A combinatorial generalization of the Springer correspondence for classical type
Slooten, K.J.

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CHAPTER 9

Remarks

In this thesis we have only considered the affine Hecke algebra with root labels \( q_k > 1 \). In some sense, this is a natural restriction on the labels, since all affine Hecke algebras that occur "in nature" (i.e., in the setting of Theorem 0.3.1), are of this type. However, from a combinatorial point of view, this assumption is unnecessarily restrictive. In this chapter we give a combinatorial generalization of our results in the more general case. As long as \( k_1k_2 \neq 0 \), our results extend easily. If one of the \( k_i \) is zero, we already have to be more careful, while the case \( k_1 = k_2 = 0 \) (i.e., \( q_1 = q_2 = 1 \)) is not accessible. This is because the ratio \( q = k_2/k_1 \), which has been the principal variable in all our considerations, is no longer defined.

9.1. Other special values

So far we have been assuming that for the root labels \( k_1, k_2 \) of the graded Hecke algebra, we have \( k_1 > 0 \). In this section we indicate how to generalize our results to the general case where \( k_i \in \mathbb{R} \). In this case, the parameters are special if ([14]):

\[
2(n-1) \\
\prod_{j=1}^{2(n-1)} (j k_1 + 2k_2)(j k_1 - 2k_2) \neq 0.
\]

Our analysis so far, i.e., the symbols that we have defined, depends on the ratio \( q = k_2/k_1 \). Clearly this ratio is only defined for \( k_1 \neq 0 \). Let us therefore denote the set of non-negative ratios of special parameters for the graded Hecke algebra of type \( B_n \) by \( S(n)^+ = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, n - 1\} \). We first treat the case where \( q = k_2/k_1 \in \pm S(n)^+ \).

9.1.1. Nonzero long root label. First we assume that \( k_1 \neq 0 \). Suppose that the parameters \( k_2 = q k_1 \) are special with \( q \in -S(n)^+ \). For a generically residual point \( c(\lambda, k_1, k_2) \), clearly we can still define the \( q \)-tableau \( T_q(\lambda) \). The following simple observation is crucial to derive the case where \( q < 0 \) from the corresponding case where \( q > 0 \).

Lemma 9.1.1. Let \( q \in S(n)^+ \) and \( \lambda \vdash n \). Then

\[
S_{-q}(\lambda) = (\xi, \eta) \iff S_q(\lambda') = (\eta, \xi),
\]

where \( \lambda' \) is the partition conjugate to \( \lambda \).

Proof: This follows from the fact that \( T_q(\lambda) \) is the conjugate tableau of \( T_{-q}(\lambda') \). The entry of square \((i, j)\) in a \((-q)\)-tableau is equal to \(|(j - i)k_1 + k_2| = |(j - i) - q|k_1 = |i - j| + q|k_1\), which is equal to the entry of square \((j, i)\) in a \(q\)-tableau. The Lemma follows since square \((i, j)\) belongs to \( T_{-q}(\lambda) \iff \) square \((j, i)\) belongs to \( T_q(\lambda') \).
Remark that, as characters of $W_0$,

$$(\xi, \eta) \otimes (-, n) = (\eta, \xi).$$

We will write this map as $\Phi : \Phi(\xi, \eta) = (\eta, \xi)$.

In particular, observe that for $q = 0$ we find $S_0(\lambda) = \Phi(S_0(\lambda'))$. We next define the $q$-symbols for the new cases. First we consider $q = 0$, where we have to be somewhat careful, since we may have either $k_1 > 0$ or $k_1 < 0$. We therefore define two symbols, the $+0$-symbol, to be applied when $k_1 > 0$ and the $-0$-symbol, to be applied when $k_1 < 0$. Both are derived from the corresponding elements in $Z^{2,0}(n, n)$, but for the $+0$-symbol of $(\xi, \eta)$ we start the symbol with $\xi_1$, whereas for the $-0$-symbol we start the symbol with $\eta_1$. Of course we have first adjusted the lengths of $\xi$ and $\eta$ such that $l(\xi) = l(\eta)$. For example, the $+0$-resp. $-0$-symbol of $(11, 2)$ are

$$\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}.$$ 

Now we define the $q$-symbols for negative $q$.

**Definition 9.1.2.** Let $(\xi, \eta)$ be a double partition of $n$. Let $q > 0 \in S(n)^+$. Then we define the $(-q)$-symbol of $(\xi, \eta)$ as the equivalence class associated to $(\xi, \eta)$ in $Z^{2,0}(n, n + q)$ if $q$ is integer, and as the equivalence class of $(\xi, \eta)$ in $Z^{2,1}(n, n + q + \frac{1}{2})$ if $q$ is not integer.

**Example 9.1.3.** The $(-2)$-symbol of $(12, 3)$ is

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 9 \end{pmatrix}.$$ 

Observe that the $(-q)$-symbol of $(\xi, \eta)$ is the $q$-symbol of $(\eta, \xi)$, up to a change of rows. In particular,

$$a_{-q}(\xi, \eta) = a_q(\eta, \xi) = a_q(\Phi(\xi, \eta)).$$

**Corollary 9.1.4.** Let $q \in S(n)^+$ and $L_- \in L(-q)$ be a residual subspace of the graded Hecke algebra $H$ with parameters $k_1$ and $k_2 = -q k_1$.

(i) Suppose $L_- \in L(-q)$ corresponds to the residual point with jumps $j_i$ in the graded Hecke algebra of type $A_\lambda \times B_1$, and labels $k_2 = -q k_1$. Then there exists a residual coset $L_+ \in L(q)$ whose center is the residual point with jumps $j_i$ in the graded Hecke algebra of type $A_\lambda \times B_1$ with labels $k_2 = q k_1$. This map induces a bijection

$$L(q) \longleftrightarrow L(-q).$$

(ii) Let $L_- \in L(q) \leftrightarrow L^+ \in L(q)$ under the above bijection. Then there is a bijection

$$C_q(L_+) \leftrightarrow C_{-q}(L_-)$$

defined by

$$L_+ \in L(q) \text{ of type } A_\lambda \times (B_1, \mu) \leftrightarrow L_- \in L(-q) \text{ of type } A_\lambda \times (B_1, \mu').$$

We denote the corresponding map also by $\Phi$, i.e., $\Phi(L)$ has type $A_\lambda \times (B_1, \mu)$ if and only if $L'$ has type $A_\lambda \times (B_1, \mu')$. Notice that $\Phi^2$ is the identity.

(iii) The set $\Sigma_{-q}(\Phi(L))$ of Springer correspondents of $\Phi(L)$ is well-defined. Let $\Phi(L) \in L(q)$ correspond to $L \in L(-q)$ under the above bijection. Then

$$\Phi(\Sigma_{-q}(L)) = \Sigma_q(\Phi(L)).$$
Proof: (i) and (ii) follow from Lemma 9.1.1, whereas (iii) also uses (9.1).

In particular, if \( q = 0 \) then any set of Springer correspondents satisfies \( \Sigma_0(L) = \Sigma_0(L) \otimes (\cdot, n) \). This is of course a trivial corollary of the definition of the \( \pm 0 \)-symbol. Notice that for all special values we find

**Lemma 9.1.5.** Let \( q \in \pm S(n)^+ \). Then \( S_q(\lambda) \) is well-defined if and only if \( c(\lambda, k, qk) \) is residual.

Proof: For \( q \leq 0 \) this follows from \( q > 0 \) for which we already know the result, and from Condition 3.1.1, which can be shown easily to hold for \( q < 0 \) if one replaces \( q \) by \( |q| \). For \( q = 0 \) it follows from the analogue of Lemma 3.3.1, which reads: let \( \lambda \vdash n \) and let the number \( l \) occur \( m_l \) times in \( T_0(\lambda) \). Then \( m_l \in \{ m_{l+1} - 1, m_{l+1}, m_{l+1} + 1 \} \) for \( l > 0 \), and \( m_0 \in \{ m_{1/2}, m_{1/2} + 1 \} \). The lemma follows from the Condition for \( c(\lambda, k, 0) \) to be residual below, in the same way it was deduced for \( q > 0 \).

**Lemma 9.1.6.** ([14, Prop. 4.6]) Let \( \lambda \vdash n \) and consider the point \( c(\lambda, k, 0) \) written in the form (3.1). Then it is residual if and only if (i) \( m_p = 1 \), (ii) \( m_l \in \{ m_{l+1}, m_{l+1} + 1 \} \) for all \( l > 0 \) and (iii) \( m_0 = \lfloor \frac{m_{1/2} + 1}{2} \rfloor \).

It follows that the Springer correspondence on the region \( k_1 k_2 < 0 \) is the mirror-image of the situation for \( k_1 k_2 > 0 \). For example, for \( n = 4, q = 1 \), the residual points with partition \( (22), (112), (1^4) \) coincide, whereas for \( q = -1 \), the generically residual points with partition \( (22), (13), (4) \) coincide:

\[
\begin{array}{ccc}
1 & 2 & 0 \\
0 & 1 & 0 \\
1 & 2 & 0 \\
1 & 0 & 2 \\
0 & 1 & 2 \\
\end{array}
\]

Their 1-symbols resp. -1-symbols are:

\[
\begin{pmatrix}
2 & 0 \\
4 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 2 \\
4 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 4 \\
2 & 0 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
2 & 0 \\
4 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 2 \\
4 & 0 \\
\end{pmatrix}, \begin{pmatrix}
0 & 4 \\
2 & 0 \\
\end{pmatrix}
\]

We thus see that the whole Springer correspondence goes through; for the unipotent classes this should then also hold and we may take \( U_q(n) = U_q(n) \). On the other hand, it depends on which of the \( k_i \) is positive, which \( \mathcal{H} \)-representations will be tempered. For example, for \( k_i > 0 \) the Steinberg representation (whose \( \mathcal{H}_0 \)-type is \( (\cdot, 1^n) \)) is a discrete series representation. On the other hand, for \( k_i < 0 \), it does not belong to the discrete series, but the trivial representation (with \( \mathcal{H}_0 \)-type \( (n, -) \)) does. For \( k_1 k_2 < 0 \), the representation which acts as the Steinberg representation on the roots with positive label, and as the trivial representation on the roots with negative labels, belongs to the discrete series. If \( k_1 > 0, k_2 < 0 \), this is the one-dimensional \( \mathcal{H} \)-representation with \( \mathcal{H}_0 \)-type \( (1^n, -) \), if \( k_1 < 0, k_2 > 0 \), its
\(\mathcal{H}_0\)-type is \((-, n)\). This should of course be taken into account when formulating the corresponding maps to \(\mathcal{U}_q(n)\), and the conjectures regarding the \(\mathcal{H}_0\)-decomposition of a tempered irreducible representation of \(\mathcal{H}\).

If \(q = 0\), we define \(\mathcal{U}_0(n)\) to be the set of partitions of \(2n\) in which even parts occur with even multiplicity. Then we define \(\psi_{+0}\) as in 5.2.1, using the \(+0\)-symbol. We define \(\phi_{+0}\) as in 5.2.8, and \(f^{bc}_{+0}\) as in 5.1.3. With these definitions, we have

**Definition 9.1.7.** Let \(q \in S(n)^+\). Let \(\mathcal{U}_{-q}(n) = \mathcal{U}_q(n)\).

(i) If \(q > 0\), define \(\psi_{-q} : \mathcal{P}_{n,2} \rightarrow \mathcal{U}_{-q}(n)\) as

\[
\psi_{-q}(\xi, \eta) = \psi_q(\eta, \xi) = (\psi_q \circ \Phi)(\xi, \eta).
\]

(ii) Define \(\phi_{-q} : \mathcal{U}_{-q}(n) \to \hat{W}_0/\sim_{-q}\) by

\[\phi_{-q} = \Phi \circ \phi_q\]

(iii) Define \(f^{bc}_{-q} : \mathcal{U}_{-q}(n) \to \mathcal{L}(-q)\) by

\[f^{bc}_{-q}(\lambda) = \Phi(f^{bc}_q(\lambda)),\]

where we use the bijection \(\mathcal{L}(-q) \leftrightarrow \mathcal{L}(q)\) from Corollary 9.1.4 (ii).

**Corollary 9.1.8.** Let \(q \in \pm S(n)^+\). Then the diagram (5.3) is commutative, i.e., for all special and finite \(q = k_2/k_1\), and \(L \in \mathcal{L}(q)\) one has

\[(f^{bc}_q \circ \psi_q \circ \Sigma_q)(L) = L.\]

**Proof:** For \(q < 0\), this is a direct consequence of the above definitions. For \(q = 0\), the proof for \(q > 0\) readily generalizes to \(q = 0\).

It remains to adapt also Conjecture 6.5.3 to the cases where \(k_2 = qk_1\) with \(q \in -S(n)^+\) and the case where \(k_2 = qk_1\) with \(q \in S(n)^+\) but \(k_i < 0\). Before doing this, let us take a look at the case where \(k_1 = 0\) and \(k_2 \neq 0\).

**9.1.2. Nonzero short root label.** Suppose that \(k_2 \neq 0\). If \(k_1 \neq 0\), then we have already described the Springer correspondence at all special values, so let us assume that \(k_1 = 0\). In this case, the ratio \(q = k_2/k_1\) is not defined; according to the sign of \(k_2\) we may view it as \(\pm \infty\). Therefore, let us define formal \(\pm \infty\)-symbols where one of the partitions has infinite length. It follows that the equivalence relation induced on \(\hat{W}_0\) is

\[(\xi, \eta)_{\pm \infty} = \{(\ast, \eta)\} \quad \text{and} \quad [(\xi, \eta)]_{-\infty} = \{(\xi, \ast)\},\]

where \(\ast\) denotes any partition of the appropriate size. On the other hand, at \(k_1 = 0\) all residual points coincide, which means that the centers of the generically residual subspaces of type \(A_\lambda \times (B_1, \mu)\) all coincide with each other for fixed \(\lambda\). We have seen in Corollary 4.5.9 that for \(0 < k_1 < (n-1)^{-1}k_2\), the Springer correspondent of the residual coset of type \(A_\lambda \times (B_1, \mu)\) is \((\mu, \lambda)\). Therefore the equivalence relation \(\sim_{-\infty}\) is indeed the one describing the confluence of all the generically residual points in each subalgebra of type \(A_\lambda \times B_1\).

Similarly, for \(k_2 < 0\), we have just seen that \(\Sigma_q(L) = \Sigma_q(L') \otimes (-, n)\), which implies that for \(0 < k_1 < -(n-1)^{-1}k_2\), the generically residual subspace of type \(A_\lambda \times (B_1, \mu)\) has Springer correspondent \((\lambda, \mu')\).

We thus have equivalence classes on \(\hat{W}_0\), it remains to define the set \(\mathcal{U}_\infty(n)\), and the maps \(\psi_\infty\) and \(\phi_\infty\). At first glance, the set \(\mathcal{U}_\infty(n)\) is not well tractable, as one would have to consider partitions of infinity, in which infinitely many odd parts occur with odd multiplicity.
9.1. Other Special Values

However, since for \( t > n - 1 \), all parameters \( k_1, k_2 \) with \( k_2 = tk_1 \) are generic, we may instead consider \( U_n(n) \), and the corresponding maps \( \phi_n \) and \( \psi_n \), if we realize the extra condition that of a double partition \( (\xi, \eta) \), only the second part \( \eta \) determines the equivalence class in \( \tilde{W}_0 \). We therefore define \( U_\infty(n) := U_n(n) \), and \( \psi_\infty = \psi_n(n) \). Of course its inverse \( \phi_\infty \) is then defined to be \( \phi_n \). Finally, it remains to define \( f^{BC}_n \). Since its domain is \( U_n(n) \), we put \( f^{BC}_n(\lambda) = [(*, \xi)]_\infty \) if \( f^{BC}_n(\lambda) \) is a residual coset \( L \) whose root system \( R_L \) is of type \( A_\xi \times B_\eta \).

**Remark 9.1.9.** It is not hard to see that the diagram (5.3) is commutative for any \( q \in \frac{1}{2} \mathbb{N} \). If we take \( q > n - 1 \), then there are \( \not\exists P_{n,2} \) \( W_0 \)-orbits of residual cosets, and so it follows that, for \( q \geq n \) integer, one has

\[
\{ \lambda = (1^{*}, 2^{*} \ldots) \mid 2n + q^2 \mid r_{\text{even}} = \text{even and } \sum_{i \text{ odd}} (r_i \mod 2) \geq q \} \leftrightarrow P_{n,2},
\]

and an analogous statement if \( q \) is half-integer.

**9.1.3. Generalization of Conjecture.** Since \( w_0 = -1 \) for type \( B_n \), the trivial and the Steinberg representations of \( H \) have the same central character, as do the one-dimensional representations with \( H_0 \)-type \( (1^n, -) \) and \( (-, n) \). It depends on the parameters which one of these four is in the discrete series. If \( q > 0 \), its central character is \( W_0r \) for \( r = \exp(c((n), k_1, k_2)) \), and for \( q < 0 \), its central character is \( W_0r \) for \( r = \exp(c((1^n), k_1, k_2)) \). We therefore adapt Conjecture 6.5.3 simply by multiplying the representations in part (iv) with the appropriate one of these four characters. For \( q \in -S(n)^+ \), the generically residual point of type \( (1^n) \) is the one which is the central character of the one-dimensional representations with \( H_0 \)-type \( (1^n, -) \) and \( (-, n) \). Therefore the generically residual point \( c((1^n), k_1, k_2) \), which indeed remains residual at all special values \( k_2 = qk_1 \) if \( q < 0 \) should correspond for \( k_2 < 0 \) to the representation with \( H_0 \)-type \( (-, n) \), and for \( k_1 < 0 \) to the representation with \( H_0 \)-type \( (1^n, -) \). Finally, for \( k_2 = qk_1 \) with \( k_i < 0 \), on the same grounds it is clear that the residual point of type \( (n) \) is the central character of the trivial representation, and that the irreducible discrete series \( H \)-representations with central character \( c(\lambda, k, qk) \) are those for \( q > 0 \), multiplied with the Steinberg representation. On the level of \( H_0 \)-types, this is multiplication with the sign representation, and thus a discrete series representation whose central character is \( c(\lambda, q, qk) \) has Springer correspondent and leading \( H_0 \)-type (the \( H_0 \)-representation in the top degree of the graded module) \( S_\lambda(\lambda) \). Therefore the part \( k_i < 0 \) of the parameter space seems to be the one for which the splitting map \( S_q \) is the most natural. We can thus generalize Conjecture (6.5.3). For \( q < 0 \), we define Green functions in the same way as before, i.e., we use the ordering on \( \tilde{W}_0 \) induced by \( a_q \). For \( q = 0 \) or \( \pm \infty \), we define Green functions as we did for the case \( q > 0 \). We will come back to this after stating the conjecture.

**Conjecture 9.1.10.** Fix special parameters \( k_2 = qk_1 \) of the graded Hecke algebra \( H \) of type \( B_n \) and let the affine Hecke algebra \( H'H \) have root labels \( q_i = \exp(k_i) \).

(i) The \( \psi^q,\alpha,^i \) are characters of \( W_0 \).

(ii) \( (\chi_\beta, \psi^q,\alpha,^i) = 0 \) unless \( \beta \succ \alpha \) and \( \alpha \asymp q \beta \), or \( \alpha = \beta \): we find an upper triangular system.

(iii) If \( i_0 \) is such that \( \psi^q,\alpha,^{i_0} \neq 0 \) and \( \psi^q,\alpha,^i = 0 \) for all \( i > i_0 \), then \( \psi^q,\alpha,^{i_0} = \chi_\alpha \). In other words, in the top degree we find with multiplicity one the irreducible representation \( \chi_\alpha \).
(iv) Let the residual coset $L$ have Springer correspondents $\Sigma_q(L)$. Then the irreducible tempered representations in $\Pi_q(L)$ are indexed by $\Sigma_q(L)$, and their $\mathcal{H}_0$-types are given by

$$\sum_i \psi_q^{,i} \gamma_i \otimes \begin{cases} 1 & \text{if } k_1 < 0 \\ \text{sgn} & \text{if } k_1 > 0 \end{cases}$$

for all $\gamma \in \Sigma_q(L)$.

Recall that the set $C_q(L)$ of generically residual cosets whose center coincides with $r_L$ for $k_2 = qk_1$ is in bijection with $\Sigma_q(L)$. Suppose that $L_{k_1}$, of type $A_{k_1} \times (B_1, \kappa^2)$ is one of these cosets, and that $\gamma \in \Sigma_q(L)$ is its Springer correspondent at $q + \epsilon$, i.e.,

$$\gamma = \Sigma_{q+\epsilon}(L) = \text{tr}_{q+\epsilon} \text{Ind}_{S_{n,1} \times W_0(B_{n,2})}^{W_0(B_n)}(\text{triv}_{\kappa^1} \otimes S_{q+\epsilon}(\kappa^2)).$$

Then the $\mathcal{H}_0$-decomposition of the irreducible tempered representation of $\mathcal{H}$ with central character $r_{L_{k_1}}$ is for $q + \epsilon$ equal to

$$\sum_i \psi_q^{,i} \gamma_i \otimes \begin{cases} 1 & \text{if } k_1 < 0 \\ \text{sgn} & \text{if } k_1 > 0 \end{cases}$$

The analogous statement holds for $q - \epsilon$.

Observe that for $k_1 = 0$, this is only well-defined if the set $\{ \sum_i \psi_0^{,i} \gamma_i \gamma | \gamma \in \Sigma_{\pm \infty}(L) \}$ of (iv) is invariant under multiplication with the sign representation.

This implies that on the lines $k_2 = \pm(n-1)k_1$, we conjecturally find as $\mathcal{H}_0$-types in the top degree of the tempered $\mathcal{H}$-representations the ones in Figure 1.

![Figure 1](image)

**Figure 1.** Leading $\mathcal{H}_0$-types of tempered representations with central character $r_L$ on the lines $k_2 = \pm(n-1)k_1$, where $L$ is a residual subspace of type $A_{\lambda} \times (B_1, \mu)$.

Finally, let us make some comments about the relation between the Green functions $P^{\pm q}$ for $q < 0$ and those for $q < 0$. Since one can use both to formulate the conjectures, there should not be a difference.

Granted conjecture 6.5.3, let $L_+$ be a residual coset for $k_2 = qk_1$ where $q > 0$, and let $V^+$ be an irreducible tempered $\mathcal{H}$-module whose central character is $r_{L_+}$, i.e., there is a $\gamma \in \Sigma_q(L)$ such that

$$V^+|_{\mathcal{H}_0} = \sum_i \psi^{,i} \gamma_i \otimes \epsilon.$$
Then there exists a module $V^-$, whose central character is $\tau_{L_-}$, such that for $k_2 = -qk_1$, $V^-$ is tempered, and whose $\mathcal{H}_0$-decomposition is

$$V^-|_{\mathcal{H}_0} = \sum_i \psi^q, \gamma_i \otimes (1^n, -),$$

if $k_1 > 0, k_2 < 0$, or

$$V^-|_{\mathcal{H}_0} = \sum_i \psi^q, \gamma_i \otimes (-, n),$$

if $k_1 < 0, k_2 > 0$. On the other hand, according to 9.1.10, the $\mathcal{H}_0$-decomposition of $V^-$ is

$$V^-|_{\mathcal{H}_0} = \sum_i \psi^{-q, \Phi(\gamma), i} \otimes (-, 1^n),$$

if $k_1 > 0, k_2 < 0$, or

$$V^-|_{\mathcal{H}_0} = \sum_i \psi^{-q, \Phi(\gamma), i}.$$

if $k_1 < 0, k_2 > 0$. Therefore, we expect that

$$\sum_i \psi^{-q, \Phi(\gamma), i} = \Phi(\sum_i \psi^q, \gamma_i)$$

for all $\gamma$ and all $q$. This is true since it holds in every graded part:

**Lemma 9.1.11.** We have

$$\psi^{-q, \Phi(\gamma), i} = \Phi(\psi^q, \gamma_i)$$

for all $q, \gamma, i$.

**Proof:** Since $a_{-q}(\xi, \eta) = a_q(\eta, \xi)$, and so choose the order $\succ_q$ on the set of 2-partitions defined by $(\alpha, \beta) \succ_q (\xi, \eta) \iff (\beta, \alpha) \succ_q (\eta, \xi)$ refines the order induced by the $a_{-q}$-value, and hence we use it to define the matrix $P^{-q}$. It follows easily from the defining equation (6.10) with $q < 0$, that we then have $P_{\alpha, \beta}(t) = P^q_{\Phi(\alpha), \Phi(\beta)}(t)$. As in the case $q > 0$, we can then multiply the matrix $P^{-q}$ with the transposed character table of $W_0$ to obtain the matrix $Q^{-q}$. Then we have

$$Q^{-q}_{\alpha, \beta}(t) = \sum_{i \geq 0} \psi^{-q, \beta, i}(\alpha) t^i$$

on the one hand, but on the other we have also

$$Q^{-q}_{\alpha, \beta}(t) = \sum_{\gamma} \chi_{\gamma}(\alpha) P^{-q}_{\gamma, \beta}(t)$$

$$= \sum_{\gamma} \chi_{\gamma}(\alpha) P^q_{\Phi(\gamma), \Phi(\beta)}(t)$$

$$= \sum_{\gamma} \chi_{(-, n)}(\alpha) \chi_{\Phi(\gamma)}(\alpha) P^q_{\Phi(\gamma), \Phi(\beta)}(t)$$

$$= \chi_{(-, n)}(\alpha) Q^q_{\alpha, \Phi(\beta)}(t)$$

$$= \chi_{(-, n)}(\alpha) \sum_i \psi^q, \Phi(\beta), i t^i.$$

Together with (9.4), this yields the required result.\[\square\]
We therefore conclude that if \( q \neq 0 \), it indeed, as it should, makes no difference if we use the Green functions for \( q \) or \(-q\).

**Corollary 9.1.12.** Given Conjecture 6.5.3, if \( q = 0 \), then the following are equivalent.

\[ \alpha = \Phi(\alpha) \iff \Phi(\psi^0,\alpha^i) = \psi^0,\alpha^i \text{ for all } i. \]

**Proof:** One side is trivial, and the other follows from the conjecture, since it claims that \( \psi^0,\alpha,\text{max} = \alpha \). It follows that \( \Phi(\alpha) = \psi^0,\alpha,\text{max} = \Phi(\psi^0,\alpha,\text{max}) = \psi^0,\alpha,\text{max} = \alpha. \)

9.1.4. **Root labels zero.** If \( k_1 = k_2 = 0 \) then \( k_2 = qk_1 \) for all special values \( q \), and the combinatorics that we have developed thus degenerate. All residual cosets coincide with each other as well. From the point of view of deformation, the group algebra of the Weyl group thus turns out to be a worst-case-scenario. We have no information to offer on this case.

9.2. **Other classical types**

To justify the title of this thesis, let us briefly take a look at the other root systems of classical type: types \( A \) and \( D \). Both these types are simply laced, which means that there is only one root label \( k \) for the graded Hecke algebra and \( q = \exp(k) \) for the affine Hecke algebra. Therefore essentially the only case that arises is the group case, where \( q \) is the cardinality of the residue field of \( F \), the field over which the corresponding \( p \)-adic group is defined. This gives rise to \( q = p^n \) for a prime \( p \) and natural number \( n \) only, but by [46], we may replace such \( q \) by powers \( q^\epsilon \), for \( \epsilon \in (0,1] \). Therefore it is sufficient to cover the group case, which means that we may consider the representations of the \( p \)-adic group with Iwahori-fixed vectors instead.

9.2.1. **Type A.** Type \( A \) has been done by Zelevinsky in [61]. In this article, he classifies the irreducible representations of \( GL_n(F) \) for a local non-archimedean field \( F \), using combinatorics on so-called segments, which are certain subsets of the set of equivalence classes of the set of irreducible representations of \( GL_n(F) \).

9.2.2. **Type D.** In this section we compare the results on the classification of the irreducible tempered representations of the affine Hecke algebra of type \( D_n \) with the results for the appropriately labeled Hecke algebra of type \( B_n \) which are predicted by Conjecture 9.1.10. We choose the lattices \( X \) of both root systems to equal the weight lattice \( P(B_n) = P(D_n) \). In \( H(B_n) \) we choose root labels \( q_2 = 1 \) and \( q_1 \in \mathbb{R}_{>1} \), which we emphasize by writing \( H(B_n)_{>1} \). Under these assumptions we may and will view \( H(D_n) \) with label \( q_1 \) as a subalgebra of \( H(B_n)_{>1} \). Recall also that the assumptions on the weight lattice are not important for any of the algebras separately, since we are concerned with real central character only.

Any representation of \( W_0(B_n) \) therefore restricts to a representation of \( W_0(D_n) \). It is known that upon restriction to \( W_0(D_n) \) the irreducible representation indexed by \( (\xi,\eta) \) coincides with the one indexed by \( (\eta,\xi) \). Moreover, upon this restriction it remains irreducible if \( \xi \neq \eta \), and if \( \xi = \eta \) it splits into two irreducibles of \( W_0(D_n) \), which we index by \( (\xi,\xi') \) and \( (\xi,\xi'') \). Since confusion may arise easily as both sets of irreducible representations are parametrized in terms of 2-partitions, we will write an irreducible character of \( W_0(B_n) \) as \( \alpha \in \mathcal{P}_{n,2} \), and its restriction to \( W_0(D_n) \) as \( \alpha_D \). We thus have \( \alpha_D = \Phi(\alpha)_D \) if \( \alpha \neq \Phi(\alpha) \), and \( \alpha_D = \alpha_D' + \alpha_D'' \) if \( \alpha = \Phi(\alpha) \). The other way around, if \( \chi \in \hat{W}_0(D_n) \) then \( \text{Ind}_{W_0(B_n)}^{W_0(D_n)}(\chi) = \alpha + \Phi(\alpha) \) if \( \chi = \alpha_D \) and \( \alpha \neq \Phi(\alpha) \), and \( \text{Ind}_{W_0(B_n)}^{W_0(D_n)}(\chi) = \alpha \) if \( \langle \alpha_D, \chi \rangle > 0 \) and
9.2. OTHER CLASSICAL TYPES

\( \alpha = \Phi(\alpha) \). Of course the same relations hold for the corresponding representations of \( \mathcal{H}_0(B_n)_{>1} \) and \( \mathcal{H}_0(D_n) \).

Since, for \( \tilde{W}_0(D_n) \), a pair of partitions is now to be regarded as unordered, it follows that the \( \pm 0 \)-symbols that we have defined will coincide into a symbol with no ordering on the rows, i.e., where we write the entries of \( \xi_1 \) above the ones of \( \eta_1 \). These are indeed the ones describing the Springer correspondence of \( SO_{2n} \), see e.g. [6].

9.2.2.1. Springer correspondence. We thus have two descriptions of the Springer correspondence, one for \( \mathcal{H}(B_n)_{>1} \) as predicted by the Conjectures, and one for \( \mathcal{H}(D_n) \) which is the classical Springer correspondence. We will compare them to each other.

We start by recalling the terminology that applies to \( \mathcal{H}(B_n)_{>1} \). The set of unipotent classes which parametrizes \( \tilde{W}_0 \)-orbits of real residual cosets is given by

\[ U_0(n) = \{ \lambda = (1^{r_1}2^{r_2} \ldots) \vdash 2n \mid r_{\text{even}} = \text{even} \} \]

The Bala–Carter map that describes this parametrization, is \( f^{bc}_0 =: f^{bc}_{+0} \) as given in section 9.1.1. Furthermore we have the map \( \phi_0 \) from \( U_0(n) \) to \( \tilde{W}_0/\sim_0 \), as described in section 9.1. Analogously we have the inverse map \( \psi_0 \). The set of Springer correspondents of a residual coset \( L \) of type \( A_\lambda \times (B_i, \mu) \) is given as the set of irreducible character of \( \tilde{W}_0(B_n) \) that occur in

\[ \sum_{\{ \nu \mid S_0(\nu) \sim_0 S_0(\mu) \}} \text{tr}^0_0 - \text{Ind}_{S_\lambda \times \tilde{W}_0(B_i)}(\text{triv} \otimes S_0(\mu)). \]

Notice that, as we have remarked also before, we have \( \Phi(\Sigma_0(L)) = \Sigma_0(L) \), since \( c(\mu, k, 0) \) is a residual point if and only if its conjugate \( c(\mu', k, 0) \) is residual, and \( S_0(\mu') = \Phi(S_0(\mu)) \). With this notation, we have, for all \( L \in L(0) \),

\[ (f^{bc}_0 \circ \psi_0 \circ \Sigma_0(L)) = L. \]

The irreducible tempered representations of \( \mathcal{H}(B_n)_{>1} \) are then parametrized by \( \tilde{W}_0(B_n) \), such that we call \( U^\alpha \) the representation whose restriction to \( \mathcal{H}_0(B_n)_{>1} \) is

\[ U^\alpha|_{\mathcal{H}_0(B_n)_{>1}} = \sum_i \psi_0^0, \alpha, i \otimes \epsilon. \]

Then all \( U^\alpha \) where \( \alpha \) ranges over its 0-similarity class, form exactly the set of irreducible tempered \( \mathcal{H}(B_n)_{>1} \)-representations whose central character is \( r_L \) with \( [\alpha]_0 = \Sigma_0(L) \).

Recall from the previous section, that for all \( \alpha \) and all \( i \), we have

\[ \Phi(\psi_0^0, \alpha, i) = \psi_0^0, \Phi(\alpha), i. \]

This means that if \( \alpha = \Phi(\alpha) \), then in \( \psi_0^0, \alpha, i \) any representation \( \gamma \) occurs as many times as \( \Phi(\gamma) \).

Now let us describe the classical Springer correspondence for \( SO_{2n}(\mathbb{C}) \). We define an equivalence relation \( \sim_D \), with equivalence classes \([\cdot]_D \), on \( \tilde{W}_0(D_n) \) as follows. If \( \chi \) is an irreducible character of \( \tilde{W}_0(D_n) \) such that \( \chi = \alpha_D \), then \( \chi \sim_D \chi' \) for all \( \chi' = \alpha'_D \) such that \( \alpha \sim_0 \alpha' \). Notice that all such \( \alpha' \) satisfy \( \Phi(\alpha') \neq \alpha' \). This means simply that we look at the 0-symbol, where the rows are regarded as unordered. If there is no such \( \alpha \), i.e., \( \chi_D \) equals \( \alpha_D' \) or \( \alpha' _D \) for some \( \alpha = \Phi(\alpha) \), then \( [\chi]_D = \{ \chi \} \) is an equivalence class by itself. Notice that for such \( \alpha \) we also have \( [\alpha]_0 = \{ \alpha \} \).

The set of unipotent classes of \( SO_{2n}(\mathbb{C}) \), which we denote by \( U_D(n) \), is parametrized by partitions of \( 2n \) in which even parts have even multiplicity, except that if such a partition
has only even parts, it parametrizes two unipotent classes in $SO_{2n}(C)$, which we write as $C_{\lambda}$ and $C_{\lambda'}$. To distinguish elements in $U_0(n)$ from those in $U_D(n)$, we write $\lambda_D$ for the corresponding element in $U_D(n)$ if $\lambda \in U_0(n)$ has an odd part, and $\lambda_D', \lambda_D''$ for the two elements in $U_D(n)$ that correspond to $\lambda \in U_0(n)$ if $\lambda$ has only even parts. The Springer correspondents of a unipotent class $C_\lambda$ for $\lambda \in U_D(n)$ are obtained by the map $\phi_D$. If $\lambda$ has an odd part, then we define $\phi_D(\lambda_D) = \phi_0(\lambda)_D$, for all $\lambda \in U_0(n)$ that have an odd part. Then it is known that $[\phi_D(\lambda)]_D$ is indeed the set of classical Springer correspondents of $C_{\lambda}$.

To describe what happens for $\lambda$ with only even parts, we observe first that

**Lemma 9.2.1.** Let $\lambda \in U_0(n)$, and let $\phi_0(\lambda) = (\xi, \eta)$. Then $\lambda$ consists of even parts if and only if $\xi = \eta$.

**Proof:** This is a straightforward calculation.  

It is also known that in the case where $\phi_0(\lambda) = (\xi, \xi)$, the Springer correspondents of $C_{\lambda}$ and $C_{\lambda'}$ are $(\xi, \xi)_D$ and $(\xi, \xi)'_D$. We must then still decide if $(\xi, \xi)'$ or $(\xi, \xi)''$ is the Springer correspondent of $C_{\lambda}$. This may be done as follows (see [6, p. 423] for this description): a result of Lusztig and Spaltenstein ([39]) shows that the Springer correspondent that belongs to the pair $(u, 1)$, is given by $j_{W, (\epsilon, j)}$, where it is assumed that $u$ lies in the Richardson class corresponding to the parabolic subgroup $P_j$, and $j$ denotes Macdonald induction (which is, like $\text{tr}_q - \text{Ind}$, also a truncated induction). This enables us to choose notation such that $(\xi, \xi)'$ corresponds to $C_{\lambda}$ and $(\xi, \xi)''$ to $C_{\lambda'}$, i.e., such that $\phi_D(\lambda') = (\xi, \xi)'$ and $\phi_D(\lambda'') = (\xi, \xi)''$. Then the map $\psi_D$ is the inverse of $\phi_D$.

Let us denote the set of real residual cosets $\mathcal{H}(D_n)$ by $\mathcal{L}(D)$. On the level of residual cosets for $\mathcal{H}(D_n)$, the corresponding result of the degeneracy of $U_0(n)$ to $U_D(n)$ is

**Lemma 9.2.2.** Let $L$ be a real residual coset of type $A_\lambda \times (B_l, \mu)$ for $\mathcal{H}(B_n)$ with label $q_2 = 1$. Then $L$ is also a residual coset for $\mathcal{H}(D_n)$, and the orbit $W_0(B_n)L$ is a single $W_0(D_n)$-orbit, unless $l = 0$ and $\lambda$ consists of even parts. In that case the orbit $W_0(B_n)L$ decomposes into two $W_0(D_n)$-orbits of residual cosets, which we denote by $L'$ and $L''$.

**Proof:** We work in the graded Hecke algebras $\mathcal{H}(B_n)$ and $\mathcal{H}(D_n)$. Let $L$ be a residual subspace of $\mathcal{H}$. Recall the expression for the center $c_L$ of a specific element of the orbit $W_0L$ given in section 3.4:

$$(\frac{\lambda_1 - 1}{2}k_1, \frac{\lambda_1 - 3}{2}k_1, \ldots, \frac{\lambda_1 - 1}{2}k_1, \ldots, \frac{\lambda_m - 1}{2}k_1, \ldots, \frac{\lambda_m - 1}{2}k_1, c(\mu, k_1, k_2)).$$

Clearly, if we substitute $k_2 = 0$, at least one of the coordinates will be zero, unless $\mu = 0$ and all $\lambda_i$ are even. The Weyl group $W_0(D_n)$ acts by permutations and even amounts of sign changes, whereas the Weyl group $W_0(B_n)$ can also produce a single sign change. Therefore, if we introduce $c'_L \in W_0(B_n)c_L$ whose coordinates differ from those of $c_L$ by an odd number of sign changes, then $c'_L \in W_0(B_n)c_L$ but $c'_L \notin W_0(D_n)c_L$. These facts, combined with the fact that since $\mathcal{H}(D_n)$ is a group case, and therefore there is one has a bijection between Weyl group orbits of real residual cosets and those of their centers, lead to the required result.  

This branching was to be expected, since of course, $\mathcal{H}(D_n)$ being a group case, we have a bijection between unipotent conjugacy classes in $G = SO_{2n}(C)$ and $W_0(D_n)$-orbits of real residual cosets of $\mathcal{H}(D_n)$. We therefore write again $L_D$ for the real residual coset in $\mathcal{L}(D)$ that corresponds to the coset $L \in \mathcal{L}(0)$ of type $A_\lambda \times (B_l, \mu)$, except if $\mu = 0$ and $\lambda$ has no
odd parts, in which case we write $L_D'$ and $L_D''$ for the cosets of $\mathcal{H}(D_n)$ that correspond to $L$, which thus has type $A_\lambda$.

In view of the preceding results, we can choose $L'$ and $L''$ so as to find the Bala–Carter map $f^{bc}_D$ such that $f^{bc}_D(\lambda') = L'$, and similarly for $L''$, i.e.,

$$(f^{bc}_D \circ \psi_D \circ \phi_D)(L) = L$$ for all $L \in \mathcal{L}(D)$.

The upshot is that the maps for $\mathcal{H}(B_n)_{>1}$, which we have defined, agree with the classical ones for $\mathcal{H}(D_n)$ in the following sense:

\[
\begin{align*}
\Sigma_0(L)_D = \Sigma_D(L_D) &= \Sigma_0(\Phi(L))_D \quad \text{or} \quad \Sigma_0(L)'_D = \Sigma_D(L_D'), \Sigma_0(L)''_D = \Sigma_D(L_D''), \\
\phi_0(\lambda)_D = \phi_D(\lambda_D) &= \phi_0(\lambda)'_D = \phi_D(\lambda_D'), \phi_0(\lambda)''_D = \phi_D(\lambda_D''), \\
\psi_0(\alpha)_D = \psi_D(\alpha_D) = \psi_0(\Phi(\alpha))_D &= \psi_0(\alpha)'_D = \psi_D(\alpha_D'), \psi_0(\alpha)''_D = \psi_D(\alpha_D''), \\
f^{bc}_0(\lambda)_D = f^{bc}_D(\lambda_D) &= f^{bc}_0(\lambda)'_D = f^{bc}_D(\lambda_D'), f^{bc}_0(\lambda)''_D = f^{bc}_D(\lambda_D'')
\end{align*}
\]

9.2.2.2. Green functions. Since $\mathcal{H}(D_n)$ is a group case, we can calculate the parametrization and the $\mathcal{H}_0(D_n)$-decomposition of $\mathcal{H}(D_n)^{\text{temp}}$ by using the Green functions for $D_n$, as explained in Chapter 6. Let us denote them by $(P^D_{\alpha_D^0}, \beta_D^0)_{\alpha_D^0, \beta_D^0 \in \mathcal{W}_0(D_n)}$. These Green functions are the solution of the matrix equation (6.8). The irreducible tempered $\mathcal{H}(D_n)$-modules are thus indexed by $\mathcal{W}_0(D_n)$ and they decompose upon restriction to $\mathcal{H}_0(D_n)$ as

$$V\chi|_{\mathcal{H}_0(D_n)} = \sum_i \psi^{D\chi,i} \otimes \text{sgn},$$

for all $\chi$ in $\mathcal{W}_0(D_n)$.

From the previous section, we know that

$$(9.6) \quad (\psi^{0,\Phi(\alpha),i})_D = (\Phi(\psi^{0,\alpha,i}))_D = (\psi^{0,\alpha,i})_D.$$

On the other hand, according to Conjecture 9.1.10, the irreducible tempered modules of $\mathcal{H}(B_n)_{>1}$ are indexed by (the 0-symbols of) $\mathcal{W}_0(B_n)$, and their restriction to $\mathcal{H}_0(B_n)_{>1}$ is

$$U^{\alpha}|_{\mathcal{H}_0(B_n)_{>1}} = \sum_i \psi^{0,\alpha,i} \otimes \text{sgn}.$$

Therefore

$$U^{\alpha}|_{\mathcal{H}_0(D_n)} = \sum_i (\psi^{0,\alpha,i})_D \otimes \text{sgn}.$$

On the other hand, if we induce $V^{\alpha}$ from $\mathcal{H}(D_n)$ to $\mathcal{H}(B_n)_{>1}$, or restrict $U^{\alpha}$ from $\mathcal{H}(B_n)_{>1}$ to $\mathcal{H}(D_n)$, then we do not change the weight spaces, so we stay in the category of tempered representations with real central character.

From (9.6) it follows that

$$U^{\alpha}|_{\mathcal{H}_0(D_n)} = U^{\Phi(\alpha)}|_{\mathcal{H}_0(D_n)}.$$

If $U^{\alpha}$ has central character $r_L$ and $\Sigma_0(L) = \alpha$, then the irreducible components of its restriction to $\mathcal{H}(D_n)$ have central character $r_{L_D'}$ if $\alpha \neq \Phi(\alpha)$, resp. $r_{L_D''}$ if $\alpha = \Phi(\alpha)$. On the other hand, inducing a $V^{\chi}$ from $\mathcal{H}(B_n)_{>1}$ to $\mathcal{H}(D_n)$ yields representations that have central character $r_L$ if $V^{\chi}$ has central character $r_{L_D'}$, or one of $r_{L_D''}$ and $r_{L_D'''}$.

First we consider $V^{\chi}$ where $\chi$ is of the form $\alpha_L'$. Inducing $V^{\chi}$ to $\mathcal{H}(B_n)_{>1}$ yields a representation $M$ with central character $r_L$, where $L$ is such that $\Sigma_0(L) = \alpha$. Notice that $\Phi(\alpha) = \alpha$, hence $[\alpha]_0 = \{\alpha\}$ and so there is only one equivalence class of irreducible
tempered \( \mathcal{H}(B_n)_{>,1} \)-modules with central character \( L \). Moreover, \( V^\alpha D \) occurs in the restriction to \( \mathcal{H}(D_n) \) of \( M \) by Frobenius reciprocity. Since the same holds for \( V^\alpha D \), the index of \( W_0(D_n) \) in \( W_0(B_n) \) is two, and there exists according to the conjectures only one irreducible tempered \( \mathcal{H}(B_n)_{>,1} \)-module with real central character \( r_L \), we conclude that \( M \) equals \( U^\alpha \), and that

\[
U^\alpha |_{\mathcal{H}_0(D_n)} = \sum_i \psi^{0,\alpha,i} = \sum_i \psi^{D,\alpha'_D,i} + \sum_i \psi^{D,\alpha''_D,i} = V^\alpha D + V^\alpha_D.
\]

Hence, up to a possible shift in the grading, we expect

\[
\psi^{0,\alpha,i} = \psi^{D,\alpha'_D,i} + \psi^{D,\alpha''_D,i}.
\]

Since \( \alpha = \Phi(\alpha) \), it follows that if \( (\xi, \eta) \) (with \( \xi \neq \eta \)) occurs in \( \psi^{0,\alpha,i} \), then \( (\eta, \xi) \) also occurs and has the same multiplicity. It thus turns out that (say) \( (\xi, \eta) \) occurs as \( (\xi, \eta)_D \) in \( \psi^{D,\alpha_D,i} \) and similarly for \( (\eta, \xi)_D = (\xi, \eta)_D \) in \( \psi^{D,\alpha''_D,i} \).

Now suppose that \( \chi = \alpha_D \), then \( \Phi(\alpha) \neq \alpha \). Let \( L \) be the real residual coset for \( \mathcal{H}(B_n)_{>,1} \) such that \( r_{LD} \) is the central character of \( V^\chi \). Then \( \{\chi\}_D = \{\chi_1, \ldots, \chi_m\} \) and we have \( \alpha_i \) such that \( \alpha_i D = \Phi(\alpha_i)_D = \chi_i \) and \( \{\alpha\}_0 = \{\alpha_1, \ldots, \alpha_m\} \). Since \( \Phi(\alpha) \neq \alpha \), also \( \Phi(\alpha_i) \neq \alpha_i \) for all \( i \). We can choose the orderings \( \succ \) on \( \tilde{W}_0(B_n) \) and \( \succ_D \) such that they are compatible with each other, i.e., such that \( \chi_i \succ \chi_j \) implies \( \alpha_i \succ \alpha_j \).

The restrictions of \( U^{\alpha_i} \) and \( U^{\Phi(\alpha_i)} \) to \( \mathcal{H}(D_n) \) yield the same tempered representation of \( \mathcal{H}(D_n) \) with central character \( r_{LD} \). It is not hard to see that it is irreducible. Since the irreducible tempered \( \mathcal{H}(D_n) \)-modules with central character \( r_{LD} \) are parametrized by \( \Sigma_D(L_D) \), the \( \mathcal{H}_0(D_n) \)-restrictions of \( U^{\alpha_i} \) and \( U^{\Phi(\alpha_i)} \) are both isomorphic to \( V^{\tilde{\chi}_i} |_{\mathcal{H}_0(D_n)} \) for some \( \tilde{\chi}_i \in \Sigma_D(L_D) \). It remains to be seen that actually \( \tilde{\chi}_i = \alpha_i, D \). This follows from the triangularity property 9.1.10(ii). In particular, as we expect, the ordering on \( \tilde{W}_0 \) inside a similarity class is not relevant. We thus expect

\[
(\psi^{0,\alpha,i})_D = \psi^{D,\alpha_D,i},
\]

for all \( \alpha \neq \Phi(\alpha) \).

Hence, the results for \( \mathcal{H}(B_n)_{>,1} \) predicted by the Conjecture are indeed compatible with the known classification of the irreducible tempered \( \mathcal{H}(D_n) \)-modules.

**Remark 9.2.3.** We have thus (almost) shown that, granted the conjectures, it follows that the Green functions of \( \mathcal{H}(B_n)_{>,1} \) agree with those of \( \mathcal{H}(D_n) \), and that the Springer correspondence for \( \mathcal{H}(B_n) \) can be obtained from the Springer correspondence for \( \mathcal{H}(D_n) \) by induction or vice versa by restriction. It seems likely that we can turn this reasoning around, thereby showing that the conjectures hold in case \( q_2 = 1 \) by showing that the Green functions of \( \mathcal{H}(B_n)_{>,1} \) and \( \mathcal{H}(D_n) \) agree. Due to time constraints we have, as of yet, not been able to pursue this investigation any further.