A combinatorial generalization of the Springer correspondence for classical type
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CHAPTER 5

"Unipotent classes" for special parameters

5.1. Definition

We have now given the definition of Springer correspondents for any parameter value. This gives us a direct way of computing them. In the group cases, we have seen that the new definition coincides with the classical one. But in these cases, there is another way of computing them, by using the following diagrams:

\[
\begin{align*}
\{\lambda \vdash 2n + 1 \mid r \text{ even } = \text{ even}\} \\
\{\psi_1 \nearrow \phi_1 \searrow f_{\lambda}^{BC}\} \\
\{(\xi, \eta) \mid (\xi, \eta) \text{ has increasing symbol}\} \quad \longleftrightarrow \quad \{\text{residual subspaces}\}/W_0
\end{align*}
\]

(5.1)

and

\[
\begin{align*}
\{\lambda \vdash 2n + 1 \mid r \text{ odd } = \text{ even}\} \\
\{\psi_\frac{1}{2} \nearrow \phi_\frac{1}{2} \searrow f_{\lambda}^{BC}\} \\
\{(\xi, \eta) \mid (\xi, \eta) \text{ has increasing symbol}\} \quad \longleftrightarrow \quad \{\text{residual subspaces}\}/W_0
\end{align*}
\]

(5.2)

The aim of this chapter is to try to generalize these diagrams to all special values of the parameters. For this, we need to define a generalization of the maps \(\phi_q, \psi_q\) and \(f_q^{BC}\), as well as of the sets \(\mathcal{U}_1(n) := \{\lambda \vdash 2n + 1 \mid r_{\text{even}} = \text{ even}\}\) and \(\mathcal{U}_\frac{1}{2}(n) = \{\lambda \vdash 2n \mid r_{\text{odd}} = \text{ odd}\}\), which parametrize the unipotent classes in the group cases. First of all let us see what the appropriate analogue of \(\mathcal{U}_q(n)\) is. This will be a subset of the partitions of a number dependent on \(n\) and \(q\).

In the group cases we have seen that a residual point with jumps \(j_i\) corresponds to the partition \(\lambda\) with parts \(\lambda_i = 2j_i + 1\). Therefore, we generalize this to all special values, associating to a residual point \(c(\mu, k_1, k_2)\) with jumps \(j_i\) the partition \(\lambda\) with parts \(2j_i + 1\). We then find that, as we would like, \(|\lambda|\) is independent of the residual point:

**Lemma 5.1.1.** Let \(k_2 = qk_1\) be special and let \(c = c(\mu, k_1, k_2)\) be residual. Let the jumps of \(c\) be \(\{j_1, j_2, \ldots\}\), and define \(\lambda\) to be the partition with parts \(2j_k + 1\). If \(q\) is integer, then \(|\lambda| = 2n + q^2\), and if it is half-integer, then \(|\lambda| = 2n + \frac{q^2}{4} - \frac{1}{2}\).

**Proof:** (i) Suppose that \(q\) is integer. First consider \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l\) with \(l \leq q - 1\). Then \(S_q(\mu) = (\mu, -1)\), and the jumps of \(c\) are \(\{q + \mu_1 - 1, q + \mu_2 - 2, \ldots, q + \mu_l - l, q - l - 1, q - l - 2, \ldots, 1, 0\}\) as can be seen from the following picture of \(T_q(\mu)\):
Therefore, we find that indeed
\[
\sum_{k} 2j_k + 1 = \sum_{i=1}^{l} (2(q + \mu_i - i) + 1) + \sum_{i=0}^{q-l-1} (2i + 1) \\
= 2ql + 2n - 2\frac{l(l+1)}{2} - 2\frac{(q-l-1)(q-l)}{2} + (q-l) \\
= q^2 + 2n
\]

Now consider the general situation, where we may (and will) assume that $T_q(\mu)$ is in standard position, i.e., $(\xi, \eta) = S_q(\mu)$ consists of first $q-1$ horizontal blocks $(\xi_1, \ldots, \xi_{q-1}) = (\mu_1, \ldots, \mu_{q-1})$, followed by a series of alternating horizontal and vertical blocks. Then $(\xi_{tr}, \eta)$, with $\xi_{tr} = (\xi_q, \xi_{q+1}, \ldots, \xi_{q+n})$, is the $q$-tableau of a residual point for $q = 1$. Therefore we know that $\sum_{k \geq q} (2j_k + 1) = 2(\sum_{k \geq q} \mu_k) + 1$. Thus
\[
\sum_{k} (2j_k + 1) = \sum_{k=1}^{q-1} (2(q + \mu_k - k) + 1) + \sum_{k \geq q} (2j_k + 1) \\
= 2q(q-1) + 2\sum_{k=1}^{q-1} \mu_k - 2\frac{(q-1)q}{2} + (q-1) + 2\sum_{k \geq q} \mu_k + 1 \\
= q^2 + 2n.
\]

(ii) If $q$ is half-integer, we find that if $l(\mu) \leq q - \frac{1}{2}$ that the jumps are $\{q + \mu_1 - 1, q + \mu_2 - 2, \ldots, q + \mu_l - l, q - l - 1, q - l - 2, \ldots, \frac{3}{2}, \frac{1}{2}\}$, which after a similar computation yields $\sum_i 2j_i + 1 = q^2 + 2n - \frac{1}{4}$. The general case is treated analogous to integer $q$ as well. □

This leads us to consider partitions of $2n + q^2$ for integer $q$, and partitions of $2n + q^2 - \frac{1}{4}$ for half-integer $q$. We can now define the set of partitions that forms the analogue of the unipotent classes.

**Definition 5.1.2.** If $q$ is integer we define
\[
U_q(n) := \{\lambda = (1^{r_1} 2^{r_2} \ldots) \mid 2n + q^2 \mid r_i \text{ is even if } i \text{ is even and } \sum_{i \text{ odd}} (r_i \text{ mod } 2) \geq q}\.
\]

That is, we consider partitions of $2n + q^2$ in which even parts have even multiplicity and which have at least $q$ distinct odd parts with odd multiplicity.

For half-integer $q$ we define
\[
U_q(n) := \{\lambda = (1^{r_1} 2^{r_2} \ldots) \mid 2n + q^2 - \frac{1}{4} \mid \\
r_i \text{ is even if } i \text{ is odd and } \sum_{i \text{ even}} (r_i \text{ mod } 2) \geq q - \frac{1}{2}\}.
\]
5.2. Generalization of the Maps $\psi_q$ and $\phi_q$

That is, we consider partitions of $2n + q^2 - \frac{1}{4}$ in which odd parts have even multiplicity, and which have at least $q - \frac{1}{2}$ even parts with odd multiplicity.

Notice that in the group case $q = 1$ we indeed recover $U_1(n)$ since a partition of $2n + 1$ always contains an odd part with odd multiplicity. In the group case $q = \frac{1}{2}$ we also recover the set $U_1$, since then the second condition in the definition is vacuous and so we find the partitions of $2n$ in which odd parts have even multiplicity.

On $U_q(n)$ we can define the map $f_q^{BC}$, associating a residual subspace of the Hecke algebra of type $B_n$ with parameters $k_2 = qk_1$ to $\lambda \in U_q(n)$.

**Definition 5.1.3.** Suppose $\lambda \in U_q(n)$.

(i) If $q$ is integer, let $l_1 > l_2 > \cdots > l_s$ be the parts that occur an odd number of times in $\lambda$, and put $n_0 := \sum_j l_j$. All $l_j$ are odd and $s \geq q$. By removing each $l_j$ once from $\lambda$, we obtain a partition where every part occurs an even number of times. If we remove each second part, we find a partition $d_1 \geq d_2 \geq \cdots \geq d_r$. The associated residual subspace $L$ is the one with

$$R_L = A_{d_1-1} \times A_{d_2-1} \times \cdots \times A_{d_r-1} \times B_{n_0},$$

and the residual point in $B_{n_0}$ has jumps $\lfloor \frac{l_j}{2} \rfloor$. Then $L := f_q^{BC}(\lambda)$.

(ii) If $q$ is half-integer, let again be $l_1 > \cdots > l_s$ be the parts of $\lambda$ that occur an odd number of times. Now all $l_i$ are even, $s \geq q + \frac{1}{2}$, and again by removing each $l_j$ once from $\lambda$ we obtain a partition where each part has even multiplicity. Remove each second part to obtain a partition $d_1 \geq d_2 \geq \cdots \geq d_r$, then $L = f_q^{BC}(\lambda)$ has root system

$$R_L = A_{d_1-1} \times A_{d_2-1} \times \cdots \times A_{d_r-1} \times B_{n_0},$$

and the residual point in $B_{n_0}$ has jumps $\frac{l_j}{2}$. Then $L := f_q^{BC}(\lambda)$.

**Lemma 5.1.4.** For every special value $k_2 = qk_1$, the set $U_q(n)$ parametrizes the set of $W_0$-orbits of residual subspaces for these parameter values.

**Proof:** Consider a residual subspace $L$ with $R_L$ of type $A_\kappa \times B_l$. The residual points in $B_l$ are characterized by their jumps $j_i$. From 3.3.3 we know that the parts $2j_i + 1$ form a partition of at least $q$ (resp. at least $q - \frac{1}{2}$) positive and distinct odd (resp. even) parts. By adding two parts $\kappa_i$ for every $A$-factor, we obtain indeed a partition $\lambda \in U_q(n)$. Clearly $f_q^{BC}(\lambda) = L$. Therefore $f_q^{BC}$ is a bijection between $U_q(n)$ and the set of orbits of residual subspaces.

Notice that for $q \in \{\frac{1}{2}, 1\}$, we recover the already existing $f_q^{BC}$, since for a distinguished unipotent class $C_\lambda$, the jumps of $s(C_\lambda)$ satisfy $\lambda_i = 2j_i + 1$, as seen in Lemma 3.5.1.

### 5.2. Generalization of the maps $\psi_q$ and $\phi_q$

Since in the group cases, $\phi_q(U_q(n))$ consists of all 2-partitions with increasing $q$-symbol, the domain of the inverse map $\psi_q$ consists of these 2-partitions.

However, in general, the $q$-similarity class of 2-partitions which are the Springer correspondents of a residual subspace, will not contain one with increasing symbol. If there would be an interpretation in terms of representations of a component group (as there is in the $B_n$- and $C_n$-group cases) this means that its trivial representation does not in general occur, contrary to the group cases!

Therefore we consider the map $\psi_q$ as being defined on any 2-partition $(\xi, \eta)$, by first replacing $(\xi, \eta)$ by the unique $(\xi', \eta') \sim_q (\xi, \eta)$ with increasing $q$-symbol.
5.2.1. Integer \( q \). We will now describe the map \( \psi_q \) for integer \( q \). Let \((\alpha, \beta)\) be a 2-partition of \( n \).

(i) We consider \( \alpha \) as having \( q \) parts more than \( \beta \). For convenience, suppose that \( \beta \) has \( n \) and \( \alpha \) has \( n + q \) parts. We now replace \((\alpha, \beta)\) by the unique 2-composition \((\xi, \eta)\) such that \((\xi, \eta) \sim_q (\alpha, \beta)\) and \((\xi, \eta)\) has increasing \( q \)-symbol.

(ii) To this 2-composition \((\xi, \eta)\) we then associate \( \xi^* \) and \( \eta^* \) as in the group cases, i.e., we take \( \xi_i^* = \xi_i + (i - 1) \) and \( \eta_i^* = \eta_i + (i - 1) \). Note that \( \xi^*_i \) still is in general a composition rather than a partition: in the last \( q \) parts of \( \xi \), we have \( \xi_i \geq \xi_{i+1} - 2 \), which leads to \( \xi_i^* = \xi_{i+1} - 1 \). On the other hand \( \eta^*_i \) is always a partition. For example, let \( n = 4, q = 3 \), and \((\alpha, \beta) = (-, 4)\). Then the 2-symbol of \((\alpha, \beta)\) is

\[
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 & 10 & 12 \\
0 & 2 & 4 & 6 & 10 & 10 & 12
\end{pmatrix}
\]

which we rearrange as

\[
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 & 10 & 12 \\
0 & 2 & 4 & 6 & 10 & 10 & 12
\end{pmatrix}
\]

so \((\xi, \eta) = (0000200, 0002)\) and \((\xi^*, \eta^*) = (0123656, 0125)\).

(iii) Next, we form the analogue of \( \lambda^* \). First we form an intermediate composition \( \mu^* \). As in the group cases, we let \( \mu_i^* \) consist of the parts \( 2\xi_i + 1 \) and \( 2\eta_i \). We order the first \( 2n \) parts in increasing order, and add the last \( q \) parts in the order they appeared in \( \xi^* \). In the previous example, we find \( \mu^* = (0, 1, 2, 3, 4, 5, 7, 10, 11, 13) \).

(iv) Now we construct from \( \mu^* \) the \( n \)-composition \( \mu \). We define \( \mu_i := \mu_i^* - (i - 1) \) for \( i = 1, 2, \ldots, 2n \). We order \( \mu \) in increasing order, and add the last \( q \) parts of \( \mu \). Thus \( \mu = (0, 0, 0, 0, 0, 1, 3, 5, 3, 5) \).

(v) Finally we remove all zeroes from \( \mu \), and replace any entries in \( \mu \) of the form \( (x+1, x-1) \) by \( (x, x) \). The end result is \( \lambda := \psi_q(\alpha, \beta) \). In the example, we thus find \( \psi_3(-, 4) = (13445) \). Note that for \( q = 1 \), we find indeed the map \( \psi_1 \). We now first prove some helpful properties of the map \( \psi_q \).

**Lemma 5.2.1.** Let \((\alpha, \beta)\) be a 2-partition of \( n \).

(i) Suppose that in the \( q \)-symbol of \((\alpha, \beta)\), one has \( e_q(\alpha_{n+i}) > e_q(\beta_n) \geq e_q(\alpha_{n+i}) \) for some \( 1 \leq i \leq q \). Let \( \alpha_{tr} = (\alpha_1, \ldots, \alpha_{n+i}) \), and suppose that \( \psi_q(\alpha, \beta) = \lambda \). Then \( \psi_q(\alpha_{tr}, \beta_{tr}) = \lambda_{tr} = (\lambda_1, \ldots, \lambda_{2(n-1)+q}) \).

(ii) Suppose that \( e_q(\beta_n) \geq e_q(\alpha_{n+q}) \). Let \( \alpha_{tr} = \alpha - \alpha_{n+q} \) and \( \beta_{tr} = \beta - \beta_n \). Suppose \( \psi_q(\alpha, \beta) = \lambda \). Then \( \psi_q(\alpha_{tr}, \beta_{tr}) = \lambda_{tr} = (\lambda_1, \ldots, \lambda_{2(n-1)+q}) \).

**Proof:** (i) This is clear, since if we denote the corresponding 2-compositions with increasing \( q \)-symbols by \((\xi, \eta) \sim_q (\alpha, \beta)\) and \((\xi_{tr}, \eta_{tr}) \sim (\alpha_{tr}, \beta_{tr})\), then \( \xi = \xi_{tr} \cup (\xi_{n+i+1}, \ldots, \xi_{n+q}) \).

(ii) Suppose that \((\xi, \eta) \sim_q (\alpha, \beta)\) and \((\xi_{tr}, \eta_{tr}) \sim (\alpha_{tr}, \beta_{tr})\) have increasing \( q \)-symbols. Then \( \xi_{tr} = (\xi_1, \ldots, \xi_n, \eta_{n-2}, \xi_{n+1}, \ldots, \xi_{n+q}) \) and \( \eta_{tr} = \eta - \eta_n \). Therefore

\[
\xi_{tr}^* = (\xi_1^*, \ldots, \xi_n^* - 1, \xi_{n+1}^* - 1, \ldots, \xi_{n+q}^* - 1) \text{ and } \eta_{tr}^* = \eta - \eta_n^*.
\]

Now there are several possibilities. Either we have \( e_q(\xi_n) = e_q(\eta_n) \) or we have \( e_q(\xi_n) < e_q(\eta_n) \) or \( e_q(\xi_n) < e_q(\eta_n) \). Let us first assume that \( e_q(\xi_n) = e_q(\eta_n) \). Let \( \xi_n = \eta_n, \mu_{2n-1} = 2\eta_n, \mu^*_n = 2\xi^*_n + 1, \) and \( \mu_{2n-1} = 2\eta^*_n - (2n - 2) = \mu_{2n} = 2\xi^*_n + 1 - (2n - 1) \). In \( \lambda = \psi_q(\alpha, \beta) \) these entries remain the same. Now lets us see what we find for \( \psi_q(\alpha_{tr}, \beta_{tr}) \). First we calculate \( \mu_{tr} = (\mu^*_1, \ldots, \mu^*_{2n-2}, 2\xi^*_n + 1, 2\eta^*_n - 1, 2\xi^*_n + 1 - (2n - 1) \).
1, \ldots, 2\xi^*_{n+q} - 1), and so \mu_{tr} = (\mu_1, \ldots, \mu_{2n-2}, 2\xi^*_n + 1 - (2n - 2), 2\eta^*_n - 1 - (2n - 2), \mu_{2n+1}, \mu_{2n+q+1}). Therefore, in \lambda_{tr}, the two entries \((2\xi^*_n + 1 - (2n - 2), 2\eta^*_n - 1 - (2n - 2))\) are replaced by the pair \((2\xi^*_n - (2n - 2), 2\eta^*_n - (2n - 2))\), which are the same entries obtained in \lambda. This proves the claim in case \(e_q(\xi_n) = e_q(\eta_n)\). Next we consider the second possibility \(e_q(\xi_n) < e_q(\eta_n) < e_q(\xi_{n+1})\). Then in \(\mu^*\) we get \(\mu_{2n-1} = 2\xi^*_n + 1\) and \(\mu_{2n} = 2\eta^*_n\), whereas \(\mu_{tr}^* = (\mu_1^*, \ldots, \mu_{2n-1}^*, 2\xi^*_n - 1, 2\xi^*_{n+1} - 1, \ldots, 2\xi^*_{n+q - 2} - 1)\), and so \(\mu_{tr} = (\mu_1, \ldots, \mu_{2n-1}, 2\eta^*_n - 1 - (2n - 2), 2\xi^*_{n+1} - 1 - 2(n - 1), \ldots, 2\xi^*_{n+q} - 1 - 2(n - 1)) = (\lambda_1, \ldots, \mu_{2n+q-2})\). Hence also \(\lambda_{tr} = (\lambda_1, \ldots, \lambda_{2n+q-q-2})\). The last possibility is that \(e_q(\eta_n) = e_q(\xi_{n+1})\), so \(\eta_n = \xi_{n+1} + 2\) and \(\eta^*_n = \xi^*_{n+1} + 1\). In that case we find in \(\mu^*\) that \(\mu_{2n}^* = 2\xi^*_{n+1} + 1\) and \(\mu_{2n+1}^* = 2\eta^*_n\), and so in \(\mu\) we get \(\mu_{2n} = \mu_{2n+1} = 2\eta^*_n - 2n\). On the other hand, \(\mu_{tr}^* = (\mu_1^*, \ldots, \mu_{2n-1}^*, 2\xi^*_n - 1, 2\xi^*_{n+1} - 1, 2\xi^*_{n+2} - 1, \ldots, 2\xi^*_{2n+q-2} - 1)\) and so \(\mu_{tr} = (\mu_1, \ldots, \mu_{2n-1}, 2\eta^*_n - 2n + 1, 2\eta^*_n - 2n - 1, \mu_{2n+2}, \ldots, \mu_{2n+q-2})\). This means that in \(\lambda_{tr}\) we find \(\lambda_{tr, 2n} = \lambda_{tr, 2n+1} = 2\eta^*_n - 2n\), which are also the corresponding entries of \(\lambda\). So we find that in every case, the statement of the lemma holds.

PROPOSITION 5.2.2. Suppose \((\alpha, \beta)\) is a 2-partition of \(n\). Then \(\psi_q(\alpha, \beta) \in \mathcal{U}_q(n)\).

Proof: Let the 2-composition \((\xi, \eta)\) of \(n\) be the one described in part (i) of the definition of \(\psi_q\). Following the procedure \(\psi_q\), we find that, since we consider \(\xi\) as having \(n + q\) parts and \(\eta\) as having \(n\) parts:

\[
|\psi_q(\alpha, \beta)| = \sum_{i=1}^{n+q} 2(\xi_i + (i - 1)) + \sum_{i=1}^{n} 2(\eta_i + (i - 1)) - \sum_{k=1}^{2n-1} k - 2nq = 2n + q^2,
\]

so indeed \(\lambda := \psi_q(\alpha, \beta) \vdash 2n + q^2\). It remains to be shown that even parts in \(\lambda\) occur with even multiplicity and that there are at least \(q\) odd parts in \(\lambda\) with odd multiplicity. Notice that although \((\xi, \eta)\) in general a 2-composition rather than a 2-partition, \(\eta\) is a partition, as is \((\xi_1, \ldots, \xi_{n+1})\). On this part, \(\psi_q\) acts as \(\psi_1\) and therefore the first \(2n + 1\) parts of \(\lambda\) form a partition where even parts occur with even multiplicity. Also, since \(\mu_{2n+i}^* = 2\xi^*_{n+i+1} + 1\) and \(\mu_{2n+i} = \mu_{2n+i}^* - 2n\), the last \(q\) parts of \(\mu\) are all odd. In the final step \(\mu \rightarrow \lambda\), we create only pairs of equal parts, which shows that indeed in \(\psi_q(\alpha, \beta)\) even parts have even multiplicity.

Now let us show that \(\psi_q(\alpha, \beta)\) contains at least \(q\) odd parts with odd multiplicity. We have seen that the parts \(\mu_{2n+1}, \ldots, \mu_{2n+q}\) are all odd, and at least equal to \(\mu_{2n}\). If \(\lambda = \mu\), then we are done. If not, then there are parts \(\xi_{n+i} = \xi_{n+i+1} + 2\) in \(\xi\), for some \(1 \leq i \leq q - 1\), giving rise to entries in the \(\eta\)-symbol of \((\xi, \eta)\) which are equal. Hence we have to show that for any such pairs of entries, there are two odd parts among \(\mu_1, \ldots, \mu_{2n}\), with odd multiplicity. Suppose that in \((\alpha, \beta)\) one has \(e_q(\beta_n) = e_q(\alpha_{n+i})\). In view of Lemma 5.2.1 (i), we may then assume that \(i = q\), since otherwise the last \(q - i\) entries of \(\lambda\) will be different odd entries occurring with multiplicity one. But then we may use 5.2.1(ii) to remove the last parts of both \(\alpha\) and \(\beta\). We can continue doing this until we reach a 2-partition of the form \((\kappa, \nu)\) with increasing \(\eta\)-symbol, for which \(\psi_q(\kappa, \nu)\) will indeed have at least \(q\) odd parts with odd multiplicity, but then so does \(\psi_q(\alpha, \beta)\).

LEMA 5.2.3. The map \(\psi_q\) is invariant under the shift operation for symbols, and therefore really is a map on \(\mathbb{Z}_n^{2,0}(n + q, n)\).

Proof: This is clear, since if we add a zero to both \(\alpha\) and \(\beta\), we add a zero to \(\xi\) and \(\eta\), which means we add two zeroes to \(\lambda\).
5.2.2. Half-integer $q$. The map $\psi_q$ for half-integer special $q$ is basically the same as for integer $q$, with the following differences: for a double partition $(\alpha, \beta)$ we consider the $q$-symbol, in which $\alpha$ has $q + \frac{1}{2}$ parts more than $\beta$. We again rearrange the symbol in increasing order and consider the associated 2-composition $(\xi, \eta)$ of $n$, but now we view $\xi$ as having $q - \frac{1}{2}$ more parts than $\eta$. We then compute $(\xi^*, \eta^*)$ and $\mu^*$ as we did for integer $q$. We obtain $\mu$ as $\mu_i = \mu_i^* - (i - 1)$ for $i = 1, 2, \ldots, 2n - 1$, and $\mu_i = \mu_i^* - (2n - 1)$ for $i = 2n, 2n + 1, \ldots, 2n + q - \frac{1}{2}$. Finally we again replace parts $(x + 1, x - 1)$ by $(x, x)$ to obtain the partition $\lambda$. Then we can prove analogously to 5.2.2 that we always find $\psi_q(\alpha, \beta) \in \mathcal{U}_q(n)$.

**Example 5.2.4.** Let $n = 3, q = \frac{3}{2}$ and $(\alpha, \beta) = (-, 12)$. Then the symbol is

\[
\begin{pmatrix}
0 & 2 & 4 & 6 & 8 \\
1 & 4 & 7
\end{pmatrix}
\]

which we rearrange into increasing order to find

\[
\begin{pmatrix}
0 & 2 & 4 & 7 & 8 \\
1 & 4 & 6
\end{pmatrix}
\]

and so the corresponding 2-composition is $(\xi, \eta) = (0010, 011)$ (notice that we have omitted the first zero of $\xi$ to give it the correct length). Therefore we have $(\xi^*, \eta^*) = (0133, 023)$, $\mu^* = (0134677), \mu = \lambda = (0011222)$.

**Example 5.2.5.** Let $n = 6, q = \frac{3}{2}, (\alpha, \beta) = (11, 4)$. Then its symbol is

\[
\begin{pmatrix}
0 & 3 & 5 \\
5
\end{pmatrix}
\]

which we rearrange as

\[
\begin{pmatrix}
0 & 5 & 5 \\
3
\end{pmatrix}
\]

so $(\xi, \eta) = (31, 2), (\xi^*, \eta^*) = (32, 2), \mu^* = (475), \mu = (464)$ and $\lambda = (455)$.

**Example 5.2.6.** Let $n = 6, q = \frac{3}{2}, (\alpha, \beta) = (111, 3)$. Then its symbol is

\[
\begin{pmatrix}
1 & 3 & 5 \\
4
\end{pmatrix}
\]

which we rearrange as

\[
\begin{pmatrix}
1 & 4 & 5 \\
3
\end{pmatrix}
\]

and so we get $(\xi, \eta) = (121, 02)$ (note that this time we have to add a zero to $\eta$ to give it the right length); which yields $(\xi^*, \eta^*) = (133, 03), \mu^* = (03677)$ and $\lambda = \mu = (02444)$.

The point of these maps is of course that they are defined to generalize the diagrams (5.1) and (5.2) into the following:

\[
\begin{array}{ccc}
\mathcal{U}_q(n) \\
\psi_q \searrow \phi_q \searrow f_q^{BC} \searrow \\
\{\tilde{W}_0/\sim\} \overset{\text{Springer}}{\longleftrightarrow} \mathcal{L}(q)/W_0
\end{array}
\]

We will define the map $\phi_q$ in a moment. First we prove
PROPPOSITION 5.2.7. For all special $k_2 = qk_1$, the diagram (5.3) is commutative, i.e.,
\[ (f_q^{BC} \circ \psi_q \circ \Sigma_q)(L) = L \text{ for all } L \in L(q). \]

Proof: Consider $L \in L(q)$ of type $A_\kappa \times (B_1, \nu)$. We prove the Proposition by induction on $l(\kappa)$.

(i) If $l(\kappa) = 0$, then we consider a residual point $c(\nu, k, qk)$, whose $q$-tableau $T_q(\nu)$ we may assume to be in standard position. We have to show that
\[ f_q^{BC}(\psi_q(S_q(\nu))) = (-, \nu). \]

Suppose that the point has jumps \(\{j_1, j_2, \ldots, j_{2m+q}\}\). Then $S_q(\nu) = (\xi, \eta)$ where $\xi = (j_1, j_2, \ldots, j_{2m+1}, j_{2m+2} - 1, j_{2m+3} - 2, \ldots, j_{2m+q} - (q - 1))$ and $\eta = (j_2 + 1, j_4 + 1, \ldots, j_{2m+1})$. This leads to $\xi^* = (j_1, j_3 + 1, j_5 + 2, \ldots, j_{2m+1} + m, j_{2m+2} + m, \ldots, j_{2m+q} + m)$ and $\eta^* = (j_2 + 1, j_4 + 2, \ldots, j_{2m} + m)$ and therefore we find that $\mu^* = (2j_1 + 1, 2j_2 + 2, 2j_3 + 3, 2j_4 + 4, \ldots, 2j_{2m+1} + 2m + 1, 2j_{2m+2} + 2m + 1, \ldots, 2j_{2m+q} + 2m + 1)$, and that $\lambda = \psi_q(\xi, \eta)$ indeed consists of the distinct odd parts $(2j_1 + 1, 2j_2 + 1, \ldots, 2j_{2m+q} + 1)$.

(ii) Now suppose that we are dealing with a subspace of type $A_\kappa \times (B_1, \nu)$ and that we know the proposition is true for every residual subspace $L$ for which $R_L$ contains at most $l(\kappa) - 1$ $A$-factors. After having added the first $l(\kappa) - 1$ $A$-factors, suppose we have a $q$-symbol of size $(m + q, m)$. We consider the effect of adding the two new entries corresponding to the $A$-factor $A_t$, as it has been described in the proof of 4.3.2. Since $S_q(\nu)$ has increasing $q$-symbol, and $\psi_q$ rearranges the entries of the $q$-symbol, there are three possibilities for their position: either they are found in position $(\xi, \eta_i)$ for some $0 \leq i \leq m$, either in position $(\xi_{i+1}, \eta_i)$ for some $0 \leq i < m$, or in position $(\xi_i, \xi_{i+1})$ for some $i > m$. This last possibility does not occur in the group cases. Let us consider these possibilities one by one, where we also have to distinguish between the case where $t$ is odd resp. even. In the first case, if $t$ is odd, we have $(\xi_t, \eta_t) = \left(\left[\frac{t}{2}\right], \left[\frac{t}{2}\right]\right)$. This implies that $\xi_t^* = \left[\frac{t}{2}\right] + (i - 1)$ and $\eta_t^* = \left[\frac{t}{2}\right] + i - 1$, hence in $\mu^*$ we find the parts $\lambda^*_{2i-1} = 2\left[\frac{t}{2}\right] + 2(i - 1) + 1 = t + 2(i - 1)$ and $\mu^*_{2i} = 2\left[\frac{t}{2}\right] + 2(i - 1) = t + 2 i - 1$. This means that indeed we find $\mu_{2i-1} = \mu^*_{2i-1} - (2i - 2) = t$ and $\mu_{2i} = \mu^*_{2i} - (2i - 1) = t$. If $t$ is even, then we find $(\xi_t, \eta_t) = \left(\left[\frac{t}{2}\right], \left[\frac{t}{2}\right]\right)$, $(\xi_t^*, \eta_t^*) = \left(\frac{t}{2} + i - 1, \frac{t}{2} + i - 1\right)$, and so $\mu^*_{2i-1} = t + 2i - 2$, $\mu^*_{2i} = t + 2i - 1$ which again yields $\mu_{2i-1} = \mu_{2i} = t$. This shows that indeed we find two equal entries $t$ in $\lambda$, which will indeed produce a factor $A_{t-1}$ under the map $f_q$. The second case where $(\xi_{i+1}, \eta_t) = \left(\left[\frac{t}{2}\right], \left[\frac{t}{2}\right]\right)$ or $(\frac{t}{2} - 1, \frac{t}{2} + 1)$ proceeds in the same way. Finally consider the third possibility, which happens when there is no 2-partition in the similarity class with increasing $q$-symbol. After rearranging the symbol into increasing order, we find in the "tail" of the symbol containing the last $q$ parts of $\xi$, two consecutive entries with difference $0$ or $1$. They correspond to the largest $A$-factor. This implies that in the proof of 4.3.2, we have found $(\xi'_t, \eta'_m) = \left(\left[\frac{t}{2}\right] - (i - m), \left[\frac{t}{2}\right] + (i - m)\right)$, giving rise to the entries $\left[\frac{t}{2}\right] + i + m$ and $\left[\frac{t}{2}\right] + i + m$. Remember that since we follow the proof of 4.3.2, now $\xi$ is numbered starting with $\xi_0$ and so is $\eta$. In the increasing $q$-symbol, these entries take position $(\xi_{i-1}, \xi_i)$. This means that $\xi_{i-1} = \left[\frac{t}{2}\right] + i + m - 2(i - 1) = \left[\frac{t}{2}\right] - i + m + 2$ and $\xi_i = \left[\frac{t}{2}\right] + i + m - 2i = \left[\frac{t}{2}\right] - i + m$. This implies $\xi_{i-1}^* = \left[\frac{t}{2}\right] + m + 1$ and $\xi_i^* = \left[\frac{t}{2}\right] + m$. So in $\mu^*$ we find two parts $t + 2m$, which means that in $\mu$ we find two parts $t$. If $t$ is even, then we find after rearranging the $q$-symbol in increasing order that $e_q(\xi_{i-1}) = e_q(\xi_i) = \frac{t}{2} + i + m$, hence $\xi_{i-1} = \frac{t}{2} - i + m + 2, \xi_i = \frac{t}{2} - i + m$, so $\xi_{i-1}^* = \frac{t}{2} + m + 1$ and $\xi_i^* = \frac{t}{2} + m$. This means that in $\mu^*$ we find two consecutive parts $t + 2m + 3$ and $t + 2m + 1$, and in $\mu$ we thus get $t + 1$ and $t - 1$. Therefore again $\lambda$ contains 2 new parts $t$ and indeed $f_q^{BC}$ finds a factor
\( A_{l-1} \). Since one easily shows that the other parts of \( \lambda \) have remained the same, this proves the induction step.

(iii) If \( q \) is half-integer, the proof is analogous.

Notice that for \( q = \frac{1}{2}, 1 \), this is essentially how we demonstrated that our general Springer correspondence reduces to the classical one in the group cases.

5.2.3. Examples. Let us consider again the examples mentioned in the definition of \( \psi_q \) and the Examples 5.2.4-6.

In the definition of \( \psi_q \) for integer \( q \) in 5.2.1, we considered \( n = 4, q = 3, (\alpha, \beta) = (-, 4) \), and found \( \lambda = (1344) \). Therefore, the last (44) gives a factor \( A_3 \), after which we are left with (135). Since \(|(135)| = 9 = q^2\), this gives a factor \( B_0 \). We conclude that \((- , 4) \) is the Springer correspondent of the residual subspace of type \( A_3 \).

In Example 5.2.4, we considered \( n = 3, q = \frac{3}{2} \) and \((\alpha, \beta) = (-, 12) \). We found \( \lambda = (11222) \). Therefore \( f_{qB}^0(\alpha, \beta) = A_1 \). We conclude that \( \lambda \) corresponds with the residual subspace of type \( A_1 \).

In Example 5.2.5, we had \( n = 6, q = \frac{3}{2} \) and \((\alpha, \beta) = (11, 4) \). We found \( \lambda = (455) \). Since \( f_{qB}^0(455) = A_4 \times (B_1, (1)) \), we see that \((11, 4) \) is the Springer correspondent of the residual subspace of type \( A_1 \times B_1 \) at \( k_2 = \frac{3}{2}k_1 \).

Finally, in Example 5.2.6, we had \( n = 6, q = \frac{3}{2} \), and \((\alpha, \beta) = (111, 3) \). We found \( \lambda = (244) \). Under the map \( f_{qB}^0 \), we find a factor \( A_3 \) and we are left with (24). This corresponds to a residual point in \( B_2 \) with jumps \( \frac{3}{2} \) and \( \frac{1}{2} \), which is (11). So \((111, 3) \) is the Springer correspondent of the residual subspace \( L \) of type \( A_3 \times (B_2, (1)) \) for \( k_2 = \frac{3}{2}k_1 \).

We can now also describe the map \( \phi_q : U_q(n) \to \tilde{W}_0 \) or \( \sim_q \) itself. First of all we must take care that we give \( \lambda \in U_q(n) \) the correct length, i.e., the length of an element of \( \psi_q(\alpha, \beta) \). For integer \( q, \psi_q(\alpha, \beta) \) has length \( 2m + q \) if the \( q \)-symbol of \((\alpha, \beta) \) has size \( (m + q, m) \). Since \( \lambda \in U_q(n) \), surely \( l(\lambda) \geq q \). Therefore we must also require that \( l(\lambda) = q (\text{mod} \ 2) \).

But clearly this is automatically true for any \( \lambda \in U_q(n) \). For half-integer \( q \), however, we find that the length \( l(\lambda) \) of \( \lambda \in U_q(n) \) must satisfy \( l(\lambda) = q - \frac{1}{2} (\text{mod} \ 2) \). This does not follow automatically, so we may have to add a zero to \( \lambda \). We can now describe the map \( \phi_q \).

**DEFINITION 5.2.8.** Let \( \lambda \in U_q(n) \). Its parts are arranged in increasing order.

(i) For integer resp. half-integer \( q \), replace in the last \( q \) resp. \( q - \frac{1}{2} \) parts of \( \lambda \) a pair of even resp. odd entries \((x, x)\) by \((x + 1, x - 1)\) to obtain an \( n \)-composition \( \mu \). If \( q \) is half-integer, possibly add a zero to make sure \( l(\mu) = q - \frac{1}{2} + 2m \) for some \( m \).

(ii) Define \( \mu_i^* = \mu_i + (i - 1) \) for \( i = 1, 2, \ldots, 2m \) and \( \mu^*_{2m+i} = \mu_{2m+i} + 2m \) for \( i = 1, 2, \ldots, q \) if \( q \) is integer; and \( \mu_i^* = \mu_i + (i - 1) \) for \( i = 1, 2, \ldots, 2m - 1, \mu^*_{2m+i} = \mu_i + (2m - 1) \) for \( i = 1, 2, \ldots, q - \frac{1}{2} \) if \( q \) is half-integer.

(iii) Form the 2-composition \((\xi^*, \eta^*)\) where \( \eta^* \) contains the \( \mu_i^* \) for even \( \mu_i \), and \( \xi^* \) contains the \( \lfloor \frac{\mu_i}{2} \rfloor \) for odd \( \mu_i \) (in the order they appear in \( \mu^* \)).

(iv) Define the 2-composition \((\xi, \eta)\) by \( \xi_i = \eta^*_i - (i - 1) \) and \( \eta_i = \eta^*_i - (i - 1) \). If \( q \) is half-integer, readjust the lengths of \( \xi \) and \( \eta \) to \( l(\xi) = l(\eta) + q + \frac{1}{2} \).

(v) Finally \( \phi_q(\lambda) := [(\xi, \eta)]_q = \{(\alpha, \beta) \mid n \mid (\alpha, \beta) \sim_q (\xi, \eta)\} \).

We have now fully completed the triangle (5.3); all maps in it are explicit bijections. For convenience, we summarize this chapter into the following
5.2. GENERALIZATION OF THE MAPS $\psi_q$ AND $\phi_q$

**Theorem 5.2.9.** Let $k_2 = qk_1$ be special. Then we have explicit bijections

$$
\mathcal{U}_q(n) \longleftrightarrow W_0 \backslash \mathcal{L}(q) \longleftrightarrow \tilde{W}_0 / \sim_q
$$

such that in the group cases $q \in \{\frac{3}{2}, 1\}$, the corresponding maps reduce to the maps that describe the classical Springer correspondence.

**Example 5.2.10.** Let $q = 3$, and $\lambda = (13447)$. Then we obtain $\mu = (13537), \mu^* = (14759), (\xi^*, \eta^*) = (0324, 2), (\xi, \eta) = (0201, 2)$ with q-symbol

$$
\begin{pmatrix}
0 & 4 & 4 & 7 \\
2 & & & \\
& 4 & & 7 \\
& & 3 & &
\end{pmatrix}
$$

Since the q-symbol of $(1, 4)$ is

$$
\begin{pmatrix}
0 & 2 & 4 & 7 \\
& 4 & & 7 \\
& & 3 & &
\end{pmatrix}
$$

and there are no other 2-partitions with similar q-symbol, we conclude $\phi_q(13447) = (1, 4)$.

**Example 5.2.11.** Let $q = \frac{3}{2}$ and $\lambda = (2444)$. Then $\mu = (02444), \mu^* = (03677), (\xi^*, \eta^*) = (133, 03), (\xi, \eta) = (121, 2)$ (after adjusting the length of $\eta$ to make sure that $l(\xi) = l(\eta) + 2$) with q-symbol

$$
\begin{pmatrix}
1 & 4 & 5 \\
3 & & \\
& 4 & 5 \\
& & 3 &
\end{pmatrix}
$$

Since $(111, 3)$ has q-symbol

$$
\begin{pmatrix}
1 & 3 & 5 \\
4 & & \\
& 4 & 5 \\
& & 3 &
\end{pmatrix}
$$

and there are no other 2-partitions with similar q-symbol, we conclude that $\phi_q(2444) = (111, 3)$.

**Example 5.2.12.** Let $q = \frac{5}{2}$ and $\lambda = (112455)$. Then $\mu = (112464), \mu^* = (124797), (\xi^*, \eta^*) = (0343, 12), (\xi, \eta) = (00220, 11)$ with q-symbol

$$
\begin{pmatrix}
0 & 2 & 6 & 8 & 8 \\
2 & 4 & & & \\
& & 8 & 6 & 8 \\
& & & 2 &
\end{pmatrix}
$$

Since $(-, 15)$ has q-symbol

$$
\begin{pmatrix}
0 & 2 & 6 & 8 & 8 \\
2 & 4 & & & \\
& & 8 & 6 & 8 \\
& & & 2 &
\end{pmatrix}
$$

and there are no other 2-partitions with similar q-symbol, we conclude that $\phi_q(112455) = (-, 15)$. 
