Bose-Einstein condensation in low-dimensional trapped gases

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Chapter 2 Overview

In this chapter we give a brief overview of literature on low-dimensional gases and introduce important concepts and methods used in this Thesis.

2.1 BEC of an ideal gas in 1D and 2D harmonic traps

The analysis of trapped gases has been extended mostly to the noninteracting case. Bagnato and Kleppner [32] have found that in a harmonically trapped ideal 2D Bose gas occupation of the ground state becomes macroscopic (ordinary BEC transition) below a critical temperature which depends on the number of particles and trap frequencies. Ketterle and van Druten have shown that the BEC-like behavior is also present in an ideal trapped 1D gas [33].

We start with thermodynamic description of an ideal 1D or 2D gas of $N$ bosons trapped in a harmonic external potential. The gas sample is assumed to be in thermal equilibrium at temperature $T$. We will calculate thermodynamic averages for the grand canonical ensemble, where the system is characterized by a fixed chemical potential $\mu$ and fluctuating number of particles $N$. In the thermodynamic limit ($N \to \infty$) this is equivalent to the description in the canonical ensemble (fixed $N$ and fluctuating $\mu$).

Generally, in an arbitrary trap of any dimension the system is characterized by a set of eigenenergies of an individual atom, $\{\epsilon_\nu\}$. The (average) total number of particles $N$ is then related to the temperature and chemical potential by the equation

$$N = \sum \nu N((\epsilon_\nu - \mu)/T),$$

(2.1.1)

where $N(z) = 1/(\exp z - 1)$ is the Bose occupation number (we use the convention $k_B = 1$). The population of the ground state ($\epsilon_0 = 0$) is

$$N_0 = \frac{1}{\exp(-\mu/T) - 1},$$

(2.1.2)

and in the thermodynamic limit can become macroscopic (comparable with $N$) only if $\mu = 0$. For a large but finite number of particles in a trap, $N_0$ becomes comparable with $N$ and one has a crossover to the BEC regime at a small but finite $\mu$.

We now determine the temperature of the BEC crossover for a 2D Bose gas confined in a circularly symmetric harmonic trap. In this case the index $\nu$ corresponds to the pair of quantum numbers $\{n_x, n_y\}$ in such a way that $\epsilon_{n_x, n_y} = \hbar \omega (n_x + n_y)$, where $\omega$ is the trap frequency. This particular level structure allows one to rewrite the sum in Eq. (2.1.1) in the form of an integral

$$N = N_0 + \int_0^\infty N((\epsilon - \mu)/T)\rho(\epsilon)d\epsilon.$$

(2.1.3)
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Here $\rho(\epsilon) = \epsilon/(\hbar \omega)^2$ is the density of states of the system and we have separated the ground state population. The transformation from the sum to the integral is justified by the inequality $\hbar \omega \ll T$ and by the fact that the relative contribution of several low-energy states is negligible (the density of states goes to zero). For a large population of the ground state Eq. (2.1.2) gives $-\mu/T \approx 1/N_0 \ll 1$. Then the first two leading terms in the expansion of the integral in Eq. (2.1.3) are

$$\int_0^\infty N((\epsilon - \mu)/T) \rho(\epsilon) d\epsilon \approx \left( \frac{T}{\hbar \omega} \right)^2 \left( \frac{\pi^2}{6} - \frac{1 + \ln N_0}{N_0} \right),$$

(2.1.4)

and Eq. (2.1.3) reduces to the form

$$N \left[ 1 - (T/T_c)^2 \right] = N_0 - (T/\hbar \omega)^2 (1 + \ln N_0)/N_0,$$

(2.1.5)

where

$$T_c = \sqrt{6N/\pi^2 \hbar \omega}.$$  

(2.1.6)

From Eq. (2.1.4) one clearly sees that there is a sharp crossover to the BEC regime at $T \approx T_c$ [32; 34]. Below $T_c$ we can neglect the last term in Eq. (2.1.5). Then we obtain the occupation of the ground state $N_0 \approx N \left[ 1 - (T/T_c)^2 \right]$ similar to that in the 3D case. Above $T_c$ the first term in the rhs of Eq. (2.1.5) can be neglected compared to the second one. Note that at $T_c$ the de Broglie wavelength of particles $\lambda \sim \sqrt{\hbar^2/MT_c}$ becomes comparable with the mean interparticle separation $\sim (N\mu^2/T_c)^{-1/2}$. In the crossover region between the two regimes all terms in Eq. (2.1.5) are equally important and we estimate the width of the crossover region as

$$\frac{\Delta T}{T_c} \sim \sqrt{\frac{\ln N}{N}}.$$  

(2.1.7)

Solutions of Eq. (2.1.5) for various $N$ are presented in Fig. 2.1.1.

For a large number of particles the relative width of the crossover region is very small. Therefore, one can speak of an ordinary BEC transition in an ideal harmonically trapped 2D gas.

In the 1D case the spectrum is $\epsilon_n = \hbar \omega n$ and the density of states is $\rho(\epsilon) = 1/\hbar \omega$. Here the integral representation Eq. (2.1.3) fails as the integral diverges for $-\mu/T \rightarrow 0$ and we should correctly take into account the lowest energy levels [33]. In the limit $\{-\mu, \hbar \omega\} \ll T$ we rewrite Eq. (2.1.1) in the form

$$N = N_0 + \frac{T}{\hbar \omega} \sum_{n=1}^M \frac{1}{n - \mu/\hbar \omega} + \sum_{n=M+1}^\infty \frac{1}{\exp(\hbar \omega n/T - \mu/T) - 1},$$

(2.1.8)

where the number $M$ satisfies the inequalities $1 \ll M \ll T/\hbar \omega$. The first sum is

$$\sum_{n=1}^M \frac{1}{n - \mu/\hbar \omega} = \psi(M+1-\mu/\hbar \omega) - \psi(1-\mu/\hbar \omega) \approx -\psi(1-\mu/\hbar \omega) + \ln(M-\mu/\hbar \omega),$$

(2.1.9)
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Figure 2.1.1: The ground state population in a 2D trap versus temperature. The curves are calculated from Eq. (2.1.5) for various $N$.

where $\psi$ is the digamma function. The second sum in Eq. (2.1.8) can be transformed to an integral

$$
\sum_{n=M+1}^{\infty} \frac{1}{\exp\left(\frac{\hbar \omega - \mu}{T}\right) - 1} \approx \frac{T}{\hbar \omega} \int_{\hbar \omega M/T}^{\infty} \frac{dx}{\exp(x - \mu/T) - 1} \approx -\frac{T}{\hbar \omega} \ln \frac{\hbar \omega (M - \mu/\hbar \omega)}{T}.
$$

Finally, eliminating the chemical potential by using the equality $-\mu \approx T/N_0$, we reduce Eq. (2.1.8) to the form

$$
N - \frac{T}{\hbar \omega} \ln \frac{T}{\hbar \omega} = N_0 - \frac{T}{\hbar \omega} \psi \left(1 + \frac{T}{\hbar \omega N_0}\right).
$$

As in the 2D case, we have two regimes, with the border between them at $T_c$ determined by $N \approx (T_c/\hbar \omega) [\ln(T_c/\hbar \omega) + 0.577]$. Indeed, below this temperature the first term in the rhs of Eq. (2.1.10) greatly exceeds the second one and the ground state population behaves as $N_0 \approx N - (T/\hbar \omega) \ln(T/\hbar \omega)$. The crossover region is determined as the temperature interval where both terms are equally important:

$$
\Delta T/T_c \sim 1/\ln N.
$$

The crossover temperature in a 1D trap is $T_c \approx N \hbar \omega / \ln N$ and, in contrast to 3D and 2D cases, is much lower than the degeneracy temperature $T_d \approx N \hbar \omega$. In Fig. 2.1.2 we present $N_0(T)$ calculated from Eq. (2.1.10).

The crossover region in the 1D case is much wider than in 2D. This is not surprising as the crossover itself is present only due to the discrete structure of the trap levels. The quasiclassical calculation does not lead to any crossover [32].

Note that an ideal 2D Bose gas in a finite box has the density of states independent of energy, just like a harmonically trapped 1D gas. Therefore, it is characterized by a similar crossover to the BEC regime [35; 33].
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2.2 Interacting Bose gas

We now consider an interacting d-dimensional Bose gas in a trap with the potential \( U(r) \). The Hamiltonian of the system in the second quantization reads (see [36])

\[
\hat{H} = \int_r \hat{\Psi}^\dagger(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + U(r) \right) \hat{\Psi}(r) + \frac{1}{2} \int_{r, r'} V(|r - r'|) \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r') \hat{\Psi}(r) \hat{\Psi}(r'),
\]

where \( V(r) \) is the potential of interaction between two atoms and \( \hat{\Psi}(r) \), \( \hat{\Psi}^\dagger(r) \) are the Boson field operators satisfying the commutation relations

\[
[\hat{\Psi}(r), \hat{\Psi}(r')] = 0, \quad [\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = \delta(r - r').
\]

In the Heisenberg representation the time derivative of \( \hat{\Psi}(r) \) is given by

\[
i\hbar \frac{\partial \hat{\Psi}}{\partial t} = -[\hat{H}, \hat{\Psi}] = \left( -\frac{\hbar^2 \nabla^2}{2m} + U(r) + \int_{r'} V(|r - r'|) \hat{\Psi}^\dagger(r') \hat{\Psi}(r') \right) \hat{\Psi}.
\]

Using the commutation relations (2.2.2) we rewrite Eq. (2.2.3) in the form

\[
i\hbar \frac{\partial \hat{\Psi}}{\partial t} + V(0) \hat{\Psi} = -\frac{\hbar^2 \nabla^2}{2m} \hat{\Psi} + U(r) \hat{\Psi} + \hat{\Psi} \int_{r'} V(|r - r'|) \hat{\Psi}^\dagger(r') \hat{\Psi}(r').
\]

The term \( V(0) \hat{\Psi} \) gives only a trivial time dependence and can be eliminated by the substitution \( \hat{\Psi} \rightarrow \hat{\Psi} \exp(iV(0)t/\hbar) \). Let us now turn to the density-phase representation of the field operators

\[
\hat{\Psi} = \exp(i\hat{\phi}) \sqrt{n}, \quad \hat{\Psi}^\dagger = \sqrt{n} \exp(-i\hat{\phi}),
\]

![Figure 2.1.2: The ground state population in a 1D trap versus temperature. The curves are calculated from Eq. (2.1.10) for various \( N \).](image)
where the density and phase operators are real and satisfy the commutation relation
\[
[n(r), \phi(r')] = i\delta(r - r'). \tag{2.2.6}
\]
Substituting Eqs. (2.2.5) into Eq. (2.2.4) and separating real and imaginary parts, we get the coupled continuity and Euler hydrodynamic equations for the density and velocity \( \dot{\psi} = (\hbar/m)\nabla \phi \):
\[
-\hbar \frac{\partial \hat{n}}{\partial t} = \frac{\hbar^2}{m} \nabla (\nabla \phi \hat{n}), \tag{2.2.7}
\]
\[
-\hbar \frac{\partial \hat{\phi}}{\partial t} = \frac{\hbar^2}{2m} (\nabla \phi)^2 - \frac{\hbar^2 \nabla^2 \sqrt{\hat{n}}}{2m \sqrt{\hat{n}}} + U(r) + \int_{r'} V(|r' - r|) \hat{n}(r'). \tag{2.2.8}
\]
Eqs. (2.2.7-2.2.8) are obtained from the Hamiltonian (2.2.1) without any approximations and, in principle, should describe any property of an interacting Bose gas. At the same time, these equations are nonlinear and very complicated. However, in a dilute ultra-cold weakly interacting gas we can use a number of simplifications. The short-range character of the interatomic potential\(^1\) allows us to rewrite the integral in Eq. (2.2.8) as
\[
\int_{r'} V(|r' - r|) \hat{n}(r') \approx g \hat{n}(r), \tag{2.2.9}
\]
where the mean-field coupling constant \( g \) can also depend on density or on average momentum of particles (see Chapters 3 and 4).

We further simplify Eqs. (2.2.7-2.2.8) assuming small fluctuations of the density. In this case, Eq. (2.2.7) shows that fluctuations of the phase gradient are also small. Representing the density operator as \( \hat{n}(r) = n_0(r) + \delta \hat{n}(r) \) and shifting the phase by \(-\mu t/\hbar\) we then linearize Eqs. (2.2.7-2.2.8) with respect to \( \delta \hat{n}, \nabla \hat{\phi} \) around the stationary solution \( \hat{n} = n_0, \nabla \hat{\phi} = 0 \). The zero order terms give the Gross-Pitaevskii equation for \( n_0 \):
\[
-\frac{\hbar^2 \nabla^2 \sqrt{n_0}}{2m \sqrt{n_0}} + U(r) + gn_0 = \mu, \tag{2.2.10}
\]
and the chemical potential \( \mu \) follows from the normalization condition
\[
\int_r n_0(r) = N. \tag{2.2.11}
\]
The first order terms provide equations for the density and phase fluctuations:
\[
\hbar \frac{\partial (\delta \hat{n}/\sqrt{n_0})}{\partial t} = (-\hbar^2 \nabla^2/2m + U(r) + gn_0 - \mu)(2\sqrt{n_0} \hat{\phi}), \tag{2.2.12}
\]
\[
-\hbar \frac{\partial (2\sqrt{n_0} \hat{\phi})}{\partial t} = (-\hbar^2 \nabla^2/2m + U(r) + 3gn_0 - \mu)(\delta \hat{n}/\sqrt{n_0}). \tag{2.2.13}
\]
Solutions of Eqs. (2.2.12-2.2.13) are obtained by representing \( \delta \hat{n}, \nabla \hat{\phi} \) in terms of elementary excitations:
\[
\delta \hat{n}(r) = n_0(r)^{1/2} \sum_{\nu} i f_{\nu}^{-}(r)e^{-i\epsilon_{\nu}t/\hbar} \hat{a}_{\nu} + \text{H.c.}, \tag{2.2.14}
\]
\[
\hat{\phi}(r) = [4n_0(r)]^{-1/2} \sum_{\nu} f_{\nu}^{+}(r)e^{-i\epsilon_{\nu}t/\hbar} \hat{a}_{\nu} + \text{H.c.}. \tag{2.2.15}
\]
\(^{1}\)In this Thesis we do not consider dipolar or charged gases with long-range interatomic forces.
From Eqs. (2.2.14-2.2.15) we then find equations for the eigenenergies $\epsilon_\nu$ and eigenfunctions $f_\nu^\pm$ of the excitations:

\begin{align}
(-\hbar^2 \nabla^2 / 2m + U(r) + gn_0 - \mu) f_\nu^+ &= \epsilon_\nu f_\nu^- , \\
(-\hbar^2 \nabla^2 / 2m + U(r) + 3gn_0 - \mu) f_\nu^- &= \epsilon_\nu f_\nu^+ .
\end{align}

The commutation relation (2.2.6) ensures that the functions $f_\nu^\pm$ are normalized by the condition

\[ \frac{1}{2} \int f_\nu^+(r) f_\nu^-*(r) + f_\nu^-(r) f_\nu^+*(r) \, dr = 1 . \tag{2.2.18} \]

Equations (2.2.16-2.2.17) are exactly the same as the Bogolyubov-de Gennes equations\textsuperscript{2} for elementary excitations of a Bose-Einstein condensate with the density profile $n_0(r)$. The functions $f_\nu^\pm$ are related to the well-known Bogolyubov $u$, $v$ functions by $f_\nu^\pm = u \pm v$. We thus see that the assumption of small density fluctuations is sufficient for having the Bogolyubov spectrum of the excitations, irrespective of the presence or absence of a true condensate.

The spectrum of elementary excitations of a trapped Bose-condensed gas has been intensively discussed in literature. In a vast majority of experiments the number of particles is very large and the chemical potential $\mu$ greatly exceeds the level spacing in the trap. In this case the kinetic energy term in Eq. (2.2.10) is much smaller than the nonlinear term and can be neglected. This approach is called the Thomas-Fermi (TF) approximation, and in a harmonic trap the density profile $n_0$ takes the well-known parabolic shape

\[ n_0 = (\mu - U(r))/g . \tag{2.2.19} \]

In this case, the low-energy excitations ($\epsilon_\nu \ll \mu$) can be found analytically [38; 39; 40]. The dependence of the excitation spectrum on the trapping geometry has been extensively studied for 3D TF condensates (see [41] for review). For example, in very elongated cigar-shaped condensates the spectrum of low lying axial excitations reads $\epsilon_j = \hbar \omega_z \sqrt{j(j+3)/2}$ [40; 42]. Stringari [42] has found the spectrum for very anisotropic pancake-shaped condensates. The spectrum of low-energy excitations has also been found for purely 2D and 1D Thomas-Fermi clouds [43].

In a homogeneous Bose gas the chemical potential $\mu = gn_0$ and Eqs. (2.2.16-2.2.18) give the well-known spectrum and wavefunctions:

\begin{align}
\epsilon(k) &= \sqrt{E(k)[E(k) + 2\mu]} , \\
f_k^\pm &= \frac{1}{\sqrt{V}} \left( \frac{\epsilon(k)}{E(k)} \right)^{1/2} e^{\pm i k \cdot r} ,
\end{align}

where $E(k) = \hbar^2 k^2 / 2m$ is the free-particle spectrum, and $V$ is the $d$-dimensional volume of the system. The Bogolyubov spectrum (2.2.20) is phonon-like, with $\epsilon(k) \approx c_s \hbar k$, for energies of the order of or smaller than $\mu$. This corresponds to momenta $k \lesssim 1/l_c$, where the healing length $l_c = \hbar / \sqrt{\mu m}$. The speed of sound is $c_s = \sqrt{\mu / m}$. For larger momenta the spectrum (2.2.20) is particle-like, with $\epsilon(k) \approx E(k) + \mu$.

\textsuperscript{2}Similar equations have been found by de Gennes [37] for inhomogeneous superconductors.
The chemical potential $\mu$ is an approximate border between two classes of excitations. In order to analyze the role of these two classes let us separate the operator of the density fluctuations into two parts: $\delta \hat{n} = \delta \hat{n}_s + \delta \hat{n}_p$, where the indices $s$ and $p$ stand for the phonon ($\epsilon < \mu$) and free-particle ($\epsilon > \mu$) parts respectively. We now use Eq. (2.2.14) and calculate the density-density correlation function $\langle \delta \hat{n}_s(r_1) \delta \hat{n}_s(r_2) \rangle$, where the symbol $\langle \ldots \rangle$ denotes the statistical average. A straightforward calculation yields

$$
\langle \hat{n}_s(r_1)\hat{n}_s(r_2) \rangle = \frac{n_0}{V} \sum_{\epsilon(k) < \mu} \frac{E(k)}{\epsilon(k)} \left[ 2N(\epsilon(k)/T) + 1 \right] \cos k \cdot (r_1 - r_2).
$$

(2.2.22)

Here we used the equality $\langle \hat{a}_k^\dagger \hat{a}_k \rangle = N(\epsilon(k)/T)$ for the equilibrium occupation number of excitations. Taking into account that this number is always smaller than $T/\epsilon(k)$ we obtain

$$
\langle \hat{n}_s(r_1)\hat{n}_s(r_2) \rangle/n_0^2 < (\mu/T_d)^{d/2} \max\{(T/\mu),1\},
$$

(2.2.23)

where $d$ is the dimension of the system, and the temperature of quantum degeneracy is $T_d = \hbar^2 n_0^{2/d}/m$. We then see that well below $T_d$ in 2D and 3D weakly interacting gases the phonon-induced density fluctuations are small and can be neglected. To ensure that they are small in the one-dimensional gas we require $T < (\mu T_d)^{1/2}$ in 1D. Then, omitting fluctuations originating from the high-energy part ($\epsilon > \mu$) of the spectrum, we represent the field operator $\Psi$ in the form

$$
\hat{\Psi} = \sqrt{n_0} \exp \left( i \hat{\phi}_s \right),
$$

(2.2.24)

In order to make sure that fluctuations coming from high-energy excitations can be omitted, we a priori assume that they are small and can be evaluated by linearizing the field operator $\Psi = \exp(i\hat{\phi}) \sqrt{n}$ with regard to the high-energy part. This is equivalent to writing the preexponential factor in Eq. (2.2.24) as $\left( \sqrt{n_0} + \hat{\Psi}' \right)$ instead of just $\sqrt{n_0}$. The operator $\hat{\Psi}'$ accounts for both density and phase fluctuations and reads

$$
\hat{\Psi}' = i \sum_{\epsilon(k) > \mu} u_k(r) \hat{a}_k e^{-i\epsilon(k)t/\hbar} - v_k(r) \hat{a}_k^\dagger e^{i\epsilon(k)t/\hbar}.
$$

(2.2.25)

At energies significantly larger than $\mu$ the function $u_k = (f_+ - f_-)/2 \to 0$, and $u_k = (f_+ + f_-)/2 \to V^{-1/2} \exp(ik \cdot r)$. The energy itself is $\epsilon(k) \approx E(k) + \mu$ and, hence, the operator $\hat{\Psi}'$ describes an ideal gas of Bose particles with chemical potential equal to $-\mu$. One thus sees that high-energy Bogolyubov excitations correspond to the incoherent non-condensed part of the gas. The density of this gas is exponentially small at $T < \mu$ and for $T \gg \mu$ it is equal to

$$
\langle \hat{\Psi}'^\dagger \hat{\Psi}' \rangle \approx \int_{\epsilon(k) > \mu} N[\epsilon(k)/T] \frac{d^d k}{(2\pi)^d} \sim n_0 \times \begin{cases} 
(T/T_d)^{3/2}, & \text{in 3D} \\
(T/T_d) \ln(T/\mu), & \text{in 2D} \\
T (\mu T_d)^{-1/2}, & \text{in 1D}.
\end{cases}
$$

(2.2.26)
At temperatures \( T \ll T_d \) (\( T \ll (\mu T_d)^{1/2} \) in the 1D case) the quantity \( \langle \hat{\Psi}^\dagger \hat{\Psi} \rangle \) is really small, which justifies Eq. (2.2.24) for the field operator.

Equation (2.2.24) allows us to calculate correlation functions at low temperatures to zero order in perturbation theory. For obtaining perturbative corrections, one should expand Eqs. (2.2.12-2.2.13) up to quadratic terms in \( \delta \hat{n} \) and \( \hat{\phi} \). This provides a correction for the stationary solution and for the chemical potential as a function of density. One should then include the low-energy density fluctuations and the high-energy fluctuations in the expression for the field operator. This is equivalent to proceeding along the lines of the perturbation theory developed by Popov (see [12]).

### 2.3 Phase fluctuations and quasicondensates in a uniform gas

Bose-Einstein condensation in a uniform gas is associated with the long-range order in the system, i.e. with the finite value of the one-particle density matrix \( \rho(r_1, r_2) = \langle \hat{\Psi}^\dagger(r_1) \hat{\Psi}(r_2) \rangle \) at \( |r_1 - r_2| \to \infty \). Using Eq. (2.2.24) the density matrix at low temperatures can be written in the form [12]

\[
\rho(r_1, r_2) = n_0 \left< e^{-i[\hat{\phi}_s(r_1) - \hat{\phi}_s(r_2)]} \right> = n_0 e^{-\frac{1}{2} \left< [\hat{\phi}_s(r_1) - \hat{\phi}_s(r_2)]^2 \right>}. \tag{2.3.1}
\]

The phase correlator is obtained from Eqs. (2.2.15) and (2.2.21). It reads

\[
\left< \left[ \hat{\phi}_s(r) - \hat{\phi}_s(0) \right]^2 \right> = \frac{1}{2n_0V} \sum_{\epsilon(k)<\mu} \frac{\epsilon(k)}{E(k)} \left[ 2N(\epsilon(k)/T) + 1 \right] (1 - \cos k \cdot r), \tag{2.3.2}
\]

In the 3D case this correlator is small even at temperatures larger than \( \mu \), where it is approximately equal to \( T/(\pi^2 T_d n_0^{1/3} l_c) \).

The two-dimensional vacuum fluctuations of the phase are negligible and in 2D one has a true condensate at zero temperature. At finite temperatures the correlator (2.3.2) takes the form (see [44])

\[
\left< \left[ \hat{\phi}_s(r) - \hat{\phi}_s(0) \right]^2 \right> \approx T/(\pi T_d) \ln(r/l_c). \tag{2.3.3}
\]

This result means that the long-wave fluctuations of the phase of the boson field provide a power law decay of \( \rho(r) \) at \( r \to \infty \) in contrast to the 3D case. This was first found by Kane and Kadanoff [44] and is consistent with the Bogolyubov theorem [10; 11] indicating the absence of a true condensate at finite temperatures in 2D. The power law behavior of \( \rho(r) \) is qualitatively different from the exponential decay at large distances in a classical gas, which indicates a possibility of phase transition at sufficiently low \( T \). The existence of a superfluid phase transition in 2D gases (liquids) has been proved by Berezinskii [45; 46]. Kosterlitz and Thouless [47; 48] found that this transition is associated with the formation of bound pairs of vortices below a critical temperature \( T_{KT} \) which is of the order of \( T_d \). Recently, the exact value of this temperature has been found using Monte Carlo simulations [49; 50].
Earlier theoretical studies of 2D systems have been reviewed in [12] and have led to the conclusion that below the Kosterlitz-Thouless transition (KTT) temperature the Bose liquid (gas) is characterized by the presence of a quasicondensate, which is a condensate with fluctuating phase (see [13]). Indeed, from Eqs. (2.3.1) and (2.3.3) we see that the density matrix decays on a length scale \( R_\phi \approx l_c \exp(\pi T_d/T) \gg l_c \). In this case the system can be divided into blocks with a characteristic size greatly exceeding the healing length \( l_c \) but smaller than the radius of phase fluctuations \( R_\phi \). Then, there is a true condensate in each block but the phases of different blocks are not correlated with each other.

The situation is similar in the 1D gas at temperatures \( T \ll (\mu T_d)^{1/2} \), except for the presence of a true condensate at \( T = 0 \). Eq. (2.3.2) gives

\[
\left\langle \left[ \hat{\phi}_s(r) - \hat{\phi}_s(0) \right]^2 \right\rangle \approx \frac{T}{\sqrt{\mu T_d}} \frac{r}{l_c} + \frac{1}{\pi} \sqrt{\frac{\mu}{T_d}} \ln \frac{r}{l_c},
\]

where the first term in the rhs comes from the thermal part and the second one from the vacuum part of the phase fluctuations. Thus, at zero temperature the density matrix undergoes a power-law decay [51; 52; 12], and there is no true Bose-Einstein condensate. This is consistent with a general analysis of 1D Bose gases at \( T = 0 \) [53]. At finite temperatures the long-range order is destroyed by long-wave fluctuations of the phase leading to an exponential decay of the one-particle density matrix at large distances [44; 12].

In this Thesis we mostly concentrate on the properties of a trapped 1D Bose gas in the weakly interacting regime, where the healing length \( l_c = \hbar/\sqrt{\pi \alpha g} \) greatly exceeds the mean interparticle separation \( 1/n \). This corresponds to small values of the parameter \( \gamma = mg/\hbar^2n \). Our previous discussions of the uniform 1D interacting gas in this chapter were related to this particular regime. In general, the 1D Bose gas with repulsive short-range interactions characterized by the coupling constant \( g > 0 \) is integrable by using the Bethe Ansatz at any \( g \) and \( n \) and has been a subject of extensive theoretical studies [54; 55; 56; 57]. The equation of state and correlation functions depend crucially on the parameter \( \gamma \). The weakly interacting regime (\( \gamma \ll 1 \)) is realized at comparatively large densities (or small \( g \)). For sufficiently small densities (or large \( g \)), the parameter \( \gamma \gg 1 \) and one has a gas of impenetrable bosons (Tonks gas), where the wavefunction strongly decreases when particle approach each other. In this respect the system acquires fermionic properties.

### 2.4 Realization of BEC’s in lower dimensions

In commonly studied BEC’s in 3D harmonic traps the mean-field interaction greatly exceeds the level spacings. In this case the Thomas-Fermi approximation can be used and the shape of the condensate density profile is given by Eq. (2.2.19). For the Thomas-Fermi condensate the chemical potential equals

\[
\mu_{3D} = \left( \frac{15N_0 g m^{3/2} \bar{\omega}^3}{\pi^{29/2}} \right)^{2/5},
\]

where \( \bar{\omega}^3 = \omega_x \omega_y \omega_z \), \( N_0 \) is the number of condensed particles, and \( g = 4\pi \hbar^2 a/m \) is the 3D coupling constant with \( a \) being the 3D scattering length.
In order to reach quasi-1D or quasi-2D regimes for BEC in harmonic traps one has to satisfy the condition $\mu \ll \hbar \omega_0$, where $\omega_0$ is the frequency of the tight confinement. In cylindrically symmetric traps ($\omega_x = \omega_y = \omega_\perp$) the approximate crossover to quasi-1D or quasi-2D, defined by $\mu_{3D} = \hbar \omega_0$, occurs if the number of condensate particles becomes

$$N_{1D} = \sqrt{\frac{32 \hbar}{225 m a^2}} \sqrt{\frac{\omega_\perp}{\omega_z}}; \quad N_{2D} = \sqrt{\frac{32 \hbar}{225 m a^2}} \sqrt{\frac{\omega_3^3}{\omega_\perp^3}}.$$ 

Görlitz et al. [6] have explored the crossover from 3D to 1D and 2D in a $^{23}$Na BEC by reducing the number of the condensed atoms. The scattering length for sodium is relatively small, $a \approx 28\,\text{Å}$, and the traps in this experiment feature extreme aspect ratios resulting in high numbers $N_{1D} > 10^4$ and $N_{2D} > 10^5$.

For the 1D case, the condition $\mu = \hbar \omega_\perp$ yields a linear density $\bar{n}_{1D} \approx 1/4a$, implying that the linear density of a 1D condensate is limited to less than one atom per scattering length independent of the radial confinement. Therefore, tight transverse confinement, as may be achievable in small magnetic waveguides [58; 59; 60; 61; 62; 63] or hollow laser beam guides [64; 65], is by itself not helpful to increase the number of atoms in a 1D condensate. Large 1D numbers may be achieved only at the expense of longer condensates or if the scattering length is smaller.

In anisotropic traps, a primary indicator of crossing the transition temperature for Bose-Einstein condensation is a sudden change of the aspect ratio of the ballistically expanding cloud, and an abrupt change in its energy. The transition to lower dimensions is a smooth cross-over, but has similar indicators. In the 3D Thomas-Fermi limit the degree of anisotropy of a BEC is independent of the number of atoms $N_0$, whereas in 1D and 2D the aspect ratio depends on $N_0$. Similarly, the release energy in 3D depends on $N_0$ [41] while in lower dimensions it saturates at the zero-point energy of the tightly confining dimension(s).

A trapped 3D condensate has a parabolic shape and its radius and half-length are given by $R_\perp = \sqrt{2 \mu_{3D}/m \omega_\perp^2}$ and $R_z = \sqrt{2 \mu_{3D}/m \omega_z^2}$, resulting in an aspect ratio of $R_\perp/R_z = \omega_z/\omega_\perp$. When the 2D regime is reached by reducing the atom number, the condensate assumes a Gaussian shape with the width $l_z = \sqrt{\hbar/m \omega_z}$ along the axial direction, but retains the parabolic shape radially. The radius of a trapped 2D condensate decreases with $N_0$ as $R_{1D2D} = (128 N_0^2 a^2 \hbar^3 \omega_z/\pi m^3 \omega_\perp^4)^{1/8}$ (see Section 3.1). Similarly, the half-length of a trapped 1D condensate is $R_{2D} = (3N_0 a \hbar \omega_\perp/m \omega_z^2)^{1/3}$ (see Chapter 5).

In a ballistically expanding cloud the size in the direction(s) of shallow confinement practically does not change in time compared to a fast expansion in the direction(s) of tight confinement. In the case of pancake Thomas-Fermi condensates the latter equals $b_z R_z$, where the parameter $b_z$ is governed by the scaling equation [66; 67; 68]

$$b_z = \omega_z^2/b_z^2. \quad (2.4.2)$$

At long times of flight $t \gg 1/\omega_z$ Eq. (2.4.2) predicts $b_z \approx \sqrt{2} \omega_z t$. Therefore, the aspect ratio is $b_z(t) R_z/R_\perp \approx \sqrt{2} \omega_z t$.

If the cloud is in the quasi-2D regime, the scaling parameter equals $b_z(t) \approx \omega_z t$ and the aspect ratio is $b_z(t) l_z/R_{1D2D} \approx (\pi \hbar \omega_z^3/128mN_0^2 a^2 \omega_\perp^4)^{1/8} \omega_\perp t$, which is larger than in the Thomas-Fermi regime. Görlitz et al. [6] have observed the crossover in the change of the aspect ratio by decreasing the number of particles in the condensate.
In cigar-shaped Thomas-Fermi condensates at $t \gg 1/\omega_\perp$ the aspect ratio equals $R_\perp b_\perp(t)/R_z \approx \omega_\perp t$. In the quasi-1D regime the radial size expands differently and the aspect ratio reads $\sqrt{\hbar/m\omega_\perp b_\perp(t)/R_{z1D}} \approx (\hbar\omega_\perp/9N_0^2a^2m\omega_z^2)^{1/6}\omega_\perp t$ and is larger than in the Thomas-Fermi case. In both cases the scaling parameter is $b_\perp(t) \approx \omega_\perp t$ (see [66]). The change in the aspect ratio with the decrease of the particle number has been observed in Ref. [6]. Schreck et al. [8] have reached the quasi-1D regime in a $^7$Li condensate taking advantage of the small scattering length. Their measurements of the radial size of the cloud agree with the time evolution of the radial ground state wave function (see Fig. 2.4.1).

An efficient way to reach the quasi-2D and quasi-1D regimes in trapped condensates is to apply a periodic potential of an optical lattice to a 3D condensate initially prepared in a usual magnetic trap. In this way an array of quasi-2D Rb condensates has been obtained by Burger et al. [7]. Greiner et al. [9] realized an array of quasi-1D Rb condensates in a two-dimensional periodic dipole force potential, formed by a pair of standing wave laser fields.

Advantages of this method are obvious: First, an optical lattice can confine a large array of 2D or 1D systems, which allows measurements with a much higher number of involved atoms with respect to a single confining potential. Second, the macroscopic population of a single quantum state in the initial 3D trap naturally transfers the whole system into 2D or 1D systems well below the degeneracy temperature.