require multiple integration).

The intention of this chapter is to investigate whether the system might serve our criterion of a lower degree of mutual cross-subsidization anyway.

In Chapter 4, the assumption of perfect and complete information will be relaxed. For two cases, it will be shown that provided some specific information is known, safe bounds of the average risk premium can be obtained by means of the majorization order.

3.2 The model and its basic assumptions

We consider a portfolio with \( n \) contracts, \( n \geq 2 \), for a certain period. For each contract the same amount at risk, denoted by \( \bar{R} \), applies. This amount is paid out at the end of the given period in case of death. The amount is negative for contracts with negative mortality risk such as annuities.

The individuals are indexed by \( \tau \) and have mortality rates \( \theta_\tau, \tau \in \{1,...,n\} \). At this stage, it is assumed that the insurer knows which mortality rates apply in the portfolio. All contracts pay the same average risk premium, denoted by \( R \). Just as for the amount at risk, this quantity may be negative too, implying that at the beginning of the period the insurer pays an amount to the insured.

The interest rate for the entire period is indicated by \( s \), which is taken to be deterministic. At the end of the period, a predetermined proportion of the mortality result, whether positive or negative, will be divided equally among either the survivors or the deaths’ heirs, provided that the number of deaths belongs to a certain set of integer numbers, denoted by \( N \). In this chapter, and also in Chapter 4, we specify \( N \) to be of one of two specific forms. If the mortality result is shared with the survivors, either \( N = \{0,...,J\} \) or \( N = \{J,...,n-1\} \). On the other hand, if the deaths’ heirs are the beneficiaries of the sharing system, either \( N = \{1,...,J\} \) or \( N = \{J,...,n\} \), with \( J \in \{1,...,n\} \). The reason why \( n \notin N \) in case survivors share in the mortality result is that, if nobody survives the given period, there are no individuals left who can share in the distribution. For a similar reason, \( 0 \notin N \) if instead the deaths’ heirs are the participants. We adopt the notation \( \overline{N} = \{0,1,...,n\} \setminus N \). This implies that if \( k \in \overline{N} \), where \( k \) denotes the actual number of deaths, no distribution will take place.
In case $k$ individuals die, the original mortality result discounted to the end of the period, denoted by $OMR(k)$, is equal to

$$OMR(k) = n \Pi r (1 + s) - k \tilde{R}. \quad (3.1)$$

The ratio of the mortality result to the amount allocated to those who share is denoted by $\rho$. In this chapter, other methods of profit sharing (such as return on investment of premiums paid) are ignored.

Suppose $k \in N$. Then the amount distributed among the beneficiaries of the sharing system is equal to $\rho OMR(k)$. If the survivors are the participants of the sharing system, each one of them will receive the following amount from the company (if the amount is negative it involves an extra premium charge):

$$\frac{\rho OMR(k)}{n-k} = \rho \left( \frac{n}{n-k} \Pi r (1 + s) - \frac{k}{n-k} \tilde{R} \right). \quad (3.2)$$

On the other hand, if the deaths' heirs are those who share, they will get:

$$\frac{\rho OMR(k)}{k} = \rho \left( \frac{n}{k} \Pi r (1 + s) - \tilde{R} \right). \quad (3.3)$$

A negative amount in either (3.2) or (3.3) implies that a benefit is transferred from the participant to the company.

In the remainder of this section, as well as in the next section, no explicit distinction will be made between the specifications "division among the survivors" and "division among the deaths' heirs", as the quantities discussed are considered from the insurer's (and not the insured's) point of view. In the rest of this chapter, we will use the quantity $R$ instead of $\tilde{R}$, where

$$R = \frac{\tilde{R}}{1 + s}. \quad (3.4)$$

Denoting the probability of $k$ deaths by $\Pr(k)$, we have that the net expected loss realized by the insurer (i.e. after subtraction of the amount to be distributed), specified by $EL$, is equal to

$$EL = (1 - \rho) \sum_{k \in N} \Pr(k) OMR(k) + \sum_{k \in N} \Pr(k) OMR(k)$$

$$= (1 + s) \left( (1 - \rho) \sum_{k \in N} [\Pr(k) (kR - n\Pi r)] + \sum_{k \in N} [\Pr(k) (kR - n\Pi r)] \right)$$

$$= (1 + s) \left\{ R \left( \sum_{i=1}^{n} \theta_i - \rho \sum_{k \in N} \Pr(k) k \right) - n\Pi r (1 - \rho) \sum_{k \in N} \Pr(k) \right\}. \quad (3.5)$$

The following average risk premium, satisfying equivalence for the whole portfolio and depending on $\theta = (\theta_1, \theta_2, ..., \theta_n)$ and $\rho$, results:

$$\Pi r = \Pi r (\theta; \rho) = R \frac{\sum_{i=1}^{n} \theta_i - \rho \sum_{k \in N} \Pr(k) k}{n (1 - \rho) \sum_{k \in N} \Pr(k)}. \quad (3.6)$$
3.2. The model and its basic assumptions

In general, \( p \) can have any real value except \( (\sum_{k \in \mathbb{N}} \Pr(k))^{-1} \), but in this chapter we will concentrate explicitly on values of \( p \in [0,1] \). We then have

\[
|\Pi_r(\theta;p)| \leq |R|. \tag{3.7}
\]

Hence, by assuming that \( 0 \leq p \leq 1 \), it is ensured that the signs of the amount at risk and the risk premium are the same. Furthermore

\[
\frac{\partial \Pi_r(\theta;p)}{\partial p} = \frac{R}{n} \sum_{k \in \mathbb{N}} \Pr(k) \left( \sum_{i=1}^{n} \theta_i - k \right)
\]

so in case the amount at risk \( R \) is of positive sign, \( \Pi_r(\theta;p) \) is monotone increasing in \( p \) if \( N \subset \{ k | k \leq \sum_{i=1}^{n} \theta_i \} \) and monotone decreasing in \( p \) if \( N \subset \{ k | k \geq \sum_{i=1}^{n} \theta_i \} \), while for negative \( R \) the opposite holds. Together with (3.7), this implies that sets \( N \), such that only positive mortality profits are divided, can always be found. This is the case for a positive amount at risk \( R \) if \( N \subset \{ k | k \leq \sum_{i=1}^{n} \theta_i \} \) (the actual number of deaths must be at most equal to the expected number of deaths) and for a negative value of \( R \) if \( N \subset \{ k | k \geq \sum_{i=1}^{n} \theta_i \} \) (the actual number of deaths must be at least equal to the expected number of deaths). The numerical examples in this chapter will be based on sharing of positive profits.

Finally, for positive \( R \), (3.6) is monotone increasing for \( N = \{0,1,\ldots,n-1\} \), as in that case

\[
\frac{\partial \Pi_r(\theta;p)}{\partial p} = \frac{R}{n} \frac{(\prod_{i=1}^{n} (1-\theta_i) \sum_{i=1}^{n} (1-\theta_i))}{(1-p(1-\prod_{i=1}^{n} (1-\theta_i)))^2} \geq 0, \tag{3.9}
\]

and monotone decreasing for \( N = \{1,\ldots,n-1,n\} \), because

\[
\frac{\partial \Pi_r(\theta;p)}{\partial p} = -\frac{R}{n} \frac{(\prod_{i=1}^{n} (1-\theta_i) \sum_{i=1}^{n} \theta_i)}{(1-p(1-\prod_{i=1}^{n} (1-\theta_i)))^2} \leq 0. \tag{3.10}
\]

Again, for negative \( R \) the opposite applies.

Remark 1 (Value of \( \Pi_r \) for special cases) Note that, in case of division among the survivors, we also have for \( N = \{0,1,\ldots,n-1\} \):

\[
\Pi_r(\theta;0) = \frac{R \sum_{i=1}^{n} \theta_i}{n}, \tag{3.11}
\]

\[
\Pi_r(\theta;1) = R. \tag{3.12}
\]

The average risk premium displayed in (3.11) involves the classical case of no distribution of the mortality result, while (3.12) shows that for \( p = 1 \), there is no insurance at all. The same conclusions hold for division among the death’s heirs and \( N = \{1,\ldots,n-1,n\} \), since in that case

\[
\Pi_r(\theta;0) = \frac{R \sum_{i=1}^{n} \theta_i}{n}, \tag{3.13}
\]

\[
\Pi_r(\theta;1) = 0. \tag{3.14}
\]

For \( N \neq \{1,\ldots,n-1,n\} \) and \( N \neq \{0,1,\ldots,n-1\} \) the insurance element in the system is in general preserved, even if \( p = 1 \).
In the next subsection it will be analyzed how charging this average risk premium influences the variance of the aggregate loss.

Remark 2 (Comparison with what is usual in practice) Note that the average risk premium derived depends on the proportion $p$. In practice, however, the risk premium is usually calculated independently of $p$. Instead it is usually based on so called first order mortality rates, i.e. mortality rates which contain a certain safety margin compared to the second order mortality rates, which are the $\theta_i$'s ($i \in \{1, \ldots, n\}$) in this section. For positive amounts at risk the first order mortality rates are higher than the corresponding second order ones, while for negative amounts at risk the opposite holds.

Suppose that, just like above, the same first order mortality rate applies to all, so that each contract pays the same average risk premium. Distribution only happens if the mortality result is positive, i.e. if there is a profit. Then there is exactly one positive value of $p$ satisfying equivalence for the entire portfolio. In other words, in practice, if there is actually proportional mortality profit sharing, usually the average risk premium will be a parameter and the proportion $p$ is an endogenous variable. On the other hand, in this chapter $p$ is the parameter while the risk premium is an endogenous variable.

The advantage of the approach used in practice is that only the average mortality rate needs to be known in order to calculate the average risk premium while this is not the case for the system considered here. We still prefer the system introduced in this chapter, however, since the solidarity measures to be derived are in line with those defined in Chapter 2, as will be clarified in Subsection 3.4.1.

Finally, referring to the system in this chapter, the set $N$ defined before can always be limited such that only positive mortality profit will be distributed. This has been mentioned already.

### 3.3 Loss variance

We just introduced a certain system of distribution of the mortality result and, as mentioned in the introduction, the main aim of this chapter is to discuss its solidarity aspects. However, apart from this, the system proves to have a nice property anyway which deserves an extra section. We will show that the system always results in a lower variance of the insurer’s aggregate loss, compared to no distribution at all.

The insurer’s aggregate loss, based on $\Pi r(\theta; p)$ derived in the previous section, will be denoted by $MR(K; \rho)$, with $K$ being the random variable representing the number of deaths in the period considered.

Since the expectation of $MR(K; \rho)$ is equal to zero, the loss variance, specified by $\text{Var}[MR(K; \rho)]$, is equal to

$$\text{Var}[MR(K; \rho)] = (1 - \rho)^2 \sum_{k \in N} \Pr(k) \left(\overline{O}MR(k)\right)^2 + \sum_{k \in N} \Pr(k) \left(\overline{OM}R(k)\right)^2.$$ 

(3.15)
3.3. Loss variance

Substitution of (3.1) yields

\[
\begin{align*}
\text{Var} [MR(K; \rho)] &= \frac{R^2 (1 + s)^2}{(1 - \rho \sum_{\ell \in N} \Pr(\ell))^2} \\
&\left\{ \sum_{k \in N} \Pr(k) \left( (1 - \rho) \mathbb{E}[K] - k - \rho \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right)^2 \\
&\quad + \sum_{k \in N} \Pr(k) \left( \mathbb{E}[K] - k - \rho \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right)^2 \right\}.
\end{align*}
\]  
(3.16)

(Note that \( \mathbb{E}[K] = \sum_{t=1}^{n} \theta_t \).)

**Theorem 3** We have

\[
\frac{\partial \text{Var} [MR(K; \rho)]}{\partial \rho} = \begin{cases} 
< 0 & \text{for } \rho \in (-\infty, 1), \\
= 0 & \text{for } \rho = 1, \\
> 0 & \text{for } \rho \in \left(1, \frac{1}{\sum_{\ell \in N} \Pr(\ell)}\right),
\end{cases}
\]  
(3.17)

and

\[
\lim_{\rho \to 1} \frac{\partial \text{Var} [MR(K; \rho)]}{\partial \rho} = \infty.
\]  
(3.18)

**Proof.** Taking the derivative of the variance with respect to \( \rho \) yields

\[
\begin{align*}
\frac{\partial \text{Var} [MR(K; \rho)]}{\partial \rho} &= \frac{2R^2 (1 + s)^2}{(1 - \rho \sum_{\ell \in N} \Pr(\ell))^3} \\
&\left\{ \sum_{k \in N} \Pr(k) (1 - \rho) \left( \mathbb{E}[K] - k - \rho \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right)^2 \\
&\quad - (1 - \rho)^2 \sum_{\ell \in N} \Pr(\ell) (\ell - k) + \\
&\quad \sum_{k \in N} \Pr(k) \left( \mathbb{E}[K] - k - \rho \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right)^2 \right\}.
\end{align*}
\]  
(3.19)

After some rewriting, using

\[
\sum_{k \in N} \Pr(k) = 1 - \sum_{k \in N} \Pr(k),
\]  
(3.20)

(3.19) proves to be equal to

\[
\begin{align*}
\frac{\partial \text{Var} [MR(K; \rho)]}{\partial \rho} &= \frac{2R^2 (1 + s)^2 (1 - \rho)}{(1 - \rho \sum_{\ell \in N} \Pr(\ell))^3} \\
&\left\{ \sum_{k \in N} \Pr(k) \left( \frac{k \sum_{\ell \in N} \Pr(\ell)(k - \ell)}{\sum_{\ell \in N} \Pr(\ell)} - (k - \mathbb{E}[K])^2 \right) \\
&\quad - \frac{(1 - \rho)^2 \sum_{\ell \in N} \Pr(\ell) (\ell - k) + \sum_{k \in N} \Pr(k) \left( \mathbb{E}[K] - k - \rho \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right)^2}{\sum_{\ell \in N} \Pr(\ell)} \right\}.
\end{align*}
\]  
(3.21)
For any real valued $\alpha$, $\sum_{k \in N} \Pr(k) (k - \alpha)^2$ is minimized for $\alpha = \frac{\sum_{k \in N} \Pr(k) k}{\sum_{k \in N} \Pr(k)}$. Hence

$$\sum_{k \in N} \Pr(k) \left( k \frac{\sum_{\ell \in N} \Pr(\ell) (k - \ell)}{\sum_{\ell \in N} \Pr(\ell)} - (k - E[K])^2 \right)$$

$$= \sum_{k \in N} \Pr(k) k^2 \left( \frac{\sum_{k \in N} \Pr(k) k}{\sum_{\ell \in N} \Pr(\ell)} \right) - \sum_{k \in N} \Pr(k) (k - E[K])^2$$

$$\leq \sum_{k \in N} \Pr(k) k^2 \left( \frac{\sum_{k \in N} \Pr(k) k}{\sum_{\ell \in N} \Pr(\ell)} \right) - \sum_{k \in N} \Pr(k) \left( k - \frac{\sum_{k \in N} \Pr(k) k}{\sum_{k \in N} \Pr(k)} \right)^2$$

$$= 0. \quad (3.22)$$

So the expression between parentheses in (3.21) is negative for $p < \frac{1}{\sum_{\ell \in N} \Pr(\ell)}$, and it approaches infinity as $p$ approaches $\frac{1}{\sum_{\ell \in N} \Pr(\ell)}$ from the left. This proves the theorem. $\blacksquare$

The consequence of Theorem 3 is that $\text{Var}[MR(K;p)]$ is minimized for $p = 1$, irrespective of $N$, and that

$$\lim_{p \to 1} \frac{1}{\sum_{\ell \in N} \Pr(\ell)} \text{Var}[MR(K;p)] = \infty. \quad (3.23)$$

Remark 4 (Minimal variance in case of no insurance) Note that the conclusion derived above concerning $p = 1$ is in agreement with the one in Remark 1 (considering two special specifications of $N$), because if there is no insurance at all, then the aggregate loss variance should be equal to zero, and hence minimal.

We will conclude this section by a numerical illustration.

Example 5 We assume that the mortality result is distributed to the survivors in case everybody survives, so $N = \{0\}$. Besides, we suppose that $n = 10$ and that the mortality rates are as follows:

$$\theta_1, \ldots, \theta_5 = 0.014; \quad \theta_6, \ldots, \theta_{10} = 0.001. \quad (3.24)$$

We take $R = 1000$ and $s = 0$. Figure 3.1 displays $\text{Var}[MR(K;p)]$ as a function of $p$. As argued before, the minimal loss variance is achieved for $p = 1$. The corresponding minimal variance $\text{Var}[MR(K;1)]$ turns out to be equal to 38.77. Since $\Pr(0) = 0.9273$, the aggregate loss variance approaches infinity as $p$ approaches 1.0784 from the left.

3.4 Solidarity aspects

Up to now, we have restricted ourselves to the mortality result on an aggregate level. Next, we are going to consider solidarity aspects and therefore we will, in order to obtain a degree of solidarity, compare the given situation with the one of equivalence on an individual level. In the latter case, the risk premiums to be paid by the respective individuals are such that for each individual the loss incurred by the insurer has expectation zero. These quantities,
3.4. Solidarity aspects

Figure 3.1: \( \text{Var}[MR(K;\rho)] \) as a function of \( \rho \).

in the remainder of this chapter defined as \textit{individual risk premiums}, will be derived in the next subsections.

Note that in the two previous sections matters were considered only from the insurer's point of view, which enabled us to consider the two sharing systems simultaneously. In this section, however, a topic will be dealt with which is also of insured's interest and therefore we will treat the two systems of division among the survivors and the deaths' heirs separately in Subsections 3.4.1 and 3.4.2.

The next two subsections have a common set-up. An overview of the several possible final states, one of which will be entered by any individual, will be given. Besides, an overview of the state-dependent transfers, defined as the individual's loss due to the insurance contract, will be displayed. If the survivors are the beneficiaries of the sharing system, there are several states "Alive" and one state "Dead", while the opposite holds in case of distribution to the deaths' heirs. In this respect, no difference is made between any insured and his or her respective heirs, who actually receive the benefits or are charged premiums in case of death. The individual risk premiums, introduced above, will be derived. Then, as in the previous chapter, each of the transfers will be split into: a) the loss applying in case of equivalence on an individual level, and b) the insurer's expected loss due to the insurance contract, irrespective of the final state being equal to the difference between the average risk premium and the individual risk premium. Just as in Chapter 2, these will be named \textit{ex post transfers} and \textit{ex ante transfers}, respectively. In both sections the average (over all individuals) of the squared \textit{ex ante transfers} will be taken as a measure for subsidizing solidarity, again just as in Chapter 2. The impact of the proportion of result sharing (which has been denoted by \( \rho \)) on this quantity will be analyzed. Finally both subsections will be concluded by a numerical example.

Without loss of generality it is assumed that actually there are \( m \) different risk classes
(1 \leq m \leq n), named 1, \ldots, m, and hence each of the n mortality rates takes the value of the mortality rate corresponding to either one of the given risk classes. There are n$_i$ individuals belonging to risk class i (i \in \{1, \ldots, m\}), so \sum_{i=1}^{m} n_i = n.

Before dealing with the two versions, one final comment needs to be made. Many phrases are used in both subsections, which may annoy someone reading the two subsections after each other. Our objective, however, is to present the systems in such a way that Subsection 3.4.2 can be read without having to rely too much on Subsection 3.4.1.

### 3.4.1 Division among the survivors

#### Final states; equivalence on an individual level; individual risk premiums

At the end of the period considered, each individual will be in one of n + 1 different states, which will be considered below:

<table>
<thead>
<tr>
<th>Table 3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Overview of the possible states and their respective meanings.</strong></td>
</tr>
<tr>
<td><strong>State</strong></td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B.\k</td>
</tr>
<tr>
<td>B.\overline{k}</td>
</tr>
</tbody>
</table>

As Table 3.1 shows, there are several states "Alive", for the following reasons:

1. a distinction has to be made between the two cases where a survivor has or has not the right to get a share of the insurer’s mortality result (symbolized by "B.\k" and "B.\overline{k}"), and

2. conditionally given that the survivors have the right to share, the actual benefit is not determined a priori, except when N consists of one element.

If \ell is the number of elements of N (m \in \{1, \ldots, n\}), there are \ell + 1 different possible financial outcomes for a survivor and hence \ell + 1 different states "Alive". The financial outcome corresponding to state B.\k is the same, irrespective of k, as long as k \in \overline{N} \setminus \{n\}.

In the remainder of this chapter, the probability of k deaths within a portfolio equal to the original one, except that one member of risk class i has been left out, will be denoted by Pr$_i$(k).

If individuals of type i pay risk premium \Pi r$_i$, the transfer for an individual of risk class i for the several states is equal to the expressions given in the right hand column of the following table. Such a transfer reflects the loss incurred by this individual due to the insurance contract.
Table 3.2

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\theta_i$</td>
<td>$\Pi r_i^* - R$</td>
</tr>
<tr>
<td>B.k</td>
<td>$(1 - \theta_i) \Pr_i (k)$</td>
<td>$\Pi r_i^* - \frac{\rho}{n-k} \sum_{j=1}^{m} n_j \Pi r_j^* + \frac{\rho}{n-k} R$</td>
</tr>
<tr>
<td>B.k</td>
<td>$(1 - \theta_i) \Pr_i (k)$</td>
<td>$\Pi r_i^*$</td>
</tr>
</tbody>
</table>

Note that the given transfers at the same time represent the profit made by the insurer for the given contract. Calculating the individual risk premiums is not as simple as it is in the ordinary case of no distribution, since the insured’s transfer depends on what happens to the other individuals within the portfolio. The individual risk premiums need to be calculated simultaneously, as will be done below.

For $i \in \{1, ..., m\}$, the following variable is specified:

$$\alpha_i = (1 - \theta_i) \sum_{k \in N} \frac{\Pr_i (k)}{n - k}.$$  

(3.25)

The set of premiums satisfying the principle of equivalence on an individual level is a solution of the matrix equation:

$$X \Pi r^* = R b,$$  

(3.26)

with

$$X = \begin{pmatrix} 1 - \rho n_1 \alpha_1 & -\rho n_2 \alpha_1 & \cdots & -\rho n_m \alpha_1 \\ -\rho n_1 \alpha_2 & 1 - \rho n_2 \alpha_2 & \cdots & -\rho n_m \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho n_1 \alpha_m & -\rho n_2 \alpha_m & \cdots & 1 - \rho n_m \alpha_m \end{pmatrix},$$  

(3.27)

$$\Pi r^* = \begin{pmatrix} \Pi r_1^* \\ \Pi r_2^* \\ \vdots \\ \Pi r_m^* \end{pmatrix}, \quad b = \begin{pmatrix} \theta_1 - \rho (1 - \theta_1) \sum_{k \in N} \frac{k \Pr_1 (k)}{n-k} \\ \theta_2 - \rho (1 - \theta_2) \sum_{k \in N} \frac{k \Pr_2 (k)}{n-k} \\ \vdots \\ \theta_m - \rho (1 - \theta_m) \sum_{k \in N} \frac{k \Pr_m (k)}{n-k} \end{pmatrix}.$$  

(3.28)

Taking into account that, for all $k \in N$,

$$\sum_{i=1}^{m} n_i (1 - \theta_i) \frac{\Pr_i (k)}{n - k} = \Pr (k),$$  

(3.29)

the following solution is obtained by using Cramer’s rule for $i \in \{1, ..., m\}$:

$$\Pi r_i^* = \frac{R}{1 - \rho \sum_{k \in N} \Pr (k)} \begin{pmatrix} \theta_1 - \rho (1 - \theta_1) \sum_{k \in N} \frac{k \Pr_1 (k)}{n-k} + \rho \sum_{j=1}^{m} n_j A_{ij} (N) \end{pmatrix},$$  

(3.30)
with
\[
A_{ij}(N) = (1 - \theta_i) \sum_{k \in N} \frac{Pr_i(k)}{n - k} \left( \theta_j + \rho (1 - \theta_j) \sum_{k \in N} Pr_j(k) \right)
- (1 - \theta_j) \sum_{k \in N} \frac{Pr_j(k)}{n - k} \left( \theta_i + \rho (1 - \theta_i) \sum_{k \in N} Pr_i(k) \right).
\] (3.31)

This solution will be denoted by \(\Pi r_i\).

Noticing that
\[
\sum_{i=1}^m n_i \Pi r_i = n \Pi r,
\] (3.32)

the values of the transfers shown in Table 3.1, prove to be as follows in case of equivalence on an individual level:

<table>
<thead>
<tr>
<th>Table 3.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transfers for an individual of risk class (i) in case of equivalence on an individual level. ((\Pi r_i, i \in {1, \ldots, m}) defined as in (3.30).)</td>
</tr>
<tr>
<td>State</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B, k</td>
</tr>
<tr>
<td>B, (\bar{k})</td>
</tr>
</tbody>
</table>

**Ex ante and ex post transfers**

The actual situation of equivalence on a group level and an average risk premium will now be reconsidered. In the given case the transfers are as given below:

<table>
<thead>
<tr>
<th>Table 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transfers for an individual of risk class (i) in case of equivalence on a group level and payment of an average risk premium. ((\Pi r) defined as in (3.6).)</td>
</tr>
<tr>
<td>State</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B, k</td>
</tr>
<tr>
<td>B, (\bar{k})</td>
</tr>
</tbody>
</table>

The transfers displayed in Table 3.3 are the losses suffered by the individual in case of equivalence on an individual level and will, just as in Posthuma (1985), be called *ex post*
3.4. Solidarity aspects

By subtracting these values from the corresponding ones in Table 3.4, one observes that the remaining part is independent of the final state. For an individual of type \( i, i \in \{1, \ldots, m\} \), it is equal to:

\[
\frac{\Pi r - \Pi r_i}{R} = 1 - \rho \sum_{k \in N} \Pr(k) \cdot \sum_{j=1}^{m} \frac{n_j}{n} \left( \theta_j - \theta_i \right) \left\{ \frac{1 - \rho \sum_{k \in N} \Pr_{ij}(k-1)}{n-k} + \rho \left( \frac{1}{n-k} \sum_{k \in N} \frac{k \Pr_{ij}(k)}{n-k} \right) \right\},
\]

(3.33)

with \( A_{ij}(N) \) as defined in (3.31). It turns out that

\[
\sum_{k \in N} \frac{k \Pr_{ij}(k)}{n-k} = n \sum_{k \in N} \frac{\Pr_{ij}(k)}{n-k} - \sum_{k \in N} \Pr_{ij}(k).
\]

(3.34)

Also, for all \( i, j \in \{1, \ldots, m\} \) and \( k \in \{0, \ldots, n - 1\} \), we have:

\[
\Pr_{ij}(k) = \Pr_{ij}(k-1) \theta_j + \Pr_{ij}(k) (1 - \theta_j),
\]

(3.35)

with \( \Pr_{ij}(k) \) denoting the probability of \( k \) deaths within a portfolio equal to the given one, except that one member of both the risk classes \( i \) and \( j \) has been left out, and \( \Pr_{ij}(-1) = \Pr_{ij}(n - 1) = 0 \), by definition. Using the above two equations, (3.33) can be rewritten as follows:

\[
\frac{\Pi r - \Pi r_i}{R} = \left( \sum_{j=1}^{m} \frac{n_j}{n} \left( \theta_j - \theta_i \right) \right) \left\{ \frac{1 - \rho \sum_{k \in N} \Pr_{ij}(k-1)}{n-k} + \rho \left( \frac{1}{n-k} \sum_{k \in N} \frac{k \Pr_{ij}(k)}{n-k} \right) \right\},
\]

(3.36)

with

\[
B_{ij}(N) = \sum_{k \in N} \frac{\Pr_{ij}(k-1) - \Pr_{ij}(k)}{n-k} \quad \forall i, j,
\]

(3.37)

and

\[
C_{ij}(N) = \left( \sum_{k \in N} \frac{\Pr_{ij}(k-1)}{n-k} \right) \left( \sum_{k \in N} \Pr_{ij}(k) \right) - \left( \sum_{k \in N} \frac{\Pr_{ij}(k)}{n-k} \right) \left( \sum_{k \in N} \Pr_{ij}(k-1) \right).
\]

(3.38)

The quantity in (3.36) represents the expected loss suffered by the individual due to the insurance contract, and will, again just as in Posthuma (1985), be called ex ante transfer.
Remark 6 (Comparison with what is usual in practice) As mentioned in Remark 2, in practice risk premiums are derived independently from the method of mortality profit sharing. Suppose that everybody pays a certain average risk premium $\Pi r$, such that

$$\Pi r > \Pi r (\theta; 0) = R \sum_{i=1}^{n} \frac{n_i \theta_i}{n}.$$  \hspace{1cm} (3.39)

(The consequence is that without distribution, the insurer's net expected profit would be positive.) Assume that an individual of risk class $i$ receives $\frac{k_i}{n-k}$ times the mortality profit if $k$ individuals die (with $k \in N$). (It is supposed that $\Pi r$ and $N$ are such that only positive mortality profits are shared.) Then for such an individual the overview of possible transfers, to be compared with Table 3.2, is as follows:

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\theta_i$</td>
<td>$\Pi r - R$</td>
</tr>
<tr>
<td>$B.k$</td>
<td>$(1 - \theta_i) \Pr_i (k)$</td>
<td>$\Pi r - \frac{k_i}{n-k} \left( n \Pi r - k R \right)$</td>
</tr>
<tr>
<td>$B.k$</td>
<td>$(1 - \theta_i) \Pr_i (k)$</td>
<td>$\Pi r$</td>
</tr>
</tbody>
</table>

Then one can derive two special values of $\rho_i$, being similar to the individual risk premium and average risk premium considered before. These are $\rho_i$, satisfying equivalence on an individual level, and an average proportion $\rho$ satisfying equivalence on the level of the portfolio. (The latter quantity should not be confused with the parameter $\rho$ in the main model in this chapter). Then the ex ante transfer for the given individual can be defined as:

$$(\rho_i - \rho) \sum_{k \in N} \Pr_i (k) \left( \frac{n \Pi r - k R}{n - k} \right).$$  \hspace{1cm} (3.40)

One can base solidarity measures on expressions like (3.40) which is in fact a counterpart of (3.36). We prefer, however, the main model in this chapter, since the ex ante transfer (3.36) is perfectly in line with Chapter 2, namely a difference between risk premiums.

The several ex ante transfers just derived are in the next definition the base of the measure of solidarity considered in Subsection 2.6 of Chapter 2.

Subsidizing Solidarity

Definition 7 The Subsidizing Solidarity is defined as the average over all individuals of the square of the ex ante transfers, in the given case equal to:

$$SS = \sum_{i=1}^{m} \frac{n_i}{n} (\Pi r - \Pi r_i)^2,$$  \hspace{1cm} (3.41)
3.4. Solidarity aspects

with \( \Pi r - \Pi r_i \) as given in (3.36).

Case 8 For \( N = \{0\} \), (3.36) reduces to

\[
\Pi r - \Pi r_i = R \left( \frac{1}{n} \sum_{j=1}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \right),
\]

hence independent of \( \rho \). So if distribution takes place only in case everybody survives, \( SS \) is constant as a function of \( \rho \), so equal to the \( SS \) corresponding to no insurance at all. This result which at least at first glance may seem remarkable, can be explained verbally. The \( SS \) can only change if the probability to be a beneficiary of the mortality result distribution differs between individuals of different risk types. In general this is the case, as mortality rates differ. If \( N = \{0\} \), however, the probability to get a share is the same for each individual, namely \( \Pr(0) \). In Subsection 3.4.2, one can read that, for similar reasons, \( SS \) does not change in case of distribution to the deaths' heirs and \( N = \{n\} \).

It will now be examined for some cases how \( \rho \) affects \( SS \). For this purpose the derivative of (3.41) will be taken with respect to \( \rho \). This yields:

\[
\frac{\partial SS}{\partial \rho} = \frac{2R^2}{(1 - \rho \sum_{k \in N} \Pr(k))^3} \sum_{i=1}^{m} \frac{n_i}{n} S_i,
\]

with

\[
S_i = \left( \sum_{j=1}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left\{ 1 - \rho \sum_{k \in N} \Pr_{ij} (k - 1) + \rho n (1 - \theta_i) (1 - \theta_j) (B_{ij} (N) - \rho C_{ij} (N)) \right\} \left( \sum_{p=1}^{m} \frac{n_p}{n} (\theta_p - \theta_i) \right) \left\{ (1 - \rho)^2 C_{ip} (N) + \sum_{k \in N} \Pr_{ip} (k) - \rho^2 (1 - \sum_{k \in N} \Pr(k)) C_{ip} (N) \right\}.
\]

An expression for the above quantity will next be displayed for the special case of distributing to the survivors in case there are survivors.
Case 9 For \( N = \{0, \ldots, n-1\} \), we get

\[
S_i = (1 - \rho) \left( \sum_{j=1 \atop j \neq i}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \left\{ 1 + \rho n (1 - \theta_i) (1 - \theta_j) \sum_{\ell=0}^{n-2} \frac{Pr_{ij}(\ell)}{(n-\ell)(n-\ell-1)} \right\} \right)
\]

\[
- \sum_{p=1 \atop p \neq i}^{m} \frac{n_p}{n} (\theta_p - \theta_i) \left\{ (1 - \rho)^2 n (1 - \theta_i) (1 - \theta_p) \sum_{\ell=0}^{n-2} \frac{Pr_{ip}(\ell)}{(n-\ell)(n-\ell-1)} - (\prod_{\tau=1}^{m} \theta_{\tau}^{\omega_{\tau}}) \right\}
\]

\[
1 + \rho^2 n (1 - \theta_i) (1 - \theta_p) \sum_{\ell=0}^{n-2} \frac{Pr_{ip}(\ell)}{(n-\ell)(n-\ell-1)}
\]

(3.45)

In practice, at least for large portfolios, \( \prod_{\tau=1}^{m} \theta_{\tau}^{\omega_{\tau}} \), the probability that all individuals die, can be neglected, so for values of \( p \) smaller than, but not very close to 1, \( \theta_{SS} \) is approximately equal to:

\[
\frac{\partial SS}{\partial \rho} = 2 R^2 \sum_{i=1}^{m} \frac{n_i}{n} \left\{ \left( \sum_{j=1 \atop j \neq i}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \left\{ 1 + \rho n (1 - \theta_i) (1 - \theta_j) \sum_{\ell=0}^{n-2} \frac{Pr_{ij}(\ell)}{(n-\ell)(n-\ell-1)} \right\} \right) \right.
\]

\[
- \sum_{p=1 \atop p \neq i}^{m} \frac{n_p}{n} (\theta_p - \theta_i) \left\{ (1 - \rho)^2 n (1 - \theta_i) (1 - \theta_p) \sum_{\ell=0}^{n-2} \frac{Pr_{ip}(\ell)}{(n-\ell)(n-\ell-1)} - (\prod_{\tau=1}^{m} \theta_{\tau}^{\omega_{\tau}}) \right\}
\]

\[
1 + \rho^2 n (1 - \theta_i) (1 - \theta_p) \sum_{\ell=0}^{n-2} \frac{Pr_{ip}(\ell)}{(n-\ell)(n-\ell-1)}
\]

(3.46)

Hence, if \( \prod_{\tau=1}^{m} \theta_{\tau}^{\omega_{\tau}} \) is actually very small, for most values of \( \rho \in [0, 1] \) always distributing as long as there are survivors is not very recommendable if one wants to decrease mutual cross-subsidization.

Note that, if \( N = \{0, \ldots, n-1\} \), in some cases the survivors will get a share of a mortality profit while in other cases they will be confronted with a mortality loss, since in practice mortality rates related to a portfolio are usually of moderate level. So the above example may not be the most fortunate one, because usually only mortality profits will be shared. However, it appears to be one of the few, besides Case 8, giving rise to unambiguous conclusions. In general it is hard to say something about the behavior of the Subsidizing Solidarity with respect to \( \rho \).
This is the reason why next our focus will be turned towards the derivative at $p = 0$, in order to investigate the behavior of $SS$ for small values of $p$. This derivative is

$$\frac{\partial SS}{\partial p (p=0)} = 2R^2 \sum_{i=1}^{m} \frac{n_i}{n} S_{i(p=0)}, \quad (3.47)$$

where

$$S_{i(p=0)} = \left( \sum_{j=1}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \right) \cdot \left( \sum_{p=1}^{m} \frac{n_p}{n} (\theta_p - \theta_i) \right) \left\{ \frac{n (1 - \theta_i) (1 - \theta_p)}{\sum_{k \in N} \Pr_{ip}(k) (\Pr_{ip}(k-1) - \Pr_{ip}(k))} \right\}. \quad (3.48)$$

Equality (3.47) can be rewritten as

$$\frac{\partial SS}{\partial p (p=0)} = 2R^2 \left\{ \sum_{i,p} \frac{n_i n_p}{n^2} (\theta_p - \theta_i)^2 F_{ip}^{\text{surv}} (N) \right\}. \quad (3.49)$$

where

$$F_{ip}^{\text{surv}} (N) = \sum_{\ell \in N} F_{ip}^{\text{surv}} (\ell), \quad (3.50)$$

with

$$F_{ip}^{\text{surv}} (\ell) = \theta_i \theta_p (\Pr_{ip} (\ell - 2) - \Pr_{ip} (\ell - 1)) + \frac{\ell}{n-\ell} (1 - \theta_i) (1 - \theta_p) (\Pr_{ip} (\ell - 1) - \Pr_{ip} (\ell)). \quad (3.51)$$

We assume $n \geq 3$. Then for $N = \{J, \ldots, n - 1\}$, $J \in \{1, \ldots, n - 1\}$ we get

$$F_{ip}^{\text{surv}} (N) = \theta_i \theta_p (\Pr_{ip} (J - 2) - \Pr_{ip} (n - 2)) + \frac{J}{n-J} (1 - \theta_i) (1 - \theta_p) \Pr_{ip} (J - 1) + (1 - \theta_i) (1 - \theta_p) \sum_{\ell=J}^{n-2} \frac{n}{(n-\ell) (n-(\ell+1))} \Pr_{ip} (\ell). \quad (3.52)$$

The last term vanishes if $j = n - 1$. In most practical situations, this expression can be considered to be positive, so $\frac{\partial SS}{\partial p} > 0$, at least for small values of $p$. 

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3.4. Solidarity aspects
On the other hand, for \( N = \{0, \ldots, J\}, J \in \{1, \ldots, n-2\} \), expression (3.50) reduces to

\[
F_{\text{surv}}^N (N) = -\theta_i \theta_p Pr_{ip} (J - 1) - \frac{J}{n - J} (1 - \theta_i) (1 - \theta_p) Pr_{ip} (J) + (1 - \theta_i) (1 - \theta_p) \sum_{\ell=0}^{J-1} \frac{n}{(n - \ell)(n-(\ell+1))} Pr_{ip} (\ell).
\]  

(3.53)

This quantity may be negative if \( n \) is large, compared to \( J \). Then \( \frac{\partial SS}{\partial p} < 0 \), at least for small values of \( p \).

In the next numerical example, we will deal with a case where

\[
\frac{\partial SS}{\partial p} (p=0) < 0.
\]

(3.54)

**Numerical example**

We take the number of risk classes, denoted by \( m \), equal to 2. Then (3.41) reduces to

\[
SS = \left( \frac{n_1 n_2}{n^2} \right) \left( \frac{R (\theta_1 - \theta_2)}{(1 - \rho \sum_{k \in N} Pr (k))} \right)^2 \left( 1 - \rho \sum_{k \in N} Pr_{12} (k - 1) + \rho n (1 - \theta_1) (1 - \theta_2) (B_{12} (N) - \rho C_{12} (N)) \right)^2.
\]

(3.55)

Let’s assume the discounted amount at risk, the size of the two risk classes and the mortality rates to be as given below:

\[
R = 1000; \quad n_1 = n_2 = 135; \quad \theta_1 = 0.014; \quad \theta_2 = 0.001.
\]

(3.56)

The mortality result is shared if the number of deaths is smaller than or equal to \( J \), with \( J \in \{1, 2\} \). Since

\[
\sum_{r=1}^{2} n_r \theta_r = 135 \cdot 0.014 + 135 \cdot 0.001 = 2.025 > 2,
\]

(3.57)

and \( R \) is of positive sign, this implies that only mortality profits are distributed. Figure 3.2 displays SS as a function of \( \rho \) for \( J = 1 \) and \( J = 2 \). One can see that in both cases, SS decreases, though the differences, compared with \( \rho = 0 \), remain small.

In the next section an alternative system, based on distribution to the deaths’ heirs, will be presented, which may also realize our purpose to diminish the Subsidizing Solidarity. This system will turn out to be the mirror image of the one that was considered in this section.
3.4. Solidarity aspects

3.4.2 Division among the heirs of the deaths

Final states; equivalence on an individual level; individual risk premiums

At the end of the period considered, each individual will be in one of $n + 1$ different states, which will be considered below:

<table>
<thead>
<tr>
<th>State</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>Alive</td>
</tr>
<tr>
<td>D,k</td>
<td>Dead, together with $k - 1$ others ($k \in N$).</td>
</tr>
<tr>
<td>D,k</td>
<td>Dead, together with $\bar{k} - 1$ others ($\bar{k} \in \overline{N \setminus {0}}$).</td>
</tr>
</tbody>
</table>

Compared to the previous subsection where for an insured several states "Alive" applied and one state "Dead", there are now one state "Alive" and several states "Dead", due to the element of mortality result sharing. If $\ell$ is the number of elements of $N$ ($\ell \in \{1, \ldots, n\}$), there are $\ell + 1$ different possible financial outcomes for a death’s heir and hence $\ell + 1$ different states "Dead". For all states D,$k$, with $\bar{k} \in \overline{N \setminus \{0\}}$, the financial outcome for the death’s heir is the same, since no distribution takes place.

If individuals of type $i$ pay risk premium $\Pi r^*_i$, the transfer for an individual of risk class $i$, reflecting the loss incurred by this individual due to the insurance contract, at the same time being the profit made by the insurer for the given contract, for the several states is equal to the expressions given in the right hand column of the following table:
Table 3.7

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$1 - \theta_i$</td>
<td>$\Pi r_i^*$</td>
</tr>
<tr>
<td>$D_i$</td>
<td>$\theta_i \Pr_i (k-1)$</td>
<td>$\Pi r_i^* - R - \frac{\rho}{k} \sum_{j=1}^{m} n_j \Pi r_j^* + \rho R$</td>
</tr>
<tr>
<td>$D_{i'c}$</td>
<td>$\theta_i \Pr_i (k-1)$</td>
<td>$\Pi r_i^* - R$</td>
</tr>
</tbody>
</table>

Just as in the case of distribution to the survivors, the individual risk premiums need to be calculated simultaneously, by solving an equality in matrix form, as will be done below.

For $i \in \{1, \ldots, m\}$, the following variable is specified:

$$\alpha_i = \theta_i \sum_{k \in N} \frac{\Pr_i (k-1)}{k} \quad (3.58)$$

The set of premiums satisfying the principle of equivalence on an individual level is a solution of the matrix equality:

$$\mathbf{X} \Pi r^* = \mathbf{R} \mathbf{b}, \quad (3.59)$$

with

$$\mathbf{X} = \begin{pmatrix} 1 - \rho n_1 \alpha_1 & -\rho n_2 \alpha_1 & \cdots & -\rho n_m \alpha_1 \\ -\rho n_1 \alpha_2 & 1 - \rho n_2 \alpha_2 & \cdots & -\rho n_m \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho n_1 \alpha_m & -\rho n_2 \alpha_m & \cdots & 1 - \rho n_m \alpha_m \end{pmatrix}, \quad (3.60)$$

$$\Pi r^* = \begin{pmatrix} \Pi r_1^* \\ \Pi r_2^* \\ \vdots \\ \Pi r_m^* \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \theta_1 (1 - \rho \sum_{k \in N} \Pr_1 (k-1)) \\ \theta_2 (1 - \rho \sum_{k \in N} \Pr_2 (k-1)) \\ \vdots \\ \theta_m (1 - \rho \sum_{k \in N} \Pr_m (k-1)) \end{pmatrix}. \quad (3.61)$$

Taking into account that, for all $k \in N$,

$$\sum_{i=1}^{m} n_i \theta_i \frac{\Pr_i (k-1)}{k} = \Pr (k) \quad (3.62)$$

the following solution is obtained by applying Cramer’s rule for $i \in \{1, \ldots, m\}$:

$$\Pi r_i^* = \frac{R}{1 - \rho \sum_{k \in N} \Pr (k)} \begin{pmatrix} \theta_i - \rho \theta_i \sum_{k \in N} \Pr_i (k-1) + \rho \sum_{j=1}^{m} n_j X_{ij} (N) \end{pmatrix}. \quad (3.63)$$
3.4. Solidarity aspects

with

\[
X_{ij}(N) = \theta_i \theta_j \left\{ \frac{\sum_{k \in N} \Pr_i(k-1)}{\Pr_j(k-1)} (1 - \rho \sum_{k \in N} \Pr_j(k-1)) \right\}.
\]

(3.64)

This solution will, just as in the case of distribution to the survivors, be denoted by \( \Pi_{r_i} \).

There is hardly any chance of misunderstanding because these quantities are only needed to discuss solidarity issues, which in turn are related to the question whether the survivors or the deaths’ heirs are the beneficiaries of the sharing system.

Noticing (again) that

\[
\sum_{i=1}^{m} n_i \Pi_{r_i} = n \Pi_r,
\]

(3.65)

the values of the ex post transfers turn out to be as follows:

<p>| Table 3.8 |
| Transfers for an individual of risk class ( i ) in case of equivalence on an individual level. (( \Pi_{r_i}, i \in {1, \ldots, m} ) as defined in (3.63).) |</p>
<table>
<thead>
<tr>
<th>State</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>( \Pi_{r_i} )</td>
</tr>
<tr>
<td>D.( k )</td>
<td>( \Pi_{r_i} - \rho \left( \frac{\sum_{i=1}^{m} n_i \theta_i - k - \rho (\sum_{i \in N} \Pr_i(\ell - k))}{k(1 - \rho \sum_{i \in N} \Pr_i(\ell))} \right) R )</td>
</tr>
<tr>
<td>D.( \overline{k} )</td>
<td>( \Pi_{r_i} - R )</td>
</tr>
</tbody>
</table>

Ex ante and ex post transfers

The situation of equivalence on a group level and an average risk premium will now be reconsidered. In the given case the aggregate transfers are as given below:

<p>| Table 3.9 |
| Transfers for an individual of risk class ( i ) in case of equivalence on a group level and payment of an average risk premium. (( \Pi_{r} ) defined as in (3.6).) |</p>
<table>
<thead>
<tr>
<th>State</th>
<th>Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>( \Pi_{r} )</td>
</tr>
<tr>
<td>D.( k )</td>
<td>( \Pi_{r} - \rho \left( \frac{\sum_{i=1}^{m} n_i \theta_i - k - \rho (\sum_{i \in N} \Pr_i(\ell - k))}{k(1 - \rho \sum_{i \in N} \Pr_i(\ell))} \right) R )</td>
</tr>
<tr>
<td>D.( \overline{k} )</td>
<td>( \Pi_{r} - R )</td>
</tr>
</tbody>
</table>

The ex ante transfers are found by subtracting the expressions in Table 3.8 from the
corresponding ones in Table 3.9 for each state. This yields:

\[
\Pi_r - \Pi_{r_i} = \frac{R}{1 - \rho \sum_{k \in N} \Pr(k)} \cdot \sum_{j \neq i} \frac{n_j}{n} \left\{ (\theta_j - \theta_i) - \rho n X_{ij}(N) \right\},
\]

(3.66)

with \(X_{ij}(N)\) as defined in (3.64). For all \(i, j \in \{1, \ldots, m\}\):

\[
\Pr_i(k) = \Pr_{ij}(k - 1) \theta_j + \Pr_{ij}(k)(1 - \theta_j),
\]

(3.67)

for all \(k \in \{0, \ldots, n - 1\}\), with \(\Pr_{ij}(k)\) denoting the probability of \(k\) deaths within a portfolio equal to the given one, except that one member of both the risk classes \(i\) and \(j\) has been left out, and \(\Pr_{ij}(-1) = \Pr_{ij}(n - 1) = 0\), by definition. This results in the following rewritten version of the ex ante transfer:

\[
\Pi_r - \Pi_{r_i} = \frac{R}{1 - \rho \sum_{k \in N} \Pr(k)} \cdot \left( \sum_{j \neq i} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left\{ 1 - \rho \sum_{k \in N} \frac{\Pr_{ij}(k - 1)}{k} \right\},
\]

(3.68)

with

\[
Y_{ij}(N) = \sum_{k \in N} \frac{\Pr_{ij}(k - 1)}{k} - \frac{\Pr_{ij}(k - 2)}{k},
\]

(3.69)

and

\[
Z_{ij}(N) = \left( \sum_{k \in N} \frac{\Pr_{ij}(k - 1)}{k} \right) \left( \sum_{k \in N} \frac{\Pr_{ij}(k - 2)}{k} \right) - \left( \sum_{k \in N} \frac{\Pr_{ij}(k - 2)}{k} \right) \left( \sum_{k \in N} \frac{\Pr_{ij}(k - 1)}{k} \right),
\]

(3.70)

for all \(i, j \in \{1, \ldots, m\}\).

**Subsidizing Solidarity**

The Subsidizing Solidarity is defined as in (3.41), so:

\[
SS = \sum_{i=1}^{m} \frac{n_i}{n} (\Pi_r - \Pi_{r_i})^2,
\]

(3.71)

with \(\Pi_r - \Pi_{r_i}\) as specified in (3.68).
3.4. Solidarity aspects

Case 10 For $N = \{n\}$, this reduces to

$$\Pi r - \Pi r_i = R \sum_{j \neq i} \frac{n_j}{n} (\theta_j - \theta_i),$$

(3.72)

independent of $\rho$. The reason is the same as stated in Case 8: if the probability to be involved in the mortality result distribution is the same for each insured, the ex ante transfers remain unaltered in value.

We now take the derivative of $SS$ with respect to $\rho$. This results in:

$$\frac{\partial SS}{\partial \rho} = -\frac{2R^2}{(1 - \rho \sum_{k \in N} \Pr(k))^3} \cdot \sum_{i=1}^{n_i} T_i,$$

(3.73)

with

$$T_i = \left( \sum_{j \neq i} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left\{ 1 - \rho \sum_{k \in N} \Pr_{ij} (k - 1) + \rho \theta_i \theta_j \right\} + \left( \sum_{k \in N} \Pr(k) - \sum_{k \in N} \Pr_{ip} (k - 1) \right)$$

+ $n_i (\theta_p - \theta_i)$

(3.74)

An expression for the above quantity will next be displayed for the special case of distributing to the deaths’ heirs in case there are deaths.

Case 11 For $N = \{1, \ldots, n\}$, we get

$$T_i = (1 - \rho) \left( \sum_{j=1}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left( 1 + \rho \theta_i \theta_j \sum_{\ell=1}^{n-1} \frac{\Pr_{ij} (\ell - 1)}{\ell (\ell + 1)} \right)$$

+ $n_i (\theta_p - \theta_i)$

(3.75)

Note that, for all $i, j \in \{1, \ldots, m\}$

$$1 + \rho \theta_i \theta_j \sum_{\ell=1}^{n-1} \frac{\Pr_{ij} (\ell - 1)}{\ell (\ell + 1)} > 0,$$

(3.76)

and for each function $g(\cdot, \cdot)$ of two arguments:

$$\sum_{i=1}^{m} n_i \left( \sum_{j=1}^{m} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left( \sum_{j \neq i} \frac{n_{ij}}{n^2} (\theta_p - \theta_i) g(\theta_i, \theta_p) \right)$$

$$= \sum_{i<p, j \neq i, p} n_i n_j n_p (\theta_p - \theta_i)^2 g(\theta_i, \theta_p) + \sum_{i<p} \frac{n_i n_p}{n^3} (\theta_p - \theta_i)^2 g(\theta_i, \theta_p).$$

(3.77)
So $\frac{\partial SS}{\partial \rho}$ is nonpositive for $\rho \in [0, 1]$ if

$$\max_{i, p, r} \left[ n\theta_i\theta_p \sum_{\ell=1}^{n-1} \frac{Pr_{ip}(\ell - 1)}{\ell (\ell + 1)} \right] \leq \left( \prod_{r=1}^{m} (1 - \theta_r)^{n_r} \right), \quad (3.78)$$

which is the case for relatively low mortality rates and a small portfolio size.

Note that a portfolio which is small in size and has "low mortality rates" usually corresponds to a portfolio where

$$\sum_{r=1}^{m} n_r \theta_r \leq 1. \quad (3.79)$$

For negative amounts at risk, this implies that in the case just discussed, only profits are shared. So if the conditions described above are satisfied, there is at least one way to decrease the subsidizing solidarity.

Still it is quite hard to draw conclusions in general and this is the reason why next our focus will be turned towards the derivative at $\rho = 0$, in order to investigate the behavior of SS for small values of $\rho$. This derivative turns out to be

$$\frac{\partial SS}{\partial \rho} (\rho = 0) = 2R^2 \sum_{i=1}^{m} \frac{n_i}{n} T_i(\rho = 0), \quad (3.80)$$

where

$$T_i(\rho = 0) = \left( \sum_{j \neq i} \frac{n_j}{n} (\theta_j - \theta_i) \right) \left( \sum_{p=1}^{m} \frac{n_p}{n} (\theta_p - \theta_i) \right) F_{ip}^{\text{death}} (N), \quad (3.81)$$

where

$$F_{ip}^{\text{death}} (N) = \sum_{\ell \in N} F_{ip}^{\text{death}} (\ell), \quad (3.82)$$

with

$$F_{ip}^{\text{death}} (\ell) = (1 - \theta_i) (1 - \theta_p) (Pr_{ip}(\ell) - Pr_{ip}(\ell - 1)) + \theta_i \theta_p \frac{n - \ell}{\ell} (Pr_{ip}(\ell - 1) - Pr_{ip}(\ell - 2)). \quad (3.83)$$

We assume $n \geq 3$. Then for $N = \{1, \ldots, J\}$, $J \in \{1, \ldots, n - 1\}$, (3.82) reduces to

$$F_{ip}^{\text{death}} (N) = (1 - \theta_i) (1 - \theta_p) (Pr_{ip}(J) - Pr_{ip}(0)) + \theta_i \theta_p \frac{n - J}{J} Pr_{ip}(J - 1) + \theta_i \theta_p \sum_{\ell=0}^{J-2} \frac{n_i}{(\ell + 1)(\ell + 2)} Pr_{ip}(\ell). \quad (3.84)$$
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The last term vanishes if \( J = 1 \). The above expression may be assumed to be positive for relatively large portfolios and small values of \( J \).

On the other hand, for \( N = \{ J, \ldots, n \} \), \( J \in \{ 2, \ldots, n - 1 \} \), (3.82) is of the following form:

\[
F_{\text{death}}^k (N) = - (1 - \theta_1) (1 - \theta_p) Pr_{ip} (J - 1) - \theta_1 \theta_p \frac{n - J}{J} Pr_{ip} (J - 2) + \theta_1 \theta_p \sum_{\ell = J - 1}^{J - 2} \frac{n}{\ell + 1} \frac{n}{\ell + 2} Pr_{ip} (\ell),
\]

(3.85)

which may be considered to be negative for relatively small mortality rates and high values of \( J \).

In the next example, which is related to the latter case, the inequality

\[
\frac{\partial SS}{\partial \rho_{(\rho=0)}} < 0,
\]

(3.86)

applies.

**Numerical example**

We take \( m \), representing the number of contracts, to be equal to 2. Then (3.71) reduces to

\[
SS = \left( \frac{n_1 n_2}{n^2} \right) \left( \frac{R (\theta_1 - \theta_2)}{(1 - \rho \sum_{k \in N} Pr (k))} \right)^2 \cdot \left( 1 - \rho \sum_{k \in N} Pr_{12} (k - 1) + \rho \theta_1 \theta_2 (Y_{12} (N) - \rho Z_{12} (N)) \right)^2.
\]

(3.87)

Let's assume the discounted amount at risk, the size of the two risk classes and the mortality rates to be as given below, just as in the illustrating example used for division among the survivors:

\[
R = -1000; \quad n_1 = n_2 = 5; \quad \theta_1 = 0.014; \quad \theta_2 = 0.001.
\]

(3.88)

The mortality result is shared if the number of deaths is greater than or equal to \( J \), with \( J \in \{ 1, 2 \} \). As

\[
\sum_{\tau = 1}^{m} n_\tau \theta_\tau = 0.075 < 1,
\]

(3.89)

and the amount at risk is of negative sign, only positive mortality profits are distributed. Figure 3.3 displays SS as a function of \( \rho \) for the two given values of \( J \). As can be read from the diagram, SS decreases as a function of \( \rho \) in both cases. Recall, however, that for the combination \( J = 1 \) and \( \rho = 1 \), there is no insurance at all.
This subsection, including the above illustrating example, demonstrates that there are certainly possibilities to decrease the mutual cross-subsidization among individuals in a portfolio, if the mortality result is distributed to the heirs. The reason appears to be the fact that mortality rates are low and in many portfolios the expected number of deaths in a year is smaller than 1. In such portfolios, if at least one individual dies during the period, there is always a mortality loss if the amount at risk is positive and a mortality profit if the amount at risk is negative.

Mortality loss sharing systems, as far as we know, do not occur in practice. One reason is from marketeer's point of view: individuals do not like to be confronted with contingent additional premium charges. Besides, it is doubtful whether an insurance company has the facility to collect an insured's share of the mortality loss from his or her heirs. This implies that the system discussed above is only feasible if the amount at risk is negative. This involves life annuities. In some respects it is comparable with an annuity treaty where (part of) the single premium, to be paid upon issue, is refunded in case of an early death.

This still is not the whole story, since, as stated in this section, for \( N = \{1, \ldots, n\} \) and \( \rho = 1 \), there is no insurance (and hence no subsidizing solidarity). This is the reason why in the next section we introduce a relative quantity for the intensity of mutual cross-subsidization, instead of the absolute one considered in this section.

### 3.4.3 Solidarity related to the volume of transfers

In this section we relate the Subsidizing Solidarity to the so called Total Solidarity, which is found as a weighted average of the squared total transfers. The total transfers are found in the right hand column of either Table 3.4 (in case of distribution of the mortality
3.4. Solidarity aspects

result to the survivors) or Table 3.9 (in case of distribution to the deaths’ heirs). As argued before, they are related to a state. The weight by which a squared total transfer is multiplied is equal to the probability to be in the corresponding state at the end of the period, averaged over all individuals in the portfolio. These probabilities can be found either in Table 3.2 (in case of distribution to the survivors) or in Table 3.7 (in case of distribution to the deaths’ heirs). So if the survivors are the beneficiaries, the Total Solidarity, denoted by $TS$, is equal to

$$TS = \sum_{i=1}^{m} \frac{n_i}{n} (\Pi_r - R)^2$$

$$+ \sum_{i=1}^{m} \frac{n_i}{n} (1 - \theta_i) \sum_{k \in N} \Pr_i (k) \left( TT^\text{surv}_i (k) \right)^2$$

$$+ \sum_{i=1}^{m} \frac{n_i}{n} (1 - \theta_i) \sum_{k \in \mathbb{N} \setminus \{n\}} \Pr_i (k) \Pi_r^2,$$  \hspace{1cm} (3.90)

with

$$ TT^\text{surv}_i (k) = \Pi_r - \rho R \left( \frac{\sum_{i=1}^{m} n_i \theta_i - k - \rho \left( \sum_{k \in \mathbb{N}} \Pr (\ell) (\ell - k) \right)}{(1 - \rho \sum_{k \in \mathbb{N}} \Pr (k)) (n - k)} \right).$$  \hspace{1cm} (3.91)

If instead the sharing system benefits the deaths’ heirs,

$$TS = \sum_{i=1}^{m} \frac{n_i}{n} (\Pi_r - R)^2$$

$$+ \sum_{i=1}^{m} \frac{n_i}{n} \sum_{k \in \mathbb{N}} \Pr_i (k - 1) \left( TT^\text{death}_i (k) \right)^2$$

$$+ \sum_{i=1}^{m} \frac{n_i}{n} \sum_{k \in \mathbb{N} \setminus \{0\}} \Pr_i (k - 1) (\Pi_r - R)^2,$$  \hspace{1cm} (3.92)

where

$$ TT^\text{death}_i (k) = \Pi_r - \rho \left( \sum_{i=1}^{m} n_i \theta_i - k - \rho \left( \sum_{k \in \mathbb{N}} \Pr (\ell) (\ell - k) \right) \right).$$  \hspace{1cm} (3.93)

For both cases, we define the Relative Subsidizing Solidarity, denoted by $RSS$, as the ratio of the Subsidizing Solidarity to the Total Solidarity:

$$RSS = \frac{SS}{TS},$$  \hspace{1cm} (3.94)

where $SS$ is defined either in (3.41) or in (3.71) and $TS$ is defined either in (3.90) or in (3.92), respectively.
Remark 12 (Impact on probabilistic solidarity) For both the sharing systems, it turns out that

\[ TS = SS + PS, \]  

(3.95)

where PS (short for Probabilistic Solidarity, which was briefly discussed in Section 2.2 of Chapter 2) is equal to the weighted average of the squared ex post transfers, averaged over all individuals. The quantity is equal to:

\[
PS = \sum_{i=1}^{m} \frac{n_i \theta_i}{n} (\Pi r_i - R)^2 \\
+ \sum_{i=1}^{m} \frac{n_i (1 - \theta_i)}{n} \sum_{k \in N} \Pr_i (k) \left( (EPT_i^{surv})^2 (k) \right) \\
+ \sum_{i=1}^{m} \frac{n_i (1 - \theta_i)}{n} \sum_{k \in N \setminus \{n\}} \Pr_i (k) \Pi r_i^2,
\]  

(3.96)

with

\[
EPT_i^{surv} (k) = \Pi r_i - \rho R \left( \frac{\sum_{i=1}^{m} n_i \theta_i - k - \rho \left( \sum_{\ell \in N} \Pr (\ell) (\ell - k) \right)}{(1 - \rho \sum_{\ell \in N} \Pr (\ell)) (n - k)} \right), \]  

(3.97)

in case of division among the survivors and

\[
PS = \sum_{i=1}^{m} \frac{n_i (1 - \theta_i)}{n} \Pi r_i^2 \\
+ \sum_{i=1}^{m} \frac{n_i \theta_i}{n} \sum_{k \in N} \Pr_i (k - 1) \left( (EPT_i^{death})^2 (k) \right) \\
+ \sum_{i=1}^{m} \frac{n_i \theta_i}{n} \sum_{k \in N \setminus \{0\}} \Pr_i (k - 1) (\Pi r_i - R)^2,
\]  

(3.98)

with

\[
EPT_i^{death} (k) = \Pi r_i - \rho \left( \frac{\sum_{i=1}^{m} n_i \theta_i - k - \rho \left( \sum_{\ell \in N} \Pr (\ell) (\ell - k) \right)}{k (1 - \rho \sum_{\ell \in N} \Pr (\ell))} \right) R_i, \]  

(3.99)

in case of division among the deaths' heirs. (Both in (3.97) and in (3.99) EPT is short for ex post transfer). The probabilistic solidarity can be considered as a "level of insurance volume". So RSS, unlike SS considered in the previous section, explicitly takes into account the insurance element in the contract.

Just as in the previous section concerning SS, one can study the behavior of RSS by taking its derivative with respect to \( \rho \). However, the derivatives related to the two sharing systems both proved to be cumbersome in the general case, so conclusions in general are
3.4. Solidarity aspects

It is hard to draw. This is the reason why we do not display the derivatives here but refer the reader to the appendix to read the results (formulas (3.109) and (3.113), respectively). In the remainder of this section, we will restrict ourselves to two relatively simple examples, one concerning distribution to the survivors and one related to distribution to the deaths' heirs. The latter case will also be accompanied by a numerical illustration.

Case 13 (Distribution to the survivors) For \( N = \{0\} \) (recall from Case 8 that \( SS \) then remains constant) formula (3.109) reduces to

\[
\frac{\partial RSS}{\partial \rho} = \frac{2 \prod_{r=1}^{m} (1 - \theta_r)^{n_r} \sum_{j,k \neq i} n_i n_j n_k (1 - \rho)}{n \left\{ \frac{\sum_{r=1}^{m} n_r (1 - \theta_r) - 2 \rho n \prod_{r=1}^{m} (1 - \theta_r)^{n_r}}{\prod_{r=1}^{m} (1 - \theta_r)^{n_r}} \right\}^2}.
\] (3.100)

The expression between large curly brackets in the denominator of the above equation is strictly positive irrespective of \( \rho \). So, if there is only distribution if everybody survives, \( \frac{\partial RSS}{\partial \rho} \) monotonously increases in \( \rho \) for the considered interval \([0, 1]\).

Case 14 (Distribution to the deaths’ heirs) For \( N = \{1, \ldots, n\} \) and \( 0 \leq \rho < 1 \), since \( RSS \) is not determined at \( \rho = 1 \) as there is no insurance in that case, formula (3.113) reduces to

\[
\frac{\partial RSS}{\partial \rho} = \frac{2 \sum_{i=1}^{m} n_i \left( \sum_{j \neq i} n_j (\theta_j - \theta_i) \left( 1 + \rho n \theta_i \theta_j \sum_{k=1}^{n-1} \frac{\text{Pr}(k)}{k} \right) \right) Z_i}{(\sum_{r=1}^{m} n_r \theta_r) \text{DEN}^2},
\] (3.101)

with

\[
\text{DEN} = \left( \sum_{r=1}^{m} n_r (1 - \theta_r) \right) + \rho^2 n \left( \sum_{r=1}^{m} n_r \theta_r \sum_{k=1}^{n} \frac{\text{Pr}(k)}{k} - \left( 1 - \prod_{r=1}^{m} (1 - \theta_r)^{n_r} \right)^2 \right),
\] (3.102)

being strictly positive, and

\[
Z_i = \sum_{j=1 \atop j \neq i}^{m} n_j (\theta_j - \theta_i) Z_{ij},
\] (3.103)

where

\[
Z_{ij} = \left( \sum_{r=1}^{m} n_r (1 - \theta_r) \right) \theta_i \theta_j \sum_{k=1}^{n-1} \frac{\text{Pr}(k)}{k} \sum_{m}^{n} \frac{\text{Pr}(\ell-1)}{\ell} \left( 1 - \prod_{r=1}^{m} (1 - \theta_r)^{n_r} \right)^2.
\] (3.104)
When studying the latter expression, one may conclude that $\frac{\partial \text{RSS}}{\partial \rho}$ is positive for some values of $\rho$ and negative for other values, but this it is hard to draw general conclusions about the behavior of the derivative. But if $m$, being the number of risk classes, is equal to 2, the right hand side of (3.101) simplifies to

$$
\frac{\partial \text{RSS}}{\partial \rho} = \frac{2n_1n_2(\theta_1 - \theta_2)^2}{(\sum_{\tau=1}^{m} n_\tau \theta_\tau) \text{DEN}^2} \left(1 + \rho n_1 \theta_1 \sum_{\ell=1}^{n-1} \frac{\text{Pr}_1(\ell-1)}{\ell(\ell+1)} \right) Z_{12},
$$

(3.105)

so the critical value of $\rho$ is

$$
\rho = \frac{(\sum_{\tau=1}^{m} n_\tau (1 - \theta_\tau)) \theta_1 \theta_2 \sum_{\ell=1}^{n-1} \frac{\text{Pr}_1(\ell-1)}{\ell(\ell+1)}}{(\sum_{\tau=1}^{m} n_\tau \theta_\tau) \sum_{k=1}^{m} \frac{\text{Pr}_1(k)}{k} - \left(1 - \prod_{\tau=1}^{m} (1 - \theta_\tau)^{n_\tau}\right)^2}.
$$

(3.106)

We denote this critical value by $\rho^*$. It should be noted that

$$
(\sum_{\tau=1}^{m} n_\tau \theta_\tau) \sum_{k=1}^{m} \frac{\text{Pr}_1(k)}{k} - \left(1 - \prod_{\tau=1}^{m} (1 - \theta_\tau)^{n_\tau}\right)^2 \geq (1 - \text{Pr}(0))^2 \left(\sum_{k=1}^{m} \frac{\text{Pr}_1(k)}{\text{Pr}(0)^k} \left(\sum_{k=1}^{m} \frac{\text{Pr}_1(k)}{\text{Pr}(0)^k} \right) - 1\right) = 0.
$$

(3.107)

The inequality-sign follows by Jensen’s inequality, applied to the convex function $\frac{1}{k}$. So $\rho^*$ is positive and RSS increases monotonously in $\rho$ for $\rho < \rho^*$ and decreases monotonously in $\rho$ for $\rho > \rho^*$. In the next example, $\rho^*$ falls in the interval $[0, 1]$.

**Example 15** As in the example dealt with in Subsection 3.4.2, the following values are taken:

$$
R = 1000; \quad n_1 = n_2 = 5; \quad \theta_1 = 0.014; \quad \theta_2 = 0.001.
$$

(3.108)

It turns out that $\rho^* = 0.8208$. Figure 3.4 displays RSS as a function of $\rho$. One can see that RSS does not change very steeply.

The above example shows a case where the relative decrease of the Subsidizing Solidarity and the Total Solidarity is about the same. Referring to Remark 12, note that in the given case, the Probabilistic Solidarity as a function of $\rho$, varies according to about the same pattern as the SS does. This is, however, only one example, and more cases will have to be worked out in order to be able to draw conclusions concerning the Relative Subsidizing Solidarity.
3.5 Conclusions, final comments and recommendations for further research

A system to share a fixed proportion of the insurer’s mortality result concerning a certain portfolio has been discussed extensively in this chapter. Two variations of mortality result sharing have been considered, namely division among the survivors and division among the deaths’ heirs.

From the insurer’s point of view, the variance of the aggregate loss suffered by the insurer is always a monotonously increasing function of the above mentioned proportion. So the higher that proportion is, the lower will be the aggregate loss variance. If the proportion is equal to one, i.e. if the entire mortality result is divided, this variance is always minimal. So the sharing system has at least one appealing property.

The main subject of this chapter, however, concerns the impact of the intensity of distribution, represented by the above mentioned fixed proportion, on the intensity by which the heterogeneous risks in the portfolio subsidize one another. This phenomenon has been quantified by the Subsidizing Solidarity (SS), discussed already in the previous chapter. Conclusions in general are difficult to draw. However, it appears that, for relatively small portfolios and small mortality rates, mortality result distribution contributes to a lower level of mutual cross-subsidization, if the mortality result is shared with the deaths’ heirs. Since in practice usually only mortality profit, and not mortality loss, can be distributed, this implies that the amount at risk must be of negative sign, which is the case for pure endowments and annuities.

Regarding the Relative Subsidizing Solidarity (RSS), it is even harder to draw inferences in general. An exception is the case of distribution to the survivors if everybody survives. While the SS remains constant, the RSS increases as the intensity of distribution
increases. Anyway, all the measures for the subsidizing solidarity derived in this section should be applied to real mortality data in order to see how the results will be in practice.

The system introduced can be made more suitable for practical applications by adding more parameters, especially if the sign of the amount at risk is negative. In that case, the amount of share that a beneficiary gets depends very much on the number of deaths. For instance, the portfolio may be such that there is already a mortality profit if only two individuals die. Still this profit is much higher if twenty persons do not survive the period. So distribution with a negative amount at risk can result in some kind of a lottery, which might be undesirable. This phenomenon can be avoided if one allows to let the proportion of the mortality result vary with the actual number of deaths at the end of the period. Another extension is to take into account different amounts at risk. The latter then provokes the question how the entire mortality result should be shared, since it sounds unfair to let every beneficiary still get the same portion of the entire mortality result.

As was already mentioned introduction, decreasing the subsidizing solidarity would be a valid justification for applying the mortality result sharing system if the information available to the insurer concerning the individuals' mortality rates was imperfect. Therefore we recommend to develop a system based on an urn-of-urns model for each insured individual, hence using an approach similar to the one in Chapter 2.
Appendix: Derivatives of RSS in Subsection 3.4.3

The derivatives are as follows:

In case of distribution to the survivors:

\[
\frac{\partial \text{RSS}}{\partial \rho} = 2 \sum_{i=1}^{m} \frac{n_i U_i}{TS'},
\] (3.109)

with

\[
TS' = \begin{cases} 
\left[ \sum_{i=1}^{m} n_i \theta_i \right] \left[ \sum_{i=1}^{m} n_i \left(1 - \theta_i\right) \right] \\
-2\rho \left[ \sum_{i=1}^{m} n_i \theta_i \sum_{k \in K} \Pr(k) (n - k) \right] \\
+ \rho^2 \left[ n \sum_{k \in K} \frac{\Pr(k)}{n-k} \left( \sum_{i=1}^{m} n_i \theta_i - k \right)^2 - \left( \sum_{k \in K} \Pr(k) \right)^2 \right] \\
+ \rho^2 \left( \sum_{i=1}^{m} n_i \theta_i \right) \left( \sum_{k \in K} \Pr(k) \right)^2 \\
-2\rho^3 n \left[ \sum_{i=1}^{m} n_i \theta_i - k \right] \sum_{k \in K} \frac{\Pr(k)}{n-k} \sum_{\ell \in N} \Pr(\ell) (\ell - k) \\
+ \rho^2 n \sum_{k \in K} \frac{\Pr(k)}{n-k} \left[ \sum_{\ell \in N} \Pr(\ell) (\ell - k) \right]^2
\end{cases}
\] (3.110)

and

\[
U_i = \left( \sum_{j=1}^{m} n_j \left( \theta_j - \theta_i \right) \right) \left( \sum_{j=1}^{m} n_j \left( \theta_j - \theta_i \right) \right) \left( \sum_{p=1}^{m} n_p \cdot X \right),
\] (3.111)
with $B_{ip}(N)$ and $C_{ip}(N)$ as defined in equations (3.37) and (3.38), respectively.
In case of distribution to the deaths’ heirs:

\[
\frac{\partial \text{RSS}}{\partial \rho} = 2 \sum_{i=1}^{m} \frac{n_i V_i}{TS''},
\]

(3.113)

with

\[
TS'' = \left\{ \begin{array}{c}
(\sum_{i=1}^{m} n_i \theta_i) (\sum_{i=1}^{m} n_i (1 - \theta_i)) \\
-2\rho (\sum_{i=1}^{m} n_i (1 - \theta_i)) \sum_{k \in \mathbb{N}} \text{Pr} \, (k) \, k
\end{array} \right\}^2
\]

(3.114)

\[
-2\rho^2 n (\sum_{i=1}^{m} n_i \theta_i) \sum_{k \in \mathbb{N}} \frac{\text{Pr}(k)}{k} \sum_{\ell \in \mathbb{N}} \text{Pr} \, (\ell) \, (\ell - k)
\]

\[
+\rho^4 n \sum_{k \in \mathbb{N}} \frac{\text{Pr}(k)}{k} \left( \sum_{\ell \in \mathbb{N}} \text{Pr} \, (\ell) \, (\ell - k) \right)^2
\]

and

\[
V_i = \left\{ \begin{array}{c}
\sum_{j \neq i} n_j (\theta_j - \theta_i) \left\{ 1 - \rho \sum_{\ell \in \mathbb{N}} \text{Pr}_{ij} (k - 1) + \rho \theta_i \theta_j (Y_{ij} (N) - \rho Z_{ij} (N)) \right\}
\end{array} \right\} \left( \sum_{p=1}^{m} \frac{n_p \cdot X_{i p}}{n} \right),
\]

(3.115)
where

$$X_{ip} = \left( \sum_{r=1}^{m} n_r (1 - \theta_r) \right) \left\{ - (\sum_{r=1}^{m} n_r \theta_r) \left( \sum_{k \in N} \Pr(k) k \right) + \eta \left( \sum_{r=1}^{m} n_r \theta_r \right) \theta_p Y_{ip}(N) \right\}$$

$$- \rho \left\{ \sum_{k \in N} \Pr(k) \left( \sum_{r=1}^{m} n_r \theta_r \right) \left( \sum_{k \in N} \Pr(k) k \right) \right\}$$

$$+ \rho^2 \left\{ 3n \theta_p Z_{ip}(N) \left( \sum_{r=1}^{m} n_r (1 - \theta_r) \right) \left( \sum_{k \in N} \Pr(k) k \right) \right\}$$

$$+ \rho^3 \left\{ \frac{3n}{m} \sum_{k \in N} \frac{\Pr(k)}{k} \left( \sum_{r=1}^{m} n_r \theta_r - k \right) \left( \sum_{k \in N} \Pr(k) k \right) \right\}$$

$$+ \rho^4 \left\{ \frac{3n}{m} \sum_{k \in N} \frac{\Pr(k)}{k} \left( \sum_{r=1}^{m} n_r \theta_r - k \right) \left( \sum_{k \in N} \Pr(k) k \right) \right\}$$

with $Y_{ip}(N)$ and $Z_{ip}(N)$ as defined in equations (3.69) and (3.70), respectively.