Appendix C  

Norms and Distances

C.1 Norms and Distances on Vectors and Matrices

**absolute value** $|x|$: For a complex value $x \in \mathbb{C}$, its absolute value, or norm, is defined by $|x| = \sqrt{x x^*}$.

**Sum norm** $\|x\|_1$: For a complex valued vector $x \in \mathbb{C}^n$, the sum norm is defined by $\|x\|_1 = \sum |x_i|$. This norm is also called the $\ell_1$, or Manhattan norm. For bitvectors $x \in \{0, 1\}^n$ the sum norm corresponds with the Hamming weight of a bit string: $\|x\|_1 = \text{"number of ones in } x$".

**Euclidean, or $\ell_2$, vector norm** $\|x\|_2$: For a complex valued vector $x \in \mathbb{C}^n$, its norm is defined by $\|x\|_2 = \sqrt{\sum x_i x_i^*}$.

**Max, or $\ell_{\infty}$, norm** $\|x\|_{\infty}$: For a complex valued vector $x \in \mathbb{C}^n$, the max norm is defined by $\|x\|_{\infty} = \max_i |x_i|$.

**Fidelity**: The *fidelity* between two mixed states $\rho$ and $\sigma$ is defined by

$$F(\rho, \sigma) = \text{tr} \left( \sqrt{\sqrt{\rho} \cdot \sigma \cdot \sqrt{\rho}} \right),$$

although the reader should be warned that some authors use the square of this value.

**Euclidean matrix norm** $\|A\|_2$: For a complex valued matrix $A \in M_n(\mathbb{C})$, the Euclidean norm is defined by

$$\|A\|_2 = \sqrt{\sum_{ij} A_{ij} A_{ij}^*} = \sqrt{\text{tr}(A \cdot A^*)}.$$

Alternative names are: $\ell_2$, *Frobenius*, *Hilbert-Schmidt*, or *Schur norm*. 

89
We call this norm unitarily invariant because \(\|U \cdot A \cdot V\|_2 = \|A\|_2\) for unitary \(U, V \in U(n)\). From this invariance it follows, using the SV decomposition, that we have

\[
\|A\|_2 = \sqrt{\sum_i \sigma_i^2},
\]

with \(\sigma_i\) the singular values of \(A\), and hence for normal matrices

\[
\|A\|_2 = \sqrt{\sum_i |\lambda_i|^2},
\]

where \(\lambda_i\) are the eigenvalues of \(A\).

**Trace norm** \(\|A\|_{\text{tr}}\): For a matrix \(A \in M_n(\mathbb{C})\), the trace norm is defined by

\[
\|A\|_{\text{tr}} = \text{tr} \left( \sqrt{A \cdot A^*} \right) = \sum_i \sigma_i,
\]

with \(\sigma_i\) the singular values of \(A\). From this definition it follows that for normal matrices the trace norm equals

\[
\|A\|_{\text{tr}} = \sum_i |\lambda_i|,
\]

where \(\lambda_1, \lambda_2, \ldots\) are the eigenvalues of \(A\).

In the case of positive, semidefinite matrices we thus have \(\|A\|_{\text{tr}} = \text{tr}(A)\), hence the name of this norm. (As a consequence, all proper density matrices obey \(\|\rho\|_{\text{tr}} = 1\).)

The usefulness of this norm lies in the distance \(\|\rho - \sigma\|_{\text{tr}}\) it defines between two density matrices \(\rho\) and \(\sigma\). For any measurement setting \(\mathcal{P} = \{P_i\}\) (with \(\sum_i P_i^* P_i = I\)), the total variation distance between \(\rho\) and \(\sigma\) is bounded from above by

\[
\|\rho - \sigma\|_{\text{tr}} \geq \sum_{P_i \in \mathcal{P}} |\text{Prob}(\rho = P_i) - \text{Prob}(\sigma = P_i)|,
\]

with \(\text{Prob}(\rho = P_i) = (F(P_i, \rho))^2\). If we choose the projectors \(P_i\) of \(\mathcal{P}\) to be the eigenvectors of \(\rho - \sigma\), then we obtain the above bound, hence

\[
\|\rho - \sigma\|_{\text{tr}} = \max_P \left( \sum_{P_i \in \mathcal{P}} |\text{Prob}(\rho = P_i) - \text{Prob}(\sigma = P_i)| \right).
\]
Both the Euclidean and the trace norm are *matrix norms* because they obey the following properties (see Chapter 5 in [54] for much more on this topic):

1. nonnegative: \( \|A\| \geq 0 \)
2. positive: \( \|A\| = 0 \) if and only if \( A = 0 \)
3. homogeneous: \( \|\alpha A\| = |\alpha| \cdot \|A\| \) for all \( \alpha \in \mathbb{C} \)
4. triangle inequality: \( \|A + B\| \leq \|A\| + \|B\| \)
5. submultiplicative: \( \|AB\| \leq \|A\| \cdot \|B\| \).

In addition, for the tensor product between two matrices, we also have the equality

- \( \|A \otimes B\| = \|A\| \cdot \|B\| \).

A very useful relation between the trace and the Euclidean norm is easily shown by the inequalities \( \frac{1}{\sqrt{n}} \sum \sigma_i \leq \sqrt{\sum \sigma_i^2} \leq \sum \sigma_i \) for any \( n \) nonnegative values \( \sigma_1, \ldots, \sigma_n \). If we take these \( \sigma_i \) to be the singular values of \( A \), we see that

\[
\|A\|_2 \leq \|A\|_{\text{tr}} \leq \sqrt{n} \cdot \|A\|_2,
\]

for all \( A \in M_n(\mathbb{C}) \).

### C.2 Norms on Superoperators

**Trace induced superoperator norm:** For a superoperator \( E : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) we can use the trace norm to define

\[
\|E\|_{\text{tr}} = \max_{A \neq 0} \frac{\|E(A)\|_{\text{tr}}}{\|A\|_{\text{tr}}}.
\]

If \( E \) is a positive, trace preserving mapping, then \( \|E\|_{\text{tr}} = 1 \). A drawback of this norm is that it can increase if we tensor \( E \) with the identity operator. Take for example the one qubit transpose, with \( T(A) = A^T \), which has \( \|T\|_{\text{tr}} = 1 \), but also \( \|T \otimes I_2\|_{\text{tr}} = 2 \).

**Diamond superoperator norm:** Let \( E : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) be a linear superoperator, the *diamond norm* can then be defined by

\[
\|E\|_o = \|E \otimes I_n\|_{\text{tr}}.
\]

The reader is referred to the original articles [3, 62] by Alexei Kitaev *et al.* for more details. One of the appealing properties of this norm is its robustness: \( \|E \otimes I\|_o = \|E\|_o \).

If \( E \) is a completely positive, trace preserving transformation, then \( \|E\|_o = 1 \).
**Euclidean induced superoperator norm:** We define a norm for a superoperator \( E : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \), by the maximization of the Euclidean norm for matrices:

\[
\|E\|_2 = \max_{A \neq 0} \frac{\|E(A)\|_2}{\|A\|_2}.
\]

It is straightforward to show that this norm is, like the diamond norm, robust:

\[
\|E \otimes I\|_2 = \|E\|_2,
\]

for the identity operator \( I \).

By the bounds of Equation C.1, we have for any superoperator \( E : M_n(\mathbb{C}) \to M_m(\mathbb{C}) \)

\[
\|E\|_2 \leq \sqrt{n} \|E\|_{\text{tr}}, \quad \text{and} \quad \|E\|_{\text{tr}} \leq \sqrt{m} \|E\|_2.
\]

Because \( \|E \otimes I\|_2 = \|E\|_2 \), we thus obtain an upper bound on the diamond norm in terms of the trace norm:

\[
\|E\|_0 = \|E \otimes I_n\|_{\text{tr}} \leq \sqrt{nm} \|E \otimes I_n\|_2 = \sqrt{nm} \|E\|_2 \leq n\sqrt{m} \|E\|_{\text{tr}},
\]

in combination with the trivial lower bound \( \|E\|_{\text{tr}} \leq \|E\|_0 \).