A combinatorial generalization of the Springer correspondence for classical type
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Citation for published version (APA):
Slooten, K. J. (2003). A combinatorial generalization of the Springer correspondence for classical type
The classical Springer correspondence

2.1. Definition

The Springer correspondence is a map from the unipotent classes of a connected reductive algebraic group $G$ to the irreducible characters of its Weyl group $W_0$. This map is obtained by defining an action of $W_0$ on the cohomology of the varieties $B_u$ of all Borel subgroups containing a given unipotent element $u$ in $G$. The existence of this action is remarkable since $W_0$ does not act on the varieties $B_u$ themselves. In the original construction of Springer in [57], $G$ was defined over a finite field, and he considered $l$-adic cohomology. Later ([58]) he found a way to carry over the construction to groups over $C$, and cohomology with coefficients in $Q$. We will review here briefly the latter construction. Since its original discovery, many constructions of the Springer representations have been obtained. An overview of these constructions, with references, can be found in [19].

Let $G$ be the simple adjoint algebraic group over $C$ whose Weyl group is $W_0$. Then Springer has shown that $W_0$ acts on each of the cohomology groups $H^i(B_u) = H^i(B_u, Q)$. The top nonvanishing cohomology group is $H^{2d_u}(B_u)$, where $d_u := \dim(B_u)$. The dimension of $H^{2d_u}(B_u)$ equals the number of components of $B_u$, which all have dimension $d_u$.

The component group $A(u)$ of the centralizer $C_G(u)$ also acts on $H^{2d_u}(B_u)$, and its action commutes with that of $W_0$. Springer has shown that $A(u) \times W_0$ acts irreducibly on each nonzero $A(u)$-isotypic subspace of $H^{2d_u}(B_u)$, and that the representations of $W_0$ thus obtained exhaust the irreducible representations of $W_0$ exactly once if $u$ runs through a set of representatives of the unipotent conjugacy classes of $G$. If $\psi$ is a character of $A(u)$, we denote by $\phi_{u,\psi}$ the $W_0$-character on the $\psi$-isotypical part of $H^{2d_u}(B_u)$. So the irreducible characters of $W_0$ can be parametrized by pairs $(C, \psi)$ where $C$ is a unipotent conjugacy class of $G$ and $\psi$ is an irreducible character of $A(u)$ for $u \in C$. We write $\phi = \phi_{C,\psi}$ if $\phi$ is parametrized by $(C, \psi)$. Then every unipotent class yields at least one character of $W_0$, but in general not all characters of $A(u)$ occur. The ones that do occur are sometimes called geometric representations, they are denoted by $\hat{A}(u)_0$. The trivial representation of a component group is always geometric.

In the literature one often multiplies Springer's original characters with the sign character $\varepsilon$. We will follow this convention. There are then two extreme cases: $\phi_{e,1} = \varepsilon$ when we consider the unipotent class $C = \{e\}$, and $\phi_{u,1} = 1$ when $u$ is a regular unipotent element (which means that $u$ is contained in only one Borel subgroup of $G$). In both of these cases, $C_G(u)$ is connected and thus $A(u)$ is trivial if we consider $G$ of adjoint type. We now define

**Definition 2.1.1.** The Springer correspondents of the unipotent class $C$ are the representations $\phi_{C,\psi}$, where $\psi$ ranges over the irreducible characters of $A(u)$ for which the isotypic subspace in $H^{2d_u}(B_u)$ is nonzero, and $u \in C$. 

29
Since the trivial representation of the component group always occurs, we define $\phi_C := \phi_{C,1}$.

2.2. Unipotent classes of $SO_{2n+1}(k)$ and $Sp_{2n}(k)$

Recall from 1.2.8 that there is a bijection between the $W_0$-orbits of real residual cosets for the affine Hecke algebra associated to a root system of adjoint type, and unipotent classes in the corresponding Langlands dual complex group, which is of simply-connected type. In this particular occurrence of the Langlands dual group its isogeny class is not relevant; an easy argument ([6, 5.1]) shows that the unipotent classes of a connected reductive group $G$ with center $Z$ are in bijection with those of $G/Z$. For the affine Hecke algebra as well, there is a bijection between the real residual cosets of $T = \text{Hom}(X, \mathbb{C}^\times)$ and those of $\text{Hom}(Q, \mathbb{C}^\times)$, since $\text{Hom}(X, \mathbb{R}_{>0}) \simeq \text{Hom}(Q, \mathbb{R}_{>0})$. We may therefore select any affine Hecke algebra attached to a root system $(R_\alpha, X, R_\check{\alpha}, Y, \Pi)$ with $\text{rank}(Q) = \text{rank}(X)$. The $W_0$-orbits of its real residual cosets are then in bijection with the unipotent conjugacy classes of the complex semisimple group $G$ with the same root system.

The aim of this section is to make this bijection explicit for the groups that have root systems of type $B$ or $C$. Therefore we review various aspects of the classification of unipotent orbits of $SO_{2n+1}(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$. A particular type of unipotent conjugacy classes turns out to play a special role. These are the so-called distinguished classes. For the reader’s convenience we repeat the definition here in case $G$ is simple.

**Definition 2.2.1.** Let $G$ be a connected simple group over an algebraically closed field $K$. Then a unipotent element $u \in G$ is called distinguished if its connected centralizer $C_G(u)^0$ does not contain a semisimple element other than the identity.

If a unipotent class contains a distinguished element, it is clear that it contains only distinguished elements. In this case we call the class itself distinguished. We remark also that there always are distinguished elements, since a regular unipotent element is always distinguished, and these form an open and dense subset of the unipotent variety.

In case $G$ is reductive, one may or may not choose to call a unipotent element for which every torus in $C_G(u)^0$ belongs to the center of $G$, distinguished. For example, in [22] these elements are indeed called distinguished. On the other hand, in the context of affine Hecke algebras, this is not preferable. The reason is that otherwise one would lose the bijection between $W_0$-orbits of real residual cosets of $H$ and unipotent classes in $G$.

### 2.2.1. Elementary divisors partitions.

Since both of the groups under consideration can be realized as subgroups of an appropriate $GL_m(\mathbb{C})$, we first look at $GL_m(\mathbb{C})$. Then the classification of unipotent classes is just the theorem of the Jordan normal form: any two unipotent elements are conjugate if and only if they have the same Jordan normal form, and therefore the unipotent orbits in $GL_m(\mathbb{C})$ are parametrized by the partitions of $m$: in the natural matrix representation, the elementary divisors of a unipotent element of type $\lambda$ are $(t - 1)^{\lambda_1}, \ldots, (t - 1)^{\lambda_{\ell(\lambda)}}$. Now consider again the case $G = SO_{2n+1}(\mathbb{C}) \subset GL_{2n+1}(\mathbb{C})$, resp. $G = Sp_{2n}(\mathbb{C}) \subset GL_{2n}(\mathbb{C})$. It turns out that any two unipotent elements in $G$ which are in the same $GL_{2n+1}(\mathbb{C})$ resp. $GL_{2n}(\mathbb{C})$-orbit, are also in the same $G$-orbit. Therefore the unipotent conjugacy classes of $G$ are also parametrized by partitions of $2n + 1$ resp. $2n$, however not every such partition occurs. One can show that, for $G = SO_{2n+1}(\mathbb{C})$, the partitions parametrizing unipotent classes are exactly those whose even parts occur with even multiplicity, whereas for $G = Sp_{2n}(\mathbb{C})$, the partitions parametrizing nilpotent orbits
are those whose odd parts occur with even multiplicity. The distinguished classes among these are those whose partition consists of distinct odd parts resp. distinct even parts for $SO_{2n+1}(\mathbb{C})$ resp. $Sp_{2n}(\mathbb{C})$. For proofs of these statements, see [22].

2.2.2. Weighted Dynkin diagrams. Another way in which to encode the classification of unipotent orbits is by means of weighted Dynkin diagrams. A weighted Dynkin diagram is by definition a Dynkin diagram with a number from the set $\{0, 1, 2\}$ attached to each node. We briefly recall the way to do this (described in [6, p. 395]). First we consider the case $G = SO_{2n+1}(\mathbb{C})$. Given a unipotent orbit corresponding to the elementary divisors partition $\lambda \vdash 2n + 1$, we take the list which is the union (with multiplicity) of all the sets $\{\lambda_i - 1, \lambda_i - 3, \ldots, -\lambda_i + 1\}$, and write its elements in decreasing order, say $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_{2n+1}$. Then $a_n \geq 0$ and the associated weighted Dynkin diagram has labels $a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n, 2a_n$. For example, if $n = 5$ the partition (155) gives the list $a = (4, 4, 2, 2, 0, 0, 0, -2, -2, -4, -4)$ and weighted Dynkin diagram labeled 02020. Recall that the double bond in the Dynkin diagram is found between the rightmost two nodes.

For $G = Sp_{2n}(\mathbb{C})$, we do the same, but the Dynkin diagram now is given the labels $a_1 - a_2, a_2 - a_3, \ldots, a_{n-1} - a_n, 2a_n$. For example, if $n = 5$, the partition (244) yields the list $\{3, 3, 1, 1, 1, -1, -1, -1, -3, -3\}$, and weighted Dynkin diagram labeled 02002.

We will come back to this description in a short while, therefore we now make the following

**DEFINITION 2.2.2.** For $G = SO_{2n+1}(\mathbb{C})$ or $G = Sp_{2n}(\mathbb{C})$, and a unipotent class $C_\lambda \subset G$, let $s(C_\lambda)$ be the vector $(a_1 / 2, \ldots, a_n / 2) \in \mathbb{R}^n$ described above.

2.2.3. The Bala-Carter theorem. The key to describe the bijection between real residual cosets of the Hecke algebra and unipotent conjugacy classes of the group $G$, is the Bala–Carter theorem. This theorem shows how one can reduce the classification of unipotent conjugacy classes of $G$ to the classification of the distinguished unipotent classes in the Levi subgroups of $G$. This is completely analogous to how one can reduce the classification of the real residual cosets of the affine Hecke algebra $\mathcal{H}$ to the classification of the real residual points in the “Levi” quotient algebras $\mathcal{H}_L$. We have already seen in 1.2.8 that there exists a bijection between the $W_0$-orbits of real residual points of $\mathcal{H}$ and distinguished unipotent conjugacy classes of $G$. Since both $\mathcal{H}$ and $G$ have the same root system $R_0$, the bijection follows.

**THEOREM 2.2.3.** (Bala–Carter) Let $G$ be a simple algebraic group over the algebraically closed field $K$, where char$(K)$ is good for $G$. Then there is a 1-1 correspondence between conjugacy classes of unipotent elements in $G$, and pairs $(L, P_L)$, where $L$ is a Levi subgroup of $G$ and $P_L$ is a distinguished parabolic subgroup of the semisimple part $L'$ of $L$.

In fact, this bijection is such that the $G$-orbit corresponding to $(L, P_L)$ contains the dense orbit of $P_L$ acting on the Lie algebra of its unipotent radical. A proof of this theorem can be found in [6, 5.9].

2.3. Bijection between real residual cosets and unipotent classes

We can now state the combinatorial formulation of the bijection between Weyl group orbits of real residual cosets of $\mathcal{H}$ (or, equivalently, residual subspaces of $\mathbb{H}$) and the unipotent conjugacy classes of $G$. At the level of residual points, we have seen in 1.2.8 that (in the group case) if $r = \exp(\gamma) \in T$ is a real residual point (chosen in its $W_0$-orbit such that $\gamma$ is
particular cases that we are considering, this means the following:

**Lemma 2.3.1.** Let \( \gamma \) be a residual point for the graded Hecke \( \mathbb{H} \) algebra of type \( B_n \) with labels \( k_1 = k_2 \) (resp. \( k_1 = 2k_2 \)). Then \( 2^x \) is equal to \( s(C_\lambda) \) (in the sense of 1.2.8) for a certain distinguished unipotent class in \( G = SO_{2n+1}(\mathbb{C}) \) (resp. \( G = Sp_{2n}(\mathbb{C}) \)). Conversely, if \( C_\lambda \) is a distinguished unipotent class in \( G \), then \( s(C_\lambda)^{\frac{1}{2}} \) is a residual point for \( \mathbb{H} \). This defines a bijection between the \( W_0 \)-orbits of residual points for \( \mathbb{H} \) and distinguished unipotent classes in \( G \).

To describe the bijection between general residual cosets and general unipotent classes we need to know a combinatorial description of the pair \( (L, P_L) \) to which a unipotent element class \( C_\lambda \subset G \) bijects through the Bala–Carter theorem. Then \( C \) corresponds to the residual coset \( M \) with \( R_M = R_L \), whose center \( c_M \) is the residual point in \( \mathcal{H}_L \) corresponding to \( P_L \).

The map we are looking for, giving the pair \( (L, P_L) \) parametrizing the unipotent class \( C_\lambda \), is known. Let us describe it here explicitly. The reader is referred to [22] for further details. Let \( G = SO_{2n+1}(k) \). Then let \( l_1 > l_2 > \cdots > l_s \) be the parts of \( \gamma \) that occur an odd number of times. All \( l_j \) are necessarily odd, and we define their sum as \( n_0 = \sum_{j=1}^{s} l_j \). Notice that \( n_0 \) is odd since \( 2n + 1 - n_0 \) is even: it is the cardinality of the remaining partition, where all parts have even multiplicity. If we remove each \( l_j \) once from \( \gamma \), we get a partition where each part occurs an even number of times. If we remove each second part, we find a partition \( d_1 \geq d_2 \geq \cdots \geq d_r \). The Levi subgroup \( L \) we were looking for is then

\[
L \cong SO_{n_0}(k) \times GL_{d_1}(k) \times GL_{d_2}(k) \times \cdots \times GL_{d_r}(k)
\]

and the distinguished parabolic subgroup is the product of a distinguished parabolic subgroup in each (semisimplified) factor. In the factor \( SO_{n_0}(k) \), it is the one corresponding to the partition \( (l_1 \geq l_2 \geq \cdots \geq l_s) \). The group \( SL_n(k) \) has only one class of distinguished parabolics, since only the regular unipotent conjugacy class, having partition \( (n) \), is distinguished.

In the case \( G = Sp_{2n}(k) \), we do the same. It now can happen that \( s = 0 \). All \( l_j \) will now be even. The corresponding Levi will be

\[
L \cong Sp_{n_0}(k) \times GL_{d_1}(k) \times GL_{d_2}(k) \times \cdots \times GL_{d_r}(k)
\]

and again, its distinguished parabolic subgroup is the product of a distinguished parabolic subgroup in each factor. In the factor \( Sp_{n_0}(k) \), it is the one corresponding to the partition \( (l_1 \geq l_2 \geq \cdots \geq l_s) \).

The parameter values \( k_2 = k_1 \) and \( k_2 = 2k_1 \) which occur in the \( B_n \)-resp. \( C_n \)-group case are special, but are not the only special parameters. Later we will generalize the map \{unipotent conjugacy classes in \( G \)\} \( \rightarrow \) \{residual cosets of \( \mathcal{H} \)\} that we have just described. We denote it by \( f_{qBC}^{\gamma} \), where \( BC \) stands for Bala–Carter, and \( q \) denotes the ratio \( k_2/k_1 \), i.e., at the moment we only have defined \( f_{qBC}^{\gamma} \) for \( q \in \{\frac{1}{2}, 1\} \). Hence, if \( \lambda \) describes a distinguished unipotent class, then \( f_{qBC}^{\gamma}(\lambda) = c \) with \( c = s(C_\lambda)^{\frac{1}{2}} \). Now consider a general \( \lambda \) parametrizing a unipotent class, and let \( (L, P_L) \) be the corresponding pair described by the Bala–Carter theorem. Let \( \mu \) be the partition which describes the distinguished parabolic \( P_L \). Then \( f_{qBC}^{\gamma}(\lambda) = M \), where \( M \) is the residual coset whose center \( c_M \) is the residual point \( c_M = f_{qBC}^{\gamma}(\mu) \) in the graded Hecke algebra \( \mathbb{H} \). In particular \( R_M = R_L \).
Later we will also give an equivalent combinatorial formulation of \( f_q^{BC} \). The image of a partition \( \lambda \) parametrizing a unipotent orbit will be a double partition \( (\xi, \eta) \) of \( n \) describing a residual coset. The partition \( \xi \) is derived from the partition \( (d_i) \) above, while the partition \( \eta \) comes from \( (l_j) \). In fact, if we take \( \xi_i = d_i \) and \( 2\eta_j + 1 = l_j \), then \( (\xi, \eta) \) characterizes the residual coset completely and it is easy to check that indeed \(|\xi| + |\eta| = n\). This is because the jumps (to be defined below) \( j_i \) of the residual point \( f_q^{BC}(l_i) \) satisfy \( 2j_i + 1 = l_i \) (see Lemma 3.5.1 below). In Chapter 7 we will give the generalization of \( f_q^{BC} \) to arbitrary special values of the parameters.

### 2.4. Symbols

Since for the groups of classical type, their unipotent classes as well as the irreducible characters of their Weyl groups admit a combinatorial parametrization, there exists in these cases a combinatorial description of the Springer correspondence. Lusztig has given such a description for type \( B \), which makes use of certain combinatorial objects called symbols. These, and other, symbols also play a role in the calculation of the Green functions associated to the finite groups of Lie type \( Sp_{2n}(\mathbb{F}_q) \) and \( SO_{2n+1}(\mathbb{F}_q) \). Later, we will introduce other symbols, which govern the (yet to be defined) Springer correspondence at other special values for the affine Hecke algebra of type \( B \). All these symbols are special cases of the general notion of symbols as introduced by Malle ([43]) and Shoji. In [43], the symbols were used to classify the “unipotent characters” of a complex reflection group \( G(e, 1, n) \), and to describe their degrees. Shoji uses them to compute Green functions associated to \( G(e, 1, n) \) in [56]. Let us review here the part we need for our case, where \( e = 2 \), since the Weyl group of type \( B_n \) is isomorphic to \( G(2, 1, n) \).

Fix \( n \geq 2 \). Let \( W_0 = \mathbb{Z}/2\mathbb{Z} \wr S_n \) be the Weyl group of type \( B_n \). A 2-partition of \( n \) is a pair of partitions \( \alpha = (\alpha^1, \alpha^2) \), such that \(|\alpha^1| + |\alpha^2| = n\). (Note that Shoji would write this pair as \((\alpha^0, \alpha^1)\)). The set \( P_{n,2} \) of 2-partitions of \( n \) parametrizes the set \( \tilde{W}_0 \) of irreducible characters of \( W_0 \). We denote by \( \chi_\alpha \) the character of \( W_0 \) corresponding to \( \alpha \). In particular, \( \chi_{(n,-1)} \) is the trivial character, and \( \chi_{(-1,n)} \) is the sign character \( \varepsilon \). Now choose integers \( m_1 \) and \( m_2 \), both at least equal to \( n \), and put \( m = (m_1, m_2) \). We denote by \( Z_n(m) \) the set of 2-partitions \( \alpha \) of \( n \), where we regard \( \alpha_i \) as an element of \( \mathbb{Z}^{m_i} \), written in the form \( 0 \leq \alpha_i \leq \cdots \leq \alpha_i \). Now fix two other integers \( r \geq s \geq 0 \). We can now introduce the symbols. First we consider the 2-partition \( \Lambda^0 = \Lambda^0(m) = (\Lambda_1, \Lambda_2) \) defined by

\[
\begin{align*}
\Lambda_1 : & \quad 0 \leq r \leq 2r \leq \cdots \leq (m_1 - 1)r \\
\Lambda_2 : & \quad s \leq r + s \leq 2r + s \leq \cdots \leq (m_2 - 1)r + s
\end{align*}
\]

Now the collection of symbols of type \((r, s, m)\) is denoted by \( Z_n^{r,s}(m) \), and consists of the 2-partitions of the form \( \Lambda(\alpha) = \alpha + \Lambda^0 \), where we take the sum entry-wise. To stress the fact that they are a generalization of the symbols introduced by Lusztig, we write them down in the same way, i.e., the entries of \( \alpha^1 \) in the top row, the entries of \( \alpha^2 \) in the bottom row, placed as in the following example, with \( n = 5 \), \( \alpha = (1^3, 2) \), \( m = (6, 5) \), \( r = 2, s = 1 \): then the symbol is

\[
\begin{pmatrix}
0 & 2 & 4 & 7 & 9 & 11 \\
1 & 3 & 5 & 7 & 11
\end{pmatrix}
\]

Usually, we will not be so much interested in the symbols themselves, but rather in a set of equivalence classes of them, which we now introduce. Putting \( m' = (m_1 + 1, m_2 + 1) \), we
have a shift operation $Z^r_m(m) \rightarrow Z^r_{m'}(m')$, which sends $\Lambda(\alpha)$ to $\Lambda'(\alpha) = \Lambda^0(m') + \alpha$, where we regard $\alpha$ as an element of $Z_n(m')$ by adding zeroes in its entries. We denote by $Z^r_m(m)$ the classes in $\bigcup_{m'} Z^r_{m'}(m')$ under the equivalence relation generated by shift operations.

We call two elements $\Lambda$ and $\Lambda'$ of $Z^r_m(m)$ similar if they have representatives in $Z^r_m(m)$ which have the same numbers as entries, and all with the same multiplicity. We call the set of symbols that are similar to a fixed symbol a similarity class. For example,

$$
\begin{pmatrix}
0 & 3 & 6 \\
2 & 5 & 6
\end{pmatrix} \quad \text{and} \quad 
\begin{pmatrix}
0 & 2 & 6 \\
3 & 5 & 6
\end{pmatrix}
$$

are similar.

Another important tool we need is the so-called $a$-function. This is a function $a : Z^r_m(m) \rightarrow \mathbb{N}$, defined on $\Lambda \in Z^r_m(m)$ by:

$$a(\Lambda) = \sum_{\lambda, \lambda' \in \Lambda} \min(\lambda, \lambda') - \sum_{\mu, \mu' \in \Lambda} \min(\mu, \mu').$$

In these sums, we assume that $\lambda \neq \lambda'$ for all pairs that are contained in the same $\Lambda$, and analogously for $\mu$ and $\mu'$. However, equality can only hold for $r = 0$, which is a case we will not consider. Since it is known that $a$ is invariant under shift operations, we get a function $a$ on $Z^r_m(m)$. Clearly, $a$ is constant on similarity classes.

If $(r, s) = (1, 0)$ and $m = (n + 1, n)$, the set $Z^1_m(1, n)$ is used by Lusztig in [30] to parametrize unipotent principal series characters of $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$. In this case the $a$-function computes the largest power of $q$ dividing the degree of the corresponding unipotent character $\rho$, which is the generic degree of the Weyl group character corresponding to $\rho$ (see [6, p. 376]). This was the original definition of $a$. If $\psi$ is a special character (see below) then the $a$-value of its $(1, 0)$-symbol coincides with $\dim(B_u)$, if $\chi \leftrightarrow (u, 1)$ under the Springer correspondence.

The unipotent characters of $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$ in the other series are parametrized by symbols with $m = (n + d, n)$, with $d = 2s + 1$ an odd integer.

If $(r, s) = (2, 1)$ (resp. $(2, 0)$), and $m = (n + 1, n)$, the set $Z^2_m(1, n)$ (resp. $Z^2_m(0, n + 1, n)$) was used by Lusztig to describe the Springer correspondence for $G = Sp_{2n}(k)$ (resp. $G = SO_{2n+1}(k)$), for $\text{char}(k) \neq 2$. In these cases, the similarity classes are in 1-1 correspondence with the unipotent classes of $G$, and the $a$-value of a symbol coincides with $\dim(B_u)$, where $u$ is in the unipotent class corresponding to the symbol. If $\text{char}(k) = 2$, Lusztig and Spaltenstein use more general symbols in [40]. The explicit form of the Springer correspondence for $\text{char}(k) \neq 2$ in terms of symbols is described in the next section.

There are also other values of $(r, s)$ which have occurred in the literature, but we will not mention them in this text, except for the family symbols discussed below.

### 2.4.1. Family symbols.

Another way to group Weyl group representations, apart from in sets of Springer correspondents, is in so-called families. This way of partitioning is closely related to the notion of special characters. These are by definition the Weyl group characters $\psi$ for which the generic degree $F_\psi(t)$ and the fake degree $\tilde{F}_\psi(t)$ have the same lowest degree. The family symbols are, in the language of Shoji, the equivalence classes $Z^1_m(1, n)$ we get from all 2-partitions. The families are then the resulting similarity classes. As is the
2.5. Combinatorics of the Springer Correspondence for Type B

For the classical groups of type $B$, $C$ and $D$, the explicit determination of the Springer correspondence was done in [52]. An alternative formulation has been given by Lusztig in [29], which can also be found in [6].

2.5.1. Finding all correspondents of a unipotent class. Recall that the irreducible representations of $W_0$ are parametrized by the 2-partitions $(\xi, \eta)$ of $n$. To each such pair, we now associate two symbols, one for $B_n$ and one for $C_n$, which will allow to decide to which unipotent orbit the Weyl group representation is associated through the Springer correspondence. In the language of section 2.4, these symbols are $\mathbb{Z}_n^{2,0}(n+1,n)$ for $B_n$ and $\mathbb{Z}_n^{2,1}(n+1,n)$ for $C_n$. Since we look at equivalence classes of symbols up to shift operations, a canonical representative of the symbol of $(\xi, \eta)$ is the one in which zeroes are added to $\xi$ or $\eta$ so as to make $\xi$ have one part more than $\eta$, in which case we get for $B_n$

$$
\begin{pmatrix}
\xi_1 & \xi_2 + 2 & \xi_3 + 4 & \cdots \\
\eta_1 & \eta_2 + 2 & \eta_3 + 4 & \cdots
\end{pmatrix}
$$

and for $C_n$

$$
\begin{pmatrix}
\xi_1 & \xi_2 + 2 & \xi_3 + 4 & \cdots \\
\eta_1 + 1 & \eta_2 + 3 & \eta_3 + 5 & \cdots
\end{pmatrix}
$$

case for the symbols governing the Springer correspondence, each of the resulting similarity classes contains an increasing symbol arising from a 2-partition.

The connection between the Springer correspondence symbols and the family symbols is the following:

**Lemma 2.4.1.** $\chi(\xi,\eta) \in \check{W}_0$ is a special character $\iff$ $\chi(\xi,\eta)$ is indexed by the trivial character of the component group in both the $B_n$ and the $C_n$-cases.

**Proof.** Translated into conditions on the symbols, the statement amounts to: for a 2-partition $(\xi, \eta)$, the family symbol is increasing $\iff$ both the Springer symbols for $B_n$ and $C_n$ of $(\xi, \eta)$ are increasing. So suppose $\xi = (\xi_1 \leq \xi_2 \leq \cdots \leq \xi_{m+1})$ and $\eta = (\eta_1 \leq \eta_2 \leq \cdots \leq \eta_m)$. Then the family symbol is increasing if and only if

(2.1) $\xi_1 \leq \eta_1 \leq \xi_2 + 1 \leq \eta_2 + 1 \leq \xi_3 + 2 \leq \cdots \leq \eta_m + (m-1) \leq \xi_{m+1} + m$,

while the $B_n$-symbol is increasing if and only if

(2.2) $\xi_1 \leq \eta_1 \leq \xi_2 + 2 \leq \eta_2 + 2 \leq \xi_3 + 4 \leq \cdots \leq \eta_m + 2(m-1) \leq \xi_{m+1} + 2m$,

and the $C_n$ symbol is increasing if and only if

(2.3) $\xi_1 \leq \eta_1 + 1 \leq \xi_2 + 2 \leq \eta_2 + 3 \leq \xi_3 + 4 \leq \cdots \leq \eta_m + 2m - 1 \leq \xi_{m+1} + 2m$.

Therefore we have to check the equivalence (2.1) $\iff$ (2.2) and (2.3). But this is easy, consider for example the direction $\Rightarrow$. Then $\eta_k + (k-1) \leq \xi_{k+1} + k \Rightarrow \eta_k \leq \xi_{k+1} + 1 \Rightarrow \eta_k + 2(k-1) \leq \xi_{k+1} + 2k$ and $\eta_k + 2(k-1) \leq \xi_{k+1} + 2k$. Similarly, $\xi_k + (k-1) \leq \eta_k + (k-1) \Rightarrow \xi_k \leq \eta_k \Rightarrow \xi_k + 2(k-1) \leq \eta_k + 2(k-1)$. The other direction can be proved in the same way. $\Box$

Then it is shown by Lusztig that two Weyl group representations labeled by $(\xi, \eta)$ and $(\xi', \eta')$ are Springer correspondents of the same unipotent conjugacy class in $SO_{2n+1}(\mathbb{C})$ (resp. $Sp_{2n}(\mathbb{C})$) if and only if the $B_n$ (resp. $C_n$)-symbols of $(\xi, \eta)$ and $(\xi', \eta')$ are similar.

Now let us describe which Weyl group representations are Springer correspondents of a given unipotent orbit. We treat both cases separately.

**2.5.1.1. $B_n$:** The unipotent orbits in $SO_{2n+1}(\mathbb{C})$ are parametrized by partitions of $2n + 1$ whose even parts occur with even multiplicity. There is a map from partitions with this property into double partitions of $n$, which is described in [6]. This map, let us call it $\phi_1$ (where again the subscript denotes the quotient $k_2/k_1$), is defined as follows. Given $\lambda \vdash 2n + 1$, we define $\lambda^* := \lambda_1 < \lambda_2 + 2 < \lambda_3 + 3 < \ldots$. We then split $\lambda^*$ into odd and even parts, which we write as $2\xi_1^* + 1 < 2\xi_2^* + 1 < \ldots$ and $2\eta_1^* < 2\eta_2^* < \ldots$. The partitions $\xi$ and $\eta$ are then defined as $\xi_i := \xi_i^* - (i - 1)$ and $\eta_i := \eta_i^* - (i - 1)$.

Notice that this map takes a particularly simple form if $\lambda$ parametrizes a distinguished unipotent orbit $C$, i.e., if it consists of distinct odd parts: $\lambda = \lambda_1 < \lambda_2 < \ldots < \lambda_{2r+1}$ for some $r \geq 0$. It follows easily that $(\xi, \eta) = \phi_1(\lambda)$ is then equal to:

$$
(\xi, \eta) = (\frac{\lambda_1 - 1}{2}, \frac{\lambda_3 - 1}{2}, \ldots, \frac{\lambda_{2r+1} - 1}{2}), (\frac{\lambda_2 + 1}{2}, \frac{\lambda_4 + 1}{2}, \ldots, \frac{\lambda_{2r} + 1}{2})
$$

**2.5.1.2. $C_n$:** We now explain the analogous procedure for the $C_n$ group case. If we have a partition of $2n$ which parametrizes a unipotent orbit in $Sp_{2n}(\mathbb{C})$, we first make sure it has an even number of parts by adding $\lambda_1 = 0$ if necessary. Then we perform the same procedure as in the $B_n$-case. Again this map, let us call it $\phi_2$, takes a simple form if $\lambda$ parametrizes a distinguished unipotent class, i.e., if it consists of distinct even parts. In this case, if $\lambda$ has $2m$ parts, we find for $(\xi, \eta) = \phi_2(\lambda)$ that

$$(\xi, \eta) = (\frac{\lambda_2}{2}, \frac{\lambda_4}{2}, \ldots, \frac{\lambda_{2m}}{2}), (\frac{\lambda_1}{2}, \frac{\lambda_3}{2}, \ldots, \frac{\lambda_{2m-1}}{2})$$

In each of these cases, the map $\phi_q$ gives us, for a partition $\lambda \vdash 2n$ or $2n + 1$, a Weyl group representation which is a Springer correspondent of the unipotent class $C_\lambda$. The other Springer correspondents of $C_\lambda$ are those representations having symbol similar to the one of $\phi_q(\lambda)$. For example, in the $B_n$-case, the distinguished unipotent class consisting of the regular unipotent elements is parametrized by $(2l + 1)$, and so has one Springer correspondent: the trivial representation indexed by $(l, -)$. One also checks easily that the Springer correspondent of the unipotent class $C = \{1\}$, which has partition $\lambda = (1^{2l+1})$, has as Springer correspondent $(-, 1^l)$, the sign representation. This is in agreement with Definition 2.1.1. In both group cases, we can therefore also associate a symbol to a unipotent class $C_\lambda$, viz. the symbol of $\phi_q(\lambda)$.

In both cases, the image of the maps $\phi_1$ and $\phi_2$ consists of those double partitions $(\xi, \eta)$ for which the symbol is increasing, i.e., for which we have $\xi_1 \leq \eta_1 \leq \xi_2 + 2 \leq \eta_2 + 2 \leq \cdots \leq \eta_k + 2(k - 1) \leq \xi_{k+1} + 2k$ in case $B_n$, and $\xi_1 \leq \eta_1 + 1 \leq \xi_2 + 2 \leq \eta_2 + 3 \leq \cdots \leq \eta_k + 2(k - 1) + 1 \leq \xi_{k+1} + 2k$ in case $C_n$. It is also known that for both types of symbols, every similarity class contains an increasing symbol $\Lambda(\alpha)$ for some 2-partition $\alpha$. This property will no longer hold for the symbols we will later associate to 2-partitions at other special values of the parameters of the Hecke algebra.

We can describe easily the inverse $\psi_q$ of the maps $\phi_q$. If $q = 1$, then we start with an increasing symbol, from which we derive a double partition $(\xi, \eta)$, and we consider $\xi$ as
having one part more than $\eta$. Then we form $(\xi^*, \eta^*), \lambda^*$ and $\lambda$ in the obvious way. To find the inverse of $\phi_\frac{1}{2}$, we do the same, but now we have to consider $\xi$ as having the same number of parts as $\eta$ (either by removing the first zero of $\xi$ or adding a zero to $\eta$), since the resulting partition $\lambda$ has an even number of parts. We extend the domain of $\psi_q$ to all 2-partitions by defining $\psi_q(\alpha) := \psi_q(\beta)$ where $\beta$ is the unique 2-partition such that the symbol of $\beta$ is similar to the one of $\alpha$, and $\beta$ has increasing symbol. Later on, we will give a description of the analogue of these maps $\psi_q$ at the other special values of $q = k_2/k_1$.

2.5.2. Relation between symbols and characters of the component group. Let $C_{\lambda} \subset G$ be a unipotent class. Let $\phi_q(\lambda) = (\xi, \eta)$, whose symbol is (as we have seen) increasing. Then it is known that $\chi_{(\xi, \eta)} = \phi_C$, i.e., $(\xi, \eta)$ corresponds to the trivial character of the component group. The fact that the trivial character always occurs in the parametrization of the Springer correspondents of a given unipotent class is therefore equivalent to the fact that every similarity class of symbols contains an increasing symbol. Moreover, given a Springer correspondent $(\alpha, \beta) = \phi_{C, \psi}$ of $C$, there exists a combinatorial method to determine the character $\psi$ of $A(u)$ from the symbol of $(\alpha, \beta)$, see e.g. [6].