On quantum computation theory

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Appendix C

Norms and Distances

C.1 Norms and Distances on Vectors and Matrices

**absolute value** $|x|$: For a complex value $x \in \mathbb{C}$, its absolute value, or norm, is defined by $|x| = \sqrt{xx^*}$.

**Sum norm** $\|x\|_1$: For a complex valued vector $x \in \mathbb{C}^n$, the sum norm is defined by $\|x\|_1 = \sum |x_i|$. This norm is also called the $\ell_1$, or **Manhattan norm**. For bitvectors $x \in \{0, 1\}^n$ the sum norm corresponds with the **Hamming weight** of a bit string: $\|x\|_1 = \text{"number of ones in } x\text{"}$.

**Euclidean, or $\ell_2$, vector norm** $\|x\|_2$: For a complex valued vector $x \in \mathbb{C}^n$, its norm is defined by $\|x\|_2 = \sqrt{\sum x_i^* x_i}$.

**Max, or $\ell_\infty$, norm** $\|x\|_\infty$: For a complex valued vector $x \in \mathbb{C}^n$, the max norm is defined by $\|x\|_\infty = \max_i |x_i|$.

**Fidelity**: The **fidelity** between two mixed states $\rho$ and $\sigma$ is defined by

$$F(\rho, \sigma) = \text{tr} \left( \sqrt{\sqrt{\rho} \cdot \sigma \cdot \sqrt{\rho}} \right),$$

although the reader should be warned that some authors use the square of this value.

**Euclidean matrix norm** $\|A\|_2$: For a complex valued matrix $A \in M_n(\mathbb{C})$, the Euclidean norm is defined by

$$\|A\|_2 = \sqrt{\sum_{i,j} A_{ij} A_{ij}^*} = \sqrt{\text{tr}(A \cdot A^*)}.$$  

Alternative names are: $\ell_2$, **Frobenius**, **Hilbert-Schmidt**, or **Schur norm**.
We call this norm *unitarily invariant* because \( \| U \cdot A \cdot V \|_2 = \| A \|_2 \) for unitary \( U, V \in U(n) \). From this invariance it follows, using the SV decomposition, that we have

\[
\| A \|_2 = \sqrt{\sum_i \sigma_i^2},
\]

with \( \sigma_i \) the singular values of \( A \), and hence for normal matrices

\[
\| A \|_2 = \sqrt{\sum_i |\lambda_i|^2},
\]

where \( \lambda_i \) are the eigenvalues of \( A \).

**Trace norm** \( \| A \|_{tr} \): For a matrix \( A \in M_n(\mathbb{C}) \), the trace norm is defined by

\[
\| A \|_{tr} = \text{tr} \left( \sqrt{A \cdot A^*} \right) = \sum_i \sigma_i,
\]

with \( \sigma_i \) the singular values of \( A \). From this definition it follows that for normal matrices the trace norm equals

\[
\| A \|_{tr} = \sum_i |\lambda_i|,
\]

where \( \lambda_1, \lambda_2, \ldots \) are the eigenvalues of \( A \).

In the case of positive, semidefinite matrices we thus have \( \| A \|_{tr} = \text{tr}(A) \), hence the name of this norm. (As a consequence, all proper density matrices obey \( \| \rho \|_{tr} = 1 \).)

The usefulness of this norm lies in the distance \( \| \rho - \sigma \|_{tr} \) it defines between two density matrices \( \rho \) and \( \sigma \). For any measurement setting \( P = \{ P_i \} \) (with \( \sum_i P_i^* P_i = I \)), the *total variation distance* between \( \rho \) and \( \sigma \) is bounded from above by

\[
\| \rho - \sigma \|_{tr} \geq \sum_{P_i \in P} |\text{Prob}(\rho = P_i) - \text{Prob}(\sigma = P_i)|,
\]

with \( \text{Prob}(\rho = P_i) = (F(P_i, \rho))^2 \). If we choose the projectors \( P_i \) of \( P \) to be the eigenvectors of \( \rho - \sigma \), then we obtain the above bound, hence

\[
\| \rho - \sigma \|_{tr} = \max_P \left( \sum_{P_i \in P} |\text{Prob}(\rho = P_i) - \text{Prob}(\sigma = P_i)| \right).
\]
Both the Euclidean and the trace norm are *matrix norms* because they obey the following properties (see Chapter 5 in [54] for much more on this topic):

1. nonnegative: $\|A\| \geq 0$
2. positive: $\|A\| = 0$ if and only if $A = 0$
3. homogeneous: $\|\alpha A\| = |\alpha| \cdot \|A\|$ for all $\alpha \in \mathbb{C}$
4. triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$
5. submultiplicative: $\|AB\| \leq \|A\| \cdot \|B\|$. 

In addition, for the tensor product between two matrices, we also have the equality 

$\|A \otimes B\| = \|A\| \cdot \|B\|$. 

A very useful relation between the trace and the Euclidean norm is easily shown by the inequalities $\frac{1}{\sqrt{n}} \sum_i \sigma_i \leq \sqrt{\sum_i \sigma_i^2} \leq \sum_i \sigma_i$ for any $n$ nonnegative values $\sigma_1, \ldots, \sigma_n$. If we take these $\sigma_i$ to be the singular values of $A$, we see that

$$\|A\|_2 \leq \|A\|_{tr} \leq \sqrt{n} \cdot \|A\|_2,$$

(C.1) for all $A \in M_n(\mathbb{C})$.

### C.2 Norms on Superoperators

**Trace induced superoperator norm:** For a superoperator $E : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ we can use the trace norm to define

$$\|E\|_{tr} = \max_{A \neq 0} \frac{\|E(A)\|_{tr}}{\|A\|_{tr}}.$$ 

If $E$ is a positive, trace preserving mapping, then $\|E\|_{tr} = 1$. A drawback of this norm is that it can increase if we tensor $E$ with the identity operator. Take for example the one qubit transpose, with $T(A) = A^T$, which has $\|T\|_{tr} = 1$, but also $\|T \otimes I_2\|_{tr} = 2$.

**Diamond superoperator norm:** Let $E : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear superoperator, the *diamond norm* can then be defined by

$$\|E\|_o = \|E \otimes I_n\|_{tr}.$$ 

The reader is referred to the original articles [3, 62] by Alexei Kitaev *et al.* for more details. One of the appealing properties of this norm is its robustness: $\|E \otimes I\|_o = \|E\|_o$.

If $E$ is a completely positive, trace preserving transformation, then $\|E\|_o = 1$. 

**Euclidean induced superoperator norm:** We define a norm for a superoperator \( E : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \), by the maximization of the Euclidean norm for matrices:

\[
\|E\|_2 = \max_{A \neq 0} \frac{\|E(A)\|_2}{\|A\|_2}.
\]

It is straightforward to show that this norm is, like the diamond norm, robust:

\[
\|E \otimes I\|_2 = \|E\|_2, \quad \text{for the identity operator } I.
\]

By the bounds of Equation C.1, we have for any superoperator \( E : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \)

\[
\|E\|_2 \leq \sqrt{n} \|E\|_{tr} \quad \text{and} \quad \|E\|_{tr} \leq \sqrt{m} \|E\|_2.
\]

Because \( \|E \otimes I\|_2 = \|E\|_2 \), we thus obtain an upper bound on the diamond norm in terms of the trace norm:

\[
\|E\|_o = \|E \otimes I_n\|_{tr} \leq \sqrt{nm} \|E \otimes I_n\|_2 = \sqrt{nm} \|E\|_2 \leq n \sqrt{m} \|E\|_{tr},
\]

in combination with the trivial lower bound \( \|E\|_{tr} \leq \|E\|_o \).