A combinatorial generalization of the Springer correspondence for classical type

Slooten, K.J.

Citation for published version (APA):
CHAPTER 1

The graded and the affine Hecke algebra

This section is mainly preparatory. In section 1.1, we introduce the graded Hecke algebra, and mention some (basic) facts about its representation theory. In section 1.2, we do the same for the affine Hecke algebra. We mainly mention results that fit into our viewpoint of studying these algebras by deformation of their parameters, and only mention very briefly the geometric approach. We continue in section 1.3 to establish a link between certain representations of the graded, and the corresponding representations of the affine Hecke algebra; to do this we need to generalize results of Lusztig, who works in a slightly different context. After having thus presented the problems we wish to consider in this thesis, we proceed in section 1.4 by computing explicitly the example of the Hecke algebra of type $G_2$, in which already many of the features of the general case are present. We conclude this chapter by stating in section 1.5 the results we have obtained for the Hecke algebra of type $B_n$, which is our main object of study in this thesis.

1.1. The graded Hecke algebra

1.1.1. Definition. Let $(X, R_0, Y, R_0, \Pi)$ be a reduced root system. This means that $X$ and $Y$ are free finitely generated abelian groups with a perfect pairing $\langle \cdot, \cdot \rangle$ between them, $R_0 \subset X$ is the set of roots, $\hat{R}_0 \subset Y$ the set of coroots, and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots. For later convenience we denote $I = \{1, 2, \ldots, n\}$. The fact that the root system is reduced means that if $\alpha \in R_0$, then the only scalar multiples of $\alpha$ which are in $R_0$ are $\pm \alpha$. Furthermore we define the vector spaces $t^* = X \otimes \mathbb{C}$ and $t = Y \otimes \mathbb{C}$. Inside $t$, we have the distinguished real form $a = t_\mathbb{R} = Y \otimes \mathbb{R}$ and similarly for $a^*$. In particular, we can speak of $Re(\lambda)$ and $Im(\lambda)$ for elements $\lambda \in t$ or $t^*$.

We do not necessarily assume that $R_0$ generates $t$. Therefore we define

$$t^I = \{\xi \in t \mid \alpha_i(\xi) = 0 \text{ for all } i \in I\},$$

$$t^I^* = \{\lambda \in t^* \mid \lambda(\alpha_i) = 0 \text{ for all } i \in I\},$$

and similarly for $a^I$ and $a^I^*$. Then we define the fundamental weights $\lambda_i \in t^*$ by demanding $\lambda_i(\alpha_j) = \delta_{ij}$ and $\lambda_i|_{t^I} = 0$ for all $i, j \in I$. Likewise, we let the fundamental coweights be $\omega_i \in t$ defined by $\alpha_i(\omega_j) = \delta_{ij}$ and $\omega_i|_{t^I^*} = 0$. We denote by $Q \subset X$ (resp. $\hat{Q} \subset Y$) the root (resp. coroot) lattice $\oplus \mathbb{Z}\alpha$ (resp. $\oplus \mathbb{Z}\alpha^*$), and by $P$ (resp. $\hat{P}$) the weight (resp. coweight) lattice $\oplus \mathbb{Z}\lambda_i$ (resp. $\oplus \mathbb{Z}\omega_i$).

Let $W_0$ be the associated Weyl group, which is generated by the simple reflections $s_\alpha, \alpha \in \Pi$. We choose formal parameters, called root labels, $k_\alpha$ such that $k_\alpha = k_\beta$ if $\alpha$ and $\beta$ are conjugate under $W_0$.

We can now define the graded Hecke algebra. This algebra was introduced independently by Lusztig in [32] and by Drinfeld in [10].
Definition 1.1.1. For the root system \((X, R_0, Y, \tilde{R}_0, \Pi)\) and root labels \(k_{\alpha}\), the associated graded Hecke algebra \(\tilde{H}\) is the tensor product of algebras

\[
\tilde{H} = C[W_0] \otimes S[t^*] \otimes C[k_{\alpha}],
\]

subject to the cross relations that the \(k_{\alpha}\) are central, and that

\[
x \cdot s_{\alpha} - s_{\alpha} \cdot s_{\alpha}(x) = k_{\alpha}(x, \alpha)
\]

for all \(\alpha \in \Pi, x \in t^*\).

The word 'graded' in the definition reflects the fact that one may view the \(k_{\alpha}\) as having degree one, as well as the elements of \(t^*\). Notice that \(\tilde{H}\) does not depend on the choice of \(X\), but only on \(\alpha = X \otimes R\).

In this thesis we will usually study a specialization of \(\tilde{H}\), obtained by sending \(k_{\alpha} \mapsto k_{\alpha} \in \mathbb{R}_+\). Suppressing \(k_{\alpha}\) from the notation, we denote this algebra by \(H\), and (by abuse of terminology) still refer to it as a graded Hecke algebra. Suppose therefore that such a specialization has been made.

As a consequence of the cross relations, one shows by induction on \(\deg(p)\) that for \(p \in S[t^*]\):

\[
p \cdot s_{\alpha} - s_{\alpha} \cdot s_{\alpha}(p) = k_{\alpha} \Delta_{\alpha}(p),
\]

where \(\Delta_{\alpha} : S[t^*] \rightarrow S[t^*]\) is the BGG-operator given by

\[
\Delta_{\alpha}(p) = \frac{p - s_{\alpha}(p)}{\alpha}.
\]

It is known ([32]) that the center \(Z(H)\) of \(H\) is equal to \(S[t^*]^{W_0}\).

1.1.2. Representation theory. Let us now review some well-known facts concerning the representation theory of \(H\).

1.1.2.1. Weight spaces, central character. Let \(V\) be a \(H\)-module, and \(\lambda \in \mathfrak{t}\). Then the generalized weight space \(V_\lambda\) is

\[
V_\lambda = \{v \in V \mid \forall x \in t^*, (x - \lambda(x))^k v = 0 \text{ for some } k \in \mathbb{N}\}.
\]

If \(V\) is finite-dimensional, then the abelian algebra \(S[t^*]\) induces a weight space decomposition

\[
V = \bigoplus_{\lambda \in \mathfrak{t}} V_\lambda.
\]

On the other hand, if \(V\) is irreducible, then by Dixmier's version of Schur's lemma, the center of \(H\) acts by a character, so \(V\) is finite-dimensional since \(H\) is finitely generated over its center. Let \(\lambda \in \mathfrak{t}\) be such that for all \(p \in S[t^*]^{W_0}, p \cdot v = p(\lambda)v\). Since \(\lambda\) is only determined up to \(W_0\)-orbit, we call (an element of) \(W_0\lambda\) the central character of \(V\).

Remark that in our definition of \(H\), a coweight can be a weight of a representation. However, all choices so far have been forced upon us by the common notational conventions for the affine Hecke algebra. Basically, this amounts to choosing the root system of the dual \(L\)-group as roots, and the root system of the \(p\)-adic group as coroots (cf. 1.2.8 below). This choice goes back to Lusztig in [32].

This weight space decomposition opens the possibility of giving a Langlands classification of the irreducible representations of \(H\). For this we need the notion of temperedness, and of parabolic subalgebras of \(H\).
1.1.2.2. **Tempered representations.** First we define what it means for a finite-dimensional representation of $\mathbb{H}$ to be tempered. We define the positive Weyl chamber

$$t^*+ = \{ \lambda \in t^* \mid \alpha_i(\lambda) \geq 0 \text{ for all } i \in I \},$$

and its antidual

$$t^*_+ = \{ \xi \in t \mid \xi(\lambda) \leq 0 \text{ for all } \lambda \in t^{*+} \}.$$

Then we define

**Definition 1.1.2.** Let $V$ be a finite dimensional $\mathbb{H}$-module.

(i) $V$ is called tempered if every weight $\gamma$ of $V$ satisfies $Re(\gamma) \in t^*_+.$

(ii) $V$ is called a discrete series representation if, for every weight $\gamma$ of $V$, $Re(\gamma)$ lies in the interior of $t^*_+.$

Notice that if $V$ is a tempered $\mathbb{H}$-module and $\gamma$ is a weight of $V$, then $Re(\gamma)|_{a^{l^*}} = 0.$ This follows from $a^{l^*} \subset t^{*+},$ which implies that $Re(\gamma) \leq 0$ on $a^{l^*},$ but then automatically $Re(\gamma) = 0$ on $a^{l^*}.$

Furthermore, it follows analogously that $\mathbb{H}$ does not have discrete series representations unless $a^{l^*} = 0,$ i.e., unless rank $(X) = \text{rank}(Q).$

1.1.2.3. **(Standard) parabolic subalgebras.** For a subset $P \subset I,$ let $\Pi P \subset \Pi$ be the corresponding simple roots, let $R_P \subset R_0$ be the roots generated by $\Pi P,$ and $W_P$ the associated Weyl group. Then $(X, R_P, Y, R_P, \Pi P)$ is also a root system, and we denote its associated graded Hecke algebra by $\mathbb{H}_P,$ using the restriction of the parameter set $k_\alpha.$ We define

$$t_P = \{ \xi \in t \mid \alpha_i(\xi) = 0 \text{ for } i \in P \} \text{ and } t_P = t_{P\perp}.$$

We have to be slightly more careful with the corresponding lattices, in order to preserve a perfect pairing. We put

$$Y_P = Y \cap \mathbb{R}R_P \text{ and } X_P = X / (X \cap Y_P^\perp),$$

notice that we then find, as we would like,

$$t_P = Y_P \otimes \mathbb{C}.$$

We then have again a perfect pairing between the lattices $X_P$ and $Y_P,$ and $\text{rank}(X_P) = \text{rank}(R_P).$ We may identify $R_P$ with its image in $X_P.$

If we then put $\mathbb{H}_P$ to be the graded Hecke algebra associated to the root system $(X_P, R_P, Y_P, \tilde{R}_P, \Pi P),$ we can write $\mathbb{H}_P = \mathbb{H}_P \otimes S[t^{P*}].$ Since $S[t^{P*}]$ is in the center of $\mathbb{H}_P,$ every irreducible representation $V$ of $\mathbb{H}_P$ can be decomposed as $V = V_P \otimes C_\nu,$ where $V_P$ is an irreducible representation of $\mathbb{H}_P$ and $C_\nu$ is the one-dimensional representation of $S[t^{P*}]$ affording the character $\nu.$ We also define

$$t^{P+} = \{ \nu \in t^P \mid Re(\nu)(\alpha_i) > 0, \forall i \in I - P \}.$$  

1.1.2.4. **Langlands classification.** These definitions now enable us to state the following theorem, the Langlands classification.

**Theorem 1.1.3.** (11) (i) Let $V$ be an irreducible $\mathbb{H}$-module. Then $V$ is a quotient of $\mathbb{H} \otimes_{\mathbb{H}_P} U,$ where $U = U_P \otimes C_\nu$ is such that $U_P$ is an irreducible tempered $\mathbb{H}_P$-module and $\nu \in t^{P+}.$

(ii) If $U$ is as in (i), then $\mathbb{H} \otimes_{\mathbb{H}_P} U$ has a unique irreducible quotient, which we denote by $J(P, U).$

(iii) If $J(P, U_P \otimes C_\nu) \cong J(P', U_P' \otimes C_{\nu'}),$ then $P = P', U_P \cong U_P',$ as $\mathbb{H}_P$-modules, and $\nu = \nu'.$
The proof of this theorem also indicates how to find $P$ for a given irreducible representation $V$ of $\mathbb{H}$. If $\lambda$ is a weight of $V$ with maximal real part in the dominance ordering, then by a lemma of Langlands it is known that there exists a unique $P \subset I$ for which

$$Re(\lambda) \in \left\{ \sum_{j \notin P} c_j \omega_j - \sum_{i \in P} d_i \alpha_i \mid c_j > 0, d_j \geq 0 \right\}.$$ 

It is this $P$ that occurs in the Langlands data of $V$.

By this theorem the classification of irreducible representations of graded Hecke algebras can be done by classifying the tempered representations. However, this classification is in general unknown. It depends heavily on the chosen parameters $k_\alpha$. One of the aims of this thesis is to gain insight for this problem for the classical series.

### 1.1.2.5 Principal series.

For any $\nu \in \mathfrak{t}$, we call the representation $M(\nu) = \mathbb{H} \otimes_{S[\epsilon]} C_{\nu}$ the (minimal) principal series representation of $\mathbb{H}$ with central character $\nu$. As a representation of $\mathbb{C}[W_0]$, it is just the regular representation. It is not hard to show that any irreducible representation of $\mathbb{H}$ is the quotient of some principal series representation. It follows that an irreducible representation is of dimension $\leq |W_0|$. Also, by [7] it is known that, although they need not be isomorphic, $M(\nu)$ and $M(\nu \psi)$ have the same composition factors for any $\nu \in W_0$. To check when a minimal principal series representation is tempered, we observe that it has weights $\nu \psi, \psi \in W_0$, after which it follows easily from the definition that $M(\nu)$ is tempered iff $Re(\nu) = 0$. Generically a principal series representation is irreducible. The precise condition is the analogue of Kato's criterion (cf. [24]) for affine Hecke algebras:

**Theorem 1.1.4.** $M(\nu)$ is irreducible if and only if $\nu(\alpha) \neq \pm k_\alpha \forall \alpha \in R$.

The hard part of this theorem is the case where $\nu$ is not regular. For a proof, see [7].

### 1.1.2.6 Calibration graph.

Suppose $V$ is a finite-dimensional $\mathbb{H}$-module, and let $\gamma \in \mathfrak{t}$ be such that $V_\gamma \neq 0$. Then we can define operators $\tau_i : V_\gamma \to V_{s_i \gamma}$, for all $i$ such that $\gamma(s_i) \neq 0$:

$$\tau_i : V_\gamma \ni v \mapsto (s_i - \frac{k_i}{\alpha_i})v \in V_{s_i \gamma}.$$ 

Notice that in general $\tau_i$ does not extend to $V$. However, it is easy to see that if $\gamma(\alpha_i) \neq \pm k_i$, then both $\tau_i : V_\gamma \to V_{s_i \gamma}$ and $\tau_i : V_{s_i \gamma} \to V_\gamma$ are invertible, which implies that $\dim(V_\gamma) = \dim(V_{s_i \gamma})$.

This leads (cf. [49]) to the definition of a graph $\Gamma(\gamma)$, that is called the calibration graph in [47]. We take as vertices the elements of $W_0 \gamma$, and place an edge between $w \gamma$ and $s_i w \gamma$ if $w \gamma(\alpha_i) \neq \pm k_i$. Then, for any finite-dimensional $\mathbb{H}$-module $V$, $\dim(V_{w \gamma})$ is constant on connected components of $\Gamma(\gamma)$.

If $\gamma$ is regular, all $\dim(V_{w \gamma}) \in \{0, 1\}$ in an irreducible $V$: its weights form a connected component of the calibration graph. On the other hand, for singular $\gamma$, the following ([27], Lemma 2.7) will be useful in computations later on:

**Lemma 1.1.5.** Let $V$ be an $\mathbb{H}$-module such that $V_\gamma \neq 0$. Let $w \in W_0$ be such that for all $\alpha > 0$ for which $\alpha w \gamma < 0$, we have $\gamma(\alpha) \notin \{0, \pm k_\alpha\}$. Suppose $\gamma(\alpha_i) = 0$. Then

(i) $\dim(V_{w \gamma}) \geq 2$.

(ii) If $V_{s_i w \gamma} = 0$ then $\langle w^{-1} \alpha_i, \alpha_i \rangle = 0$ and $(w \gamma)(\alpha_j) = \pm k_j$. 

1.1.2.7. Residual subspaces. In our approach towards the classification of the tempered representations of the graded Hecke algebra, we use certain affine subspaces $L \subset \mathfrak{t}$. We will give the precise statement below and first give the definition. Let $L \subset \mathfrak{t}$ be an affine subspace. Then we define a parabolic root subsystem $R_L$ of $R_0$ as

$$R_L = \{ \alpha \in R_0 \mid \alpha(L) = \text{constant} \}.$$ 

Then we call $L$ a residual subspace if and only if

$$\# \{ \alpha \in R_L \mid \alpha(L) = k_{\alpha} \} = \# \{ \alpha \in R_L \mid \alpha(L) = 0 \} + \text{codim}(L).$$

In particular, $\mathfrak{t}$ itself is residual. A residual point is sometimes also called a distinguished point. It is clear that the notion of residual subspace is $W_0$-invariant, since $w(R_L) = R_{wL}$. If $L$ is a residual subspace, then if we define $t_L = \text{span}(R_L)$, we have $L = c_L + t_L$, where $c_L = L \cap t_L$, and $t_L = t_L^L$. Notice that if $R_L$ is a standard parabolic root system in $R_0$, we retrieve the notions defined in 1.1.2.3 for $R_P = R_L$. We call $c_L$ the center of $L$. The determination of the residual subspaces boils down to the classification of residual points and is completely described in [14]. By Lemma 7.10 of [46] and induction on the rank of $R_0$, it is not hard to see that there are only finitely many residual subspaces. The importance of the residual subspaces lies in the following fact, that we will prove below using the affine Hecke algebra. For a residual subspace $L = c_L + t_L$, let $L^{\text{temp}} = c_L + i a L$ be the corresponding tempered form of $L$, which we call a tempered residual subspace. Then we will show later in this introduction, using theorems from [46], [9] and [32], that

**Theorem 1.1.6.** The collection $\bigcup L^{\text{temp}}$ of all tempered residual subspaces of $\mathbb{H}$ is equal to the set of central characters of irreducible tempered representations of $\mathbb{H}$.

**Remark 1.1.7.** In [37], a classification of the tempered and discrete series representations of a certain class of graded Hecke algebras is obtained. The labels $k_{\alpha}$ are defined in terms of a connected reductive algebraic group $G$, and belong to a discrete set of possibilities. Lusztig's geometric approach results in a classification in terms of certain geometric data. Our more elementary approach is based on deformation of the parameters, a point of view which is in some sense complementary to Lusztig's geometric approach, and sheds a different light on the classification problem in the case of a Hecke algebra with "unequal labels". It would be interesting to compare these two approaches, in particular to check if Lusztig's data classifying the irreducible tempered representations of the graded Hecke algebra can be translated into the data developed in this thesis, which lead to the conjectures stated in 6.5.3.

### 1.2. The affine Hecke algebra

The graded Hecke algebra can be viewed as an infinitesimal version of the affine Hecke algebra, if one so wishes, much like the relation between Lie groups and Lie algebras. Important results of Lusztig ([32]) reduce questions about the representation theory of the affine Hecke algebra to the simpler case of the graded Hecke algebra. Below, we will discuss a variation of such a reduction, that plays an important role in this thesis. However, the affine Hecke algebra is in many ways a more natural object than its graded version, and often offers a guideline for definitions and results. For example, the affine Hecke algebra comes equipped with a natural trace $\tau$, which was studied in [46] and [45]. It gives rise to a natural Hilbert algebra structure. The canonical central decomposition of $\tau$ defines a Plancherel measure $\mu_{\text{Pl}}$ on the irreducible spectrum. In [46], the projection onto the spectrum of the center was expressed in terms of
tempered residual cosets, which are the analogue of the tempered residual subspaces defined in 1.1.2.7. In addition, in [9] it was shown that the tempered irreducible spectrum of the affine Hecke algebra is precisely the support of the Plancherel measure $\mu_{P_{L}}$. This provides a natural interpretation of the notions of residual subspace, temperedness and discrete series for the graded Hecke algebra.

Let us now discuss the affine Hecke algebra and some of the above mentioned issues in more detail.

1.2.1. Definition. Consider again the reduced root system $(X, R_0, Y, \hat{R}_0, \Pi)$. Form the affine Weyl group $W = W_0 \ltimes X$. The affine roots are by definition the elements of $R = \hat{R}_0 \times \mathbb{Z} \subset Y \times \mathbb{Z}$. If $a = (\alpha, k)$ is an affine root, we denote by $a + n$ the affine root $(\alpha, k + n)$. An affine root $a = (\alpha, k) \in R$ defines an affine reflection $s_a \in W$ by

$$s_a(x) = x - a(x)\alpha = x - ((x, \alpha) + k)\alpha.$$  

The simple affine roots are then by definition

$$\Pi^{\text{aff}} = \{ (\alpha, 0) \mid \alpha \in \Pi \} \cup \{ (\tilde{\alpha}, 1) \mid \tilde{\alpha} \in S^m \}$$

where $S^m$ is the set of minimal coroots in $\hat{R}$, with respect to the dominance ordering on $Y$. Then the group $W^{\text{aff}}$ generated by the reflections in $S^a = \{ s_a \mid a \in \Pi^{\text{aff}} \}$, is a normal subgroup of $W$. It is in fact a Coxeter group on these generators. Although $Q$ need not have a complementary subgroup in $X$, there is one for $W^{\text{aff}}$ in $W$: it is well known that one has the decomposition $W = W^{\text{aff}} \ltimes \Omega$, where $\Omega \simeq W/W^{\text{aff}} \simeq X/Q$.

As in the graded case, we define a set of formal parameters, also called root labels, $q_a$ for every affine root, such that $q_{w\alpha} = q_\alpha$ for all $w \in W$. This implies that for $a = (\alpha, k)$, $q_\alpha = q_{(\alpha, 0)} =: q_\alpha$, except when $\alpha \in 2Y$, in which case $q_\alpha = q_{(\alpha, k \text{ mod } 2)}$. Let $l$ be the length function on $W$. We sometimes write these labels as a function on $W$, being the length-multiplicative extension of $q_{(s_a)} = q_{a+1}, s \in S^a$, i.e., $q(ww') = q(w)q(w')$ if $l(ww') = l(w) + l(w')$.

The affine Hecke algebra $\hat{H}(W, q)$ is then the complex associative unital algebra with generators $T_w, w \in W$ such that:

$$\begin{cases}
\text{If } l(ww') = l(w) + l(w') \text{ then } T_w T_{w'} = T_{ww'}.
\text{For } s \in S^a, (T_s + 1)(T_s - q(s)) = 0.
\end{cases}$$  

As in the graded case, we will work only with a specialization of $\hat{H}$, obtained by sending $q_{\hat{a}} \mapsto q_\alpha \in \mathbb{R}_{>1}$. Once such a specialization is made, we denote the specialized algebra by $\mathcal{H}$ by suppressing notation. Let us assume from now on that this has indeed been done.

The analogy with the graded Hecke algebra is exhibited by the Bernstein decomposition ([32]):

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{A}$$

where $\mathcal{H}_0$ is the Hecke algebra of the finite Weyl group $W_0$, and $\mathcal{A}$ is isomorphic to the group algebra $\mathbb{C}[X]$. $\mathcal{A}$ has a basis $\theta_x, x \in X$. The cross relations are given by the Bernstein–Zelevinsky–Lusztig relations:

$$(1.3) \quad \theta_x T_s - T_s \theta_x = \begin{cases}
(q_\alpha - 1) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}} & \text{if } \alpha \notin 2Y \\
((q_\alpha - 1) + \theta_{-\alpha}(q_{\alpha/2, 1/2} \cdot q_{\alpha/2, 1/2} - q_{\alpha/2, 1/2} \cdot q_{\alpha/2, 1/2})) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}} & \text{if } \alpha \in 2Y
\end{cases}$$

for a simple reflection $s = s_\alpha \in S$ and $x \in X$. 

1.2.2. Representation theory. We now review some basic facts concerning the representation theory of $\mathcal{H}$. This treatment is analogous to what we have mentioned for $\mathbb{H}$. In particular, the center of $\mathcal{H}$ is equal to $\mathbb{Z} = A W_0$ (unpublished result of Bernstein, see [32, Prop. 3.11]), and since $\mathcal{H}$ is finitely generated over $\mathbb{Z}$, an irreducible representation of $\mathcal{H}$ must be finite-dimensional, since the action of $\mathbb{Z}$ is then scalar. Let $T = \text{Hom}(X, \mathbb{C}^*) = Y \otimes \mathbb{Z} \mathbb{C}^*$. Then $\text{Spec}(\mathbb{Z}) \cong W_0 \setminus T$. Let $(\pi, V_\pi) \in \hat{\mathcal{H}}$ be an irreducible representation with character $\chi_\pi$. We denote by $W_0 t_\pi \in W_0 \setminus T$ the character of $\mathbb{Z}$ such that $\chi_\pi(x) = \dim(\pi) x(t_\pi)$, and call it, or any of the elements of $W_0 t_\pi$, the central character of $(\pi, V_\pi)$. As in the graded case, a finite-dimensional $\mathcal{H}$-module $V$ decomposes into (generalized) weight spaces $V_t = \{ v \in V \mid \forall \alpha \in \mathcal{A}, (a - a(t))k v = 0 \text{ for some } k \}$ according to the action of $\mathcal{A}$. We now define

\textbf{Definition 1.2.1.} Let $(\pi, V)$ be a finite dimensional representation of $\mathcal{H}$.

(i) We call $V$ a tempered representation if all $\mathcal{A}$-weights $t$ of $V$ satisfy $|t(x)| \leq 1$ for all $x \in X^+ = \{ x \in X \mid \alpha(x) = \langle x, \alpha \rangle \geq 0 \forall \alpha \in \Pi \}$.

(ii) We call $V$ a discrete series representation if all $\mathcal{A}$-weights of $V$ satisfy $|t(x)| < 1$ for all $x \in X^{++} = \{ x \in X \mid \alpha(x) = \langle x, \alpha \rangle > 0 \forall \alpha \in \Pi \}$.

1.2.3. Parabolic induction. Recall the root systems $(X, R_P, Y, \tilde{R}_P, \Pi_P)$ and $(X_P, R_P, Y_P, \tilde{R}_P, \Pi_P)$. We have associated tori $T_P = \text{Hom}(X_P, \mathbb{C}^*)$ and $T^P = (T^{W_P})^0$. We denote the associated affine Hecke algebras by $\mathcal{H}_P$ and $\mathcal{H}_P$, respectively. We now define parabolic induction. Let $t^P \in T^P$. We then have a map

$$\phi_{t^P} : \mathcal{H}_P \rightarrow \mathcal{H}_P : \begin{cases} T_w \mapsto T_w & \text{for } w \in W_P \\ \theta_x \mapsto t^P(x)\theta_x & \text{for } x \in X \end{cases}$$

In this formula, $\bar{x}$ denotes the image of $x$ under the projection $X \rightarrow X_P$. The map $\phi_{t^P}$ is a Hecke algebra homomorphism. Now suppose that $(\delta, V)$ is a representation of $\mathcal{H}_P$ with central character $W_{P \cap P}$. Then, for $t^P \in T^P$, we call

$$\text{Ind}_{\mathcal{H}_P}^{\mathcal{H}_P}(\delta \circ \phi_{t^P})$$

a parabolically induced representation.

1.2.4. The Plancherel formula. So far, we have seen how an irreducible representation of the graded Hecke algebra has Langlands parameters $(P, U_P \otimes \mathbb{C}_\nu)$, where $U_P$ is an irreducible tempered representation of the parabolic subalgebra $\mathbb{H}_P$. In this section, we will take a closer look at the tempered representations of the affine Hecke algebra. The classification, although still largely an open problem, of tempered representations also uses parabolic induction: it is shown that every irreducible tempered representation $V$ arises as a summand of the induction of a discrete series representation $U_P$ of a parabolic Hecke subalgebra $\mathcal{H}_P$. However, this description is, compared to the Langlands classification, incomplete in the sense that there is in general not a unique tempered summand in such an induced representation, so several tempered representations may correspond to the same data. On the other hand, the induction data $(P, U_P)$ of a tempered representation are unique (up to conjugacy by $W_0$). Also, a classification of the discrete series representations is unknown, apart from the classification of their central characters. These are precisely the residual points, which we will define below.
One of the main themes in this thesis, is the conjecture that the combinatorics we will develop, leads to a classification of the discrete series representations with real central character, for the root system of type B, and arbitrary root labels $q_\alpha > 1$.

Let us now describe the results in [46] in more detail. First of all, we make the Hecke algebra into a *-algebra, with the anti-linear anti-involution $T_w^* = T_w^{-1}$. Then we define the trace functional $\tau$ by $\tau(T_w) = \delta_{w,e}$, and the associated hermitian inner product

\[(h_1, h_2) := \tau(h_1^* h_2).\]

The aim is then to decompose $\tau$ as an integral where we integrate the irreducible characters of $H$ against a certain measure on the space of irreducible representations of $\mathcal{C}$, where $\mathcal{C}$ is the associated $C^*$-algebra. It is by definition the closure of $H$ in the operator norm $||h||_o := ||\lambda(h)|| = ||\rho(h)||$, $\lambda(h)$ resp. $\rho(h)$ being the left resp. right multiplication operator on $H$ by $h \in H$.

Then the decomposition of $\tau$ is

\[(1.5) \quad \tau(h) = \int_{\mathcal{C}} \chi_{\pi}(h) d\mu_{PL}(\pi).\]

Notice that only irreducible representations of $H$ that extend to $\mathcal{C}$ appear in this decomposition. Moreover, $\text{supp}(\mu_{PL}) = \mathcal{C}$. One of the results of [9] is that the irreducible representations of $H$ that indeed extend to $\mathcal{C}$ are precisely the irreducible tempered representations.

Let $p_z : \mathcal{C} \to W_0 \backslash T$ denote the projection of an irreducible representation of $\mathcal{C}$ onto its central character. Then (1.5) can be rewritten, using $p_z$, to the integral (cf. [46, 3.23]):

\[(1.6) \quad \tau(h) = \int_T \chi_t(h) d\nu(t).\]

In this formula, all $\chi_t$ are positive, central functionals satisfying $\chi_t(1) = 1$. The probability measure $\nu$ is the push-forward $p_{z,*}(\dim \cdot \mu_{PL})$.

It turns out ([46, Theorem 3.29]) that the support of $\nu$ is equal to the union of certain compact cosets in $T$, called tempered residual cosets. We will pause here to describe them in more detail.

### 1.2.5. Residual cosets

To describe which central characters arise as central character of an irreducible tempered representation of $H$, we need the notion of residual coset, analogous to the notion of residual subspace for the graded Hecke algebra.

**Definition.** We only treat the case where $q_\alpha = q_{\alpha+1}$ for all $\alpha \in R$; for the general case, see [46]. As we will explain below, this assumption does not restrict the generality, if we restrict ourselves to representations with real central character. Now, for a coset $L = \tau_L T^L$ of a subtorus $T^L \subset T$, we define

\[R_L = \{ \alpha \in R_0 \mid \alpha(T^L) = 1 \}, \quad W_L = W_0(R_L),\]

notice that $R_L \subset R_0$ is a parabolic root subsystem.

We then define

\[R^P_L = \{ \alpha \in R_L \mid \alpha(L) = q_\alpha \}\]

and

\[R^e_L = \{ \alpha \in R_L \mid \alpha(L) = 1 \}\]

If we put $i_L = |R^P_L| - |R^e_L|$, then the residual cosets are those $L$ for which

\[i_L = \text{codim}(L).\]
Let $L$ be a residual coset, and $T_L \subset T$ the subtorus whose Lie-algebra is $C \tilde{R}_L$ (the torus "perpendicular" to $T^L$). Then we can write $L = r_LT^L$, where $r_L \in L \cap T_L$. In this case, $r_L$ is a residual point in $T_L$, w.r.t. the root system $(X_L, R_L, Y_L, \tilde{R}_L, \Pi_L)$. The point $r_L$ is determined up to multiplication with elements of the finite group $T^L \cap T_L$ only. This means that in the polar decomposition $r_L = sLC_L \in T_{L,u}T_{L,rs}$ (the subscripts $u$ and $rs$ standing for 'unitary' and 'real split'), $c_L$ is independent of the choice of $r_L$. We say that a residual coset is real if $r_L$ can be chosen in $T_{L,rs}$.

Denote the collection of residual cosets by $\mathcal{L}$, and let, for a residual coset $L = r_LT^L$, the tempered form be $L^{temp} = r_LT^L_u$. Then we have the following

**Theorem 1.2.2.**

$$\bigcup_{L \in \mathcal{L}} L^{temp} = \{\text{central characters of irreducible tempered representations of } \mathcal{H}\}.$$ 

**Proof:** We have already mentioned that according to [46, Theorem 3.29], the support of the measure $\nu$ is equal to the union of the tempered residual cosets. Moreover, by [9], the set of central characters of tempered representations of $\mathcal{H}$ is equal to the support of $\nu$. 

### 1.2.5.2. Classification of residual cosets via residual subspaces.

We have already seen that, since for a residual coset $L = r_LT^L$, $r_L$ is a residual point in $T_L$, it suffices to classify the residual points in all parabolic algebras, if one wants to classify all residual cosets. This last problem however can be reduced to the same problem in a family of graded Hecke algebras. We still assume that there are no $\tilde{\alpha} \in 2\mathbb{Y}$.

Let $r \in T$ be a residual point with polar decomposition $r = sc$. Define

$$R_s := \{\alpha \in R_0 \mid \alpha(s) = 1\}.$$ 

$R_s$ is a sub root system of $R_0$ with $\text{rank}(R_s) = \text{rank}(R_0)$.

Then $\gamma = \log(c) \in \text{Lie}(T_{rs})$ is a residual point for the graded Hecke algebra associated to the root system $(X, R_s, Y, \tilde{R}_s, \Pi_s)$ (where $\Pi_s \subset R^+_0$), with root labels $k_\alpha = \log(q_{\tilde{\alpha}})$. Similarly the point $c$ itself is residual for the affine Hecke subalgebra of $\mathcal{H}$ attached to $(X, R_s, Y, \tilde{R}_s, \Pi_s)$.

Conversely, given $s \in T_u$ such that $R_s$ has maximal rank, and a residual point $\gamma \in \text{Lie}(T_{rs})$ w.r.t. root labels $k_\alpha$ as above, the point $r = sexp(\gamma) \in T$ is residual for $\mathcal{H}$. This gives a 1-1 correspondence between $W_0$-orbits of residual points in $T$ and the collection of pairs $(s, \gamma)$ where $s$ ranges over the $W_0$-orbits of points in $T_u$ such that $R_s$ has rank equal to $\text{rank}(X)$, and $\gamma$ ranges over the $W_0(R_s)$-orbits of residual points for the graded Hecke algebra associated to $(X, R_s, Y, \Pi_s)$ with labels $k_\alpha$.

For real residual cosets, it follows that we have a bijection

$$\{\text{real residual cosets for } \mathcal{H}\} \longleftrightarrow \{\text{residual subspaces for } \mathbb{H}\}.$$ 

In fact, for a real residual coset $L \subset T$, $\log(r_LT^L_{rs}) \subset t$ is the real part $\gamma_L + n^L$ of a residual subspace for $\mathbb{H}$.

### 1.2.6. The Plancherel measure.

We now return to the discussion of the Plancherel measure, or rather of its push-forward $\nu$. Recall that the support of $\nu$ is the union of all tempered residual cosets. On such a coset, $\nu$ can be computed (almost) explicitly. Let $L$ be a residual coset and $d^L t$ the normalized Haar measure on $T^L_u$, transported to $L^{temp}$. Then the
measure $\nu_L$ on $L^{\text{temp}}$ equals $d\nu_L(t) = \lambda_L m_L(t) d^L(t)$, where $\lambda_L \in \mathbb{Q}$ is a constant and $m_L$ equals (in the case where there are no $\alpha \in 2Y$):

$$m_L(t) = q(w_0) \frac{\prod_{\alpha \in R_0} (1 - \alpha(t))}{\prod'_{\alpha \in R_0} (1 - q_\alpha \alpha(t))}.$$ 

The function $m_L$ can be proven ([46, Theorem 3.25]) to be smooth, which implies that $\nu_L$ is absolutely continuous w.r.t. the Haar measure on $T_u^L$.

We will need the following theorem below.

**Theorem 1.2.3.** (i) If $r \in T$ is a residual point, then there exists a non-empty and finite set $\Delta_r$ such that $\chi_r$ can be written as

$$\chi_r = \sum_{\delta \in \Delta_r} \chi_{r,\delta} d_{r,\delta},$$

where the $\chi_{r,\delta}$ are characters of irreducible discrete series representations with central character $r$, and $d_{r,\delta} \in \mathbb{R}_+$. (ii) Conversely, if $W_0r$ is the central character of an irreducible discrete series representation, then $r$ is a residual point.

**Proof:** (i) is Corollary 3.30 in [46]. (ii) follows from Theorem 3.29 in [46] and the smoothness of $m_L$. 

The $\chi_t$ in general are finite linear combinations of irreducible tempered characters, by [46, Prop. 4.20]. These irreducible tempered characters can generically be obtained by unitary parabolic induction from discrete series representations.

For $t \in T$, we define the parabolic root subsystem $R_t = R_0 \cap \text{Lie}(T_{rs})t$, with $\text{Lie}(T_{rs})$ the subspace spanned by the roots for which either $\alpha(t) = q_\alpha$, $\alpha(t) = 1$ or $\alpha(t) = -1$ and $\alpha \notin 2X$. For a parabolic root subsystem $R_P$, we say that $t$ is $R_P$-generic if $R_t \subset R_P$. Now let $L = r_L T_u^L$ be a residual coset for which $R_L \subset R_0$ is a standard parabolic root subsystem. Then we have the following theorem:

**Theorem 1.2.4.** ([46, Theorem 4.23]) Let $L$ be as above, and let $\Delta_L$ be the set of inequivalent discrete series representations of $\mathcal{H}_L$ with central character $r_L$, then $\chi_{L, r_L} = \sum_{\delta \in \Delta_L} \chi_{L, \delta} d_{L, \delta}$ (see Theorem 1.2.3 above). Choose $t^L \in T^L$ such that $t = r_L t^L$ is $R_L$-generic. Then

(i) the parabolically induced representations $\text{Ind}_{\mathcal{H}_L}^{L} (\delta \circ \phi_{t^L})$ are irreducible, unitary, tempered and mutually inequivalent.

(ii) We have

$$\chi_t = \frac{|W_L|}{|W_0|} \sum_{\delta \in \Delta_L} \chi_{L, \delta, t^L} d_{R_L, \delta},$$

where $\chi_{L, \delta, t^L}$ denotes the character of $\text{Ind}_{\mathcal{H}_L}^{L} (\delta \circ \phi_{t^L})$.

Since for any $P$, the set of $R_P$-generic points in $L^{\text{temp}}$ forms an open dense subset of $L^{\text{temp}}$, the smoothness of $m_L$ implies that the non-generic points do not play a role in the Plancherel formula. In its present state, the theory does not yet provide the means to study the reducibility behaviour of the representations (1.4) at these non-generic points. What seems to be needed is an analogue of Arthur's theory of analytical $R$-groups, cf. [1].
1.2.7. Special parameter values. From the above, it is clear that the classification of the discrete series representations of \( \mathcal{H} \), and therefore of all tempered representations, depends strongly on the choice of the parameters \( q_\alpha \). We can make a distinction however between two types of situations. Remember that \( \mathcal{H} \) denotes the generic affine Hecke algebra, where the \( q_\alpha \) are formal parameters. Then the set of "generically" residual points of \( \mathcal{H} \) defines a (finite) collection of parameter families \( \{ r_c^I(q_c) \mid c \in C_I \} \). Let us denote the corresponding collection for a parabolic algebra \( \mathcal{H}_P \) by \( C_P \). Then we call the parameters \( q_\alpha \in \mathbb{R}_{>1} \) generic if, for all \( P \subset I \), the evaluation map \( r_c^P(q_\alpha) \mapsto r_c^P(q_\alpha) \) is a bijection, and in addition each of the \( r_c^P(q_\alpha) \) is a residual point for \( \mathcal{H}_P \). If the parameters \( q_\alpha \) are not generic, we call them special.

**Example 1.2.5.** The simplest example which makes this distinction clear comes about when one considers the affine Hecke algebra of type \( A_1 \), although not among the representations with real central character. Let the positive root of this root system be \( \alpha \). Then the affine root system has generators \( s_{(-\alpha,1)} \) and \( s_{(\hat{\alpha},0)} \) which we denote by \( s_0 \) and \( s_1 \). Therefore, if \( X = Q \), \( s_0 \) and \( s_1 \) are not conjugate under the affine Weyl group, and we choose labels \( q_0, q_1 \) for the reflections \( s_0 \) and \( s_1 \). Then one can check easily from the general definition of residual cosets (see [46]) that \( \mathcal{H} \) has two discrete series representations, having central characters \( r_1, r_2 \) satisfying \( \alpha(r_1) = -\sqrt{q_0/q_1} \) and \( \alpha(r_2) = \sqrt{q_0/q_1} \). It is now clear that when \( q_0 = q_1 \), \( r_1 \) is no longer the central character of a discrete series representation. One checks also that for \( q_0 = q_1 \), the unitary minimal principal series representation with central character \(-1\) becomes reducible. Therefore we call parameters \( q_0, q_1 \) generic if \( q_0 \neq q_1 \) and special if \( q_0 = q_1 \). Of course, if we consider more generally parameters \( q_\alpha \in \mathbb{R}_{>0} \), we also find \( q_0 = q_1^{-1} \) as special parameters; these two cases can then even occur simultaneously if \( q_0 = q_1 = 1 \), in which case the Hecke algebra reduces to the group algebra \( \mathbb{C}[W] \).

1.2.8. Unipotent classes versus residual cosets in the equal label case. Let \( G \) be a split semisimple algebraic group of adjoint type defined over \( F \), where \( F \) is a \( p \)-adic field. Suppose that \( G \) has root system \( (\tilde{R}_0, \tilde{Q}, R_0, P, \Pi) \). If \( I \) is an Iwahori subgroup of \( G = G(F) \), then the centralizer algebra \( End_1(1^G_I) \) is isomorphic to the affine Hecke algebra \( \mathcal{H}(W,q) \) where \( q \) is the cardinality of the residue class field of \( F \), and \( W = W_0 \times P \). In this situation, Kazhdan and Lusztig [26] have given a complete classification of the irreducible representations of \( \mathcal{H} \), and one can therefore calculate the relation between their parameters and the residual cosets; see [46], Appendix. Here, we only describe the connection between the residual cosets for \( \mathcal{H} \) and the unipotent conjugacy classes in the Langlands dual group \( G \) of \( G \), which is the simply-connected complex group with root system \((R_0, P, \tilde{R}_0, \tilde{Q}, \Pi)\). We can view \( T = \text{Hom}(P, \mathbb{C}^*) \) as a maximal torus in \( G \). Let us denote the Lie algebra of \( G \) by \( \mathfrak{g} \). Then the relation between distinguished unipotent classes of \( G \) and residual points of \( T \) is as follows. Suppose \( r = sc \) is a residual point in \( T \), chosen in its orbit under \( W_0 \) such that \( \gamma = \log(c) \) is dominant. Then the algebra \( C_\gamma(s) \) is semisimple (not just reductive) and \( 2\gamma/k \) \( (k = \log(q)) \) is the weighted Dynkin diagram of a distinguished nilpotent orbit of \( C_\gamma(s) \). By this we mean that the vector \( \frac{2}{k}(\alpha_1(\gamma), \ldots, \alpha_n(\gamma)) \) contains the labels of the weighted Dynkin diagram. For the groups that concern us, we will make this explicit in sections 2.2 and 2.3.

The converse construction gives us a residual point in \( T \), see [46], Appendix. In general we obtain a bijection

\[
\{ W_0 - \text{orbits of real residual cosets of } \mathcal{H} \} \leftrightarrow \{ \text{unipotent classes in } G \}.
\]
We will later consider only the case where the root system $R_0$ is of type $B_n$, and we have labels $q_1$ for the long roots and $q_2$ for the short roots. Then the $W_0$-orbits of real residual cosets of $\mathcal{H}$ in case $q_1 = q_2$ correspond bijectively to unipotent conjugacy classes in $G = Spin_{2n+1}(\mathbb{C})$. Since the classification of unipotent classes does not depend on the isogeny class, we can also choose $G = SO_{2n+1}(\mathbb{C})$, the group of adjoint type. We will call this case “the $B_n$-group case”.

In the same way, the real residual cosets of $\mathcal{H}$ in the case $q_1 = q_2^2$, correspond bijectively to unipotent conjugacy classes in $Spin_{2n}(\mathbb{C})$. We will call this case “the $C_n$-group case”.

This is because in the graded Hecke algebra with root system of type $B_n$, choosing $k_1 = k_2$ produces the equal label graded Hecke algebra for type $B_n$, whereas choosing $k_1 = 2k_2$ produces the equal label graded Hecke algebra for type $C_n$.

These two cases are special parameter choices for the affine Hecke algebra of type $B_n$. In general, for root labels $q_\alpha > 1$, there are $2(n - 1)$ special parameter half-lines $q_1 = q_2^n$. If one considers all $q_\alpha \in \mathbb{R}_{>0}$, there are $4n - 2$ special lines, intersecting in the case $q_1 = q_2 = 1$, where the Hecke algebra reduces to the group algebra $\mathbb{C}[W]$.

For a general root system, the “equal label case” is the best-studied case of affine Hecke algebras. This case was the first to be considered, and furthermore it can be studied using geometry arising from the Langlands dual group $G$. However these methods are not easy to generalize to the general case where such a group is no longer around. See [38] for an account of the general theory, as a generalization of the equal label case.

### 1.2.9. Discrete series with regular central character.

In this section we prove that if a residual point is regular, there is only one discrete series representation of $\mathcal{H}$ with central character $r$. First remark that the set of weights of any representation with central character $r$ forms a union of components of the calibration graph $\Gamma(r)$ of $r$ (the obvious affine analog of 1.1.2.6).

However, the weights are also grouped together in another manner (cf. [46]). Let us review this here. For a root $\alpha$, we define $L_\alpha = \{ t \in T \mid \alpha(t) = q_\alpha \}$. This is a codimension-$1$ residual coset, which we can write as $L_\alpha = c_\alpha T^\alpha$. We define its real part as $L_{\alpha,rs} = c_\alpha T_{rs}^\alpha$. We use these to define the following configurations in $T_{rs}$:

**Definition 1.2.6.** Let $r = sc \in T_0 T_{rs}$ be residual. Then we denote by $\mathcal{L}^r$ the configuration 

\[ \{ cT_{rs}^\alpha \mid r \in L_\alpha \} . \]

$\mathcal{L}^r$ divides $T_{rs}$ into $2^n$ open chambers. For a chamber $C$, let its antidual be the open chamber

\[ C^{ad} = \{ t = c \cdot \exp(v) \in T_{rs} \mid (v,w) < 0 \text{ for all } w \in \log(c^{-1}C) \} . \]

We then find a collection of $2^n$ open chambers as antidual chambers to $\mathcal{L}^r$.

Now fix a point $t_0 \in T_{rs}$ which lies far into the negative chamber, i.e., $\alpha(t_0) < q_\alpha^{-1}$ for all $\alpha \in R_0^+$. It is clear that $t_0$ does not lie on any of the cosets defining $\mathcal{L}^r$, hence $t_0$ lies in a well-defined chamber, denoted by $C^r(t_0)$.

The point of these chambers is the following proposition:

**Proposition 1.2.7.** [46, Prop. 3.12] Let $r$ be a residual point and $V^r$ a discrete series representation with central character $r$. If $e \notin C^{wr}(t_0)^{ad}$, then the weight space $V^r_{wr} = 0$.

We will now see what the relation is between the calibration graph $\Gamma(r)$ and the chambers induced by $\mathcal{L}^r$. Remember that $r$ is regular.
**Lemma 1.2.8.** Let $\alpha$ be a simple root. Then $\alpha(\omega r) = q_\alpha^{\pm 1}$ if and only if $w^{-1}(t_0)$ and $w^{-1}s_\alpha(t_0)$ lie in different chambers of $\mathcal{L}^r$.

**Proof:** $\Rightarrow$: Suppose $\alpha(\omega r) = q_\alpha^{\pm 1}$. Then, since the coroot labels are $W_0$-invariant, $w^{-1}\alpha(r) = q_\alpha^{\pm 1}$ and so one of $L_{\pm w^{-1}\alpha, rs}$ is a wall of the arrangement $\mathcal{L}^r$. But since $w^{-1}\alpha(w^{-1}t_0) = \alpha(t_0)$ and $w^{-1}\alpha(w^{-1}s_\alpha t_0) = \alpha(t_0)^{-1}$, we see that indeed the two points $w^{-1}(t_0)$ and $w^{-1}s_\alpha(t_0)$ lie on different sides of the walls $L_{\pm w^{-1}\alpha, rs}$.

$\Leftarrow$: If $w^{-1}(t_0)$ and $w^{-1}s_\alpha(t_0)$ lie in different chambers of $\mathcal{L}^r$, then $t_0$ and $s_\alpha(t_0)$ lie in different chambers of $\mathcal{L}^{\omega r}$. The only positive root whose sign on $\log(t_0)$ differs from that on $\log(s_\alpha t_0)$ is $\alpha$, so $L_{\pm \alpha, rs}$ must be a wall of $\mathcal{L}^{\omega r}$, hence $\alpha(\omega r) = q_\alpha^{\pm 1}$.

But this implies the two notions are the same:

**Lemma 1.2.9.** Let $w, w' \in W_0$. Then $w^{-1}(t_0)$, $w'^{-1}(t_0)$ are in the same chamber of $\mathcal{L}^r$ if and only if $w, w'$ are in the same connected component of $\Gamma(r)$.

**Proof:** $\Rightarrow$: This follows from Lemma 1.2.8, which can be reformulated as: $w(r)$, $s_\alpha w(r)$ lie in the same component of $\Gamma(r)$ if and only if $w^{-1}(t_0)$, $w^{-1}s_\alpha(t_0)$ lie in the same chamber of $\mathcal{L}^r$. We thus need to check that if $w^{-1}t_0, w'^{-1}t_0$ lie in the same chamber of $\mathcal{L}^r$, there is a chain inside this chamber of the form $w^{-1}t_0, w^{-1}s_{i_1}t_0, \ldots, w^{-1}s_{i_k} \ldots s_{i_1}t_0 = w'^{-1}t_0$, where all $s_{i_j}$ are simple. This can be seen as follows: suppose $w^{-1}t_0$ and $w'^{-1}t_0$ lie in adjacent $W_0$-conjugates of $D = \{x \in T_{rs} | \alpha_i(x) < 1 \forall i\}$ (since $\alpha_i(t_0) << q_\alpha^{-1}$, we can shift the chambers in $\mathcal{L}^r$ to have center 1), then $w'^{-1}(t_0) = s_\alpha w^{-1}(t_0)$ for a certain root $\alpha$. Since $D$ is bounded by hyperplanes corresponding to simple roots, $w(\alpha)$ is simple, say $w(\alpha) = \alpha_i$. Then $w'^{-1}(t_0) = s_\alpha w^{-1}(t_0) = w^{-1}w_\alpha w^{-1}(t_0) = w^{-1}s_i(t_0)$. Since $w' = s_i w$, Lemma 1.2.8 implies that indeed $w r$ and $w' r$ lie in the same component of $\Gamma(r)$. This argument immediately generalizes to any two $t_0$-conjugates in the same $\mathcal{L}^r$-chamber.

$\Leftarrow$: This also follows from the previous Lemma. For this direction, we need to check that for two elements $w r, w' r$ in the same component of $\Gamma(r)$, we can find a sequence $w r, s_{i_1} w r, \ldots, s_{i_k} \ldots s_{i_1} w r = w' r$, with all the $s_{i_j}$ simple and the whole sequence inside the same component of $\Gamma(r)$. We use that $r$ is regular, as it then follows immediately from the definition of $\Gamma(r)$.

**Remark 1.2.10.** For non-regular points, the Lemma is false. See Remark 1.4.2 for a counter-example.

**Corollary 1.2.11.** If $r$ is a regular distinguished point, there is a unique irreducible discrete series representation $V^r$ of $\mathcal{H}$ with central character $r$.

**Proof:** By [46, 3.31], there exists a discrete series representation $V^r$ with central character $r$. Take $r$ in its orbit such that $V^r_r \neq 0$. Denote the component of $\Gamma(r)$ containing $r$ by $\Gamma$, and the set of weights of $V^r$ by $Wt(V^r)$. Then we have $\Gamma \subset Wt(V^r)$ since we know that $\dim(V^r_{\omega r})$ is constant when $w r$ ranges over a component of $\Gamma(r)$. On the other hand, $Wt(V^r) \subset \{w r \mid e \in C_{\omega r}(t_0)^d\} = \{w r \mid e \in C^r(w^{-1}(t_0))^d\} = \{w r \mid w^{-1}(t_0) \text{ lies in the same } \mathcal{L}^r - \text{chamber as } t_0\} = X$. By the previous Lemma, $X = \Gamma$. Hence any irreducible discrete series representation $U$ of $\mathcal{H}$ with central character $r$ has the same set of weight spaces, which are all one-dimensional. This implies easily that $U \cong V^r$. 

\qed
1.3. Lusztig’s reduction theorems.

In [32], Lusztig establishes the connection between the representation theory of the affine and graded Hecke algebra. Since these reductions play an essential role in this thesis, we will explain them in some detail. It will be necessary to slightly adapt his constructions. Let \( \mathcal{H} \) have root labels \( q_\alpha > 1 \). Then we consider the graded Hecke algebra with labels \( k_\alpha \) determined by \( q_\alpha \). In our situation, we take (cf. [46, (7.8)]):

\[
k_\alpha = \begin{cases} 
\log(q_\alpha) & \text{if } \alpha \notin 2Y \\
\log(q_{\alpha/2}^{1/2}) & \text{if } \alpha \in 2Y 
\end{cases}
\]

Now fix a real central character \( W_0 t \) of an irreducible representation of \( \mathcal{H} \). Denote the corresponding maximal ideal of \( \mathbb{Z} \) by \( I_t \), and let \( \hat{Z} \) be the \( I_t \)-adic completion of \( \mathbb{Z} \). Furthermore we put

\[
\hat{\mathcal{H}} = \mathcal{H} \otimes \hat{Z}.
\]

Denote the set of irreducible representations of \( \mathcal{H} \) with central character \( W_0 t \) by \( \text{Irr}_t(\mathcal{H}) \). Then

\[
(1.10) \quad \text{Irr}_t(\mathcal{H}) \longleftrightarrow \text{Irr}(\hat{\mathcal{H}}),
\]

which follows from the fact that

\[
\mathcal{H}/I_t \mathcal{H} \cong \hat{\mathcal{H}}/I_t \hat{\mathcal{H}},
\]

where \( I_t \) denotes the maximal ideal of \( \hat{Z} \) corresponding to \( I_t \).

Analogously, let \( W_0 \gamma \) be a \( W_0 \)-orbit in \( t \), then after carrying out the analogous constructions for \( \hat{\mathbb{H}} \) we have

\[
(1.11) \quad \text{Irr}_\gamma(\mathbb{H}) \longleftrightarrow \text{Irr}(\hat{\mathbb{H}}).
\]

Let \( \hat{\mathcal{H}}_{\text{rec}} \) denote the irreducible representation of \( \mathcal{H} \) with real central character, and analogously for \( \hat{\mathbb{H}}_{\text{rec}} \). We can now prove the following important fact:

**Theorem 1.3.1.** There exists a natural bijection

\[
\{ \text{irreducible representations in } \hat{\mathcal{H}}_{\text{rec}} \} \longleftrightarrow \{ \text{irreducible representations in } \hat{\mathbb{H}}_{\text{rec}} \}.
\]

**Proof:** This is basically Lusztig’s second reduction theorem (Theorem 9.3) in [32]. We explain how to adapt his construction, since he works with the different assumptions that the root labels \( q_\alpha \) are of the form \( q_\alpha = q^{na} \) for some \( q \in \mathbb{C}^* \), and moreover that for all \( t' \in W_0 t \), one has \( \alpha(t') \in < q > \) (the group in \( \mathbb{C}^* \) generated by \( q \)) if \( \alpha \notin 2Y \), and \( \alpha(t') \in \pm q > \) if \( \alpha \in 2Y \).

Fix a real central character \( W_0 t \), i.e., \( t \in T_r s \). Since the exponential map, restricted to \( a \), yields a \( W_0 \)-equivariant isomorphism \( \exp : a \rightarrow T_r s \), it gives rise to a bijection \( W_0 \gamma \rightarrow W_0 t \), where \( \gamma \in a \) is such that \( \exp(\gamma) = t \). In Lusztig’s notation, this means that \( t_0 = 1 \). Our assumption that all \( q_\alpha > 1 \) then implies that Lemma 9.5 still holds. Hence, Theorem 9.3 still holds: the algebras \( \hat{Z} \) and \( \hat{Z}(\mathbb{H}) \) are isomorphic, and moreover

\[
\hat{\mathcal{H}} \cong \hat{\mathbb{H}}.
\]

This isomorphism is compatible with the \( \hat{Z} \cong \hat{Z}(\mathbb{H}) \)-structures. We therefore find a natural bijection between the irreducible representations of \( \hat{\mathcal{H}} \) and the irreducible representations of
Combining with (1.10), (1.11) and the fact that this holds for any real central character $W_0t$, we find the desired result.

It is clear from (the proof of) this theorem, that if the $H$-module $V$ corresponds to the $H$-module $U$ under this bijection, that $V$ is a tempered (resp. discrete series) representation if and only if $U$ is a tempered (resp. discrete series) representation.

**Remark 1.3.2.** In general, the irreducible representations of $H$ with central character $W_0$ correspond even bijectively to the irreducible representations of a 'smaller' graded Hecke algebra, but twisted by a finite group that acts on it via diagram automorphisms. This follows from Lusztig’s first reduction theorem (Theorem 8.6 in [32]): to $t$ one can associate a subalgebra $H_c$ and a finite group $\Gamma(c)$ such that $H$ is Morita-equivalent to $\hat{H}_c[\Gamma(c)]$. Combined with the second reduction theorem this means that the classification of all irreducible representations of $H$ can be reduced to the same problem for the collection of $\mathbb{H}(R', k'_\alpha)[\Gamma]$, where $R' \subset R$ is a sub root system, the $k'_\alpha$ depend on $q_\alpha$ as explained in [46], and $\Gamma$ is a finite group acting via diagram automorphisms on $\mathbb{H}(R', k'_\alpha)$.

**Remark 1.3.3.** In fact, 1.3.1 exhibits a natural equivalence between the category of $H$-modules with fixed real central character $W_0$, and the category of $H$-modules with fixed real central character $W_0 \gamma$, where $\gamma \in \alpha$, i.e., $V$ has real central character.

As a consequence, when we restrict ourselves to the study of $\mathcal{H}_{\text{red}}$, we can work in the graded Hecke algebra with labels $k_\alpha$, where the $k_\alpha$ depend on the $q_\alpha$ as in (1.9). In view of this fact, we will mostly work in the context of the graded Hecke algebra in the remainder of this thesis.

Let us now turn to the proof of Theorem 1.1.6. First we prove a Lemma:

**Lemma 1.3.4.** Let $V$ be an irreducible discrete series representation of $\mathbb{H}$ with central character $\gamma$. Then $\gamma \in \alpha$, i.e., $V$ has real central character.

**Sketch of Proof:** We apply the analogue of Lusztig's first reduction theorem ([32, Theorem 8.6]) to $\mathbb{H}$ (instead of $\mathbb{H}$). We adapt his definition of the root subsystem $R_\alpha$, and use the parabolic root system $R_\gamma$ of [46, 4.15] instead. Explicitly, $R_\gamma = R_0 \cap a_\gamma^*$, where $a_\gamma^*$ is the $\mathbb{R}$-span of $\{\alpha \in R_0 \mid \alpha(\gamma) \in \{0, \pm k_\alpha\}\}$. We choose $\gamma$ in its $W_0$-orbit such that $R_\gamma = R_\gamma$, a standard parabolic subsystem and put $c = W_P \gamma$. Let $\Gamma \subset W_0$ be the group $\{w \in W_0 \mid w(c) = c \text{ and } w(R_0^+) = R_0^+\}$, and $n = |W_0 \gamma|/|W_\gamma|$. Then, according to [32] (see also [46, Theorem 4.10]),

$$\mathbb{H} \cong (\mathbb{H}^P[\Gamma])_n,$$

where $\mathbb{H}$ is the completed algebra as in (1.11), and the right hand side denotes the algebra of $n \times n$-matrices with entries in $\mathbb{H}^P[\Gamma]$. The upshot is that $V = \text{Ind}_{\mathbb{H}^P[\Gamma]}^H U$, where $U$ is an irreducible $\mathbb{H}^P[\Gamma]$-module. Furthermore, by Lusztig’s explicit description of the isomorphism 1.12, we obtain the following formula for the set $Wt(V)$ of weights of $V$:

$$Wt(V) = \bigcup_{w \in W^P} w \cdot Wt(U),$$

where $W^P$ denotes the set of coset representatives of $W_0/W_P$ of minimal length.

Suppose that $\gamma \notin \alpha$, then $R_\gamma(= R_P) \neq R_0$. The assumption that $V$ is a discrete series module implies that $\gamma \in Wt(U)$ satisfies $Re(\gamma) \in \sum_{i \in I} \mathbb{R}_{<0} \alpha_i$. But for $w^P$ (the longest
element of $W^P$) we then find that $w^P \cdot \gamma \notin \sum_{i \in I} \mathbb{R} < 0 \alpha_i$ (since $w^P(I - P) \subset R^+_0$); while on the other hand by (1.13), $w^P \cdot \gamma \in Wt(V)$. This is a contradiction, so the Lemma follows. □

**Proof of Theorem 1.1.6:** Let $L^\text{temp} = \gamma_L + i\alpha_L$ be a tempered residual subspace of $H$, and let $\gamma_L + i\alpha \in L^\text{temp}$. Then $r_L = \exp(\gamma_L)$ is a residual point for $H_L$, and by [46] it is then the central character of a non-empty set $\Delta_L$ of discrete series representations of $H_L$. Since $r_L$ is real, we can invoke 1.3.1 to see that $\gamma_L$ is the central character of a set of discrete series representations of $H_L$, which is in natural bijection with $\Delta_L$. It follows that $\gamma_L + i\alpha$ is the central character of a tempered representation of $H_L$, and (by induction) also of $H$.

Conversely, let $V$ be an irreducible tempered representation of $H$ with central character $W_{\gamma_L}$. Let the set of weights of $V$ be $Wt(V)$. We then define, analogous to the analysis in [9] of (weak) constant terms of tempered representations of the affine Hecke algebra, the sets

$$E^0_P(V) = \{ \lambda \in Wt(V) \mid Re(\lambda) \in \mathbb{R} \hat{R}_P \},$$

where $P \subset I$. Now let $P$ be minimal such that $E^0_P(V) \neq \emptyset$. Then it is easy to see that

$$V^P = \sum_{\lambda \in E^0_P(V)} V_\lambda$$

is a non-zero tempered $H^P$-module, and minimality of $P$ implies that for all $\lambda \in Wt(V^P) = E^0_P(V)$, we have $\lambda \in \sum_{\alpha \in H_P} \mathbb{R} < 0 \alpha$. Now take an irreducible subquotient $U$ of $V^P$, then $Wt(U) \subset Wt(V^P)$, so $U$ is an irreducible, tempered $H_P = H_P \otimes S[t^P \ast]$-module, which implies that $U = U_P \otimes C_\nu$, where $U_P$ is an irreducible discrete series module of $H_P$. Since $U$ is a tempered $H^P$-module, $Re(\nu)|_{t^P \ast} = 0$, so $\nu = i\alpha_P$. By the above Lemma, we now know that $U_P$ has real central character $W_P \lambda$ where we can choose $\gamma$ in its $W_0$-orbit such that $\gamma = \lambda + \nu$. We can thus apply Lusztig’s theorem 1.3.1, combined with Theorem 1.2.3, to conclude that $\lambda$ is a residual point for $H_P$. Hence, $\gamma \in \lambda + i\alpha_P$, i.e., indeed $\gamma$ lies in a tempered residual subspace of $H$.

□

**1.4. Example: $H(G_2)$**

In this section, we will illustrate the preceding theory with the example of the root system of type $G_2$. Choose simple roots $\alpha_1, \alpha_2$ such that $\alpha_1$ is long. This means that $R_0^+ = \{ \alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2 \}$. Then the Dynkin diagram of $W_0$ is

$$\begin{array}{c}
\alpha_1 \\
\alpha_2
\end{array}$$

The affine Weyl group $W = W_0 \ltimes X$ is generated by the simple reflections $s_0, s_1, s_2$ in the affine roots $(\tilde{\theta}, 1) (\tilde{\theta} = -(2\alpha_2 + 3\alpha_1)$ being the lowest coroot), $(\tilde{\alpha_1}, 0), (\tilde{\alpha_2}, 0)$ respectively. $W$ operates as affine linear transformations of $X$. We then have the following Dynkin diagram for $W$:

$$\begin{array}{c}
a_0 \\
a_1 \\
a_2
\end{array}$$

because the taking of coroots has interchanged root lengths. On the other hand, the Weyl group $W_0 \ltimes Y$, which operates as affine linear transformations on $Y$, can be generated by the affine reflections $t_0, t_1, t_2$ in the affine roots $(\tilde{\theta'}, 1), (\tilde{\alpha_1}, 0), (\tilde{\alpha_2}, 0)$ where $\theta'$ is the negative of the highest root: $\theta' = -(2\alpha_1 + 3\alpha_2)$. This group has Dynkin diagram

$$\begin{array}{c}
b_0 \\
b_1 \\
b_2
\end{array}$$
1.4.1. Residual cosets. In this section we compute the residual cosets for $\mathcal{H}(G_2)$ for all parameter values. First we fix a set of root labels for the affine root system. This means choosing $q_1,q_2 \in \mathbb{R}_{>1}$ where $q_1 = q_{a_1} = q(\alpha_1,0)$ and $q_2 = q_{a_2} = q(\alpha_2,0)$. Since $\theta$ is conjugate to $\alpha_2$ under $W_0$, we get $q_{a_0} = q_2$.

The residual cosets are the cosets $r_i T^L \subseteq T$ for which $i_L = |R^p_{L_L}|-|R^u_{L_L}| = \text{codim}(L)$. This means that the classification of 1- and 2-dimensional residual cosets is easy: we have the torus $T$ itself, and all cosets of the form

$$L_\alpha = \{ t \in T \mid \alpha(t) = q_\alpha \}$$

These are residual, and of course there are no other 1-dimensional residual cosets, since $|R^p_L| > 1$ forces $\dim(L)=0$.

It remains to classify the residual points. As we have seen in 1.2.5.2, this means that we first locate the $s \in T_u$ for which $\text{rank}(R_s) = 2$. Let us first see which points $s \in T_u$ meet this requirement. In general, we have the isomorphism $T_u \cong Y_R/Y$, where on an element $s = [\sum_i r_i \alpha_i]$, a root $\alpha$ takes value $\alpha(s) = \exp(2\pi i \sum_i r_i (\alpha, \alpha_i))$. In this example, $Y = \hat{Q}$ since $Q = P$, hence $\hat{Q} = \hat{P}$ for the root system of type $G_2$.

Since we are interested in $W_0$-orbits, we need to look at $T_u/W_0 \cong Y_R/(W_0 \times Y)$, i.e. we need to look at the fundamental alcove for the action of the affine Weyl group $W_0 \times Y$ on $Y_R$. This fundamental alcove $D$ is given by the equations $D = \{ y \in Y_R \mid 0 \leq \alpha_1(y) \leq 1, 0 \leq \alpha_2(y) \leq 1, 0 \leq (2\alpha_1 + 3\alpha_2)(y) \leq 1 \}$. It is now clear that in order to produce an element $s \in T_u$ such that $R_s$ is a rank-2 root system, we need to take the points in the corners of the fundamental alcove. This gives us 3 representatives of $W_0$-orbits of points $s_1, s_2, s_3$ on which we have:

$$\begin{align*}
\{ \alpha_1(s_1) = 1 \}, & \{ \alpha_1(s_2) = -1 \}, & \{ \alpha_1(s_3) = 1 \\
\{ \alpha_2(s_1) = 1 \}, & \{ \alpha_2(s_2) = 1 \}, & \{ \alpha_2(s_3) = \exp(2\pi i/3) \}
\end{align*}$$

Remark 1.4.1. Notice that the order of $s_i$ is equal to $i$. This follows also from a theorem of Borel and de Siebenthal (cf. [4]): the determination of the $s_i$ is equivalent to the determination of the subgroups of a simple compact Lie group of type $G_2$ with maximal rank. Since the highest root is expressed in the simple roots as $2\alpha_1 + 3\alpha_2$, it follows from their theorem that the centre of the maximal subgroups corresponding to the deletion of $\alpha_1$ resp. $\alpha_2$ from the affine diagram, are cyclic groups of order 2 and 3 respectively. These centers are exactly the groups generated by $s_2$ and $s_3$.

By 1.2.5.2, the residual points in $T$ are all $W_0$-conjugates of points of the form $s_i \exp(\gamma)$, where $i \in \{1,2,3\}$ and $\gamma$ is a residual point for the graded Hecke algebra with root system $(X, R_{s_i}, Y, \hat{R}_{s_i}, \Pi_{s_i})$ and labels $k_i = \log(q_i)$.

- First we treat the point $s_1 = 1$. Since $R_{s_1} = R_0$, we are looking for residual points in $\mathbb{H} = \mathbb{C}[W_0] \otimes S[\text{Lie}(T)]$. The number of such points depends on the parameters $q_i$. Since $D = \{ t \in T \mid \alpha_i(t) \geq 0 \}$ is a fundamental domain for the action of $W_0$ and we are only looking for $W_0$-orbits, we will restrict attention to points in $D$. Remember that the labels of $\mathbb{H}$ are $k_i = \log(q_i)$. Let us now see how the classification of the residual points depends on the $k_i$. Since $\dim(\text{Lie}(T_{r_0})) = 2$, we can picture $\text{Lie}(T_{r_0}) = Y \otimes_{\mathbb{R}} \mathbb{R}$ as done below. The full lines denote the sets $L^\pm_\alpha = \{ t \in \text{Lie}(T_{r_0}) \mid \alpha(t) = \pm k_\alpha \}$ for $\alpha \in R^+_0$, and the dotted lines the sets $L^0_\alpha = \{ t \in \text{Lie}(T_{r_0}) \mid \alpha(t) = 0 \}$ for $\alpha \in R^+_0$. The vertical dotted line is the line $\{ \alpha_2(t) = 0 \}$; the dotted line on its right is the line $\{ \alpha_2(t) = k_2 \}$. Then the
residual points can simply be found by inspection: \( t \in \text{Lie}(T_{rs}) \) is residual if and only if 
\[ \# \{ \alpha \in R_0^+ \mid t \in L_\alpha \} - 2 \# \{ \alpha \in R_0^+ \mid t \in L_0 \alpha \} = 2. \]
In this way we can treat all parameter values. We then find that there are four special parameter values: \( \frac{k_2}{k_1} \in \{ 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3} \} \). At generic parameter values, there are three orbits of regular residual points \( W_{0\gamma_1}, W_{0\gamma_2}, W_{0\gamma_3} \) where we choose the \( \gamma_i \) in their orbit by \( \alpha_1(\gamma_1) = -k_1, \alpha_2(\gamma_1) = -k_2; \alpha_1(\gamma_2) = k_1, \alpha_2(\gamma_2) = -k_2; \alpha_1(\gamma_3) = k_1, \alpha_2(\gamma_3) = \frac{1}{2}(k_2 - k_1). \)
At the special values \( k_2 = k_1 \) and \( k_2 = \frac{1}{3}k_1 \), there is one regular residual orbit \( W_{0\gamma_1} \), and one singular residual orbit, which is the confluence of \( W_{0\gamma_2} \) and \( W_{0\gamma_3} \). At \( k_2 = \frac{2}{3}k_1 \) and \( k_2 = \frac{1}{2}k_1 \), only \( \gamma_1 \) and \( \gamma_3 \) remain residual, \( \gamma_2 \) is no longer. However, at these parameter values, \( \gamma_2 \) coincides with the center of a one-dimensional residual coset \( L_\alpha \). All the possible choices of parameters are depicted in the pictures \( a - i \).

\begin{itemize}
  \item For \( s = s_2 \), we find \( R_s^+ = \{ \alpha_2, 2\alpha_1 + 3\alpha_2 \} \) so \( R_s \) has type \( A_1 \times A_1 \). Therefore in the graded Hecke algebra we find only one orbit of residual points, namely the points in the orbit of \( \gamma_4 \) satisfying \( \alpha_2(\gamma_4) = k_1, (2\alpha_1 + 3\alpha_2)(\gamma_4) = k_1 \). This means that \( \alpha_1(\gamma_4) = \frac{1}{2}(k_1 - 3k_2) \) and \( \alpha_2(\gamma_4) = k_2 \). Observe that \( \gamma_4 \in W_{0\gamma_3} \).
  \item Finally we consider \( s = s_3 \). Then \( R_s \) consists of the long roots, hence has type \( A_2 \), and we obtain the residual point \( \gamma_5 \) with \( \alpha_1(\gamma_5) = k_1, \alpha_2(\gamma_5) = 0 \).
\end{itemize}

1.4.2. Comparison with literature for group case. In the group case \( q_1 = q_2 \), the orbits of real residual cosets are in bijection with the unipotent conjugacy classes in the group.
1.4. EXAMPLE: \( \mathcal{H}(G_2) \)

\[ e. q_1^{1/2} < q_2 < q_2^{2/3} \]

\[ f. \text{Confluence: } q_2 = q_1^{1/2} \]

\[ g. q_1^{1/3} < q_2 < q_1^{1/2} \]

\[ h. q_2 = q_1^{1/3} \]

\[ i. q_2 < q_1^{1/3} \]

\( G_C \) with root system of type \( G_2 \). On the level of residual points, this bijection is as follows (see [46]): If \( r = sc = \exp(\gamma) \) is a residual point, then \( 2\gamma/k \) is a distinguished Dynkin diagram of \( C'_g(s) \).

It is known that \( G_C \) has 5 unipotent classes, see e.g. [6, p. 410]. A common way to label a unipotent class \( C \) is by \( P_L \), if \( C \) corresponds to \( (L, P_L) \) by the Bala–Carter theorem (see section 2.2 below). Let us therefore choose a notation for the residual cosets such that in
1.4.2: Centers of residual cosets

<table>
<thead>
<tr>
<th>Name</th>
<th>Generically</th>
<th>group case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>(0,0)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$A_1$</td>
<td>(1,0)</td>
<td>(0,0)</td>
</tr>
<tr>
<td>$\tilde{A}_1$</td>
<td>(0,0)</td>
<td>(0,1)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$(2,0)$</td>
<td>$\zeta(0,0)$</td>
</tr>
<tr>
<td>$A_1 \times \tilde{A}_1$</td>
<td>$(2,0)$</td>
<td>$(-1,1)$</td>
</tr>
<tr>
<td>$G_2(b_1)$</td>
<td>$(2,0)$</td>
<td>$(-2,2)$</td>
</tr>
<tr>
<td>$G_2(a_1)$</td>
<td>$(2,0)$</td>
<td>$(-1,1)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$(2,0)$</td>
<td>(0,2)</td>
</tr>
</tbody>
</table>

the group case we recover this. For the residual points, it is clear that we have to define $c_1 = \exp(\gamma_1)$ as having type $G_2$, and since the orbits of $\gamma_2$ and $\gamma_3$ coincide for $k_1 = k_2$, let us say $c_2 = \exp(\gamma_2)$ has type $G_2(b_1)$ and $c_3 = \exp(\gamma_3)$ has type $G_2(a_1)$. The other residual points do not correspond to a unipotent class in $G_C$, let us say that $c_4 = s_2\exp(\gamma_4)$ has type $A_1 \times \tilde{A}_1$ and $c_5 = s_3\exp(\gamma_5)$ has type $A_2$. We use $\tilde{A}_1$ to denote a root subsystem of type $A_1$ in $R_0$, consisting of short roots.

Likewise, the one-dimensional residual cosets of type $L_{\pm \alpha} = \exp(\pm \alpha)$ are said to have type $A_1$ if $\alpha$ is long or $\tilde{A}_1$ if $\alpha$ is short. Finally, the torus $T$ itself is of type $\emptyset$. We now summarize the results in Table 1.4.2 as follows. For a residual coset $L = r_L T^L$, chosen in its orbit such that $|\alpha_i(L)| \geq 1$ if $q_1 > q_2$, the column "generically" contains the entry

$$\eta_1(x_{1,1}, x_{1,2}) \eta_2(x_{2,1}, x_{2,2})$$

if $\alpha_i(r_L) = \eta_i q_1^{1/2} x_{i,1} q_2^{1/2} x_{i,2}$, with $\eta_i \in S^1$.

The column "group case" contains the limit case $q_1 = q_2$. Notice that in this case $G_2(a_1) = G_2(b_1)$, and moreover the real parts of the two non-real residual points coincide with them as well.

1.4.3. Weights of discrete series for generic parameters. Now we use the results of 1.2.9 to obtain information on the discrete series representations. Since this section deals with regular central character only, we choose generic root labels. Consider the calibration graphs
\( \Gamma(r) \) of the residual points. For a point \( r \) of type \( G_2(b_1) \) with \( \alpha_1(r) = q_1, \alpha_2(r) = q_2^{-1}, \) \( \Gamma(r) \) has connected components \( \Gamma(r)_1 = \{ r \}, \Gamma(r)_2 = \{ s_1 s_1 s_2 r, s_1 s_2 s_1 r, s_2 s_1 s_2 r \}, \) \( \Gamma(r)_3 = w_0 \Gamma(r)_2, \) and \( \Gamma(r)_4 = \{ w_0 r \}. \) As usual, \( w_0 \) denotes the longest element in \( W_0. \) The component that contains the weights of the discrete series representation with central character \( r \) depends on the parameters \( q_1, q_2, \) and can be calculated or simply read off from the Figures. One may check that for \( q_2^{3/2} > q_1, \) it is \( \Gamma(r)_1, \) for \( q_2^3 > q_1 > q_2^{3/2} \), we get \( \Gamma(r)_2, \) and finally for \( q_1 > q_2^2 \) we get \( \Gamma(r)_4 = \{ w_0 r \}. \) For the point \( r \) of type \( G_2(a_1) \) with \( \alpha_1(r) = q_1, \alpha_2(r) = \sqrt{q_2/q_1}, \) the graph \( \Gamma(r) \) has components \( \{ r, s_2 r, s_1 s_2 r \}, \{ s_2 s_1 s_2 r, s_1 s_2 s_1 r, s_2 s_1 s_2 s_1 r, s_2 s_1 s_2 s_1 r, \} \), \( \{ s_1 s_2 s_1 r, s_2 s_1 r, s_1 r \}. \) The component graph for the point \( A_1 \times A_1 \) with \( \alpha_1(r) = q_1, \alpha_2(r) = -\sqrt{q_2/q_1} \) is the same one as for \( G_2(a_1), \) and the one \( r \) of type \( A_2 \) with \( \alpha_1(r) = q_1, \alpha_2(r) = \zeta, \) consists of \( \{ r, s_2 r \}, \{ s_1 s_2 r, s_1 s_2 s_1 r, s_2 s_1 s_2 s_1 r, \} \), \( \{ w_0 r, s_2 w_0 r \}, \{ s_1 r, s_2 s_1 r, s_1 s_2 r, s_2 s_1 s_2 r \}. \)

One checks easily that the following numbers of weights satisfy the chamber condition 4.7.2 for the various choices of parameters:

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>( k_2 )</th>
<th>( \frac{1}{3} )</th>
<th>( \frac{2}{3} )</th>
<th>( \frac{3}{3} )</th>
<th>( \frac{4}{3} )</th>
<th>( \frac{5}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( G_2(a_1) )</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( G_2(b_1) )</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( A_1 \times A_1 )</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

This means that indeed in all generic cases, the representations \( V^r \) are irreducible for all residual points \( r \), and the above table lists the dimensions of \( V^r \) for all \( r \) and all generic parameter values.

**Remark 1.4.2.** To see that Lemma 1.2.9 no longer holds for a non-regular point, take \( r \) to be of type \( G_2(a_1) \) and \( q_1 = q_2. \) Then we have \( s_1 r = s_1 s_2 r, \) so in the same component of \( \Gamma(r), \) but on the other hand \( s_1(t_0) \) and \( s_2 s_1(t_0) \) do not lie in the same chamber of \( L^r. \)

**1.4.4. Restriction to \( \mathcal{H}_0, Springer correspondence.** Knowing the weights of the discrete series characters of all parabolic subalgebras now enables us to compute the decomposition of the representations

\[ \text{Ind}_{\mathcal{H}_0}^{\mathcal{H}_L}(\delta \circ \phi_L) \]

when we restrict them to \( \mathcal{H}_0. \) This decomposition is independent of \( t^L \in T^L_u, \) we therefore assume \( t^L = 1. \)

The irreducible characters of \( \mathcal{H}_0 \) are in bijection with those of \( W_0; \) this bijection is determined by the condition that if \( \phi \) is an irreducible character of \( \mathcal{H}_0, \) then \( \psi \) defined by \( \psi(w) = \phi(T_w) \mid \sqrt{q(w^{-1})} \) is the corresponding irreducible character of \( W_0. \) A common way to label the irreducible characters of \( W_0 \) is (\( [6] \)), by the set \( \{ \phi_{1,0}, \phi_{2,1}, \phi_{2,2}, \phi_{1,3}^L, \phi_{1,3}^L, \phi_{1,6} \}. \) The first subscript equals the dimension of the corresponding representation, the second the lowest degree of the coinvariant algebra in which it occurs. These two numbers do not distinguish between the two one-dimensional representations which act as the trivial representation for roots of one length and as the sign representation for roots of the other length; the convention is to denote by \( \phi_{1,3}^L \) the representation which acts trivially on reflections about a short root. This means that by \( \phi_{1,3}^L, \) we denote the one-dimensional representation \( V \) such that \( T_1 v = -v, T_2 v = q_2 v. \) (A word of caution: in [6], \( \alpha_2 \) is the long root, but still \( A_1 \) denotes a system
1.4.4a: Springer correspondence for \( q_2 > q_1^{2/3} \)

<table>
<thead>
<tr>
<th>Center</th>
<th>Corresponding ( W_0 )-representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>( \phi_{1,0} + 2\phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi''</em>{1,3} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( \phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( A_1 \times A_1 )</td>
<td>( \phi_{2,2} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( G_2(b_1) )</td>
<td>( \phi_{1,3} )</td>
</tr>
<tr>
<td>( G_2(a_1) )</td>
<td>( \phi_{2,1} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \phi_{1,6} )</td>
</tr>
</tbody>
</table>

1.4.4b: Springer correspondence for \( q_1^{2/3} > q_2 > q_1^{1/2} \)

<table>
<thead>
<tr>
<th>Center</th>
<th>Corresponding ( W_0 )-representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>( \phi_{1,0} + 2\phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi''</em>{1,3} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( \phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( A_1 \times A_1 )</td>
<td>( \phi_{2,2} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( G_2(b_1) )</td>
<td>( \phi_{2,1} + \phi_{2,2} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( G_2(a_1) )</td>
<td>( \phi_{2,1} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \phi_{1,6} )</td>
</tr>
</tbody>
</table>

1.4.4c: Springer correspondence for \( q_1 > q_2^2 \)

<table>
<thead>
<tr>
<th>Center</th>
<th>Corresponding ( W_0 )-representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \emptyset )</td>
<td>( \phi_{1,0} + 2\phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi''</em>{1,3} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( \phi_{2,1} + 2\phi_{2,2} + \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( A_1 \times A_1 )</td>
<td>( \phi_{2,2} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( \phi'<em>{1,3} + \phi</em>{1,6} )</td>
</tr>
<tr>
<td>( G_2(b_1) )</td>
<td>( \phi_{1,3} )</td>
</tr>
<tr>
<td>( G_2(a_1) )</td>
<td>( \phi_{2,1} + \phi_{1,6} )</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( \phi_{1,6} )</td>
</tr>
</tbody>
</table>

of long roots, and \( \tilde{A}_1 \) one of short roots. This means that in the character table, p. 412, we need to interchange his \( \phi'_{1,3} \) and \( \phi''_{1,3} \), to obtain those in our setting. However, for the table of Springer correspondents, p. 427, this is not necessary, since this table is labeled in terms of root subsystems.)

We now claim that

**PROPOSITION 1.4.3.** For generic root labels, the \( \mathcal{H}_0 \)-restriction of the tempered series for each residual coset is as shown in Tables 1.4.4a-c.

**Proof:** We come to these tables as follows:
1.4. EXAMPLE: \( \mathcal{H}(G_2) \)

- For the coset \( T \) of type 0, this means simply that we restrict the (irreducible as \( \mathcal{H} \)-module) minimal principal series representation \( \text{Ind}_A^\mathcal{H}(\mathbb{C}_1) \), but this is of course the regular representation for \( \mathcal{H}_0 \).
- Consider a residual coset \( L \) of type \( A_1 \). We thus get as parabolic Hecke algebra \( \mathcal{H}^L = \mathbb{C}[[1, s_1]] \otimes A \). The Hecke algebra \( \mathcal{H}_L \) has only one discrete series representation, which is its Steinberg representation. Restricted to \( W_L \) this is just the sign representation \( s_1 \mapsto -1 \). By Frobenius reciprocity, the desired table entry thus consists of \( W_0 \)-representations having a vector that is sent to its negative by \( s_1 \). As \( \mathcal{H} \)-representation, this is a six-dimensional irreducible representation, as can be checked e.g. by using the component graph.
- The table entries for \( A_1 \) are found analogously.
- Now consider \( L \) of type \( A_1 \times A_1 \). The quotient algebra \( \mathcal{H}_L \) has only one residual point, which is the central character of its one-dimensional discrete series representation. Restricted to the Weyl group \( W_0(A_1) \times W_0(A_1) \), this representation sends \( s_1 \mapsto -1, s_2 \mapsto -1 \). Again Frobenius leads to the desired table entries.
- For a residual coset of type \( A_2 \), the table entry is computed analogously.

It now remains to treat the real residual points.

- Consider the one-dimensional discrete series representation with central character of type \( G_2 \). Only the weight \( r \) with \( \alpha_i(r) = q_\alpha^{-1} \) appears in this representation. Therefore we have \( \theta_{\alpha_i}(r) = q_\alpha^{-1} \), which implies that \( T_i \) acts as \( q_\alpha^{-1} \). Therefore, upon restriction to \( \mathcal{H}_0 \), we find \( \phi_{1,6} \).
- Next we compute the restriction of the three-dimensional discrete series representation of \( \mathcal{H} \) with central character of type \( G_2(a_1) \). One can check easily that the weights that occur in this representation are independent of the parameter values, hence so is the \( \mathcal{H}_0 \)-type of this representation. To do this, we try to compute the desired \( \mathcal{H}_0 \)-type explicitly. Let us denote by \( V \) the discrete series representation of \( \mathcal{H} \) with central character \( G_2(a_1) \), then \( V \) has weights \( r, s_1 r, s_1 s_2 r \), where \( r \) satisfies \( \alpha_1(r) = q_1^{-1}, \alpha_2(r) = \sqrt{q_2/q_1} \). We thus have

\[
V = V^r \oplus V^{s_2 r} \oplus V^{s_1 s_2 r}.
\]

The action of \( A \) is diagonal, and using this action and the cross relations (1.3) thus enables us to compute several entries of the matrices of the action of \( T_i = T_i^* \), on this basis. We omit the details of these calculations, of which the result is

\[
T_1 \to \begin{pmatrix}
-1 & 0 & 0 \\
0 & -\sqrt{q_1/(q_1-1)} & x_1 \\
0 & y_1 & \frac{q_2/(q_1-1)}{q_2/\sqrt{q_2-\sqrt{q_1}}} \\
\end{pmatrix}
\]

(1.15)

\[
T_2 \to \begin{pmatrix}
-\frac{(q_2-1)/\sqrt{q_1}}{\sqrt{q_2-\sqrt{q_1}}} & x_2 & 0 \\
y_2 & \frac{(q_2-1)/\sqrt{q_2}}{\sqrt{q_2-\sqrt{q_1}}} & 0 \\
0 & 1 & -1 \\
\end{pmatrix}
\]

(1.16)

where the \( x_i \) and \( y_i \) are still unknown. The quadratic relation \( T_i^2 = (q_i - 1)T_i + q_i \) then implies that \( x_i \) can be expressed in terms of \( y_i \), leaving us with only one unknown in (1.15) and (1.16). However, this knowledge is already sufficient to
compute the eigenvalues of $T_1$ and $T_2$. These turn out to be $-1$ with multiplicity two and $q_1$ with multiplicity one for both $T_i$. Comparing this to the possible eigenvalues of any three-dimensional representation of $\mathcal{H}_0(G_2)$, implies that $V|\mathcal{H}_0$ is isomorphic to either $\phi_{2,1} + \phi_{1,6}$, to $\phi_{2,2} + \phi_{1,6}$ or to $\phi_{1,6} + \phi_{1,3} + \phi_{1,3}$. However, in this last representation the actions of $T_1$ and $T_2$ commute, and one can check from (1.15) and (1.16) that they do not. By also computing the character of $T_1T_2$, we conclude that $V|\mathcal{H}_0 \cong \phi_{2,1} + \phi_{1,6}$. Notice also that the matrix entries of $T_2$ have, as is to be expected, singularities at $q_1 = q_2$.

- Finally, to compute the discrete series representation with central character of type $G_2(b_1)$, we have to distinguish between the different possible parameter values.

First let $q_2 > q_1^{2/3}$. Then the representation is one-dimensional, with the only occurring weight $r$ satisfying $\alpha_1(r) = q_1$ and $\alpha_2(r) = q_2^{-1}$. Since $\alpha_2$ is short, we find that the restriction to $\mathcal{H}_0$ of this representation produces $\phi_1''$. Similarly, if $q_1 > q_2^2$, we obtain $\phi_{1,3}'$. It remains to compute the decomposition of the five-dimensional discrete series representation with central character $r$ that occurs for $q_1^{1/2} > q_2 > q_1^{1/2}$. We do this in the same way as above for type $G_2(a_1)$. We then find $\phi_{2,1} + \phi_{2,2} + \phi_{1,6}$. Again, we find that the matrices representing the action of the $T_i$ contain singularities at $q_1 = q_2^2$ and $q_1^2 = q_2^3$. At these parameter values, the point $G_2(b_1)$ is no longer residual but coincides respectively with the center of the residual coset $\tilde{A}_1$.

1.4.4.1. At special parameter values. At generic parameter values, we have seen that all representations listed in the Tables 1.4.4a-c are irreducible as $\mathcal{H}$-modules. Let us now see what happens at special parameter values.

- First consider the group case $q_1 = q_2$. Then the situation is as in Table 1.4.4a, but the residual points of type $G_2(a_1)$ and $G_2(b_1)$ coincide. We now compute the discrete series representation with this central character. Therefore, let $r$ be such that $\alpha_1(r) = q^{-1}$, $\alpha_2(r) = 1$. The only weights that can appear in a discrete series representation with central character $r$ are $r$ and $s_1r$. Clearly, there is a one-dimensional discrete series representation with weight $s_1r$. On the other hand, suppose that $V$ is an irreducible $\mathcal{H}$-module with central character $r$ and $V_r \neq 0$. Thus $V$ is a subquotient of the minimal principal series module $M_r$ with central character $r$. In this module, all weight spaces are two-dimensional. Then, by the affine analog of Lemma 1.1.5, we see that $\dim(V_r) = 2, \dim(V_{s_1r}) = 1$. We therefore consider the induced module

$$M := \text{Ind}_{\mathcal{H}(A_1)}^{\mathcal{H}(A_1)}(C_r),$$

where $\mathcal{H}(A_1)$ is the parabolic subalgebra corresponding to the long root $\alpha_1$, and $C_r$ denotes the representation space on which $T_1$ acts as $-1$ and $\theta(x)$ as $x(r)$. The representation $M$ is six-dimensional, with weight spaces for $r, s_1r, s_2s_1r, s_1s_2s_1r, s_2s_1s_2s_1r$ and $s_1s_2s_1s_2s_1r$, the first being two-dimensional. By Frobenius reciprocity, we see that $V|\mathcal{H}_0 \cong \phi_{2,1} + \phi_{2,1} + \phi_{1,3}' + \phi_{1,6}$. Choosing a basis $v$ for $C_r$, we take as basis for $M$ the vectors $\{T_2T_1T_2T_1T_2v, T_1T_2T_1T_2v, T_2T_1T_2v, T_1T_2v, T_2v, v\}$. The relations (1.2) and (1.3) then allow us to compute the action of $\mathcal{H}$ on $M$ explicitly. We then find that $M$ has the following structure:

$$M^1 \subset M^2 \subset M,$$
where $M^1$ is a one-dimensional subrepresentation of $M$, affording $\phi'_{1,3}$, on the weight space $M_{s_1 s_2 s_3 r}$. $M^2$ is a three-dimensional subrepresentation of $M$, on the weight spaces $M_{s_1 s_2 s_3 r} \oplus M_{s_2 s_1 s_3 r} \oplus M_{s_3 s_1 s_2 r}$. The restriction to $\mathcal{H}_0$ of this representation is $\phi_{2,2} + \phi_{1,3}$. Notice that $M^1$ has a complement in $M^2$ for the action of $\mathcal{H}_0$ but not for $\mathcal{H}$; as $\mathcal{H}$-module we find that the quotient $M^2/M^1$ is an irreducible two-dimensional representation, whose restriction to $\mathcal{H}_0$ is equal to $\phi_{2,2}$. This is because the two linearly independent $w_1, w_2$ w.r.t which $\mathcal{H}_0$ acts as $\phi_{2,2}$, also have a component in $M_{s_1 s_2 s_3 r}$. Finally, $V' := M/M^2$ is a three-dimensional $\mathcal{H}$-module, with $\dim(V_{s_1 r}) = 2$ and $\dim(V_{s_2 r}) = 1$. Therefore we see that $V' = V$, and that $V_{\mathcal{H}_0} \cong \phi_{2,1} + \phi_{1,6}$. We thus obtain two discrete series representations with central character $r$, and their restrictions to $\mathcal{H}_0$ are precisely the representations we also find at regular parameter values close to the group case.

- At the special value $q_1 = q_1^3$, the computations are completely analogous.
- Now consider the case $q_2 = q_2^{1/3}$. As we have seen, at these parameter values, the point $G_2(b_1)$ is no longer residual, but coincides with the center of the one-dimensional coset of type $\tilde{A}_1$. We therefore investigate the tempered representations with central character $r$, where $r$ satisfies $\alpha_1(r) = q_2^{1/2} = q_1, \alpha_2(r) = q_2^{-1}$. Clearly we find a one-dimensional tempered representation whose restriction to $\mathcal{H}_0$ is $\phi_{1,3}''$. But on the other hand, suppose that $V$ is an irreducible $\mathcal{H}$-module for which $V_{s_1 r} \neq 0$. Then by using Lemma 1.1.5 we find that $\dim(V_{s_1 r}) = 2$ and $\dim(V_{s_2 r}) = 1$. Since $V$ is a constituent of the induced representation $\text{Ind}_{\mathcal{H}((\tilde{A}_1))}(C_r)$, this is an irreducible representation, whose restriction to $\mathcal{H}_0$ must then be equal to $\phi_{2,1} + \phi_{2,2} + \phi_{1,6}$.

- At the special value $q_1 = q_2^2$, the considerations are analogous and we conclude that $\mathcal{H}$ has two irreducible tempered representations with central character $r$, a one-dimensional one whose restriction to $\mathcal{H}_0$ is $\phi_{1,3}'$ and a five-dimensional one whose restriction to $\mathcal{H}_0$ is $\phi_{2,1} + \phi_{2,2} + \phi_{1,6}$.

### 1.4.5. Springer correspondence

Looking at the Tables 1.4.4a-c and the $\mathcal{H}_0$-decomposition of tempered representations at special parameter values, we observe that for any choice of parameters, we can partition $W_0$ into parts labeled by the real residual cosets, such that for each irreducible tempered $\mathcal{H}$-representation with real central character $r_L$, exactly one of the $W_0$-representations associated to $L$ occurs in its restriction to $\mathcal{H}_0$. The remarkable fact is that this can be done in a unique way. The reader is encouraged to verify this using the Tables 1.4.4a-c, in which we have written the correspondent of a real residual coset in boldface. In fact, we believe that this is not a coincidence, but holds for any root system, for any choice of parameters. Returning now to our example, note that for generic parameter values, this means we get a bijection: for every real residual coset $L$, there is one irreducible tempered representation with central character $r_L$, and the set of real residual cosets is in bijection with the irreducible representations of $W_0$. The computations done for special parameter values show that we can extend this map to all parameter values. For example, in the group case $q_1 = q_2$, the subregular residual point of type $G_2(a_1)$ has one correspondent for each discrete series representation with central character $G_2(a_1)$, and it follows from our analysis above that we may still define $\phi_{2,1}$ and $\phi_{1,3}'$ as Springer correspondents in this case. We will call our map the (generalized) Springer correspondence, since at the group case, we find (via the bijection between real residual cosets of $T$ and unipotent classes in the complex simple group
of type $G_2$, described in [46] and above) that the (just defined) Springer correspondents of $L$ are precisely the (classical) Springer correspondents of the unipotent class associated to $L$, as can be checked for example from the list in [6, p. 427], (but notice that one has to tensor all representations listed there with the sign representation $\phi_{1,6}$). Notice also the interaction between $G_2(b_1)$ and $\tilde{A}_1$ at $q_2 = q_1^{2/3}$ and between $G_2(b_1)$ and $A_1$ at $q_1 = q_2^2$: the Springer correspondents are exchanged, resulting in a walk of the correspondent $\phi_{2, 2}$ from $\tilde{A}_1$ via $G_2(b_1)$ to $A_1$.

As we have remarked above, this map is for all parameter values a bijection between

\[ \{ \text{irreducible tempered representations of } \mathcal{H} \text{ with real central character} \} \leftrightarrow \{ \hat{W}_0 \}. \]

Another way to say this, is that apparently at all parameter values, the characters of the restriction to $\mathcal{H}_0$ of the irreducible tempered representations of $\mathcal{H}$ with real central character form a basis for the class functions (traces) on $\mathcal{H}_0$.

It is this map that we will study in the main part of this thesis for the Hecke algebra with root system of type $B_n$.

1.4.6. **Plancherel measure.** As an aside, let us compute (up to a rational constant) the Plancherel measures of some of the discrete series representations. This measure is given by the formula (1.8). For example, for $r$ a residual point of type $G_2(a_1)$, we obtain:

\[
m_{\{r\}}(r) = \frac{q_1 q_2 (q_2 - 1)(q_1 - 1)}{(q_1 q_2 + q_1 q_2 + 1)(q_1 + \sqrt{q_1 q_2})(q_2 + 1)(q_1 + 1)}
\]

Notice that this is indeed finite and positive, for all choices of $q_1 > 1$. On the other hand, if we take $r$ to be a residual point of type $G_2(b_1)$, then we obtain

\[
m_{\{r\}}(r) = \frac{(q_2^2 - q_1)(q_1^2 - q_1^3)}{(q_2^2 + q_1 q_2 + q_1^2)(q_2^2 + q_2 + 1)(q_2 + 1)(q_1 + 1)}
\]

and we indeed see that this measure becomes zero at the special parameter values $q_1 = q_2^2$ and $q_1^3 = q_2^2$.

It is possible to compute the rational constants $\lambda_{\{r\}}$, and also to compute the Plancherel measures of all series of representations. However, since we will not go into this for type $B_n$, we also omit it in this example. See also [48] for a calculation of formal degrees of certain discrete series representations.

1.5. **Outline of the results for type $B_n$**

In this thesis we attempt to generalize to $B_n$ what we have done for $G_2$. We work in the graded Hecke algebra, but in view of Lusztig's theorem 1.3.1 our results can also be viewed as results in the affine Hecke algebra, both for $X = Q$ and $X = P$, see (1.9).

We have seen in the $G_2$-example that for special parameters two types of phenomena occur: either several generically different residual points coincide but remain residual, or a residual point ceases to be residual. In the latter case, at the special parameter value it coincides with the center of a higher-dimensional residual coset. However, the generalized Springer correspondence we have defined remains a bijection between the irreducible tempered representations of $\mathcal{H}$ with real central character and the irreducible representations of $\hat{W}_0$.

In type $B_n$, these two types of degeneracy remain essentially the same, although they might occur simultaneously. One of the first things we prove is (cf. Proposition 3.4.3):
PROPOSITION 1.5.1. Let $c$ be a generically residual point. Then, at all values of the parameters, $c$ is the center of a residual subspace.

Our next goal is to define the generalized Springer correspondence. To this end, we first introduce for any special value of the parameters, say $k_2 = qk_1$ where $k_1$ is the label of the long roots, an equivalence relation $\sim_q$ on $\check{W}_0$. We do this in terms of symbols that generalize the symbols defined by Lusztig. By 1.2.8, the following theorem then shows the possible existence of the correspondence we seek to define:

THEOREM 1.5.2. (cf. Theorems 3.8.17 and 3.8.18) Let $C$ be a distinguished unipotent class in $G$, where $G = SO_{2n+1}(\mathbb{C})$ or $G = Sp_{2n}(\mathbb{C})$. Suppose $C$ has a collection $\{\chi_1, \ldots, \chi_l\}$ of Springer correspondents. Let $c$ be the residual point in $T$ which corresponds to $C$. Then there is a natural bijection

$$\chi_i \leftrightarrow c_i,$$

where the set $\{c_i\}$ consists of the generically residual points which coincide into $c$ in the special parameter case $k_1 = k_2$ or $k_1 = 2k_2$.

This bijection is in terms of two operations on Young tableaux: a splitting procedure $S_q$ and a joining procedure $J_q$.

We proceed to define a set of Springer correspondents for a residual subspace $L$ for any special value $k_2 = qk_1$ of the parameters. We then generalize the preceding theorem into (cf. Theorem 4.3.5):

THEOREM 1.5.3. Let $k_2 = qk_1$ be special. Then we have a bijection

$$\{W_0 - \text{orbits of residual subspaces of } \mathbb{H}\} \leftrightarrow \check{W}_0 / \sim_q.$$

Furthermore, the set of Springer correspondents of $L$ exhausts exactly one equivalence class in $\check{W}_0 / \sim_q$. Let $\Sigma_q(L)$ be this equivalence class. Then we have a bijection between

$$\Sigma_q(L) \leftrightarrow \{L' \text{ generically residual subspace } | c_{L'} = c_L \text{ at } k_2 = qk_1\}.$$

We then show that indeed this general correspondence reduces to the classical one in the group cases, i.e., at the special values $k_1 = k_2$ and $k_1 = 2k_2$ we have (cf. Theorem 4.4.3):

THEOREM 1.5.4. Let $k_1 = k_2$ or $k_1 = 2k_2$, and let $L$ be a residual subspace, corresponding to the unipotent class $C$ in $G$. Let the (classical) Springer correspondents of $C$ be $\{\chi_1, \ldots, \chi_l\}$. Let $\Sigma_q(L)$ be the equivalence class in $\check{W}_0 / \sim_q$ associated to $L$ by our general correspondence. Then $\Sigma_q(L) = \{\chi_1, \ldots, \chi_l\}$.

To fully generalize the (combinatorial description of) the group cases, we then define a combinatorial analogue $U_q(n)$ of the unipotent classes. We then show that (cf. Theorem 5.2.9):

THEOREM 1.5.5. There are explicit bijections

$$U_q(n) \leftrightarrow \{W_0 - \text{orbits of residual subspaces of } \mathbb{H}\},$$

and therefore also between

$$U_q(n) \leftrightarrow \check{W}_0 / \sim_q.$$

This concludes the combinatorial part of this thesis. We then switch back to the affine Hecke algebra $\mathcal{H}$ and state conjectures regarding the meaning of this correspondence for the representation theory of $\mathcal{H}$. We do not repeat the full statement here (see Conjecture 6.5.3) but
mention only that we expect to find the full $\mathcal{H}_0$-type decomposition of any tempered representation of $\mathcal{H}$ with real central character. Furthermore, we conjecture an explicit bijection between irreducible tempered representations of $\mathcal{H}$ with real central character, and $\tilde{W}_0$. The conjectures require the use of generalized Green functions. This generalization has been obtained recently by Shoji, using certain symbols defined by Malle and himself. We would like to express our thanks to Gunter Malle for computing them for several low rank cases, thereby allowing us to check the validity of these conjectures up to $B_3$ in full, and partially for $B_4$. 