A combinatorial generalization of the Springer correspondence for classical type
Slooten, K.J.

Citation for published version (APA):
Slooten, K. J. (2003). A combinatorial generalization of the Springer correspondence for classical type
CHAPTER 8

Examples

In this section, we illustrate Conjecture 6.5.3, by calculating explicitly the irreducible tempered representations with real central character of several affine Hecke algebras, and checking that they indeed confirm the Conjecture. We use Lusztig's theorem 1.3.1 to switch back to the affine Hecke algebra with $Q = P$ and root labels $q_i = \exp(k_i)$.

If $L \subset T$ is a residual coset, we are interested in the $\mathcal{H}_0$-decomposition of the tempered representation (see 1.4):

$$\text{Ind}_{\mathcal{H}_L}^{\mathcal{H}_0}(\delta \circ \phi_{\ell L}),$$

where $\delta$ is a discrete series representation of $\mathcal{H}_L$ with central character $W_L \tau_L$, $t^L \in T^L_u$, and we let $r = r_L t^L$. Since the $\mathcal{H}_0$-decomposition of the induced representation does not depend on $t^L$, we take $t^L = 1$ (and thus obtain a representation with real central character) and write $\chi_r$ for its character.

The central characters of $\mathcal{H}_0^{\text{temp}}$ are thus exactly those of the form $W_0 \tau_L$, where $\tau_L$ is a residual point of the parabolic algebra $\mathcal{H}_L$. In each example, we list these centers by the type of the root system $R_L$, together with a partition of $l$ for the factor $B_1$ occurring in $R_L$, since such a root system has a generically residual point for each partition of $l$. For notational convenience we write $(\xi, \eta)$ for the irreducible $\mathcal{H}_0$-character indexed by the double partition $(\xi, \eta)$. Recall that this indexation is as in [15], i.e., $(n, -)$ is the trivial character and $(-, 1^n)$ is the sign character.

First we prove a lemma, which will be useful in computations later. It is a generalization of Lemma 1.1.5.

**Lemma 8.0.3.** Let $\mathbb{H}$ be the graded Hecke algebra with root labels $k_\alpha$, and let $\gamma \in \mathfrak{t}$. Let the stabilizer $W_\gamma$ contain the standard parabolic subgroup $W_J$ for $J \subset I$. Let $M$ be an $\mathbb{H}$-module such that $M_\gamma \neq 0$. Then

1. $\dim(M_\gamma) \geq |W_J|$.
2. Suppose that $M_{\alpha_i \gamma} = 0$. Then $\langle \alpha_i, \alpha_j \rangle = 0$ for all $j \in J$.

**Proof:** (i) Let $\mathbb{H}^J = \mathbb{C}[W_J] \otimes S[\mathfrak{t}^*]$. Let $C_\gamma$ be the one-dimensional representation of $S[\mathfrak{t}^*]$ affording the character $\gamma$. Then, by Kato's theorem 1.1.4, the induced module $M_J(\gamma) = \text{Ind}_{S[\mathfrak{t}^*]}^{\mathbb{H}^J}(C_\gamma)$ is irreducible, of dimension $|W_J|$. If we choose a basis of $M_J(\gamma)$ consisting of vectors $w \cdot 1$ ordered in the Bruhat-ordering, then an element $x \in \mathfrak{t}^*$ acts by $\gamma(x)$ times a unipotent upper-triangular matrix. Choose a non-zero vector $v_\gamma \in M_\gamma$. Then there exists a unique homomorphism of $\mathbb{H}_i^J$-modules

$$M_J(\gamma) \rightarrow M$$

$$1 \mapsto v_\gamma$$

Since $M_J(\gamma)$ is irreducible, this homomorphism is injective.
(ii) Recall the intertwiner (1.1) from section 1.1.2.6; \( \tau_i : M_\gamma \rightarrow M_{s_i \gamma} \). Suppose that \( M_{s_1 \gamma} = 0 \), then \( s_1(\gamma) \neq \gamma \), so \( \gamma(\alpha_i) \neq 0 \). Since \( M_{s_1 \gamma} = 0 \), \( \tau_i \) is the zero operator, which implies that \( s_i \) acts as \( \frac{\Delta_i}{\alpha_i} \) on \( M_\gamma \). Let \( v \in M_\gamma \) be a pure weight vector. The two-dimensional subspace of \( M_\gamma \) spanned by \( v \) and \( s_j v \) is left invariant by \( \alpha_i \), hence also by \( s_i \). Since \( s_i \) acts as \( \frac{\Delta_i}{\alpha_i} \) on \( M_\gamma \), its action on the indicated two-dimensional space is given by \( \gamma(\alpha_i)^{-1} \) times a unipotent matrix which contains a non-zero multiple of \( \langle \alpha_i, \alpha_j \rangle \) above the diagonal. On the other hand, \( s_i^2 = 1 \), so we conclude \( \langle \alpha_i, \alpha_j \rangle = 0 \). This goes for any \( j \in J \), so the Lemma follows. \( \square \)

8.1. \( B_2 \)

First we consider the simplest example, \( B_2 \). We treat it in the same way as \( G_2 \), i.e., we first study the generic parameter values, and then see what happens at special values. Note that we only have two special values here, only the two group cases occur.

8.1.1. Generic parameter values.

**Proposition 8.1.1.** In the following table, we have put in the first column the type of the real residual coset \( L \). In the remaining columns we give the \( \mathcal{H}_0 \)-types of the irreducible tempered representations of \( \mathcal{H} \) with central character \( W_0 T_L \) at the various generic values of the parameters \( k_i = \log(q_i) \). These representations are the following:

<table>
<thead>
<tr>
<th>center ( k_1 &lt; k_2 )</th>
<th>( k_2 &lt; k_1 &lt; 2k_2 )</th>
<th>( 2k_2 &lt; k_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_2, (2) )</td>
<td>( (-1,^2) )</td>
<td>( (-1,^2) )</td>
</tr>
<tr>
<td>( B_2, (1^2) )</td>
<td>( (-,2) )</td>
<td>( (1,1) + (-,11) )</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>( (1^2,^2) ) + ( (1,1) + (-,12) )</td>
<td>( (1^2,^2) ) + ( (1,1) + (-,12) )</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>( (1,1) + (-,2) + (-,12) )</td>
<td>( (1,1) + (-,2) + (-,12) )</td>
</tr>
<tr>
<td>( \emptyset )</td>
<td>regular(( (2,^2) ))</td>
<td>regular(( (2,^2) ))</td>
</tr>
</tbody>
</table>

Note that we have written \( (\lambda, \mu) \) for \( \chi(\lambda, \mu) \). The Springer correspondent, tensored with the sign representation, is accentuated in boldface.

**Proof:** (i) \( \emptyset \) corresponds to the center \( 1 \in T \) of the residual coset \( L = T \). Therefore the \( \mathcal{H}_0 \)-type of the irreducible \( \mathcal{H} \)-module \( \text{Ind}_{C(T)}^H(C_1) \) is just the regular representation of \( \mathcal{H}_0 \).

(ii) Types \( A_1 \) and \( B_1 \). The associated Hecke algebras \( \mathcal{H}_L \) have only one residual point. In both case they are the central character of the Steinberg representation of \( \mathcal{H}_L \). Therefore, to compute the \( \mathcal{H}_0 \)-type of the irreducible tempered representation with central character \( W_0 T_L \), it suffices to induce the sign representations of \( W_0(A_1) \) resp. \( W_0(B_1) \) to \( W_0(B_2) \).

(iii) The residual points \( B_2, \lambda \). Since both are regular, one can easily calculate the dimension of the discrete series representations we are looking for, by using Proposition 1.2.7. We will only do this computation in detail in the next example where we consider \( \mathcal{H}(B_3) \). We first consider \( r = \exp(c((2), k_1, k_2)) \). One checks that the only point in the orbit \( W_0 T_L \) satisfying the condition in this Proposition is \( t = w_0 r \), which verifies \( \alpha_i(t) = q_i^{-1} \). Therefore the associated one-dimensional discrete series representation is the Steinberg representation, the restriction of which to \( \mathcal{H}_0 \) is its sign representation \( (-,1^2) \). As for \( \lambda = (1^2) \), the chamber condition yields dimensions 1,3,1 in the respective parameter regions. The one-dimensional representations are again easy to determine. The three-dimensional representation can be computed in the same way as we did for \( G_2 \), i.e., by computing the matrices of \( T_i \) acting on this three-dimensional space, and comparison with the character table of \( W_0 \). \( \square \)
8.2. B₃

8.1.2. Special parameter values.

- First we consider the $B_2$-group case, arising if $k_1 = k_2$. The point $r$ of type $B_2, (11)$ ceases to be residual but coincides with the center of the one-dimensional residual coset of type $B_1$. Suppose that we have chosen $r$ in its $W_0$-orbit such that $\alpha_1(r) = q^{-1}, \alpha_2(r) = 1$. It follows from Lemma 1.1.5 that if $V$ is a $H$-module with $V_r \neq 0$, then $\dim(V_r) = 2, \dim(V_{s_1 r}) = 1$. On the other hand, it is easy to see that there is a one-dimensional tempered representation with weight $s_1 r$. Its restriction to $H_0$ is $(-, 2)$. To compute the $H_0$-restriction of $V$, we consider $\text{Ind}_{H(A_1)}^H(C_r)$, whose restriction to $H_0$ is $(1, 1) + (-, 11) + (11, -)$. One checks that the weights of this representation are $r, s_1 r, s_2 s_1 r$ and thus $V$ is a three-dimensional subrepresentation. On the other hand, there exists a one-dimensional representation of $H$ with weight $s_2 s_1 r$. Its $H_0$-decomposition is $(11, -)$. It follows that the $H_0$-decomposition of $V$ is $(1, 1) + (-, 11)$.

We conclude that there exist two irreducible tempered representations with central character $r$, whose $H_0$-types are $(1, 1) + (-, 11)$ and $(-, 2)$.

- Next, we consider the $C_2$-group case $q_2^2 = q_1$. Completely analogous, we find that there are two irreducible tempered representations with central character $r$, where $r$ denotes the center of the one-dimensional coset of type $A_1$. They are three-, resp. one-dimensional, with $H_0$-type $(1, 1) + (-, 11)$ resp. $(11, -)$.

REMARK 8.1.2. For all generic parameters, the $W_0$-characters corresponding to the irreducible discrete series characters of $H$ (by taking, of course, the limit $\sqrt{q_i} \to 1$ in the $H_0$-character) form an orthonormal basis for the elliptic pairing on $\hat{W}_0$. This can be verified either by direct calculation in all the components of regular parameter space. Otherwise, for $k_1 < k_2$ it follows from Corollary 7.2.4, while at every transition, we replace the representation $V$ by a representation $V'$ where $V \oplus V'$ is a parabolically induced representation.

Therefore $[V \oplus V'] = 0$ in $\hat{\mathcal{R}}(\hat{W}_0)$.

REMARK 8.1.3. Since there are no other special values than the group case, we could also have calculated $\mathcal{H}^\text{temp}_{\text{reg}}$ for these parameters by using the Green functions, which are listed in section 6.4. In any case, we easily check that indeed Conjecture 6.5.3(iv) holds for $B_2$ (parts (i)-(iii) are known to be true).

8.2. B₃

8.2.1. Regular parameter values. We can now do the same calculations for $B_3$. First we treat the generic parameter values. This yields:

PROPOSITION 8.2.1. Suppose that the parameters are generic. Then the irreducible discrete series representations with real central character of $H$ have the following $H_0$-decomposition:

$$
\begin{align*}
\text{center} & \quad k_1 < \frac{1}{2} k_2 & \frac{3}{2} k_2 < k_1 & \frac{1}{2} k_2 < k_1 < \frac{3}{2} k_2 & \frac{3}{2} k_2 < k_1 < k_2 \\
B_3, (3) & \quad (-, 1^5) & (-, 1^3) & (-, 1^3) \\
B_3, (21) & \quad (-, 21) & (-, 21) & (-, 21) \\
B_3, (1^3) & \quad (-, 3) & (1, 2) + (-, 21) & (1^2, 1) + (-, 1^3) + (1, 1^2) \\
\text{center} & \quad k_2 < k_1 < 2k_2 & 2k_2 < k_1 \\
B_3, (3) & \quad (-, 1^3) & (-, 1^3) \\
B_3, (21) & \quad (1, 1^2) + (-, 1^3) & (1, 1^2) + (-, 1^3) \\
B_3, (1^3) & \quad (1^3, -) & (1^3, -)
\end{align*}
$$
The \( H_0 \)-types of the remainder of \( \hat{H}_{\text{rec}} \) can be determined by induction. Note that we have again written \( (\lambda, \mu) \) for \( \chi(\lambda, \mu) \). The Springer correspondent, tensored with the sign representation, is accentuated in boldface.

**Proof:** All three residual points are regular, and therefore section 1.2.9 applies. We use it to calculate the dimension of the discrete series representations. We only explicitly treat the case \( r = \exp(c) \) with \( c = c((1^3), k_1, k_2) \), the computations for the other points being similar. The discrete series module \( V_r \) is a quotient of the principal series module \( \text{Ind}_{C(t)}^G(\mathbb{C}_r) \), and therefore is a direct sum of one-dimensional (since \( r \) is regular) weight spaces \( V_r' \), with \( r' \in W_0r \). Since it is easier to work in vector spaces, from now on we work in \( t := \text{Lie}(T_r) \).

We define a hyperplane arrangement \( \mathcal{C} \) in \( t \), by taking the logarithms of \( \alpha \), i.e.,

\[
C_r = \{ c \in t \mid \text{Lie}(T_r^\alpha) \cap \mathcal{C} = \emptyset \}.
\]

Then \( t - \mathcal{C} \) consists of \( 2^3 = 8 \) open chambers, since \( r \) lies on 3 residual hyperplanes \( L_\alpha \). If \( x \in t \), we denote by \( C(x) \) the chamber in which \( x \) lies. For a chamber \( C \), we define its anti-dual \( C^{ad} \) to be

\[
C^{ad} := \{ x \in t \mid \langle x - r, C_i - r \rangle \leq 0 \}.
\]

We know from Proposition 1.2.7, that for \( \gamma_0 := \log(t_0) \):

\[
0 \notin C^{wr}(\gamma_0)^{ad} \Rightarrow V_r^{wr} = 0.
\]

**Lemma 8.2.2.** \( 0 \in C^{wr}(\gamma_0)^{ad} \iff 0 \in C^{r}(w^{-1}\gamma_0)^{ad} \).

**Proof:** Since \( C^{wr}(\gamma_0) = w(C^{r}(w^{-1}\gamma_0)) \), we have

\[
0 \in C^{wr}(\gamma_0) \iff \langle -wr, C^{wr}(\gamma_0) - wr \rangle \leq 0
\]

\[
\iff \langle -wr, w(C^{r}(w^{-1}\gamma_0) - wr) \rangle \leq 0
\]

\[
\iff \langle -r, C^{r}(w^{-1}\gamma_0) - r \rangle \leq 0
\]

\[
\iff 0 \in C^{r}(w^{-1}\gamma_0)
\]

which is what we wanted to show. \( \square \)

So we have to determine

\[
|\{w \in W_0 \mid 0 \in C^{r}(w\gamma_0)^{ad}\}|.
\]

First we determine for which chamber \( C \) we have \( 0 \in C^{ad} \) (in general, this chamber will depend on the parameters), and then we determine how many \( W_0 \)-conjugates \( w^{-1}t_0 \) of \( t_0 \) are in \( C_i \). We also know that the weights of \( V_r \) form a connected component of the calibration graph \( \Gamma(r) \). Since \( V_r^{wr} = 0 \) if \( 0 \notin C^{wr}(\gamma_0) \), and there exists a discrete series representation with central character \( r \), it follows that

\[
(8.1) \quad \dim(V_r) = |\{w \in W_0 \mid w\gamma_0 \in C_i\}| \text{ if } 0 \in C^{ad}_i.
\]

Let us now do these calculations for the case \( B_3, (1^3) \), i.e., we take \( c = (-2k_1 + k_2, -k_1 + k_2, k_2) \) \( \in t \). Then \( c \) lies on the 3 residual hyperplanes \( L_1 = L_{-\alpha_1} = (-k_1 + k_2, 0) + \{ (x, y, z) \mid x, y \in \mathbb{R} \} = r_1 + V^1 \), \( L_2 = L_{-\alpha_2} = (0, -k_1 + k_2, 0) + \{ (x, y, z) \mid x, y \in \mathbb{R} \} = r_2 + V^2 \), \( L_3 = L_{-\alpha_3} = (0, 0, k_2) + \{ (x, y, z) \mid x, y \in \mathbb{R} \} = r_3 + V^3 \). Therefore the chamber in which a point \( x \) lies is determined by the signs of the inner products \( \langle x, V^\perp \rangle \). We thus
find chambers
\[
C_1 = \{(x_1, x_2, x_3) \mid x_1 - x_2 > -k_1, x_2 - x_3 > -k_1, x_3 > k_2\},
\]
\[
C_2 = \{(x_1, x_2, x_3) \mid x_1 - x_2 < -k_1, x_2 - x_3 > -k_1, x_3 > k_2\},
\]
\[
C_3 = \{(x_1, x_2, x_3) \mid x_1 - x_2 < -k_1, x_2 - x_3 < -k_1, x_3 > k_2\},
\]
\[
C_4 = \{(x_1, x_2, x_3) \mid x_1 - x_2 < -k_1, x_2 - x_3 < -k_1, x_3 < k_2\},
\]
as well as four others that we will not need in the sequel. One then checks that the anti-duals are given by
\[
C^{ad}_1 = \{(x_1 - 2k_1 + k_2, x_2 - k_1 + k_2, x_3 + k_2) \mid x_1 < 0, x_1 + x_2 < 0, x_1 + x_2 + x_3 < 0\}
\]
and similarly for the other \(C^{ad}_i\), and knowing this we find that
\[
0 \in C^{ad}_1 \quad \text{if} \quad k_1 < \frac{1}{2} k_2
\]
\[
0 \in C^{ad}_2 \quad \text{if} \quad \frac{1}{2} k_2 < k_1 < \frac{5}{3} k_2
\]
\[
0 \in C^{ad}_3 \quad \text{if} \quad \frac{5}{3} k_2 < k_1 < k_2
\]
\[
0 \in C^{ad}_4 \quad \text{if} \quad k_2 < k_1
\]
Using (8.1), we then find:
\[
\dim(V_r) = 1 \quad \text{if} \quad k_1 < \frac{1}{2} k_2
\]
\[
\dim(V_r) = 5 \quad \text{if} \quad \frac{1}{2} k_2 < k_1 < \frac{5}{3} k_2
\]
\[
\dim(V_r) = 7 \quad \text{if} \quad \frac{5}{3} k_2 < k_1 < k_2
\]
\[
\dim(V_r) = 1 \quad \text{if} \quad k_2 < k_1
\]
The computations for the other residual points are entirely analogous.

Next, we want to find explicitly the required \(\mathcal{H}_0\)-characters \(\chi_r\) for the three residual points.

- All the one-dimensional representations are easy to determine directly by their central character.
- The \(\mathcal{H}_0\)-types of the other higher-dimensional representations can be computed analogous to the way we solved this problem in the \(G_2\)-example: we choose for each \(\mathcal{H}\)-module a basis consisting of weight vectors, and use the cross relations (1.3) to determine the matrices of the action of the \(T_\gamma\). Even though this does not allow us to compute all matrix entries, one may check that still one can compute the character table of all \(T_w, w \in W_0\), and thus the table entries.

**Remark 8.2.3.** 1. Notice that again, in every sector of the parameter values we find that the \(\chi_r\) for the three residual points indeed form an orthonormal basis for the elliptic pairing on Weyl group representations.

2. If we define \(f_V\) as in 7.3.1 for \(V = (-, 21)\), we find indeed that \(\tau(f_V) = \mu_{P_l}(r_{(21)})\). For \(V = (1, l^2) + (-, 1^3)\), we find \(\tau(f_V) = -\mu_{P_l}(r_{(21)})\).

**8.2.2. Special parameter values.**

- First we consider the special value \(k_2 = 2k_1\). Let \(q_1 = q\), then \(q_2 = q^2\). In this situation, the point \(r\) of type \(B_3, (1^3)\) is no longer residual, but coincides with the center of the coset of type \(B_2(1^2)\), which is still residual. Choose \(r\) in its orbit such that \(\alpha_1(r) = q, \alpha_2(r) = q, \alpha_3(r) = q^{-2}\). First of all, there clearly exists a one-dimensional module \(\mathbb{C}_r\) with weight \(r\), whose restriction to
\( \mathcal{H}_0 \) is \((-3,3)\). One checks that this is indeed a tempered module: the fundamental weights take values \(1, q^{-1}, q^{-3/2}\) respectively. Now suppose that \(V\) is a \(\mathcal{H}\)-module with \(V_{s_2 s_1 r} \neq 0\). Then it follows from the affine analog of 1.1.5 that \(\dim(V_{s_2 s_1 r}) = 2\), \(\dim(V_{s_1 r}) = 2\), \(\dim(V_r) = 1\). We thus find a \(\mathcal{H}\)-module \(V\) which is at least five-dimensional. On the other hand, let \(M\) be the discrete series representation of \(\mathcal{H}_L = \mathcal{H}(B_2)\) with central character \(W_L r\), then \(V\) is a summand of the induced representation \(M^{\text{ind}} = \text{Ind}_{\mathcal{H}_L}(M \circ \phi_1)\). By Frobenius reciprocity and since we have calculated the restriction of \(M\) to \(\mathcal{H}_0(B_2)\), we see that \(M^{\text{ind}}|_{\mathcal{H}_0} \cong (-3,3) + (1,2) + (-,21)\). Choose a basis vector \(v\) for \(M\), then \(M^{\text{ind}}\) has basis \(\{v, T_1 v, T_2 T_1 v, T_3 T_2 T_1 v, T_1 T_2 T_3 T_2 T_1 v, T_1 T_2 T_3 T_2 T_1 v\}\), which means that \(M^{\text{ind}} \cong V \oplus C_r\) as \(C[\theta_2]\)-modules. Since the dimension of the weight space of \(r\) in the minimal principal series representation with central character \(r\) is two, we conclude that \(V_{\mathcal{H}_0} \cong (1,2) + (-,21)\). Again, one checks that \(V\) is a tempered \(\mathcal{H}\)-module.

**8. EXAMPLES**

- Next, consider the case \(k_2 = \frac{3}{2} k_1\). The situation is analogous: this time the point \(r\) of type \(B_3(1^3)\) is no longer residual either, and coincides with the center of the residual coset of type \(A_1 \times B_1\). Along the same lines as above, we find that there are two tempered \(\mathcal{H}\)-modules with central character \(r\), which have \(\mathcal{H}_0\)-restriction \((2,1) + (-,21)\) and \((1^2,1) + (1,1^2) + (-,1^3)\) respectively.

- Now consider the \(B_3\)-group case \(q_1 = q_2\). The points of type \(B_3(21)\) and \(B_3, (1^3)\) are no longer residual, and coincide with the centers of the residual cosets of type \(B_2, (2)\) and \(A_2\) respectively. First consider \(r \in T\) of type \(B_3(21)\). We choose \(r\) such that \(\alpha_1(r) = q^0, \alpha_2(r) = q^{-1}, \alpha_3(r) = q^{-1}\). The parabolic subalgebra of type \(B_2\) has a one-dimensional representation \(M\) sending \(T_2 \mapsto -1, T_3 \mapsto -1, \theta_x \mapsto r(x)\). The induction of \(M\) to \(\mathcal{H}\) yields a six-dimensional representation \(M^{\text{ind}}\) with weight spaces for \(r, s_2 r\) and \(s_1 s_2 r\), all two-dimensional. By using Frobenius reciprocity, one checks that as a \(\mathcal{H}_0\)-module, \(M^{\text{ind}}\) is isomorphic to \((-21) + (1,1^2) + (-,1^3)\). Now we need to determine the \(\mathcal{H}\)-decomposition of \(M^{\text{ind}}\). First we observe, by computing the action of the \(T_i\), that there exists a two-dimensional representation \(N^1\) of \(\mathcal{H}\) with weights \(r\) and \(s_2 r\). As a \(\mathcal{H}_0\)-module, this representation is isomorphic to \((-21)\). This means that we find a \(\mathcal{H}\)-subrepresentation in \(W^{\text{ind}}\) with \(\mathcal{H}_0\)-type \((-21)\). Now let \(N^2\) be a \(\mathcal{H}\)-module with \(N^{2}_{s_2 s_1 r} \neq 0\). Then again an application of 1.1.5 implies that \(\dim(N^2_{s_2 s_1 r}) = 2, \dim(N^2_{s_1 r}) = \dim(N^2_r) = 1\). This means that \(M^{\text{ind}}/N^1 \cong N^2\), and thus that \(N^2\) has \(\mathcal{H}_0\)-type \(\phi_{(1,1^2)} + \phi_{(-,1^3)}\). Another way to compute the representation \(N^2\) is as follows. Consider the one-dimensional representation of \(\mathcal{H}(A_2)\) with central character \(r\), and basis \(v\), defined by \(T_1(v) = -v, T_2(v) = -v, \theta_x \mapsto s_2 s_1 r(x)v\). Denote the induction of this module to \(\mathcal{H}\) by \(P\). Then \(P\) has basis \(\{v, T_3 v, T_2 T_3 v, T_3 T_2 T_3 v, T_1 T_2 T_3 T_2 T_3 v, T_1 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v, T_3 T_2 T_3 T_2 T_3 v\}\). The cross relations (1.3) then enable us to compute the matrices of the action of \(T_i\) and \(\theta_4\). With this explicit description at hand, one verifies that \(P\) has a four-dimensional subrepresentation whose restriction to \(\mathcal{H}_0\) is isomorphic to \(V(1^3, -) \oplus V(1^2, 1)\), and a four-dimensional tempered quotient module whose \(\mathcal{H}_0\)-restriction is isomorphic to \(V(1^3, -) \oplus V(1^2, 1)\). This last module indeed has weights \(r, s_1 r, s_2 s_1 r\). We thus have obtained two irreducible tempered \(\mathcal{H}\)-representations whose central character is \(r\).
Now we consider the central character of type $B_3(1^3)$. We choose $r$ of this type such that $\alpha_1(r) = q^{-1}, \alpha_2(r) = q^{-1}, \alpha_3(r) = q$. There is a one-dimensional module $M$ of the parabolic subalgebra $\mathcal{H}(A_2) = \mathcal{H}_0(A_2) \otimes \mathcal{A}$ in which $T_1 \mapsto -1, T_2 \mapsto q$ and $\theta(x)$ acts as $r(x)$. Its $\mathcal{H}_0(A_2)$-type is $\begin{pmatrix} 1^3 \end{pmatrix}$, hence by Frobenius reciprocity, induction of this module to $\mathcal{H}$ yields an eight-dimensional representation $M^{\text{ind}}$ whose $\mathcal{H}_0$-type is $\begin{pmatrix} 1^3, - \end{pmatrix} + \begin{pmatrix} 1^2, 1 \end{pmatrix} + \begin{pmatrix} 1, 1^2 \end{pmatrix} + \begin{pmatrix} -, 1^3 \end{pmatrix}$. One can show that is tempered either by calculating the occurring weights of $M^{\text{ind}}$, or by remarking that it is actually the induction of a discrete series representation, and therefore surely tempered. Moreover, one sees immediately that there is a one-dimensional $\mathcal{H}$-module with central character $r$: we can make $M$ into an $\mathcal{H}$-module by putting $T_3 \mapsto q$. The $\mathcal{H}_0$-type of this representation is $\begin{pmatrix} 1^3, - \end{pmatrix}$. One checks that $M^{\text{ind}}$ has weights $r, s_3 r, s_2 s_3 r$ with multiplicity (resp.) 2, 2, 4. Suppose that $V$ is an $\mathcal{H}$-module such that $V_{s_2 s_3 r} \neq 0$. Since $\alpha_1(s_2 s_3 r) = \alpha_3(s_2 s_3 r) = 1$, it follows from 8.0.3 that $\dim(V_{s_2 s_3 r}) = 4$. By calculating the action of the intertwiner $\tau_2 : V_{s_2 s_3 r} \rightarrow V_{s_3 r}$ one can then check that $\dim(V_{s_3 r}) = 2$, and similarly that $\dim(V_r) = 1$. Summarizing, we find indeed a one-dimensional and a seven-dimensional irreducible $\mathcal{H}$-representation with central character $r$, with $\mathcal{H}_0$-types (resp.) $\begin{pmatrix} 1^3, - \end{pmatrix}$ and $\begin{pmatrix} 1^2, 1 \end{pmatrix} + \begin{pmatrix} 1, 1^2 \end{pmatrix} + \begin{pmatrix} -, 1^3 \end{pmatrix}$. It is not hard to see that there are no irreducible tempered representations of $\mathcal{H}$ with the central characters that we have considered.

- Finally we consider the $C_3$-group case where $q := q_1 = q_2^2$. Then the two generically residual points $(12)$ and $(1^3)$ coincide into the same residual point. Selecting $r$ in its orbit such that $\alpha_1(r) = q^{-1}, \alpha_2(r) = q^{-1}, \alpha_3(r) = q^{1/2}$, it is easy to see that there is a one-dimensional discrete series representation with central character $r$ and $\mathcal{H}_0$-type $\begin{pmatrix} 1^3, - \end{pmatrix}$, as well as a four-dimensional irreducible discrete series representation with weights $r, s_1 r, s_2 s_1 r$ and $\mathcal{H}_0$-type $\begin{pmatrix} 1, 1^2 \end{pmatrix} + \begin{pmatrix} -, 1^3 \end{pmatrix}$.

Again, strictly speaking it would not have been necessary to perform these calculations for the two group cases, but we have chosen to treat all parameter choices equally, and to illustrate that the all special cases are essentially equally tractable.

### 8.2.3. Green functions.

Since there are four special values, we also have four types of Green functions for type $B_3$. We restrict ourselves to the special case $k_2 = 2k_1$, i.e., $q = 2$. Then we order the double partitions of 3, such that $\alpha \succ \beta$ implies $a_2(\alpha) \leq a_2(\beta)$, and such that similarity classes form intervals. In this case, we have as $a_2$-values

<table>
<thead>
<tr>
<th>$(\xi, \eta)$</th>
<th>$a_2(\xi, \eta)$</th>
<th>$(\xi, \eta)$</th>
<th>$a_2(\xi, \eta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-, 111)$</td>
<td>12</td>
<td>$(11, 1)$</td>
<td>3</td>
</tr>
<tr>
<td>$(-, 12)$</td>
<td>8</td>
<td>$(111, -)$</td>
<td>3</td>
</tr>
<tr>
<td>$(1, 11)$</td>
<td>6</td>
<td>$(2, 1)$</td>
<td>2</td>
</tr>
<tr>
<td>$(-, 3)$</td>
<td>5</td>
<td>$(12, -)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>4</td>
<td>$(3, -)$</td>
<td>0</td>
</tr>
</tbody>
</table>

and also $(11, 1) \sim_2 (111, -)$; we therefore reverse the ordering in the table. We then calculate the matrix $Q^2_{\alpha, \beta}$ from the matrix $P^2_{\alpha, \beta}$ computed for us by G. Malle. The corresponding
characters $\psi^{2, \alpha, i}$ are then as follows. The coset $L$ is calculated as $f_q^{BC}(\psi_q(\xi, \eta))$.

<table>
<thead>
<tr>
<th>$(\xi, \eta)$</th>
<th>$L$</th>
<th>$i$</th>
<th>$\psi^{2, (\xi, \eta), i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-, 111)$</td>
<td>$\emptyset$</td>
<td>3</td>
<td>$(3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$(2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>$(1, 2) + (12, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>$(-, 3) + (11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>$(1, 11) + (1, 2) + (12, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>$(-, 12) + (11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9</td>
<td>$(1, 11) + (11, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>$(-, 12) + (11, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>$(1, 11)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>$(-, 111)$</td>
</tr>
<tr>
<td>$(-, 12)$</td>
<td>$A_1$</td>
<td>3</td>
<td>$(3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$(2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>$(1, 2) + (12, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>$(-, 3) + (11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>7</td>
<td>$(1, 11) + (1, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td>$(-, 12)$</td>
</tr>
<tr>
<td>$(1, 11)$</td>
<td>$B_1$</td>
<td>2</td>
<td>$(12, -) + (3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>$(11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$(1, 2) + (111, -) + (12, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>$(11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>$(1, 11)$</td>
</tr>
<tr>
<td>$(-, 3)$</td>
<td>$A_2$</td>
<td>2</td>
<td>$(3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>$(2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
<td>$(-, 3)$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$A_1 \times B_1$</td>
<td>2</td>
<td>$(12, -) + (3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>$(11, 1) + (2, 1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>$(1, 2)$</td>
</tr>
<tr>
<td>$(11, 1)$</td>
<td>$B_2, (11)$</td>
<td>2</td>
<td>$(12, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>$(11, 1)$</td>
</tr>
<tr>
<td>$(111, -)$</td>
<td>$B_2, (11)$</td>
<td>3</td>
<td>$(111, -)$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$B_2, (2)$</td>
<td>1</td>
<td>$(12, -) + (3, -)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>$(2, 1)$</td>
</tr>
<tr>
<td>$(12, -)$</td>
<td>$B_3, (12)$</td>
<td>1</td>
<td>$(12, -)$</td>
</tr>
<tr>
<td>$(3, -)$</td>
<td>$B_3, (3)$</td>
<td>0</td>
<td>$(3, -)$</td>
</tr>
</tbody>
</table>

Notice that the decomposition into graded parts of the regular representation, which we obtain for the coset $T$ (type $\emptyset$) is not the one of the coinvariant algebra! More generally, even if the $\mathcal{H}_0$-type of the tempered representation with real central character $\tau_L$ is the same for different special values of the parameters, its decomposition into graded parts needs not be the same. Since we have also calculated the decomposition of the tempered characters with real central character, we can check now that indeed Conjecture 6.5.3 holds for $B_3$. Points (i),(ii) and
(iii) are readily verified by inspection. For part (iv), all residual cosets but $B_2, (11)$ have only one Springer correspondent. We treat only this coset. Then $(\xi, \eta) = (11, -)$, and $f_q^{bc}(\psi_q(111, -)) = (1, 11)$. Remember also that $[(111, -)]_2 = \{(111, -), (11, 1)\}$. From the above table of the corresponding functions $\psi^2(\xi, \eta)^i$, we find that according to the conjecture, for the residual coset $L$ of type $B_2, (11)$ there are two tempered representation of $\mathcal{H}$ with central character $r_L$, one with $\mathcal{H}_0$-type $[(11, 1) + (12, -)] \otimes \text{sgn} = (1, 2) + (-, 12)$ and one with $\mathcal{H}_0$-type $(111, -) \otimes \text{sgn} = (-, 3)$. This is indeed correct, as we verified above. Similarly, we can check (iv) for all cosets.

8.3. $B_4$

This time, we find:

PROPOSITION/CONJECTURE 8.3.1. (i) At the residual points, the $\mathcal{H}_0$-types of the tempered representations are the following, for generic parameter values:

<table>
<thead>
<tr>
<th>center</th>
<th>$k_1 &lt; \frac{1}{2} k_2$</th>
<th>$\frac{1}{2} k_2 &lt; k_1 &lt; \frac{2}{3} k_2$</th>
<th>$\frac{2}{3} k_2 &lt; k_1 &lt; \frac{1}{2} k_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_4, (4)$</td>
<td>$(-, 1^4)$</td>
<td>$(-, 1^4)$</td>
<td>$(-, 1^4)$</td>
</tr>
<tr>
<td>$B_4, (31)$</td>
<td>$(-, 21^2)$</td>
<td>$(-, 21^2)$</td>
<td>$(-, 21^2)$</td>
</tr>
<tr>
<td>$B_4, (22)$</td>
<td>$(-, 22)$</td>
<td>$(-, 22)$</td>
<td>$(-, 22)$</td>
</tr>
<tr>
<td>$B_4, (21^2)$</td>
<td>$(-, 31)$</td>
<td>$(-, 31)$</td>
<td>$(-, 31)$</td>
</tr>
<tr>
<td>$B_4, (1^4)$</td>
<td>$(-, 4)$</td>
<td>$(1, 3) + (-, 31)$</td>
<td>$(1^2, 2) + (1, 21) + (-, 21^2)$</td>
</tr>
</tbody>
</table>

(ii) For special parameters $k_2 = qk_1$, the $\mathcal{H}_0$-types of the irreducible tempered representations with central character $B_4, \lambda$ are the $\mathcal{H}_0$-types of the irreducible discrete series representations with central character $B_4, \lambda$ for $k_2 = (q \pm \epsilon)k_1$.

We do not repeat the irreducible tempered $\mathcal{H}$-modules whose central character is not the specialization of a generically residual point since these can be determined by induction.

Note that we have written $(\lambda, \mu)$ for $\chi_{(\lambda, \mu)}$. The Springer correspondent is accentuated in boldface.

Sketch of proof: Generic parameters: All residual points except $B_4, (22)$ are regular for generic parameters, and therefore we can compute the dimension of the representations $\chi_r$ for those points, as we did for type $B_3$. These computations indeed yield the dimensions of the representations in the tables. To compute the actual representations, one may proceed as in the cases for $B_2$ and $B_3$. One checks that this indeed produces all representations in the table. We have thus settled the generic case, apart from $B_4(22)$. For this central character,
we will only prove that the two irreducible tempered representations that occur for $k_1 = 2k_2$ indeed have the $H_0$-type that occurs in the table.

**Special parameters:** Let us look at the residual points one by one. The point of type (4) is always residual and thus needs no further computations: it is always the central character of the Steinberg representation. The point of type (31) is residual unless $k_1 = k_2$. In this case it coincides with the center of the one-dimensional coset of type $B_3/(3)$. The Hecke subalgebra $H_0(B_3) \otimes \mathbb{C}[X]$ has a one-dimensional representation with this central character, which is the induction of the Steinberg representation of the Hecke algebra of type $B_3$. Inducing this representation to $H$ yields an eight-dimensional representation whose $H_0$-type is $(1,1^3) + (-,112) + (-,1^4)$. Choosing $r$ such that $\alpha_2(r) = \alpha_3(r) = \alpha_4(r) = q^{-1}$, it has weights $r, s_{1r}, s_{2s_1r}, s_{3s_2s_1r}$, all with multiplicity two. By Lemma 1.1.5, a module $V$ which has a non-zero weight space for $s_{3s_2s_1r}$ necessarily satisfies $\dim(V_{s_{3s_2s_1r}}) = 2$. It then follows that $\dim(V_{s_{2s_1r}}) = \dim(V_{s_1r}) = \dim(V_r) = 1$. Similarly one shows that there is another $H$-module $V'$ such that $\dim(V'_{s_{2s_1r}}) = \dim(V'_{s_1r}) = \dim(V'_r) = 1$. Thus, the induced representation decomposes into two irreducible tempered representations, with $H_0$-types $(-,112)$ and $(1,1^3) + (-,1^4)$.

The point of type (22) is not residual if $k_1 = 2k_2$ only. In this case, it coincides with the center of the residual coset of type $A_1 \times B_2/(2)$. The corresponding Hecke algebra has a one-dimensional discrete series representation (the Steinberg) which we induce to $H$. This yields a 24-dimensional representation. If we choose $r$ such that $\alpha_1(r) = \alpha_3(r) = -k$ and $\alpha_4(r) = -\frac{1}{2}k$, the induced modules has weights $r, s_{2}r, s_{1}s_{2}r, s_{4}s_{3}s_{2}r, s_{3}s_{1}s_{2}r, s_{4}s_{3}s_{2}r, s_{2}s_{4}s_{3}s_{2}r$ with multiplicities (resp.) 2,2,2,4,4,6,2. First one checks that there exists a two-dimensional $H$-module $M$ with weights $r$ and $s_2r$, whose $H_0$-type is $(-,22)$. Suppose that $V$ is an $H$-module such that $V_{s_4s_3s_1s_2r} \neq 0$, then Lemma 8.0.3 implies that $\dim(V_{s_4s_3s_1s_2r}) = 6$. Suppose $\dim(V_{s_4s_3s_2r}) = 0$. Then, since $s_4s_3s_2 = s_1s_4s_3s_1s_2$, it follows that $\langle \alpha_1, \check{\alpha}_2 \rangle = 0$, which is not the case. Since $\alpha_3(s_4s_3s_2r) = 1$, it follows that $\dim(V_{s_4s_3s_2r}) \geq 2$.

On the other hand, if we restrict $V$ to an $H(A_1(\alpha_2))$ module, where $A_1(\alpha_2)$ is the root system of $\alpha_2$, then the composition factors of $V_r$ are three copies of the minimal principal series representation of $H(A_1(\alpha_2))$. Then Lemma 1.1.5 implies that $\dim(V_{s_4s_3s_2r}) \geq 3$. But the dimension of $V_{s_4s_3s_2r}$ must be even, since its basis vectors occur in pairs. Therefore, it follows that $\dim(V_{s_4s_3s_2r}) = 4$.

In the same fashion we conclude that $\dim(V_{s_3s_1s_2r}) = 4$. But then also $\dim(V_{s_2s_3s_2r}) \geq 2$ since $\langle \alpha_2, \check{\alpha}_3 \rangle \neq 0$ and $\alpha_1(s_2s_4s_3s_2r) = 1$. Another application of Lemma 1.1.5 then also shows that $\dim(V_{s_3s_2r}) \geq 1$ and $\dim(V_{s_1s_2r}) \geq 1$. By arguments similar to the ones for $V_{s_4s_3s_2r}$, i.e., by viewing $V_{s_4s_3s_2r}$ as a $H(A_1(\alpha_3))$-modules, we even find that $\dim(V_{s_3s_2r}) = 2$, and similarly that also $\dim(V_{s_1s_2r}) = 2$.

By looking at the calibration graph it is clear that the weight spaces of $r$ and $s_2r$ must be of equal dimension. Now suppose that $V$ is not 22-dimensional. We have already shown that $\dim(V) \geq 20$. Therefore, let $N \neq M$ be an $H$-module with $N_r \neq 0$, then $\dim(N) \geq 2$. However, $N$ is also a module for $H_0$, and the total $H_0$-type of the induced representation that we are decomposing is $(-,22) + (1^2,1^2) + (1,12) + (1,1^3) + (-,112) + (-,1^4)$. These constituents have dimension (resp.) 2,6,8,4,3,1; and the constituent $(-,22)$ has already been localized. This is a contradiction and it follows that $\dim(V) = 22$.

Of course, since we are dealing here with the $C_n$-group case, we could have also used the fact that we know that the Green functions describe the irreducible tempered representations
8.3. $B_4$

and their $\mathcal{H}_0$-decomposition to come to this conclusion. But as in the previous cases we have chose to do these calculation, for uniformity of the exposition.

The central characters of the remaining two types are dealt with using similar arguments. We do not repeat them here.

\[ \square \]

**Remark 8.3.2.** These results are a confirmation of the Conjectures 6.5.3.

**Remark 8.3.3.** (i) As in the previous examples, for generic parameters the $\mathcal{H}_0$-types of the irreducible discrete series representations are an orthonormal set w.r.t. the elliptic pairing on $\hat{\mathcal{W}}$. If we assume this in advance, we can use this assumption to calculate the irreducible tempered representations with central character $B_4(22)$ in another way. Let us explain this in detail. We have already obtained four orthonormal $\mathcal{H}_0$-representations for all generic parameters. It remains to find the one with central character $B_4(22)$. Consider the virtual character $a_1(1^2, 1^2) + a_2(1, 21) + a_3(1, 1^3) + a_4(-, 22) + a_5(-, 21^2) + a_6(-, 1^4)$, then the characters of the discrete series only form an orthonormal basis (for all possible generic parameters) for the elliptic pairing if $a_1 = a_2$ and $a_3 = a_5 = a_6$, and $(a_1 - a_4)^2 = 1$.

(ii) Let $q_1 = q_2^2$. If we calculate $f_V$ for $V = (-, 22)$ using Lemma 7.3.2, then indeed we find $r(f_V) = \mu_{P_1}(r)$, as in (1.1). This could have lead us to the same conclusion for the point (22) and special parameters $q_1 = q_2^2$ as well.