Quantization of Hamiltonian Loop Group Actions
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Citation for published version (APA):

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CHAPTER 6

Index theory

In the previous chapters we have constructed the representation theory of loop groups by quantization from the geometry of moduli spaces. In particular, we constructed positive energy representations that could be considered as the $L^2$-kernel

$$H^\xi_{\Sigma} = L^2 - \ker (\bar{\partial}_{\mathcal{M}(\Sigma)}) \in \text{Rep}_\xi(LG)$$

of the $\bar{\partial}$-operator on $\mathcal{M}(\Sigma)$. Inspired by the corresponding situation of quantization of Hamiltonian actions of compact Lie groups on compact Kähler manifolds, one might ask whether there exists an associated Spin$_C$-Dirac operator $\mathcal{D}_{\mathcal{M}(\Sigma)}$ on $\mathcal{M}(\Sigma)$ of which one can take the "$L\!G$-equivariant index"

$$\text{Index}_{L\!G} \left( \mathcal{D}_{\mathcal{M}(\Sigma)} \right) \in R_\ell(G),$$

which takes values in the fusion ring. Assuming "vanishing of higher cohomology" this index should give the character of the positive energy representation $H^\xi_{\Sigma}$. In this chapter we will construct a suitable Dirac operator on homogeneous spaces of loop groups, and consider its $L\!G$-equivariant index. For the general moduli space of flat connections $\mathcal{M}(\Sigma)$ the existence of a Dirac operator remains conjectural, although its index in twisted $K$-theory should fit nicely with the axioms for topological quantum field theory.

6.1. Dirac Induction

In the previous chapters we have constructed the representation theory of loop groups from the geometry of moduli spaces, in particular of homogeneous spaces of loop groups. In this section we do the converse, namely we try to reconstruct the geometry of homogeneous spaces for loop groups from its representation theory. In this procedure, two ingredients are crucial: First, the idea, coming from elliptic cohomology [S1], that supersymmetry can be used to construct the Dirac operator on infinite dimensional manifolds. The second ingredient is the fact that the Dirac operator gives rise to a spectral triple from the point of view of noncommutative geometry [Con].

6.1.1. The Clifford algebra $\text{Cliff}(Lg)$. We have already encountered the real Clifford algebra of a real Hilbert space and its irreducible representations, c.f. section 5.1.4. We now apply this construction to the Lie algebra $Lg$. Consider $Lg$ with its $L^2$-inner product (1.9). The associated real Clifford algebra $\text{Cliff}(Lg)$ is generated by elements $\psi(\xi), \xi \in Lg$, subject to

$$\psi(\xi)\psi(\eta) + \psi(\eta)\psi(\xi) = \langle \xi, \eta \rangle_{Lg} \cdot \xi, \eta \in Lg.$$
With respect to the natural algebraic basis \( X'(n) = X'z^n \), \( n \in \mathbb{Z} \) of \( \mathfrak{Lg} \), where \( X', i = 1, \ldots, \dim \mathfrak{g} \) is a basis of \( \mathfrak{g} \), we will abbreviate the corresponding elements of the Clifford algebra by \( \psi'_n = \psi(X'(n)) \). These elements then satisfy the anticommutation relations \( \psi'_m \psi'_n + \psi'_n \psi'_m = \delta_{m,n} \delta_{m,0} \). As we have seen in section 5.1.4, the natural polarization of \( \mathfrak{Lg} \) by the subspace of positive and negative Fourier frequencies gives a representation of this algebra on the Hilbert space

\[
\mathcal{S}_{\mathfrak{Lg}} = \Lambda^*(\mathfrak{Lg}^+_C),
\]

where \( \mathfrak{Lg}^+_C \) denotes the Hardy subspace of the complexified Lie algebra. Since the adjoint action of \( G \) on its Lie algebra \( \mathfrak{g} \) defines a real representation \( G \to \text{SO}(\mathfrak{g}) \) which is orthogonal with respect to the Killing form, by Proposition 5.22 we find a positive energy representation of \( \mathfrak{Lg} \) on \( \mathcal{S}_{\mathfrak{Lg}} \). By definition, the Dynkin index of the adjoint representation is twice the dual Coxeter number, c.f. section 5.2, and therefore this representation is at level \( c \). In fact, this representation is \( \mathbb{Z}_2 \)-graded and therefore splits as \( \mathcal{S}_{\mathfrak{Lg}} = \mathcal{S}^+_{\mathfrak{Lg}} \oplus \mathcal{S}^-_{\mathfrak{Lg}} \), where

\[
\mathcal{S}^+_k = \bigotimes_{k = \text{even/odd}} \Lambda^*(\mathfrak{g}^*_Cz^k).
\]

### 6.1.2. The noncommutative geometry of \( G \)

Our strategy in the rest of this section is to first perform all constructions in the finite dimensional case, for a compact Lie group \( G \), and subsequentially generalize to the infinite dimensional situation for the loop group \( \mathfrak{Lg} \).

From the point of view of noncommutative geometry, a smooth manifold is described by a so called *spectral triple*. A spectral triple \((\mathcal{H}, \mathcal{A}, D)\) consists of a Hilbert space \( \mathcal{H} \) that carries a representation of a smooth algebra \( \mathcal{A} \), and \( D \) is a specific selfadjoint operator on \( \mathcal{H} \) that satisfies certain conditions concerning its interaction with \( A \) [Con].

The prototype of such a geometry is given by the triple \((L^2(M, \mathcal{S}), C^\infty(M), \hat{\mathcal{G}})\), where \( M \) is a smooth Riemannian spin-manifold, \( \mathcal{S} \) the associated spinor bundle with Dirac operator \( \hat{\mathcal{G}} \). This triple encaptures the whole geometry of \( M \). From a physical point of view, this triple describes supersymmetric quantum mechanics on \( M \), with the Dirac operator acting as the Hamiltonian. A specific example is given when \( M \) is a compact Lie group \( G \). In that case, trivializing the spinor bundle by the \( G \) action, one has

\[
L^2(G, \mathcal{S}) \cong L^2(G) \otimes \mathcal{S}_0,
\]

where \( \mathcal{S}_0 = \Lambda^* \mathfrak{g}_C \) is the unique irreducible Clifford module of the Clifford algebra \( \text{Cliff}(\mathfrak{g}) \) defined by the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{g} \). The Dirac operator \( \hat{\mathcal{G}} \) can be described entirely in terms of representation theory, and this construction works for any representation of \( G \) [Ko]: Let \( V \) be a finite dimensional representation of \( G \). The infinitesimal left action of \( G \) extends to the universal enveloping algebra \( U(\mathfrak{g}) \), and combined with the Clifford algebra, one finds an action of the so-called noncommutative Weil algebra [AM]

\[
W_G = U(\mathfrak{g}) \otimes \text{Cliff}(\mathfrak{g}).
\]

In fact, the noncommutative Weil algebra is simply the universal enveloping algebra of the \( \mathbb{Z} \)-graded super-Lie algebra \( \mathfrak{g}_{\text{super}} \) associated to \( \mathfrak{g} \) [GS2], defined by

\[
\mathfrak{g}_{\text{super}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}_1.
\]
where the subscript denotes the grading. Here \( g_0 \) is simply a copy of \( g \), whereas \( g_{-1} \) is isomorphic to \( g \) as a vector space, but has zero bracket. The structure of this algebra is easily read off from the following canonical representation of \( g_{\text{super}} \):

Let \( M \) be a smooth \( G \)-manifold. Then \( g_{\text{super}} \) acts on \( \Omega^*(M) \) as follows: \( g_0 \) acts by the Lie derivative on forms \( L_X : \Omega^*(M) \rightarrow \Omega^*(M) \), for \( X \in g \) and we denote the associated generating vector field on \( M \) by \( X \) as well. Furthermore, the action of \( R_1 \) is generated by the exterior derivative \( d : \Omega^*(M) \rightarrow \Omega^{*+1}(M) \). Finally \( g_{-1} \) acts by the contraction \( \iota_X : \Omega^*(M) \rightarrow \Omega^{*-1}(M) \). All the supersymmetry relations now have a geometric interpretation: In degree 0 we have an ordinary representation of \( g \), i.e., \([L_X, L_Y] = L_{[X,Y]}\), \( \forall X, Y \in g \). In degree \(-1\) the super Lie bracket gives \( \iota_X \iota_Y + \iota_Y \iota_X = 0 \), \( \forall X, Y \in g \), and we have of course Cartan’s formula

\[
d X + \iota_X d = L_X, \quad X \in g.
\]

As is well-known, a representation \( V \) of \( G \), can be thought of as a quantum-mechanical system whose “Hamiltonian” is given by the Casimir operator. Tensoring with the Spin representation \( S \), one introduces supersymmetry, i.e., extends the representation of \( g \) to a representation of \( g_{\text{super}} \). The presence of supersymmetry on \( V \otimes S \) allows one to take a square root of the Hamiltonian, and introduce the Dirac operator \([\text{Ko}]\)

\[
(6.1) \quad \phi^2 = \sum_{a=1}^{\dim g} X_a \otimes \psi^a + 1 \otimes \frac{1}{12} \sum_{a,b,c=1}^{\dim g} f_{abc} \psi^a \psi^b \psi^c.
\]

Here \( X^a, \ a = 1, \ldots, \dim g \) is an orthonormal basis of \( g \), and \( X^a \) simply stands for \( X^a \) acting on \( V \), while \( \psi^a \) gives the Clifford action of \( X^a \) on \( S \). The last term can be recognized as the image of the fundamental three form \((1.6)\) on \( G \), using the isomorphism \( g \cong g^* \) and the Chevalley identificiation \( \text{Cliff}(g) \cong \Lambda^* g \) given by \( \psi^a \mapsto \psi^a1 \). Abstractly, \( V \otimes S \) is a representation of the noncommutative Weil algebra and \( \phi \) is a particular element of the latter, satisfying \([\text{AM}]\):

\[
(6.2) \quad \frac{1}{2} \phi^2 = -\Delta - \frac{1}{24} \text{tr}_g \Delta_{\text{ad}}.
\]

Interpreting the Casimir operator as the “Laplacian”, this equation gives a Weitzenböck formula from the point of view of representation theory. Notice that for \( V = V_\lambda \) irreducible with highest weight \( \lambda \in \Lambda^* \), one has

\[
\phi^2 = \frac{1}{2} ||\lambda + \rho_g||^2.
\]

Applying this construction of the Dirac operator to the regular representation, \( V = L^2(G) \), one finds exactly the geometric Dirac operator on the spin or bundle \( S \) associated to the so called reductive connection. (The reductive connection is defined on Riemannian symmetric spaces of the form \( G/H \), for \( H \subseteq G \) a closed subgroup of \( G \). For our case, put \( G \subset G \times G \).) In this case the Casimir gives exactly the Laplacian on \( G \), and the equation above is exactly the usual Weitzenböck formula. In conclusion, this gives a construction of the spectral triple

\[
(L^2(G,S), \phi, C^\infty(G))
\]

of the manifold \( G \), from the point of view of supersymmetry. From a physical point of view, this triple describes supersymmetric particles moving on \( G \). Observe that
the general philosophy of noncommutative geometry, applied to this case, now implies that the representation theory of $G$ completely describes the geometry of the manifold underlying $G$.

6.1.3. The noncommutative geometry of $LG$. Just like the quantization of supersymmetric particles on $G$ captures the geometry of $G$, the quantization of the supersymmetric Wess–Zumino Witten model should be related to the geometry of $LG$. Notice that, applied to the loop group, the philosophy to use supersymmetry to construct the Dirac operator coincides with the approach to elliptic cohomology, see [S1], except for the fact that we twist with a line bundle, namely the central extension of $LG$.

First we will explain how the notion of supersymmetry combines with the theory of Kac–Moody algebras, or loop groups. Let $E$ be a positive energy representation of $LG$ at level $\ell$, and consider the tensor product

$$E \otimes S_LG.$$ 

This Hilbert space carries a positive energy representation of $LG$ at level $\ell + c$, which, together with the action of the Clifford algebra $\text{Cliff}(Lg)$ gives a representation of the nonabelian Weil algebra of loop groups

$$W_{LG} = U(Lg) \otimes \text{Cliff}(Lg).$$

Again this representation comes from an action of the super Kac–Moody algebra

$$Lg_{\text{super}} = Lg_{-1} \oplus Lg_0 \oplus \mathbb{R}_1.$$

By the Segal Sugawara construction, the action of $Lg$ extends to $\text{Vir} \times Lg$. In the supersymmetric case above, the Virasoro algebra is replaced by the super-Virasoro algebra, and in total one finds an action of the semi-direct product of the super-Virasoro algebra and the super-Kac–Moody algebra, commonly referred to as the $N = 1$ superconformal algebra [KT]. Without fully explaining the proof of this statement, let us simply observe that in the super-Virasoro algebra, besides the usual generators $L_n$, $n \in \mathbb{Z}$ given by the Segal–Sugawara construction, one also has generators $Q_n$, $n \in \mathbb{Z}$ satisfying certain graded commutation relations. For us, the important element of this algebra is the odd degree generator $Q_0$ for $n = 0$, which write as:

$$\phi_{Lg} := \sum_{n \in \mathbb{Z}} \sum_{a=1}^{\dim g} J_{-n}^a \otimes \psi_n^a - 1 \otimes \frac{1}{12} \sum_{k,m \in \mathbb{Z}} \sum_{a,b,c=1}^{\dim g} f_{abc} \psi_n^a \psi_m^b \psi_{-m-n}^c.$$

Observe that this operator has the same structure as the Dirac operator (6.1) for $G$, including the familiar "cubic term". Indeed, at the zero energy level, restricting to "zero momentum", that is, setting $n = 0$, one finds exactly the Dirac operator $\phi_g : E(0) \otimes S_g \rightarrow E(0) \otimes S_g$ given by (6.1). The graded commutation relations in the $N = 1$ superconformal algebra imply the following well known properties of $\phi_{Lg}$:
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Lemma 6.1. The Dirac operator $\mathcal{D}_{Lg}$ satisfies the following relations:

\[
\mathcal{D}_{Lg}^2 = (\ell + c)L_0 - \frac{c}{24}, \quad c = \frac{1}{2} \dim g + \frac{\ell \dim g}{(\ell + c)}
\]

\[
[\mathcal{D}_{Lg}, \xi] = 0, \quad \forall \xi \in g
\]

\[
[\mathcal{D}_{Lg}, L_0] = 0
\]

Of course, the third follows immediately from the first equation. The second shows in fact that $\mathcal{D}_{Lg}$ is only $G$-equivariant, not $LG$-equivariant. It follows from the second and third equation that the kernel and cokernel of this operator carry representations of the product $G \times T$. As we have seen in the previous chapters, it is useful to think of $L_0$ as the “Hamiltonian of the quantum mechanical system given by a positive energy representation. By analogy with the finite dimensional case for a compact Lie group $G$, we therefore interpret $\mathcal{D}_{Lg}$ as the Dirac operator for loop groups.

Consider the case of an irreducible representation $E = H_\lambda$, $\lambda \in \Lambda^*_t$. Notice that the $L_0$ above consists of a sum of two operators, the Segal–Sugawara operator associated to the representation of $LG$ on $E$ and the energy operator $D$ in the spin representation $S_{Lg}$. Therefore, in this case the operator $L_0$ differs from the total energy operator $d$ by

\[
\Delta_\lambda = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(\ell + c_g)}
\]

Therefore, in the case of an irreducible representation one easily finds that

\[
\mathcal{D}_\lambda^2 = (\ell + c)d + \frac{1}{2}||\lambda + \rho||^2,
\]

see also [L2]. From this formula, it is clear that the Dirac operator $\mathcal{D}_{Lg}$ is a positive operator and one has $\ker \mathcal{D}_\lambda = \ker \mathcal{D}_\lambda^2 = 0$. Let us finally construct the spectral triple that describes the geometry of $LG$. Recall that the Hilbert space for the ordinary WZW-model is given by

\[
H_{WZW}^t = \bigoplus_{\lambda \in \Lambda^*_t} H_\lambda \otimes \overline{H}_\lambda.
\]

Therefore the supersymmetric model is described by the Hilbert space

\[
H_{SWZW}^t = H_{WZW}^t \otimes S_{Lg}.
\]

We now apply the previous construction of the Dirac operator, even though the representation on $H_{WZW}^t$ is not a positive energy representation. The resulting operator $\mathcal{D}_{Lg}$ is therefore no longer Fredholm. This gives us two ingredients of the spectral triple, namely the Hilbert space $H_{SWZW}^t$ and the dirac operator $\mathcal{D}_{Lg}$ which is unbounded, but has dense domain of definition given by the vectors of finite energy with respect to the natural bigrading on the Hilbert space. The last ingredient is given by a suitable algebra on $H_{SWZW}^t$. See also [FG] for this.

Cyclic cohomology. The Dirac operator $\mathcal{D}_{Lg}$ is also important for cyclic cohomology since it gives rise to a cyclic cocycle called the JLO-cocycle [JLO]. For an involutive algebra $A$ acting on $H = E \otimes S_{Lg}$ by bounded operators, the pair $(H, \mathcal{D}_{Lg})$ gives rise to a $\theta$-summable $K$-cycle over $A$, since

\[
\text{tr}_H \left( e^{-s \mathcal{D}_{Lg}^2} \right) = \text{tr}_H \left( e^{-s(L_0 - \frac{\imath}{2\pi} \text{tr}_S \Delta_{ad})} \right) < \infty.
\]
The Chern character $\text{Ch}^n(a_0, \ldots, a_n)$ of this spectral triple in the entire cyclic cohomology of $A$ is defined by

$$\text{Ch}^n(a_0, \ldots, a_n) = \int_{\Delta_n} \text{tr} \left( a_0 e^{-s_0 a_1} [\theta, a_1] e^{-s_1 a_2} \ldots [\theta, a_n] e^{-s_n a_n} \right) d^n s.$$ 

Recall that entire cyclic cohomology is well defined on the category of Banach algebras, so we can take as our algebra the von Neumann algebra $M$ used in the previous chapter.

### 6.1.4. Dirac induction for compact Lie groups

Again we first start with the finite dimensional situation. Dirac induction is a well known technique in the representation theory of semi-simple Lie groups to construct irreducible representations. In the case of a compact Lie group, the relevant Dirac operator on homogeneous spaces can in fact be constructed with the methods of section 6.1.2, and gives a very simple construction of the induction homomorphism [L1].

Let $H \subseteq G$ be a closed subgroup of the compact Lie group $G$ with Lie algebra $\frak{h} \subseteq \frak{g}$. Let $\frak{p} \cong \frak{g}/\frak{h}$ be the ortho-complement with respect to the Killing form on $\frak{g}$, so that one has the decomposition $\frak{g} \cong \frak{p} \oplus \frak{h}$ as $H$-representations induced by the adjoint action of $G$ on $\frak{g}$. This decomposition induces an isomorphism $\text{Cliff}(\frak{g}) \cong \text{Cliff}(\frak{p}) \otimes \text{Cliff}(\frak{h})$, and the representation of $H$ on $S_\frak{g}$ can be decomposed as

$$S_\frak{g} \cong S_\frak{p} \otimes S_\frak{h},$$

where $S_\frak{p}$ and $S_\frak{h}$ are the irreducible modules of $\text{Cliff}(\frak{h})$ and $\text{Cliff}(\frak{p})$. Now, let $V$ be a representation of $G$ and consider the tensor product

$$V \otimes S_\frak{p}.$$ 

On this space, the representation of $\frak{g}$ only extends to a representation of $U(\frak{g}) \otimes \text{Cliff}(\frak{p})$ instead of the full nonabelian Weil algebra. However, just like the nonabelian Weil algebra, this algebra contains an operator of Dirac type [Ko]. For this consider the tensor product $V \otimes S_\frak{p}$, with the Dirac operator $\hat{\theta}_\frak{p}$. Given the decomposition of the spin representation $S_\frak{g} \cong S_\frak{p} \otimes S_\frak{h}$, one can also consider the Dirac operator $\hat{\theta}_\frak{h}$, given by writing $V \otimes S_\frak{g} \cong (V \otimes S_\frak{p}) \otimes S_\frak{h}$, where $\frak{h}$ acts on $V \otimes S_\frak{p}$ by the tensor product representation on both factors. Now we write $\hat{\theta}_{\frak{g}/\frak{h}} := \hat{\theta}_\frak{g} - \hat{\theta}_\frak{h}$. This turns out to be an element of $U(\frak{g}) \otimes \text{Cliff}(\frak{p})$ and can be written out as

$$\hat{\theta}_{\frak{g}/\frak{h}} = \sum_{a=1}^{\dim \frak{p}} X_a \otimes \psi^a + 1 \otimes \sum_{a,b,c=1}^{\dim \frak{p}} f_{abc} \psi^a \psi^b \psi^c,$$

where this time the $X_a$ only form a basis of $\frak{p}$. Again, the formula has the same structure as (6.1). This is the operator that we use to construct the Dirac operator on the homogeneous space $G/H$.

Using the transitive $G$-action, one can trivialize the tangent bundle to $G/H$, with fiber given by $\frak{p} \cong \frak{g}/\frak{h}$, which in turn induces a trivialization of the spin bundle $S$ with fiber given by $S_\frak{p}$. Now, given any representation $V$ of $H$, form the induced bundle $E = G \times_H V$ over $G/H$ with fiber $V$. Consider the space of $L^2$-sections of the tensor product of this vector bundle $E$ with the spinor bundle $S$. By the transitive $G$-action, this bundle is trivializable, and therefore we can also consider its space of $L^2$-sections. The operator $\hat{\theta}_{\frak{g}/\frak{h}}$ acts on these sections, and its spectrum gives the desired representation of $G$.
6.1. Dirac Induction for loop groups. In this section we generalize the constructions of Kostant and Landweber to homogeneous spaces for loop groups which should give a construction of the Dirac induction homomorphism. Such a procedure assigns a positive energy representation of $LG$ to a representation of a subgroup $H \subseteq G$. Two cases are of special importance: When $H = T$, the maximal torus in $G$, this is simply the Borel--Weil theorem for loop groups, however using Spin$^c$-quantization instead of ordinary geometric quantization. (This causes a shift in the level by the dual Coxeter number.) This means that we use the Dirac operator on coadjoint orbits, rather than the ordinary $\partial$-operator, and this accounts for an extra shift by the dual Coxeter number in the level. On the other hand, when $H = G$, Dirac induction will turn out to describe the fact that the fusion ring of $LG$, i.e., the Grothendieck ring of the modular tensor category $\text{Rep}_{\ell}(LG)$, is a quotient of the representation ring $R(G)$ of $G$.

Again, let $H \subseteq G$ be a closed subgroup. We want to consider the index of the Dirac operator over $LG/H$. The decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ induces a decomposition $LG = Lg/h \oplus \mathfrak{h}$, and the Clifford algebra factors as $\text{Cliff}(Lg) = \text{Cliff}(Lg/h) \otimes \text{Cliff}(\mathfrak{h})$. Eventually, one finds an $H$-equivariant decomposition of the Spin representation $S_{Lg} = S_{Lg/h} \otimes S_{\mathfrak{h}}$, where

$$S_{Lg/h} = S_{\mathfrak{p}} \otimes \bigotimes_{k > 0} \Lambda^* \mathfrak{g} \otimes z^k.$$

Now let $E$ be a positive energy representation of $LG$ at level $\ell$, and consider the tensor product

$$E \otimes S_{Lg/h}.$$

Notice that this Hilbert space carries a positive energy representation of $LG$ at level $\ell$, by letting $LG$ act trivially on the second component, whereas taking the tensor product of the $\text{Rot}(S^1)$ representations: The energy eigenspaces are finite dimensional since this is the case for the actions on both $E$ and $S_{Lg/h}$. Notice that, just as in the finite dimensional case, $E \otimes S_{Lg/h}$ does carry a representation of $H$ which is nontrivial on both factors. On the tensor product $E \otimes S_{Lg}$ we now take the difference $\partial_{Lg} - \partial_{\mathfrak{h}}$.

action, this Hilbert space is identified with

$$L^2(G/H, E \otimes S) \cong (L^2(G) \otimes V \otimes S_p)^H.$$ 

The left action induces a unitary representation $G$ on this Hilbert space. Since the operator $\partial_{Lg/h}$ is $H$-equivariant [L1], we can use the previous construction to find that the dirac operator, denoted by $\partial_L$, acts on this space, commuting with the representation of $G$. The $G$-index of this operator is defined as

$$\text{index}_G(\partial_L) = [\ker(\partial_L)] - [\text{coker}(\partial_L)] \in R(G).$$

Observing that also the spin bundle $S_p$ is an induced bundle, one can apply Bott's index theorem for homogeneous differential operators [B2] to compute this virtual representation [L1]. When $H = T$, the maximal torus, and $V$ is an irreducible representation of $T$ labeled by a weight $\lambda \in \Lambda^*$, one finds the Borel--Weil theorem shifted by $\rho/2$, where $\rho$ is the sum of positive roots.

$$\text{index}_G(\partial_L) = [\ker(\partial_L)] - [\text{coker}(\partial_L)] \in R(G).$$

Notice that this Hilbert space carries a positive energy representation of $LG$ at level $\ell$, by letting $LG$ act trivially on the second component, whereas taking the tensor product of the $\text{Rot}(S^1)$ representations: The energy eigenspaces are finite dimensional since this is the case for the actions on both $E$ and $S_{Lg/h}$. Notice that, just as in the finite dimensional case, $E \otimes S_{Lg/h}$ does carry a representation of $H$ which is nontrivial on both factors. On the tensor product $E \otimes S_{Lg}$ we now take the difference $\partial_{Lg} - \partial_{\mathfrak{h}}$, 

$$\text{index}_G(\partial_L) = [\ker(\partial_L)] - [\text{coker}(\partial_L)] \in R(G).$$

Notice that this Hilbert space carries a positive energy representation of $LG$ at level $\ell$, by letting $LG$ act trivially on the second component, whereas taking the tensor product of the $\text{Rot}(S^1)$ representations: The energy eigenspaces are finite dimensional since this is the case for the actions on both $E$ and $S_{Lg/h}$. Notice that, just as in the finite dimensional case, $E \otimes S_{Lg/h}$ does carry a representation of $H$ which is nontrivial on both factors. On the tensor product $E \otimes S_{Lg}$ we now take the difference $\partial_{Lg} - \partial_{\mathfrak{h}}$. 

$$\text{index}_G(\partial_L) = [\ker(\partial_L)] - [\text{coker}(\partial_L)] \in R(G).$$
and we write this out as

\[(6.4) \quad \tilde{\varphi}_{Lg/h} = \varphi_{g/h} + \sum_{n \neq 0} \sum_{a=1}^{\dim g} J^a_n \otimes \psi^a_n - 1 \otimes \frac{i}{6} \sum_{k \neq m \neq 0} \sum_{a,b,c=1}^{\dim g} f_{abc} \psi^a_n \psi^b_m \psi^c_{m-n}.\]

This seems a natural generalization of Kostant's Dirac operator to homogeneous spaces of loop groups of the form \(LG/H\). Indeed, on the energy zero level one recovers the operator \(\tilde{\varphi}_{g/h} : E(0) \otimes \mathcal{S}_p \to E(0) \otimes \mathcal{S}_p\). First notice,

**Lemma 6.2.** \([\tilde{\varphi}_{Lg/h}, \xi] = 0, \forall \xi \in h.\]

**Proof.** As remarked, \(\varphi_{Lg} - \varphi_{Lg/h} = \varphi_g - \varphi_{g/h} =: \varphi'_h\) holds in \(U(\widetilde{Lg}) \otimes \text{Cliff}(Lg)\), for example by letting it act on \(E \otimes \mathcal{S}_{Lg}\). This operator \(\varphi'_h \in U(g) \otimes \text{Cliff}(g)\) is described in [L2] and is known to be \(h\)-invariant, as remarked in the previous section. The result now follows from the fact that \(\varphi_{Lg}\) commutes with the representation of \(g\), c.f. Lemma 6.1.

For the sake of future index calculations it is convenient to compute the square of this operator:

**Proposition 6.3.** The square of the Dirac operator is given by

\[\frac{1}{2} \left(\tilde{\varphi}_{Lg/h}\right)^2 = L_0 - \Delta_h - \frac{1}{24} (c - \text{tr}_h \Delta_h).\]

**Proof.** As we have seen above, \(\varphi_{Lg/h} = \varphi_g - \varphi'_h\), where \(\varphi'_h\) is the Dirac operator associated to the \(h\)-representation on \(E \otimes \mathcal{S}_{Lg/h}\). We now show that \(\varphi_{Lg/h}\) and \(\varphi'_h\) anticommute, i.e.,

\[\{\varphi'_h, \varphi_{Lg/h}\} = 0.\]

From (6.1) we see that the Dirac operator \(\varphi'_h\) consists of two terms. The second term clearly anticommutes with \(\varphi_{Lg/h}\), as both are odd degree operators acting on different spaces, namely \(S_h\) and \(E \otimes \mathcal{S}_{Lg/h}\). For the first term we compute

\[\left\{\varphi_{Lg/h}, \sum_{a=1}^{\dim h} \pi'(X_a) \otimes \psi^a\right\} = \sum_{a=1}^{\dim h} \left[\varphi_{Lg/h}, \pi'(X_a) \right] \otimes \psi^a = 0.\]

by Lemma 6.2. Using this anticommutation rule, one finds

\[\varphi_{Lg/h}^2 = \varphi_g^2 - \varphi'_h^2.\]

and the proposition now follows from Lemma (6.1) and (6.2).

Clearly, the Dirac operator \(\varphi_{Lg/h}\) commutes with the energy operator, so we can decompose \(E \otimes \mathcal{S}_{Lg/h}\) with respect to the \(\text{Rot}(S^1) \times H\)-action, to elucidate the structure of \(\varphi_{Lg/h}\). For an irreducible positive energy representation we find:

**Corollary 6.4.** For \(E_\lambda, \lambda \in \Lambda_h^+\) irreducible, one has, restricted to the isotypical summand labeled by \((\mu, n) \in \Lambda_h^+ \times \mathbb{Z}_+\),

\[\varphi_{Lg/h}^2 \bigg|_{(\mu, n)} = (\ell + c_g)n + ||\lambda + \rho_g||^2 - ||\mu + \rho_h||^2.\]
6.2. $K$-theory

PROOF. This follows from Proposition 6.3, equation (6.3) and the fact that in the irreducible summand labeled by $\mu \in \Lambda^*_h$ one has

$$\Delta_h = \frac{1}{2} \langle \mu, \mu + \rho_h \rangle.$$ 

Putting this together, the result follows. \qed

This result can be used to prove a generalization of the Weyl-Kac character formula, decomposing an irreducible positive energy representation $E_\lambda$ into characters in $R(H)[[q]]$, by viewing $E_\lambda$ as a $(H \times \mathbb{T})$-module. For $H = T$ the maximal torus, one finds back the original Weyl-Kac formula. We now proceed as in the finite dimensional case. For any representation $V$ of $H$, we consider the Hilbert space

$$(H^{WZW}_H \otimes V \otimes S_{L_0/h})^H.$$ 

Since the Dirac operator $\tilde{\phi}_{L_0/h}$ commutes with the $H$-action, it descends to an unbounded operator on the above Hilbert space. So far, we have only used the right $LG$-action on $H^{WZW}_H$, which was supersymmetric, so the Dirac operator by construction commutes with the left $LG$-action. The spin representation

$$S_{L_0/h} = S^{+}_{L_0/h} \oplus S^{-}_{L_0/h}$$

is $\mathbb{Z}_2$-graded and the Dirac operator commutes with the grading. Therefore, we want to consider the component $\tilde{\phi}^\Gamma_{L_0/h} : E \otimes S^{+}_{L_0/h} \to E \otimes S^{-}_{L_0/h}$ and take its $LG$-index

$$\text{index}_{LG}(\tilde{\phi}^\Gamma_{L_0/h}) = [\ker \tilde{\phi}^\Gamma_{L_0/h}] - [\coker \tilde{\phi}^\Gamma_{L_0/h}] \in R_{t}(G).$$

To show that this is indeed well defined, one should use the fact that the Dirac operator essentially squares to $L_0$, and that the Hilbert space $H^{WZW}_H$ is bigraded by the left and right $\text{Rot}(S^1)$-action and finite dimensional in each bidegree. Then this would give an induction homomorphism

$$\text{Ind}^{LG}_H : R(H) \to R_{t}(G).$$

This essentially defines Dirac induction. The difficult task is then to show that for $H = T$, the maximal torus, this defines (up to a possible shift) the usual Borel-Weil theorem for loop groups.

6.2. $K$-theory

The main tool in proving index theorems is given by $K$-theory. Therefore it is natural to speculate about the nature of the underlying $K$-theory of the index computations of the previous section. It turns out that there is a natural, geometric $K$-theory in which the index classes should live. This gives rise to a general framework in which to prove a general $LG$-equivariant index theorem over the moduli space $\mathcal{M}(\Sigma)$. We sketch the general picture from the point of view of topological quantum field theory.
6.2.1. Quantization as an index. In the previous chapters we have defined quantization of a symplectic manifold \((M, \omega)\) as the vector space \(H^0(M, L)\) where \(L\) is a prequantum line bundle and \(M\) is equipped with a compatible complex structure. We now turn to a different definition, usually attributed to Bott. See [Sj]: Fix in addition a \(\text{Spin}_c\) structure on \(M\), and denote by \(\mathcal{D}_L\) the Dirac operator on \(M\), twisted by the line bundle \(L\). The \(\text{Spin}_c\)-quantization of \(M\) is defined by

\[
\mathcal{Q}(M) = \text{index}(\mathcal{D}_L) = \pi_![L] \in K^0(pt.) = \mathbb{Z},
\]

where \(\pi : M \to pt\). maps \(M\) to a point. The second equality is essentially the Atiyah-Singer index theorem. When \(M\) is Kähler and therefore \(L\) holomorphic, it computes the Euler characteristic \(\chi(L)\) of \(L\). In case the higher cohomology of \(L\) is zero, for example by Kodaira-vanishing, this definition therefore simply computes the dimension of the Quantum Hilbert space we have used before. Since the index of \(\mathcal{D}_L\) is quite a coarse invariant, this definition is much less dependent on the polarization used, in fact it only depends on its homotopy class.

In the case of a Hamiltonian \(G\)-manifold, where \(G\) is a compact Lie group, one uses of course the push forward in equivariant \(K\)-theory. This index may be computed by the equivariant index theorem of Atiyah-Segal, and is an element of the equivariant \(K\)-theory of a point. \(K_G(pt.) \cong R(G)\), i.e., a virtual representation of \(G\). For a coadjoint orbit \(O_\lambda, \lambda \in \Lambda^*\), this gives, by the Lefschetz fixed point formula, precisely the Weyl character formula for \(\chi(V_\lambda)\), see e.g. [PS] chapter 14. It is in this sense that the definition of Quantization as an index encaptures the usual Kirillov correspondence between symplectic geometry and representation theory.

6.2.2. Twisted \(K\)-theory. Let \(X\) be a topological space with a degree three integral cohomology class \(\chi \in H^3(X, \mathbb{Z})\). Recall that \(H^3(X, \mathbb{Z})\) classifies projective Hilbert bundles: Since \(T \cong \mathbb{R}/\mathbb{Z}\) is a \(K(\mathbb{Z}, 1)\)-space, it follows from the long exact sequence of homotopy groups associated to the canonical central extension

\[
1 \to T \to U(H) \to PU(H) \to 1,
\]

together with the contractibility of the unitary group of an infinite dimensional Hilbert space \(H\), that \(PU(H)\) is a \(K(\mathbb{Z}, 2)\). Since \(PU(H)\) is a topological group, it then follows that its classifying space is a \(K(\mathbb{Z}, 3)\), and therefore

\[
H^3(X, \mathbb{Z}) = [X, K(\mathbb{Z}, 3)] = [X, BPU(H)],
\]

i.e., classes in \(H^3(X, \mathbb{Z})\) correspond to \(PU(H)\)-bundles over \(X\). Since \(PU(H) \cong \text{Aut}(K(H))\), the automorphism group of the ideal of compact operators, there is an associated bundle of Fredholm operators \(\text{Fred}_\chi\) over \(X\) associated to \(\chi \in H^3(X, \mathbb{Z})\). The twisted \(K\)-group is defined to be the space of homotopy classes of sections of this bundle. Notice that when \(\chi = 0\) the bundle is trivial and by the Atiyah–Jänich theorem, this definition coincides with the ordinary \(K\)-theory of \(X\) in the untwisted case.

6.2.3. \(KK\)-theory. We close by giving a conjectural picture from the point of view of \(KK\)-theory. This combines the isomorphism between the fusion rules and the equivariant twisted \(K\)-theory ring \(K^*_G(G, \chi)\) with the general properties of topological quantum field theory and should offer a general framework for index theory over the moduli spaces \(\mathcal{M}(\Sigma)\).
As before, we let $L\mathfrak{g}^*$ be the dual of the Lie algebra, fibered over $G$ by the holonomy map $\text{Hol} : L\mathfrak{g}^* \to G$, see section 1.2.1. The coadjoint action of $LG$ turns this into a $G$-equivariant principal $\Omega G$-bundle, in fact the universal one. For a representation $V$ of $G$, consider the projective representation at level $d_V$ of $LG$ on the Fock space $\mathcal{F}_V$ constructed in Proposition 5.1. Using the homomorphism $\pi : LG \to PU(\mathcal{F}_V) \cong \text{Aut}(K(\mathcal{F}_V))$, one obtains a $G$-equivariant bundle of compact operators over $G$

$$\mathcal{B} = L\mathfrak{g}^* \times_{LG} K(\mathcal{F}_V),$$

associated to the universal fibration $\text{Hol} : L\mathfrak{g}^* \to G$. Let $A$ be the algebra of $G$-equivariant continuous sections of this bundle, equipped with pointwise multiplication. Define a norm on $A$ by

$$||s|| := \sup_{g \in G} ||s(g)||, \ s \in A,$$

so that it becomes a $C^*$-algebra. It is not difficult to observe that this is a continuous trace $C^*$-algebra [RW], and its spectrum is given by $\text{Spec}(A) \cong G$, with the induced $G$-action by conjugation. Continuous trace $C^*$-algebras $A$ are classified, up to Morita equivalence, by degree three integral cohomology classes on the spectrum, called the Dixmier–Douady class $\delta(A) \in H^3(\text{Spec}(A), \mathbb{Z})$. In this case we therefore find an equivariant cohomology class in $H^3_G(G, \mathbb{Z})$. In this case, since the central extension of $LG$ defines a $G$-equivariant gerbe over $G$ with associated Chern class $d_V \chi$, where $\chi$ is the canonical generator of $H^2_G(G, \mathbb{Z}) \cong \mathbb{Z}$, the Dixmier–Douady invariant of the continuous trace $C^*$-algebra $A$ is also given by $d_V \chi \in H^3_{LG}(G, \mathbb{Z})$.

Given a $G$-equivariant $C^*$-algebra, one can study its $G$-equivariant $K$-theory defined in terms of projective modules, see e.g. [Con, Bl]. In this case, it was proved in [R] that the $G$-equivariant $K$-theory is isomorphic to the $K$-theory of the spectrum, twisted by the Dixmier–Douady class:

$$K^G(A) \cong K^G(\text{Spec}(A), \delta(A)).$$

The advantage of having a $C^*$-algebra is that we can now evoke the powerful machinery of $KK$-theory, a very efficient way to induce homomorphisms of $K$-groups. This leads to the following, conjectural picture:

Let $\Sigma$ be a topological cobordism from $\partial \Sigma_{\text{in}} \cong C_m$ to $\partial \Sigma_{\text{out}} \cong C_n$. Given a level $\ell_\chi \in H^2_G(G, \mathbb{Z})$, one constructs two $G$-equivariant continuous trace $C^*$-algebras $A_{\text{in}}$ and $A_{\text{out}}$ with $\text{Spec}(A_{\text{in}}) = G^m$ and $\text{Spec}(A_{\text{out}}) = G^n$ and Dixmier–Douady invariant given by $\ell_\chi$. The quantization functor now should assign to a circle $S^1$ the $K$-group $K^G_*(A)$, and to a cobordism $\Sigma$ the element

$$Q(\mathcal{M}(\Sigma)) \in KK(A_{\text{in}}, A_{\text{out}}).$$

Recall from [Bl] that elements in $KK$-theory are given by so called Kasparov-triples. Basically, this consists of a $C^*$-module over $A_{\text{out}}$ with a left action of $A_{\text{in}}$ and an operator almost intertwining the actions. This operator on the module, the essential part of the triple, should somehow be related to the Dirac operator on $\mathcal{M}(\Sigma)$. Now, $KK$-theory has the fundamental property that there exists an intersection product

$$\otimes_B : KK(A, B) \times KK(B, C) \to KK(A, C),$$
by which the $KK$-groups can be view as homomorphisms between $K$-groups. From this point of view, functoriality of quantization is given by

$$Q(M(\Sigma_1)) \otimes_A Q(M(\Sigma_2)) = Q(M(\Sigma)).$$

for $\Sigma = \Sigma_1 \cup_{S^1} \Sigma_2$. This fits exactly into the quantization scheme proposed in [La3], when applied to the moduli space of flat connections. By Theorem 2.9, such a quantization gives a Frobenius algebra structure on the $K$-group $K^*_G(A)$, i.e., the twisted equivariant $K$-theory $K^*_G(G, \tilde{\chi})$. Under the isomorphism with the fusion ring $R_\ell(G)$, the ring structure should coincide with the fusion product.

Unfortunately, this analytical theory remains conjectural. It is not even clear how the Dirac operator of the previous section fits into this machinery, giving rise to a special kind of Fredholm module. Notice, see [FHT], that there is considerable progress on topological side of this theory.