Quantization of Hamiltonian Loop Group Actions
Posthuma, H.B.

Citation for published version (APA):
CHAPTER 2

TQFT and CFT

In this chapter we discuss the axioms for Topological Quantum Field Theory (TQFT), the connection with special kinds of tensor categories, and finally the axiomatic approach to Conformal Field Theory (CFT). These subjects are all closely related as we will explain. A general reference for these topics and their interrelations is the recent book [BK], although we follow a different approach. Historically, Topological Quantum Field Theories were introduced by Witten using the language of path integrals. His approach was mathematically axiomatized by Atiyah [A], who was inspired by a similar mathematical abstraction of the path integral approach to Conformal Field Theory developed by G. Segal [S2].

2.1. Modular tensor categories

In this section we state the definition of a modular tensor category. This inevitably leads to an overwhelming number of definitions. However notice that the theory of TQFT leads to a conceptually satisfying picture of such categories, as we will explain in later sections. A general reference for categories is of course [Mac].

2.1.1. Braided tensor categories. First, recall that an additive category is a category in which the Hom-sets are vector spaces over a field \( k \), for which the composition is bilinear, and in which there exists a zero object, as well as finite direct sums. We will always be concerned with the ground field \( k = \mathbb{C} \). An abelian category is an additive category in which all morphisms have kernels and cokernels, and every monomorphism is a kernel and every epimorphism a cokernel. An object \( V \) is called simple if every injection \( U \rightarrow V, \in \text{Ob}(C) \) is either 0 or an isomorphism. A category \( C \) is said to be semi-simple if the set of isomorphism classes of simple objects is finite and any object of \( C \) is isomorphic to a direct sum of simple objects. We now start with the definition of a tensor category [JS]:

**Definition 2.1.** A tensor category is a category \( C \) equipped with

1. a bifunctor \( \otimes : C \times C \rightarrow C \),
2. associativity isomorphisms
   \[
   \alpha_{UVW} : (U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W), \ U, V, W \in \text{Ob}(C)
   \]
   natural in \( U, V \) and \( W \),
3. a unit object \( 1 \in \text{Ob}(C) \) with natural isomorphisms
   \[
   \lambda_V : 1 \otimes V \xrightarrow{\cong} V, \ \rho_V : V \otimes 1 \xrightarrow{\cong} V,
   \]
such that the so called pentagon diagram

\[
((V_1 \otimes V_2) \otimes V_3) \otimes V_4 \xrightarrow{\alpha_{123} \otimes id_4} (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4
\]

\[
\alpha_{1 \otimes 2,3,4}
\]

\[
(V_1 \otimes V_2) \otimes (V_3 \otimes V_4)
\]

\[
\alpha_{1,2,3} \otimes 4
\]

\[
V_1 \otimes ((V_2 \otimes V_3) \otimes V_4)
\]

\[
id_1 \otimes \alpha_{234}
\]

\[
V_1 \otimes (V_2 \otimes (V_3 \otimes V_4))
\]

commutes for all \( V_1, V_2, V_3, V_4 \in \text{Ob}(C) \), as well as the triangle axiom:

\[
(V_1 \otimes 1) \otimes V_2 \xrightarrow{\alpha_{1,0,2}} V_1 \otimes (1 \otimes V_2)
\]

\[
\rho_1 \otimes id_2
\]

\[
id_1 \otimes \lambda_2
\]

\[
V_1 \otimes V_2
\]

The easiest example of a tensor category is the category of finite dimensional vector space \( \text{Vect}_C \), where the monoidal structure is given by the tensor product of vector spaces. Another example is given by \( \text{Rep}(G) \), the category of finite dimensional representations of a compact Lie group \( G \), where the monoidal structure is given by taking the tensor product of representations. Notice that in these examples, there is an isomorphism \( V_1 \otimes V_2 \cong V_2 \otimes V_1 \), for any pair of objects \( V_1 \) and \( V_2 \). This motivates the following

**Definition 2.2.** A tensor category is said to be **braided** if it comes equipped with a natural isomorphisms

\[
\beta_{VW} : V \otimes W \xrightarrow{\cong} W \otimes V,
\]

such that the hexagon diagram

\[
V_1 \otimes (V_2 \otimes V_3) \xrightarrow{\beta_{1,2} \otimes 3} (V_2 \otimes V_3) \otimes V_1
\]

\[
\alpha_{1,2,3}^{-1}
\]

\[
(V_1 \otimes V_2) \otimes V_3
\]

\[
\beta_{1,2} \otimes id_3
\]

\[
(V_2 \otimes V_1) \otimes V_3
\]

\[
\alpha_{2,1,3}
\]

\[
V_2 \otimes (V_3 \otimes V_1)
\]

\[
id_2 \otimes \beta_{1,3}
\]

\[
V_2 \otimes (V_1 \otimes V_3)
\]

commutes for all \( V_1, V_2, V_3 \in \text{Ob}(C) \), and so does the same diagram with \( \beta \) replaced by its inverse.

Notice that a braiding is simply a natural isomorphism of functors \( \beta : \otimes \to \otimes \tau \), where \( \tau : C \times C \to C \times C \) is the flip, i.e., \( \tau(V, W) = (W, V) \). A tensor category with a braiding is called a braided tensor category, or a monoidal category. It is called symmetric if \( \beta_{VW} \circ \beta_{WV} = 1_{W \otimes V}, \forall V, W \in \text{Ob}(C) \). The name braided category comes from the "universal" example, the category of braids, see e.g. [Ka].
2.1.2. Ribbon categories. A further refinement of a braided category is given by so-called Ribbon category. Again, the name is derived from the “universal” example, the category of Ribbons, see e.g. [Ka, Tu]. First, we need to introduce duals.

**Definition 2.3.** A monoidal category is called **rigid** if there exists an equivalence of categories \( \ast : \mathcal{C} \rightarrow \mathcal{C}^{op} \) together with morphisms

\[
e_v : V^* \otimes V \rightarrow 1, \quad i_v : 1 \rightarrow V \otimes V^*,
\]

for every object \( V \) of \( \mathcal{C} \), such that

\[
(id_V \otimes e_V)(i_V \otimes id_V) = id_V, \quad (e_V \otimes id_V \cdot)(id_V \otimes i_V) = id_V^*.
\]

**Remark 2.4.** In the literature, see [Ka, Tu, BK], one usually introduces right and left duals and defines a braided category to be rigid when it has both left and right duals. The definition we give is adapted for later use, but is equivalent to the usual one: The functor above gives the right duals whereas its inverse gives the left duals. Following [BK], we call an equivalence of categories \( \ast : \mathcal{C} \rightarrow \mathcal{C}^{op} \) a **weak** duality, if \( V^* \), for every object \( V \) of \( \mathcal{C} \), represents the functor \( \text{Hom}(1, V \otimes -) : \mathcal{C} \rightarrow \text{Vect}_k \).

Next, a twist in a braided tensor category is given by a family of natural isomorphisms \( \{ \Theta_{V,W} \} \), indexed by the objects of \( \mathcal{C} \), such that

\[
\Theta_{V^*W} = (\Theta_{V^*W} \otimes \Theta_{W})\beta_{W,V} \beta_{V,W}
\]

**Definition 2.5.** A (weak) **Ribbon category** is a (weakly) rigid braided tensor category with a twist.

2.1.3. Modular tensor categories. Finally we come to the definition of a modular tensor category:

**Definition 2.6.** A **modular tensor category** is a semi-simple ribbon category with finitely many isomorphism classes of simple objects, \( \text{End}_\mathcal{C}(1) \cong k \), and the property that for all simple objects \( V \neq 1 \), \( \exists W \in \text{Ob}(\mathcal{C}) \) such that

\[
\beta_{VW} \neq \beta_{WV}^{-1}.
\]

A modular tensor category is thus, by definition, “maximally non-symmetric”. The definition we have given here differs from the standard one, see e.g. [Tu, BK], and first appeared in [BB] where the two were shown to be equivalent. See also [Mu, KeLu].

The name modular tensor category is explained by the following proposition:

**Proposition 2.7.** Let \( \mathcal{C} \) be a modular tensor category. Then the \( k \)-vector space

\[
\bigoplus_{i \in I} \text{Hom}_\mathcal{C}(1, V_i \otimes V_i^*),
\]

where the \( V_i \) form a complete set of simple objects, carries a projective representation of \( SL(2,\mathbb{Z}) \).

This can be proved explicitly, see e.g. [Tu, BK]. However, a conceptual explanation for this result will be given by TQFT.
2.1.4. The Grothendieck group. For any abelian category $\mathcal{C}$, its Grothendieck group $K(\mathcal{C})$ is the quotient of the free abelian group on the set of isomorphism classes of objects in $\mathcal{C}$ modulo the relation $[V] = [U] + [W]$ for every short exact sequence
\[ 0 \to U \to V \to W \to 0. \]
When $\mathcal{C}$ is a rigid monoidal category we can make $K(\mathcal{C})$ into a ring by
\[ [U] \cdot [V] = [U \otimes V]. \]
This definition works, since in a rigid tensor category, the functor $- \otimes V : \mathcal{C} \to \mathcal{C}$ is exact, see e.g. [BK] prop. 2.1.8., and gives an associative ring with unit. The Grothendieck group is commutative when $\mathcal{C}$ is braided.

2.2. Topological Quantum Field theory

2.2.1. Cobordism categories. Historically, the notion of cobordism emerged in the fifties in the work of Thom on the so-called cobordism group. Later, it became clear that cobordism is the natural language from the point of view of Morse theory [M]: A Morse function $f : M \to \mathbb{R}$ on an $(n+1)$-dimensional manifold gives a decomposition of $M$ into $n$-dimensional manifolds $M_a$, $a \in \mathbb{R}$ of critical points of $f$, connected by "simple" $n+1$-dimensional manifolds $M_{ab}$. From this decomposition one wants to read off the global topology of $M$. The cobordism category allows one to write the decomposition of $M$ as a composition of morphisms $M = M_{ab} \circ \ldots \circ M_{cd}$ between (isomorphism classes of) $n$-dimensional manifolds.
First, construct the following bicategory (see [Mac] for the definition of a bicategory, also called a weak 2-category): Its objects are given by smooth compact $n$-dimensional oriented manifolds, and arrows are given by cobordisms between such manifolds: A cobordism from $M_1$ to $M_2$ consists of a triple $(N, f_1, f_2)$ where $N$ is a smooth oriented $(n+1)$-dimensional manifold with boundary $\partial N = \partial N_{in} \sqcup \partial N_{out}$ and diffeomorphisms $f_1 : M_1 \to \partial N_{in}$, $f_2 : M_2 \to \partial N_{out}$, where $f_1$ is orientation reversing and $f_2$ preserves the orientation.
Composition of cobordisms is given by the following construction: Given cobordisms $(N, f_1, f_2)$ from $M_1$ to $M_2$ and $(K, g_2, g_3)$ from $M_2$ to $M_3$, define
\[ N \cup_{g_2 f_2^{-1}} K = N \sqcup K / \{(x, y) : f_2(x) = g_2(y)\}. \]
This space has a unique topology by requiring the obvious projection $\pi : N \sqcup K \to N \cup_{g_2 f_2^{-1}} K$ to be an open, continuous map. There is even a unique smooth structure turning $\pi$ into a smooth map [M]. In this way, the triple $(N \cup_{g_2 f_2^{-1}} K, f_1, g_3)$ forms a cobordism from $M_1$ to $M_3$, called the composition of $N$ and $K$, denoted $N \circ K$.
One might hope to obtain a category in this way, with unit given by $I \times I$, where $I = [0, 1]$, but unfortunately this is not the case. This is because of the fact that the composition of cobordism is not associative, but merely associative up to diffeomorphism, i.e.,
\[(N \circ K) \circ L \cong N \circ (K \circ L).\]
One can add such diffeomorphisms as 2-arrows (i.e., "arrows between arrows") to obtain a bicategory. Of course, we require such diffeomorphisms between two cobordisms $N$ and $N'$ to commute with the embeddings $f_i : M_i \to N$ and $f_i' : M_i \to N'$, $i = 1, 2$. 

On the other hand, taking diffeomorphism classes of cobordisms, one obtains a true category $\text{Cob}$, called the $n$-dimensional cobordism category. Its objects are given by $n$-dimensional compact manifolds, and arrows by diffeomorphism classes of cobordisms. Composition of arrows is induced by composition of cobordisms, and the unit is given by the diffeomorphism class of the cobordism $M \times I$.

Cobordism is an equivalence relation between manifolds, which is weaker than diffeomorphism: Suppose $f : M_1 \to M_2$ is a diffeomorphism, then the cobordism $(M_1 \times I, id_{M_1}, f^{-1})$ is invertible in $\text{Cob}$ with inverse $(M_2 \times I, id_{M_2}, f)$. The objects $M_1$ and $M_2$ are therefore isomorphic in $\text{Cob}$.

Finally, observe that the disjoint union of $n$-dimensional manifolds induces a symmetric tensor structure on $\text{Cob}$ such that it becomes a monoidal category. Changing the orientation, denoted by $N \mapsto \overline{N}$, gives a $*$-structure. In this way, the cobordism category becomes a symmetric monoidal $*$-category.

### 2.2.2. Axioms for TQFT

Consider the $n$-dimensional cobordism category $\text{Cob}$ constructed in the previous section. Inspired by the path integral approach to quantum field theory, Atiyah [A] gave the following definition of a Topological Quantum Field Theory (TQFT):

**Definition 2.8.** An $n$-dimensional TQFT is a monoidal $*$-functor $Z : \text{Cob} \to \text{Vect}_k$.

One can think of this as a "representation" of the cobordism category. Let us write out what this all means:

An $n$-dimensional TQFT consists of the following data:

1. A finite dimensional vector space $Z(N)$ for every $(n-1)$-manifold $N$.
2. A vector $Z(M)$ in the vector space $Z(\partial M)$.

Such that,

- **a)** $Z$ is functorial w.r.t. diffeomorphisms of $N$ and $M$. This means that any diffeomorphism $f : N \to N'$ induces an isomorphism $Z(f) : Z(N) \to Z(N')$, such that $Z(fg) = Z(f)Z(g)$. If $f$ extends to a diffeomorphism $M \to M'$, where $\partial M \cong N$ and $\partial M' \cong N'$, then $Z(f) : Z(N) \to Z(N')$ should map $Z(M) \mapsto Z(M')$.

- **b)** $Z$ is involutive, i.e., $Z(\overline{N}) = Z(N)^*$.

- **c)** $Z$ is multiplicative, i.e., $Z(N_1 \amalg N_2) = Z(N_1) \otimes Z(N_2)$.

- **d)** $Z$ is transitive when composing cobordisms, i.e., for $M_1$, $M_2$ with $\partial M_1 \cong \overline{N}_1 \amalg N_2$ and $\partial M_2 \cong \overline{N}_2 \amalg N_3$ we have

\begin{equation}
Z(M_1 \cup_{N_2} M_2) = Z(M_2)Z(M_1),
\end{equation}

where we have used condition b) and c) to view $Z(M_1) : Z(N_1) \to Z(N_2)$ as a linear map.

- **e)** $Z$ is nontrivial, i.e., $Z(\emptyset) = k$ and $Z(N \times I) = id_{Z(N)}$.

The vector $Z(M)$ is called the *partition function* on $M$, and depends, by definition, only on the diffeomorphism class of the manifold. It is only in this sense that the theory is called topological, because after all it is defined for *smooth* manifolds. Since a closed $d$-dimensional manifold is simply a cobordism from the empty set $\emptyset$ to itself, the partition function assigns an invariant $Z(M) \in k$ to $d$-dimensional manifolds. These are the celebrated manifold invariants associated to TQFT. Relation (2.2) can be looked upon as giving a calculation device when cutting a manifold into simple
pieces, similar to the Mayer-Vietoris sequence for ordinary cohomology. However notice that, in contrast to cohomology, a TQFT is always tied to a specific dimension. The main examples of manifold invariants that fit into this scheme, such as, for example, the Reshetikhin-Turaev invariant of 3-manifolds, have made contributions to low-dimensional topology. Notice that, because of the h-cobordism theorem [M], in higher dimensions, the structure of a TQFT will simplify.

One can modify these axioms by altering the target category of the functor, as long as it remains a symmetric monoidal *-category. For example, the category Hilb of finite dimensional Hilbert space gives rise to unitary TQFT's and the category of $\mathbb{Z}_2$-graded vector spaces give supersymmetric TQFT's etc.

### 2.2.3 2d-TQFT

An often used, but illuminating simple example of a TQFT is given by TQFT in 2 dimensions [Q]. Recall the notion of a Frobenius algebra: A Frobenius algebra $A$ is a commutative, associative, unital algebra over a field $k$ equipped with a nondegenerate trace $\text{tr} : A \to k$. The following is well known:

**Theorem 2.9.** There is a bijective correspondence between 2d-TQFT's and finite, dimensional Frobenius algebras.

The proof is easy: First we define

$$A := Z(S^1).$$

The algebra multiplication is induced by a pair of pants surface

$$Z \left( \begin{array}{c} \includegraphics[width=1cm]{pants_surface} \end{array} \right) : A \otimes A \to A.$$ 

Associativity of this multiplication follows from the diffeomorphism

$$\begin{array}{c} \includegraphics[width=1.5cm]{assoc_diffeomorphism} \end{array} \sim \begin{array}{c} \includegraphics[width=1.5cm]{assoc_diffeomorphism} \end{array}$$

The unit is given by

$$Z \left( \begin{array}{c} \includegraphics[width=1cm]{unit_surface} \end{array} \right) = 1 : k \to A$$

as one can check pictorially. Dually, the trace is given by

$$Z \left( \begin{array}{c} \includegraphics[width=1cm]{trace_surface} \end{array} \right) = \text{tr} : A \to k.$$ 

Conversely, to prove that a finite dimensional Frobenius algebra gives rise to a TQFT in 2 dimensions, one uses the fact that any 2-dimensional surface can be cut into a union of disks, cylinders and three-holed spheres (in fact only pair of pants surfaces are needed). The trace, the unit and the algebra multiplication then associate a linear map to any such surface together with a pair of pants decomposition. To show that this map is independent of the decomposition, one checks invariance under four "simple moves" taking one decomposition into the other. It is known [HT] that this is sufficient to prove invariance of the decomposition. Consult [BK] for more details.
2.2. TOPOLOGICAL QUANTUM FIELD THEORY

**Remark 2.10.** The axiom that the functor of 2d-TQFT should map to the category of vector spaces is in fact redundant. From a monoidal functor $Z$ to abelian groups with the additional requirement that $Z(\emptyset) = k$ is a field, one deduces from the axioms that the abelian group associated to a circle is in fact a $k$-module, i.e. a vector space.

**Remark 2.11.** Theorem 2.9 is pleasing in the sense that it relates a TQFT in a specific dimension to a known algebraic structure, a Frobenius algebra. Going up to higher dimensions, one may wonder if there are similar “algebraic gadgets” that correspond to TQFT’s. In dimension 3 this is known to be a modular tensor category [Tu], and shows that for increasing dimensions, this type of questions tends to be a painful exercise in higher category theory. We will focus on $d = 3$ and show how the connection with category theory is made. This relation was first observed in [MS], in the context of conformal field theory.

### 2.2.4. Two dimensional cobordisms.

Of special interest for us is the two dimensional cobordism category, $\text{Cob}_2$. First notice that every object is isomorphic to a disjoint union of circles. Let us simply denote by $C_n$ the disjoint union of $n$ such circles. A cobordism from $C_m$ to $C_n$ is called an extended surface:

**Definition 2.12.** An extended surface is a smooth oriented 2-dimensional surface with parameterized boundaries isomorphic to a disjoint union of circles.

Up to isomorphism such a cobordism is determined by its genus $g$. Its automorphisms are given by the orientation preserving diffeomorphisms that fix the boundary, $\text{Diff}^+(X, \partial X)$. Its group of components

$$\Gamma(X, \partial X) := \pi_0(\text{Diff}^+(X, \partial X))$$

is called the mapping class group of $X$. It is not difficult to see that the two dimensional cobordism bicategory is equivalent to the category where the 2-cells are given by isotopy classes of diffeomorphisms. In this category, the group of automorphisms of a extended surface is given by its mapping class group. This bicategory is closely related to the Teichmüller tower of groupoids used in [BK]. The essential point is the existence of a “horizontal” homomorphism of the mapping class groups of $X_1$ and $X_2$ to the mapping class group $X_1 \cup_S, X_2$ when gluing.

In the following we will need a certain central extension of the 2-dimensional cobordism (bi)category. This is an extension by $\mathbb{Z}$, and we write, symbolically,

$$1 \to \mathbb{Z} \to \overline{\text{Cob}}_2 \to \text{Cob}_2 \to 1.$$

For an extended surface $X$, consider the first homology group $H_1(X, \mathbb{R})$. It carries an skew-symmetric pairing

$$H_1(X, \mathbb{R}) \times H_1(X, \mathbb{R}) \to \mathbb{R},$$

given by intersection. Only when $X$ is closed, this pairing is nondegenerate and $H_1(X, \mathbb{R})$ is a $2g$-dimensional symplectic vector space. In general, the kernel of the intersection pairing is exactly the image of the inclusion $H_1(\partial X, \mathbb{R}) \hookrightarrow H_1(X, \mathbb{R})$.

The extension $\overline{\text{Cob}}_2$ has the same objects as $\text{Cob}_2$, namely compact 1-manifolds, but a morphism is given by a pair $(X, L)$, consisting of an extended surface $X$ and a Lagrangian subspace $L \subset H_1(X, \mathbb{R})/H_1(\partial X, \mathbb{R})$. Finally, a 2-arrow between $(X_1, L_1)$ to $(X_2, L_2)$ is given by a pair $(f, [\gamma])$, where $f : X_1 \to X_2$ is a smooth map and $[\gamma]$ is a homotopy class maps $\gamma : I \to \text{Lag}(H_1(X, \mathbb{R}))$ connecting $L_2$ with $f_\ast L_1$. Since
\[ \pi_1(\text{Lag}(V)) \cong \mathbb{Z}, \] for any symplectic vector space. This defines an extension by \( \mathbb{Z} \). The essential point of this definition is that it gives a coherent system of central extensions of the mapping class groups of all two-dimensional surfaces, i.e., of the Teichmüller tower [BK]. The extension is trivial when restricted to genus zero surfaces.

### 2.3. Modular functors

A fundamental concept underlying the relationship between TQFT’s and modular tensor categories is that of a modular functor. We give a definition which differs from the literature, but in the end is equivalent. Let \( \textbf{Ab} \) be the bicategory with semi-simple Abelian categories as objects, functors between abelian categories (i.e., \( k \)-linear on the vector space of morphisms) as 1-arrows and 2-arrows given by natural transformations. This bicategory has a monoidal structure given by Deligne’s tensor product \( \boxtimes \) of abelian categories [D]. Recall that, for two abelian categories \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), the objects of \( \mathcal{C}_1 \boxtimes \mathcal{C}_2 \) are given by finite sums of the form

\[
\bigoplus_{i \in I} X_i \boxtimes Y_i, \quad X_i \in \text{Ob}(\mathcal{C}_1), \quad Y_i \in \text{Ob}(\mathcal{C}_2).
\]

while morphisms from \( X \boxtimes Y \) to \( X' \boxtimes Y' \) are given by elements in

\[
\text{Hom}_{\mathcal{C}_1}(X, X') \otimes \text{Hom}_{\mathcal{C}_2}(Y, Y').
\]

There is a \( * \)-structure that sends a category \( \mathcal{C} \) to its dual \( \mathcal{C}^* \), the category of all functors \( \mathcal{C} \to \text{Vect}_k \) and natural transformations. With these structures, we can state:

**Definition 2.13.** A topological modular functor is a monoidal weak \( * \)-functor

\[
Z : \widetilde{\text{Cob}}_2 \to \text{Ab}.
\]

A genus zero modular functor is defined to be a monoidal functor from the subcategory whose morphisms are restricted to be of genus zero. The category \( Z(S^1) \) associated to the circle is called the circle category.

**Remark 2.14.** This definition differs from the one given in [BK], although it will turn out to be equivalent. For related definitions of this kind, see [Be, Ti].

Such a bifunctor consists of assignments

<table>
<thead>
<tr>
<th>1-manifold ( S^1 )</th>
<th>( \leadsto ) category ( \mathcal{C}_{S^1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>cobordism ( \Sigma ) from ( \partial \Sigma_{\text{in}} ) to ( \partial \Sigma_{\text{out}} )</td>
<td>( \leadsto ) functor ( U_{\Sigma} : \mathcal{C}<em>{\partial \Sigma</em>{\text{in}}} \to \mathcal{C}<em>{\partial \Sigma</em>{\text{out}}} )</td>
</tr>
<tr>
<td>smooth map ( f : \Sigma_1 \to \Sigma_2 )</td>
<td>( \leadsto ) natural transformation ( T_f : U_{\Sigma_1} \to U_{\Sigma_2} )</td>
</tr>
</tbody>
</table>

subject to some natural relations, reflecting the geometry of the cobordism category. This implies that if the composition of two cobordisms \( \Sigma_1 \) and \( \Sigma_2 \) is given by \( \Sigma = \Sigma_1 \cup_{S^1} \Sigma_2 \), then there is a natural isomorphisms of functors

\[ U_{\Sigma} \cong U_{\Sigma_2} \circ U_{\Sigma_1}. \]

compatible with the action of the diffeomorphism group by natural transformations. This gives the following

**Proposition 2.15 ([MS]).** There is a bijective correspondence between topological genus zero modular functors and weakly semi-simple Ribbon categories with finitely many isomorphism classes of simple objects.
Remark 2.16. With a different (but equivalent) definition of modular functor, this is Theorem 5.4.1 of [BK]. It seems worthwhile to give the proof from the point of view of Definition 2.13, since it is more geometric. Roughly speaking, the proof given below gives a geometric construction of the “structure functors” of the ribbon category. The equivalence with [BK] follows when one realizes that such functors are representable. The resulting objects in \( \mathcal{C} \) constitute a modular functor in the sense of [BK].

**Proof.** \( \Rightarrow \) Recall the central extension of the cobordism category is trivial over the genus zero component of the morphism spaces. We will show that \( \mathcal{C} := Z(S^1) \) is a semi-simple Ribbon category. A pair of pants surface defines a functor

\[
Z \left( \begin{array}{c} \hline \hline \end{array} \right) : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C},
\]

which we use to define a monoidal structure on \( \mathcal{C} \). The diffeomorphism on a four-holed sphere exchanging the two ways in which it can be cut into two 2-holed spheres, i.e.,

\[
\begin{array}{c}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
Therefore any functor $C \to \text{Vect}_k$ is isomorphic to one of the form $\text{Hom}(\cdot, A)$. $A \in \text{Ob}(C)$. Combining the equivalences $C \simeq C^* \simeq C^{\text{op}}$, one finds an equivalence of categories $* : C \to C^{\text{op}}$, characterized on objects by the equality

$$\langle A, B \rangle = \text{Hom}_C(A^*, B) \quad \forall B \in \text{Ob}(C).$$

This defines the weak duality. The twist $\theta$ is given by the natural transformation of the identity, induced by a Dehn twist $T$ of the cylinder that rotates one of the boundary components over $\pi$. That this defines a twist can be checked geometrically in the diffeomorphism group of a three-holed sphere. In conclusion, $C$ is a semisimple weak Ribbon category.

Let $C$ be a semi-simple Ribbon category with finitely many isomorphism classes of simple objects. Define $Z(S^1) := C$, and the monoidal structure gives a functor associated to a pair of pants, the unit object the functor associated to a disk. Since $C$ is semi-simple and has a finite set of isomorphism classes of simple objects, one can identify functors $U_\Sigma : C_{\partial \Sigma_{\text{in}}} \to C_{\partial \Sigma_{\text{out}}}$ with objects in $C_{\partial \Sigma_{\text{in}}}^{\text{op}} \boxtimes C_{\partial \Sigma_{\text{out}}}$. Using this identification, one finds functors for any genus zero extended surface with a pants decomposition. Now the diffeomorphism group acts by permuting pants decompositions, and its action on the set of pants decompositions of a surface factors over its group of connected components, the mapping class group. This group is generated by Dehn twists around closed noncontractible loops and braiding isomorphisms around two boundary components of a pair of pants surface in the decomposition. Now, for the first, the Dehn twist, we use the twist isomorphism $\theta$ and for the latter we use the braiding isomorphism $\beta$ in $C$ to define a natural transformation associated to an element of the mapping class group. To show that this assignment of functors and natural transformations is independent of the choices made, i.e., of the pants decomposition and the presentation of the mapping class group used, one uses the result of [BK] chapter 5.

Remark 2.17. Analogous to the 2d-case, definition 2.13 implies that the circle category is a "module" over $\text{Vect}_k$, more precisely, it is a module category. In the literature, such a category is called a 2-vector space.

Theorem 2.18 ([MS]). There is a bijective correspondence between topological modular functors and weakly modular tensor categories.

Proof. We have already seen in Proposition 2.15 that the restriction to genus zero gives a Ribbon category. The modularity will come from higher genus surfaces. Actually, we only need to prove the "maximally non-symmetric" condition in Definition 2.6. For this, it is enough to observe that the braiding $\beta$ induced from a diffeomorphism on a three holed sphere, induces an element of the mapping class group of a one holed torus by gluing a cylinder to a three holed sphere.

![Diagram of a one holed torus]

and that this element does not square to 1. This proves the condition in Definition 2.6 for the simple objects, and therefore, by semi-simplicity of $C$, for all objects. For the implication in the other direction, consult [BK] Theorem 5.7.11.
Corollary 2.19. In a modular tensor category $\mathcal{C}$, the $k$-vector space $\text{Hom}_\mathcal{C}(E, U_\Sigma(F))$, for $\Sigma$ a cobordism from $\mathcal{C}_{\text{in}}$ to $\mathcal{C}_{\text{out}}$ carries a projective representation of the mapping class group $\Gamma(\Sigma, \partial \Sigma)$, for any $E \in \mathcal{C}_{\text{in}}$ and $F \in \mathcal{C}_{\text{out}}$.

Proof. This follows, given the previous equivalence of MTC’s and modular functors, from the fact that a central extension of mapping class group acts on $U_\Sigma$ by natural transformations.

This gives a natural proof of the representation of $SL(2, \mathbb{Z})$ of Proposition 2.7. Taking into account the composition of the cobordisms, the mapping class groups for all cobordisms constitute a so called tower of groupoids, see [BK], called the Teichmüller tower. (The idea for this goes back to Grothendieck.) In this sense, a modular tensor category gives rise to a representation of this tower.

Unitary modular functors. As we have seen in Remark 2.17, the circle category is automatically a 2-vector space. Analogous to the situation in TQFT, it seems natural to define a unitary modular functor to be a bifunctor from the two dimensional cobordism category to the subcategory of 2-Hilbert spaces. Recall, see [B], that a 2-Hilbert space $\mathcal{C}$ is a module category over the symmetric monoidal category of finite dimensional Hilbert spaces, together with an “inner product”, i.e., a functor

$$\langle \ , \rangle : \mathcal{C} \otimes \mathcal{C} \to \text{Hilb}.$$  

A functor between 2-Hilbert spaces has an adjoint, and a modular functor functor as defined in Definition 2.13 is said to be unitary if the circle category $Z(S^1)$ is a 2-Hilbert space and the adjoint of the functor $U_X$ associated to a 2d cobordism is given by $U_X^* \cong U_X$. In this case, the resulting modular tensor category is called unitary [Tu], and the representation of the mapping class group is unitary.

Grothendieck group. Consider the functor of taking the Grothendieck group of an abelian category. This takes a functor between abelian categories to a homomorphism to abelian groups, and functors that are naturally isomorphic induce the same homomorphism. Therefore, composing the bifunctor in Definition 2.13 with the functor that takes an abelian category to its Grothendieck group, one obtains a true functor from the two-dimensional cobordism category to abelian groups, that is, a 2d-TQFT! (see Remark 2.10). By Theorem 2.9, together with theorem 2.18, we observe that the Grothendieck group of a modular tensor category is a Frobenius algebra, so that we find

Proposition 2.20. The Grothendieck ring of a modular tensor category is in a natural way a Frobenius algebra over $\mathbb{Z}$.

Remark 2.21. Because of this proposition, it is tempting to think of a modular tensor category as a “categorical version” of a Frobenius algebra. This statement can be made precise and identifies a modular tensor category as a “Frobenius object” in $\text{Ab}$. A Frobenius object in $\text{Vect}_\mathcal{C}$ is simply an ordinary Frobenius algebra, i.e., a 2d-TQFT, see [Fr2, Ti].

2.4. Extended TQFT

The previous section explains how modular tensor categories are related to modular functors, but its connection to TQFT remains mysterious. It was shown in [Tu]...
that a modular tensor category uniquely determines a 3d-TQFT, generalizing earlier work [RT], which had only produced the corresponding 3-manifold invariant. This approach uses the fact that any 3-manifold can be obtained by surgery in $S^3$ along a framed link, as well as the corresponding Kirby calculus to check independence of the link that is used.

Conversely, it is believed that a general 3d-TQFT induces a topological modular functor, see [BK]. From the point of view of the previous section, all this implies that we should be able to extend the functor giving the topological modular functor, to a collection of assignments

<table>
<thead>
<tr>
<th>1-manifold $S^1$</th>
<th>$\sim$</th>
<th>category $C_{S^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cobordism $\Sigma$ from $\partial \Sigma_{in}$ to $\partial \Sigma_{out}$</td>
<td>$\sim$</td>
<td>functor $U_\Sigma : C_{\partial \Sigma_{in}} \rightarrow C_{\partial \Sigma_{out}}$</td>
</tr>
<tr>
<td>cobordism $M$, $\partial M = \Sigma_1 \amalg \Sigma_2$</td>
<td>$\sim$</td>
<td>natural transformation $T_M : U_{\Sigma_1} \rightarrow U_{\Sigma_2}$</td>
</tr>
</tbody>
</table>

subject to natural conditions reflecting the geometry. This scheme differs from the one following definition 2.13 by the fact that we now allow “cobordisms between cobordisms” to induce natural transformations between functors. Notice that such cobordisms necessarily are manifolds with corners, and one has to be careful with the exact definition. Also, framing this assignment into categorical language, i.e. as a “functor” from some “cobordism category” to the “category of abelian categories” is not so obvious. For this we refer to [KeLu], which uses so called double categories. For our purposes, the two dimensional modular functor will suffice, so we will not discuss this any further.

2.5. Conformal Field theory

One can think of conformal field theory as a slightly more involved version of 2-dimensional TQFT, in the sense that partition functions depend on more structure of the surface, most notably on a conformal structure [S2]. Therefore, let us first investigate the complex cobordism “category”.

2.5.1. The moduli space of extended Riemann surfaces. Let $X$ be an extended surface, and let us have a look at the “moduli space” $\mathcal{E}_X$ of different complex structures one can put on $X$. Equipped with a complex structure, $X$ will be called an extended Riemann surface, that is, a smooth surface with parameterized boundaries, together with a conformal structure on the interior. It is natural to call two extended Riemann surfaces equivalent if there is a diffeomorphism from one to the other that preserves the boundary parameterizations and is biholomorphic on the interior. This equivalence relation comprises the definition of the moduli space of extended Riemann surfaces.

More concretely, denote by $J(X)$ the space of almost complex structures on $X$. Recall from [ES], that this is a contractible space. In one complex dimension, any almost complex structure is automatically a complex structure, and two almost complex structures determine the same conformal structure on $X$ iff they differ by a diffeomorphism in $\text{Diff}^+(X, \partial X)$, which acts on $J(X)$ by pull-back. Therefore, the moduli space can be identified with

\[
\mathcal{E}_X = J(X)/\text{Diff}^+(X, \partial X).
\]

(2.3)
First observe that $\text{Diff}^+(X, \partial X)$ acts freely, for suppose that an almost complex structure $J \in J(X)$ is a fixed point of $f \in \text{Diff}^+(X, \partial X)$. This means that $f$ is a biholomorphic map of $X_J$ onto itself, which equals the identity on $\partial X$. But this implies that $f$ must be the identity map.

This free action can be used to give the quotient (2.3) the structure of a smooth infinite dimensional Fréchet manifold. Notice that this is not as straightforward as it may seem, since we are not dealing with the action of a Banach–Lie group on a Banach manifold, and we cannot use the implicit function theorem. However, its structure as an inverse limit of Hilbert manifolds will suffice, similar to the construction of the (finite dimensional) Teichmüller space as an infinite dimensional quotient. We will often confuse a complex Riemann surface $\Sigma$ with the topology of $X$ with its image in $\mathcal{E}_X$ and write $\Sigma \in \mathcal{E}_X$.

Given this fact, we can easily read off the topology of the moduli space $\mathcal{E}_X$. The contractibility of $J(X)$ implies that $\mathcal{E}_X$ is a model for the classifying space of the mapping class group, i.e., it has homotopy type

$$
\pi_i(\mathcal{E}_X) = \begin{cases} 
\Gamma(X, \partial X), & i = 1 \\
0, & i \neq 1.
\end{cases}
$$

The action of $\text{Diff}^+(X)$ on $J(X)$ quotients to an action of $\text{Diff}^+(\partial X)$ on $\mathcal{E}_X$. This gives a Lie algebra homomorphism $\text{Vect}(\partial X) \to T\mathcal{E}_X$, which extends to the complexification $\text{Vect}_c(\partial X)$, and turns out to be surjective. The kernel of this map at a given point $\Sigma \in \mathcal{E}_X$ is obviously given by the set of vector fields that have an extension to a holomorphic vector field on $\Sigma$ and in this way one finds that the tangent space to $\Sigma \in \mathcal{E}_X$ identifies with $[\text{BS}, \text{KNTY}]$

$$
T_{\Sigma}\mathcal{E}_X = \text{Vect}_c(\partial \Sigma)/\text{Vect}_c(\Sigma),
$$

where $\text{Vect}_c(\Sigma) \subset \text{Vect}_c(\partial \Sigma)$ denotes the space of complex vector fields on $\partial \Sigma$ that have an extension to a holomorphic vector field on $\Sigma$. By a Mayer–Vietoris argument, this isomorphism is compatible with the Kodaira–Spencer isomorphism for closed surfaces under the gluing map defined in section 2.5.3.

**Remark 2.22.** Recall that there is no group that integrates the Lie algebra $\text{Vect}_c(S^1)$, and therefore it is most natural to consider the pair $(\text{Diff}^+(S^1), \text{Vect}_c(S^1))$ as a *Harish-Chandra pair*, see [FZ]. Then the above identification of the tangent space gives the moduli space $\mathcal{E}_X$ a $(\text{Diff}^+(S^1), \text{Vect}_c(S^1))$-structure: There is a surjective homomorphism $\text{Vect}_c(S^1) \to T\mathcal{E}_X$, compatible with the action of $\text{Diff}^+(S^1)$ on $\mathcal{E}_X$.

**Remark 2.23.** There are several other pictures of this moduli space. The first is as the moduli space of closed Riemann surfaces $\Sigma$ with $n$ holomorphically embedded disks $f_i : D \to \Sigma$, $i = 1, \ldots, n$. One can also develop a "metric picture" as a space of conformal equivalence classes of constant curvature metrics for which the boundaries are geodesics.

**2.5.2. Examples.** Some examples can be explicitly constructed. For the easiest, consider the disk $D$. By the Riemann mapping theorem, any complex disk can be mapped holomorphically onto the unit disk in the complex plane, by a map that is unique up to an automorphism of the unit disk, i.e., an element of $\text{PSL}(2, \mathbb{R})$. Therefore, one has

$$
\mathcal{E}_D = \text{Diff}^+(S^1)/\text{PSL}(2, \mathbb{R}).
$$
Next, consider a cylinder $C$. Ignoring the parameterization of the boundary, any complex cylinder is conformally equivalent to an annulus $A_q = \{ z \in \mathbb{C}, \, q \leq |z| \leq 1 \}$, with $q \in (0, 1)$ [Ab]. The only automorphisms of such a surface are given by the rigid rotations $T$, and we find

\[(2.6) \quad \mathcal{E}_C = (0, 1) \times (\text{Diff}^+(S^1) \times \text{Diff}^+(S^1))/T.\]

### 2.5.3. Gluing.

As we have seen, an important property of extended surfaces is that they can be glued. Now, can this still be done when they are endowed with a conformal structure?

**Proposition 2.24.** Suppose that topologically $X = X_1 \cup_{S^1} X_2$, and consider $\Sigma_1 \in \mathcal{E}_{X_1}$ and $\Sigma_2 \in \mathcal{E}_{X_2}$. Then $X$ has a unique conformal structure, denoted by $\Sigma_1 \cup_{S^1} \Sigma_2$, whose structure sheaf is given by

$$f \in \mathcal{O}_{\Sigma_1 \cup_{S^1} \Sigma_2} \iff f|_{\Sigma_1} \in \mathcal{O}_{\Sigma_1} \text{ and } f|_{\Sigma_2} \in \mathcal{O}_{\Sigma_2}. $$

**Proof.** The proof of the conformal structure follows the construction of the Schottky double of a Riemann surface in [Ab]: The essential point is to show that the sheaf above defines a conformal structure in the neighbourhood of the closed curve $\gamma$ in $X$ along which $X_1$ and $X_2$ are glued. Therefore, consider a function $f$ in an open neighbourhood $U \subset X$ such that $U \cap \Sigma_1 \neq \emptyset$ and $U \cap \Sigma_2 \neq \emptyset$. Using the coordinate charts on $\Sigma_1$ and $\Sigma_2$, one can map $U$ onto $\mathbb{C}$, such that $U \cap \Sigma_1$ is mapped onto the upper half plane, $U \cap \Sigma_2$ onto the lower half plane, and $U \cap \gamma \subset \mathbb{R}$. By definition of the conformal structures $\Sigma_1$ and $\Sigma_2$, this map is holomorphic on the interior of $U \cap \Sigma_1$ and $U \cap \Sigma_2$. Therefore, $f$ is continuous on $U \subset \mathbb{C}$ and holomorphic on the complement of $U \cap \mathbb{R}$. It then follows that $f$ is in fact holomorphic in the whole of $U$, i.e., also on $U \cap \mathbb{R}$: Draw a rectangle $C$ in $U \subset \mathbb{C}$ intersecting both the upper and lower half plane. By Cauchy's formula, one has

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

for $z \notin \mathbb{R}$ inside $C$. However, the right hand side represents an analytic function for all $z$ inside $C$, and since the left hand side is continuous on the whole of $U$, the left and right hand sides remain equal also for $z \in \mathbb{R}$. Therefore, $f$ is holomorphic on $U$. $\square$

**Remark 2.25.** Sometimes it is convenient to consider complex surfaces with boundary parameterizations that are analytic, that is, the parameterization map $f : S^1 \rightarrow \partial \Sigma$ extends to a holomorphic map $f : A_q \rightarrow \Sigma$ for some small annulus $A_q$. For such surfaces the above proposition is immediate.

### 2.5.4. The complex cobordism category.

Of course, up to diffeomorphism, extended surfaces are classified by the number of boundary components and the genus, and the moduli space for two diffeomorphic surfaces is isomorphic. The complex cobordism category $\mathcal{E}$ has the same class of objects as the topological cobordism category, but the morphisms are now cobordisms endowed with a conformal structure. So the objects are given by $C_n$, and a morphism from $C_m$ to $C_n$ is an equivalence class of extended Riemann surfaces $\Sigma$ with $\partial \Sigma \cong C_m \amalg C_n$. Again, we will refer to $C_m$ as the incoming boundary $\partial \Sigma_{\text{in}}$ and to $C_n$ as the "outgoing boundary" $\partial \Sigma_{\text{out}}$. 
The space of all morphisms from $C_m$ to $C_n$ is denoted by $\mathcal{E}_{m,n}$. As we have seen in section 2.5.1, these $\mathcal{E}_{m,n}$ have a natural topological, and even smooth structure. With respect to this smooth structure, the space of morphisms is the disjoint union of its connected components $\mathcal{E}_{m,n}(g)$ of extended Riemann surfaces of genus $g$. Gluing of extended Riemann surfaces, as in proposition 2.24, induces the composition

\begin{equation}
\mathcal{E}_{m,k} \times \mathcal{E}_{k,n} \to \mathcal{E}_{m,n}.
\end{equation}

With these structure maps, the complex cobordism category is not quite a category in the true sense, since it has no unit morphisms. Whereas a category is usually considered to be a generalization of a monoid, the structure we have here is analogous to a semigroup, c.f. section 2.5.7. However, in what follows it is useful to think of $\mathcal{E}$ as if it were a category, giving the obvious meaning to functors etc. With this in mind, observe that just like the topological cobordism category, the complex cobordism is a symmetric monoidal $*$-category:

1. There is a monoidal structure, by taking the disjoint union of 1-manifolds, $C_m \sqcup C_n = C_{m+n}$.
2. By reversing the parameterization at a boundary component, one obtains a map $\mathcal{E}_{m,n} \to \mathcal{E}_{m+1,n-1}$.
3. Complex conjugation $\Sigma \mapsto \Sigma$ induces a map $\mathcal{E}_{m,n} \to \mathcal{E}_{n,m}$.

2.5.5. Central extensions. As for TQFT, we actually need to consider central extensions of the cobordism category. Since $\mathcal{E}$ is a topological category, i.e., the morphism spaces have a topology such that composition is continuous, it is natural to require that the morphism spaces of a central extension of $\mathcal{E}$ by $\mathbb{C}^*$ actually form a complex line bundle over the morphism spaces of $\mathcal{E}$. Such central extensions can be classified [S2], and turn out to be isomorphic to a tensor power of a certain universal extension, which we will now describe.

This central extension is given by the determinant line bundle. Consider the $\bar{\partial}$-operator associated to the Riemann surface $\Sigma$. This operator is not Fredholm since $\Sigma$ has a boundary, and we have to add boundary condition in order to make it so. The parameterization at each boundary allows us to impose the following boundary conditions: We consider the operator $\bar{\partial} : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma)$ acting on sections that have negative Fourier coefficients at the incoming boundaries and have nonnegative Fourier coefficients at the outgoing boundaries. With these boundary conditions, the $\bar{\partial}$-operator becomes Fredholm, and we can associate its determinant line

\[ \text{Det}(\bar{\partial}) = \Lambda^{\text{max}}(\ker \bar{\partial})^* \otimes \Lambda^{\text{max}}(\text{coker} \bar{\partial}), \]

where $\bar{\partial}$ now denotes, in sloppy notation, the $\bar{\partial}$-operator with the boundary conditions above. We will see later that this indeed defines a central extension of $\mathcal{E}$, i.e., that there are coherent isomorphisms of determinant line bundles inducing composition in the category, see also [S2, H].

2.5.6. Axioms for CFT. Consider the "category" of Hilbert spaces and trace class maps. Similar to the complex cobordism category, this is not a true category, since for an infinite dimensional Hilbert space the identity operator is not trace class. However, in a similar spirit, it is a symmetric monoidal category under the Hilbert space tensor product, and the space of morphisms carries a $*$-operation that takes
an operator to its adjoint. With this at hand, a conformal field theory is nothing but a “projective representation” of the complex cobordism category:

**Definition 2.26** (G. Segal [S2]). A conformal field theory is given by a continuous projective monoidal $*$-functor $T$ from the complex cobordism category to the category of complex Hilbert spaces and trace class maps.

Concretely, this means that one has

1. A Hilbert space $H$, such that $T(C_n) = H^\otimes n$, i.e., $T$ is monoidal.
2. A trace class operator $T_\Sigma : H^{\otimes m} \to H^{\otimes n}$, determined up to scalar $c \in \mathbb{C}^*$, for each conformal equivalence class $\Sigma \in \mathcal{E}_{m,n}$. Alternatively, this can be thought of as a ray $[v_\Sigma] \in H_{in} \otimes H_{out}$, using the correspondence

$$T \in B_1(H_{in}, H_{out}) \mapsto v = \sum_n \bar{e}_n \otimes T e_n \in H_{in} \otimes H_{out},$$

where $\{e_n\}$ is a basis for $H_{in}$, between trace class operators and vectors.

3. The equality

$$T_\Sigma = T_{\Sigma_1} \circ T_{\Sigma_2},$$

up to a complex scalar, whenever $\Sigma = \Sigma_1 \cup_{C_k} \Sigma_2$, for $\Sigma_1 \in \mathcal{E}_{m,k}$ and $\Sigma_2 \in \mathcal{E}_{k,n}$. This means that the vector determined by the trace class operator $T_{\Sigma_1} \circ T_{\Sigma_2}$ is an element of the ray determined by $\Sigma$. The complex numbers involved in the equality should satisfy obvious cocycle conditions obtained from associativity.

4. $T_\Sigma^* = T_\Sigma$, i.e., $T$ is a $*$-functor.

**Remark 2.27.** These axioms were originally inspired by the path integral approach to conformal field theory. For a very clear exposition of this, see [Ga].

**Remark 2.28.** The collection of moduli spaces $\{\mathcal{E}_{m,1}(g)\}_{g,n \geq 0}$ forms a so-called modular operad [GK]. In particular, restricting to genus zero one obtains a true operad. Using this language, the axioms above imply that, for zero central charge, the Hilbert space $H$ of the conformal field theory forms an algebra over this operad. In that case, a three-holed sphere $\mathbb{P}^3$ induces a multiplication operation

$$T_{\mathbb{P}^3}: H \otimes H \to H,$$

which is almost associative. (It is associative only if the theory does not depend on the conformal structure of the surfaces, i.e., is topological.) This multiplication is also known as the operator product expansion. The resulting algebra is closely related to the theory of vertex operator algebras [H].

### 2.5.7. The semigroup of annuli

The genus zero component of $\mathcal{E}_{1,1}$, i.e., of Riemann surfaces that are homeomorphic to an annulus, forms a semigroup $\mathcal{S}$ under the gluing operation (2.7). Notice that it is not a monoid since the complex cobordism category is not a true category, i.e., $\mathcal{S}$ has no unit. The semigroup $\mathcal{S}$ carries an involution by the complex conjugation $A \mapsto \bar{A}$. $A \in \mathcal{S}$. For $q \in \mathbb{C}_{<1}$, define $A_q = \{z \in \mathbb{C}, |q| \leq |z| \leq 1\}$, parameterized by $z$ and $qz$. One easily checks that $A_{q_1} A_{q_2} = A_{q_1 q_2}$, so one finds that $\mathcal{S}$ contains the semigroup $\mathbb{C}_{<1}$ as a subsemigroup.

It is clear from the axioms that a CFT gives, in particular, an involutive representation of the semigroup $\mathcal{E}$ on $H$ by contraction operators. The point of view expressed in [S2] is that the semigroup of annuli should be considered to be the complexification of
the diffeomorphism group of the circle, and that the representation of this semigroup given by a CFT is therefore related to the action of the Virasoro algebra in the physics literature. Recall, see e.g. [PS], that there is no complex group integrating the Virasoro algebra.

2.5.8. Teichmüller space. Recall that for closed Riemann surfaces it is convenient to view the moduli space as a quotient of its universal cover, called Teichmüller space, by the mapping class group. This is motivated by the fact that the moduli space is singular and shows that in general it will be an orbifold. Although the extended moduli space \( E_X \) is already smooth itself, it will be useful in the sequel to introduce its universal cover

\[
T_X := J(X)/\text{Diff}^+_0(X, \partial X),
\]

where \( \text{Diff}^+_0(X, \partial X) \) is the connected component of the identity in \( \text{Diff}^+(X, \partial X) \). Again this is smooth manifold, evidently simply connected, equipped with a canonical projection onto \( E_X \) that makes it its universal cover. Observe that \( T_X \) now carries a smooth action of the group

\[
(2.8) \quad \mathcal{K}_X := \text{Diff}^+(X)/\text{Diff}^+_0(X, \partial X).
\]

This is a covering group of the \( \text{Diff}^+(S^1) \) and its action covers the action of the latter on \( E_X \). By definition, the group \( \mathcal{K}_X \) fits into an exact sequence

\[
(2.9) \quad 1 \to \Gamma(X, \partial X) \to \mathcal{K}_X \to \text{Diff}^+(\partial X) \to 1.
\]

The mapping class group \( \Gamma(X, \partial X) \) acts on \( T_X \) by deck transformations, and the fibration \( T_X \to E_X \) is a model for the universal fibration \( E\Gamma(X, \partial X) \to B\Gamma(X, \partial X) \) over the classifying space.

To distinguish complex Riemann surfaces under the equivalence relation given by Teichmüller space, one can consider surfaces with a marking. In fact, inspired by the Fenchel–Nielsen coordinates, one may consider complex surfaces with a pair of pants decomposition and a marking on each of the components, that is a trivalent graph with one vertex and three vertices ending on a different boundary component. The pants decomposition is given by specifying \( 3g-3+2n \) closed curves \( C = \{ \alpha_1, \ldots, \alpha_{3g-3+2n} \} \) on \( X \). Notice that this specifies a gluing map

\[
\pi_C : \underbrace{E_P \times \cdots E_P}_{2g-2+n \text{ times}} \to E_X
\]

Two surfaces are said to be equivalent if they are biholomorphic by a map that preserves the marking up to isotopy. It is not difficult to see that the space of such equivalence classes of surfaces is exactly Teichmüller space \( T_X \).