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Duality in Power-Law Localization in Disordered One-Dimensional Systems

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The transport of excitations between pinned particles in many physical systems may be mapped to single-particle models with power-law hopping, $1/r^a$. For randomly spaced particles, these models present an effective peculiar disorder that leads to surprising localization properties. We show that in one-dimensional systems almost all eigenstates (except for a few states close to the ground state) are power-law localized for any value of $a > 0$. Moreover, we show that our model is an example of a new universality class of models with power-law hopping, characterized by a duality between systems with long-range hops ($a < 1$) and short-range hops ($a > 1$), in which the wave function amplitude falls off algebraically with the same power γ from the localization center.

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Long-range interactions are crucial in many disordered systems. In particular, dipole-induced transport of excitations, with a hopping amplitude decaying with interparticle distance r as $1/r^3$, plays a key role in disparate scenarios, including magnetic atoms [1], polar molecules [2], Rydberg atoms [3], nitrogen vacancy centers in diamonds [4], and nuclear spins in solid-state systems [5]. Interestingly, systems with tunable power-law interactions $1/r^a$ have been recently proposed. In the presence of laser-driven coupling, trapped ions may realize power-law-decaying spin interactions with tunable $0 < a < 3$ [6,7]. Spin models with an arbitrary power-law interaction can be also engineered between atoms trapped in a photonic crystal waveguide [8]. These tunable systems can be realized in low dimensions, opening intriguing questions on transport with truly long-range interactions, i.e., for $a < d$, with d as the dimensionality.

Most of the previous works considered excitation transport among regularly spaced particles, e.g., by pinning them in an uniformly filled lattice with on-site disorder. In addition, long-range hopping has been considered either anisotropic with zero angular average (e.g., dipole hopping [9,10]) or random with zero ensemble average [11]. Under such conditions, the interplay of on-site disorder and long-range hops may result in localization, critical behavior, or fully extended states. In his seminal work, Levitov suggested that these cases occur for $a > d$, $a = d$, and $a < d$, respectively [9]. Later studies for $d = 1$ using a supersymmetric nonlinear sigma-model approach confirmed this suggestion [11]. Intense efforts have been devoted to the critical power-law banded random matrix (PLBRM) ensemble ($d = a = 1$) [12–18]. Dipolar excitations in a 2D lattice with on-site disorder ($d = a = 2$) exhibit a purely critical behavior only for the time-reversal-invariant case and may have a "metal-insulator" transition if the invariance is broken [10].

Notwithstanding these details, the common wisdom until recently was that, for noninteracting particles with on-site disorder, long-range hopping is a delocalizing factor that destroys localization for $a \leq d$. Only few recent works have stood out of this paradigm. Surprisingly, all states appeared to be localized for the exotic case of linear hopping ($a = -1$) between randomly placed points in a 1D system [19]. A second example is provided by a simplex ($a = 0$), where all excited states were found to be localized [20]. This peculiar behavior was linked in Ref. [20] to the macroscopic degeneracy of the perturbation matrix. The role of degeneracy in this model was recently further emphasized and extended to $a > 0$ in Refs. [21,22], where it was argued that the long-range hopping has no effect on the system dynamics as long as the width of the disorder-broadened $(N-1)$-fold degenerate level (with N as the number of sites) is smaller than the gap between the ground and excited states for $a = 0$. Since this gap is proportional to N^{1-a} and the bandwidth is N independent, the long-range hopping is "shielded" in the thermodynamic limit for all $0 < a < 1$, and the localization of excited states is preserved.

In this Letter, we consider the physically relevant case of excitation transport due to power-law hops among randomly

pinned particles. We call these models power-law Euclidean (PLE) models, in connection with the Euclidean random matrices with no diagonal elements [23]. Disorder in PLE models is purely off diagonal given by the Poisson distribution of the particle positions. However, it is crucially different from the off diagonal disorder in nearest-neighbor hopping models [24,25], since long-range hops break down the chiral (sublattice) symmetry [26] of the nearest-neighbor hopping case. It also differs from the off diagonal disorder of Ref. [11] because the angular or ensemble average of the hopping amplitude is not zero.

We focus on the 1D case, showing that PLE models possess features radically different from those with random-sign long-range hopping [11]. Similar to the Anderson model on a lattice with diagonal disorder and nonrandom long-range hopping [21], almost all states (except for a few states close to the lower edge of the spectrum; see Supplemental Material [27]) are localized for any $a > 0$. However, in contrast to the conventional Anderson model with exponential localization, in the PLE model, localization is algebraic with a power-law decrease of wave functions, $\exp(\overline{\ln|\psi(n)|^2}) \sim |n-n_0|^{-\gamma}$ at sites n away from the localization center n_0, where γ depends on a. This remains valid for the model of Ref. [21], even for arbitrarily small but finite a. As discussed below, power-law localization emerges because long-range hops are not fully shielded, but rather become effectively short range.

The main result of this Letter is a surprising duality,

$$\gamma(a) = \gamma(2-a), \tag{1}$$

for $0 < a < 1$. We obtain this duality numerically for $0 < a < 1$ (Fig. 1 and Fig. 2) and analytically for the PLE model at finite but small a [29]. Strikingly, this duality is more general than our PLE models. It holds for the model of Ref. [21], which presents on-site disorder in a regular lattice with long-range hops. It also holds for the PLE model with additional on-site disorder. Moreover, the exponent γ depends only on a but not on other details of the models, including the eigenstate energy (see Fig. 3 and Fig. S6 in the Supplemental Material [27]), except for a small fraction (~3%) of the states at the edges of the spectrum. Thus, the considered models are not only experimentally relevant but represent a new class of long-range models where localization is algebraic and the duality Eq. (1) holds.

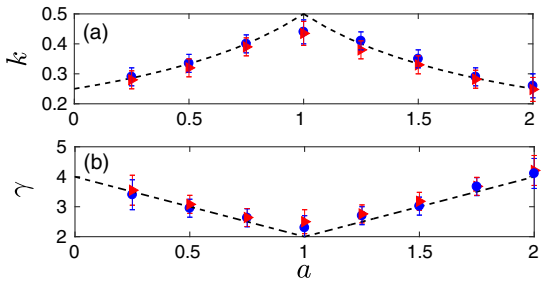

FIG. 1. Duality for $0 < a < 1$ and $1 < a < 2$ of the localization properties of the PLE model (blue circles) and the model of Refs. [20,21] with diagonal disorder and deterministic sign-definite, long-range hopping (red triangles): (a) slope k of the MFS extrapolated to $N \to \infty$; (b) exponent γ of the algebraic localization of $|\psi|^2$. Dashed lines correspond to $\gamma = 1/k = 2a$ for $1 < a < 2$ or to $\gamma = 4 - 2a$ for $0 < a < 1$. The error bars mainly stem from the extrapolation to infinite size systems.

Model.—We consider N particles pinned at random positions $\{r_n\}$ by an external 1D potential. Each particle has two internal states $\{\uparrow, \downarrow\}$, which can be treated as (pseudo)spin states. The particles experience power-law spin exchange described by the XY Hamiltonian

$$H_{XY} = -\frac{J}{2}\sum_{n,m}\frac{1}{|r_n - r_m|^a}(S_n^+ S_m^- + S_n^- S_m^+), \tag{2}$$

with $S_n^{\pm}$ as the spin operators associated with particle n. Assuming that all particles are in the state $\downarrow$, we denote $|n\rangle$ the state with a single spin-flipped particle $\uparrow$ at the site n placed at position r_n. The propagation of a single excitation among the particles, determined by the XY exchange, can be modeled by the Hamiltonian

$$H = \sum_n \sum_{m\neq n} H_{n,m}|n\rangle\langle m|. \tag{3}$$

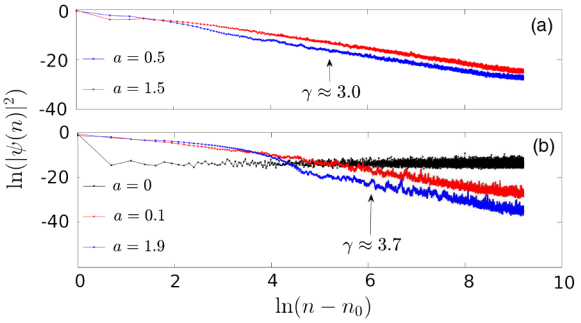

FIG. 2. Average of $\ln(|\psi(n)|^2)$ for 1D chain of $N = 2 \times 10^4$ sites vs logarithm of the distance n from the localization center. (a) PLE model with random site positions and deterministic hopping function $1/r^a$ for $a = 0.5$ and $a = 1.5$. The duality Eq. (1) manifests itself in the same slope $-\gamma(a)$ of the curves at the tail. (b) Model of Ref. [21] on regular lattice with diagonal disorder and deterministic power-law hopping for $a = 0$, $a = 0.1$, and $a = 1.9$. Notice a drastic difference between the exactly solvable [30] case $a = 0$ and the two dual cases with a close to zero and two ($a = 0.1$ and $a = 1.9$). At $a \to 0$, the eigenstates are algebraically localized with $\gamma \to 4$, while at $a = 0$ they are "critically multifractal" with $f(\alpha) = \alpha/2$ for $0 < \alpha < 2$ and have a form of a sharp peak on the top of a background [31]. In contrast to the eigenstates of the standard Anderson model, where the background is exponentially small, the critically multifractal states are characterized by much stronger background $\sim N^{-2}$.

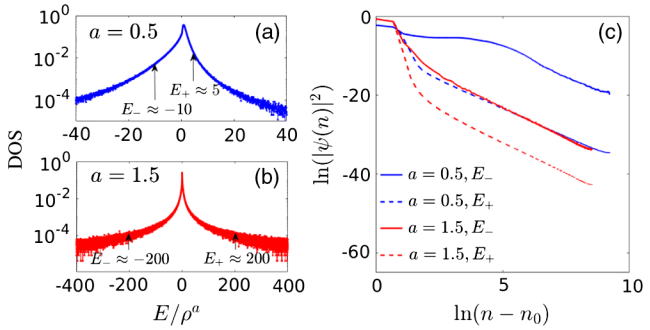

FIG. 3. Dependence of γ on the energy for the PLE model. (a) DOS for $a = 0.5$. (b) DOS for $a = 1.5$. (c) Average of $\ln(|\psi(n)|^2)$. For all energies $E_- < E < E_+$, in both cases $a = 0.5$ and $a = 1.5$, the slope of the curves $-\gamma \approx -3$ is the same. The fraction of states beyond this energy interval, i.e., at small and large energies, is $\sim 3\%$ in both cases.

The hopping of the excitation between the nth and mth particles follows a power law

$$H_{n,m} = -J|r_n - r_m|^{-a}. \tag{4}$$

Below we assume for simplicity $J = 1$. Since the positions r_n are randomly distributed, the excitation experiences a peculiar randomness that may significantly handicap its propagation in the system. The single-particle model is expected to describe well the case of more than one excitation, as long as the gas of excitations is dilute enough.

Localization and multifractality.—We are interested in the localization properties of the eigenstates $|\psi_s\rangle$ of H, $H|\psi_s\rangle = E_s|\psi_s\rangle$, close to the maximum of the density of states (see Supplemental Material [27]) and, in particular, how these properties depend on a. The eigenfunctions and eigenenergies were obtained by exact diagonalization of model (3) with N up to 8.4×10^4.

We characterize the eigenstates $|\psi_s\rangle = \sum_n \psi_s(n)|n\rangle$ by the moments $I_q(s) = \sum_n |\psi_s(n)|^{2q} \propto N^{-\tau(q)}$, where $D(q) = \tau(q)/(q-1)$ are the so-called fractal dimensions. Localized states are characterized by $D(q) = 0$, ergodic extended states by $D(q) = 1$, while extended, but nonergodic, multifractal states have a set of nontrivial fractal dimensions $0 < D(q) < 1$ [17,32–35]. The Legendre transform $\tau(q) = q\alpha - f(\alpha)$ defines the multifractal spectrum (MFS) $f(\alpha)$ [32], which characterizes the Hausdorff dimension of the manifold of sites, where $|\psi_s(n)|^2 = N^{-\alpha}$. Normalization $\sum_n |\psi_s(n)|^2 = 1$ requires $\alpha \geq 0$.

An important difference between power-law- and short-range-hopping models is that, in the former case, localized states, if they exist at all, have power-law-decaying tails $|\psi_s(n)|^2 \propto 1/|n - n_0|^\gamma$; i.e., they are algebraically localized. One can show that the corresponding $f(\alpha)$ has a triangular form, where $f(\alpha) = k\alpha$ for $0 < \alpha < 1/k$ with $k = d/\gamma$. This is in contrast to exponentially localized states for a short-range hopping, where $k = 0$. Note that $|\psi_s(n)|^2 \propto |H_{n+n_0,n_0}|^2 \propto n^{-2a}$ in the perturbative regime at $a > d$, and hence γ cannot be smaller than $\gamma_a = 2a > 2d$. Therefore, for algebraically localized states in 1D, one has $k < 1/2$ like for the exponentially localized states on the Bethe lattice [34].

We evaluate the spatial distribution of the eigenstates by setting the localization center placed at the maximum value of $|\psi|^2$ at index $n = n_0$ and averaging $\ln|\psi(n)|^2$ over disorder realizations and over an energy window. For the localized states discussed below, the typical average $\exp(\overline{\ln|\psi(n)|^2})$ is well fitted by the expected algebraic dependence $|n - n_0|^{-\gamma}$ as seen in Fig. 2, except for the special point $a = 0$ [see Fig. 2(b)], where the PLE model is disorder free and the model [21] is exactly solvable [30].

Multifractal spectrum.—We obtain the MFS in the thermodynamic limit by means of a linear extrapolation of our finite-size calculations in terms of $1/\ln N$ [34,35]. For $a > 1$, we expect localized states and, indeed, we find linear $f(\alpha)$ with a slope $k < 1/2$. At $a = 1$, for all eigenstates, the MFS displays a triangular shape with the slope $k_c \simeq 0.42$ [Fig. 4(b)]. Since k_c is very close to the critical value $1/2$, the eigenstates for $a = 1$ are either weakly localized or critical. For $a < 1$, one would naively

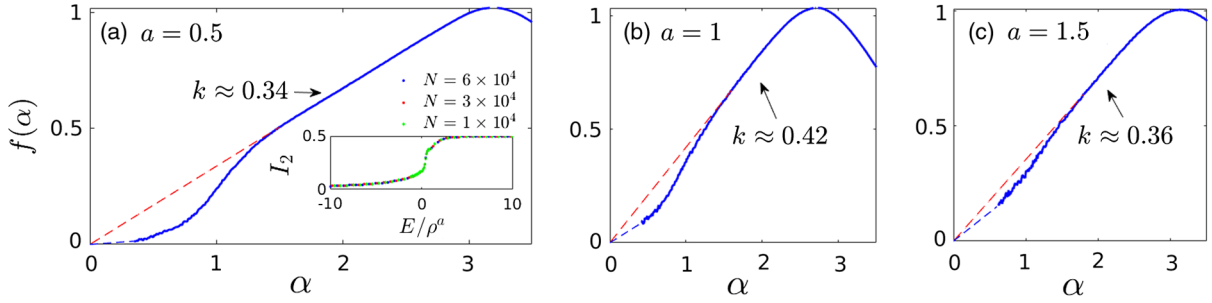

FIG. 4. Multifractal spectrum $f(\alpha)$ (blue curve) obtained from the distribution function $P(|\psi_s(n)|^2)$, extrapolated to $N \to \infty$, for (a) $a = 0.5$; (b) $a = 1$, and (c) $a = 1.5$. We consider eigenstates at the maximum of the density of states. The inset of (a) shows that the inverse participation ratio I_2 does not depend on N, thus proving insulating behavior. Note that the concave feature in $f(\alpha)$ appears in all cases (see text). The red dashed line in each case shows the triangular $f_{\rm tr}(\alpha)$ with $k < 1/2$ that corresponds to the same moments I_q as $f(\alpha)$. The blue dashed line indicates that $f(0) = 0$, which is confirmed by our numerics.

expect extended eigenstates. However, this expectation appears to be wrong. The eigenstates inside a band emerging due to disorder broadening of the macroscopically degenerate level remain localized for all $0 < a < 1$.

Note that the slope k of the linear part of $f(\alpha)$ shown in Fig. 4 reflects the algebraic decay of $|\psi_s(n)|$ far from the localization center. For small α, the MFS becomes concave, implying a reduced probability to find intermediate values $|\psi_s(n)|^2 \propto N^{-\alpha}$. A similar concave MFS has been discussed for the insulating phase of the Rosenzweig-Porter random matrix ensemble [36]. The moments $I_q \propto N^{-\tau(q)}$ are, however, determined by finding the tangent $-\tau(q) + q\alpha$ to $f(\alpha)$ that does not cross the MFS at any point. Hence, $\tau(q)$ and fractal dimensions for $f(\alpha)$ with a concave segment are the same as those for the MFS in which the concave part is replaced by the red dashed line shown in Fig 4.

Duality.—The localization properties of the eigenstates of model (3) for $0 < a < 1$ and for $1 < a < 2$ exhibit a striking duality given by Eq. (1). The slope k of the MFS is depicted in the upper panel of Fig. 1. The lower panel shows the exponent γ extracted from the average $\overline{\ln(|\psi(n)|^2)} \propto -\gamma \ln|n - n_0|$ (see Fig. 2). Both graphs are in good mutual agreement given by the relation $k = 1/\gamma$. Remarkably, a similar duality is present for the model of Ref. [21], where disorder is purely diagonal (see Fig. 1).

Proof of duality for $a \to 0$ and $a \to 2$.—As was already mentioned, for $a = 0$, there is an $(N-1)$-fold degeneracy of the excited states, which may be chosen both extended and localized on equal footing. The localization properties of degenerate states are controlled by the perturbation $V(r) = r^{-a} - 1$, namely, by the properties of the "good zero-order wave functions" [37], which are eigenvectors of the perturbation operator in a certain convenient basis. Let us select a basis of $(N-1)$ states including $N/2$ nonoverlapping dimers $|\psi_{n,n+1}\rangle = (|n\rangle - |n+1\rangle)/\sqrt{2}$, $N/4$ nonoverlapping tetramers $|\psi^{n+2,n+3}_{n,n+1}\rangle = (|n\rangle + |n+1\rangle - |n+2\rangle - |n+3\rangle)/2$, $N/8$ octamers, and so on. The basis constructed in this way is orthonormal and complete for any $N = 2^k$, where k is a positive integer. Moreover, the n-mers are compact and thus represent localized states for any $n = \mathcal{O}(1)$. The perturbation results in hopping between n-mers. The hopping amplitude $V_{n,m}$ from the dimer $|\psi_{n,n+1}\rangle$ to the dimer $|\psi_{m,m+1}\rangle$, located far away (and $m \gg n$), in the limit of small a is given by

$$V_{n,m} = \left(\frac{1}{|r_{m+1} - r_n|^a} - \frac{1}{|r_m - r_n|^a} + \frac{1}{|r_m - r_{n+1}|^a} - \frac{1}{|r_{m+1} - r_{n+1}|^a}\right) \propto \frac{a}{R^2_{nm}}, \quad (5)$$

where $R_{nm} = (r_{m+1} + r_m - r_{n+1} - r_n)/2$. A similar dependence holds for the hopping between tetramers and all n-mers (see the Supplemental Material [27]). Therefore good zero-order wave functions in the basis of n-mers obey the same equation as eigenvectors of the Hamiltonian with hopping $\sim 1/r^2$. Thus, we have reduced the problem of localization close to the point of macroscopic degeneracy $a = 0$ to the problem of localization in a system of n-mers with hopping amplitude $\propto 1/R^2_{nm}$ for which localization is well understood. This proves the localization in our model for $a \to 0$ case [29] and its duality to localization at $a = 2$.

Note finally that for $a < 1$ at the edge of the spectrum at negative energy there are also delocalized states (see the Supplemental Material [27]). The measure of the number of such states is zero, but they will be responsible for the transport in the system.

Role of sign alteration.—The localization properties of our model and, in particular, the duality (2) are drastically different from those of PLBRM [11] and Levitov's scenario [9]. We show at this point that this difference is related to the sign randomness in the long-range hopping. To this end, we modify our model in such a way that $H_{n,m}$ acquires a randomness $\eta_{n,m}$,

$$H_{n,m} = -(J + \eta_{n,m})/r^a_{nm}, \quad (6)$$

where $\eta_{n,m}$ are random bounded numbers, $|\eta_{n,m}| < W$, and $r_{nm} = |r_n - r_m|$. Model (3) corresponds to $W = 0$, $J = 1$, while that of Ref. [11] corresponds to $W = 1, J = 0$. Model (6) can be physically realized for a spatially pinned two-component system, in which the spinlike excitations may be transferred either between particles of the same or of different species (see the Supplemental Material [27]).

Another possible modification is the staggering model,

$$H_{n,m} = (-1)^{r_{nm}}/r^a_{nm}, \quad (7)$$

for which the sign of the product $H_{m,j_1}H_{j_1,j_2},\ldots,H_{j_{\ell-1},n}$ over a path of ℓ hops from m to n is independent of the number of hops and equal to $(-1)^{r_{nm}}$.

The analysis presented in the Supplemental Material [27] shows that, even at small $W = 0.1$, model (6) exhibits for $J = 1$ the same properties as PLBRM and the eigenstates are extended for $a < 1$. In contrast, model (7) is in the same universality class as model (3). This shows that localization in model (3) is due to interference of long-range paths involving one or more hops, which is not affected by staggering, but destroyed by sign randomness.

Conclusions.—Systems with power-law $\propto 1/r^a$ spin exchange between randomly spaced particles represent a peculiar form of off diagonal disorder that leads to surprising localization properties. We have shown that 1D spin excitations remain algebraically localized for any value of the hopping power $a > 0$. Moreover, we show that our model is a representative of a new universality class of models with power-law hopping, characterized by a duality Eq. (1) between models with $0 < a < 1$ and $2 > a > 1$.

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[29] Note that the limit $a \to 0$ does not commute with the limit $N \to \infty$ of infinite system size. The duality holds only if the limit $N \to \infty$ is taken first. The opposite order of limits corresponds to the exceptional point $a = 0$, where the PLE model is disorder free and macroscopically degenerate, whereas for the model [21] with diagonal disorder, the wave functions in the bulk of the spectrum are critically multifractal. In this case, the problem allows for an exact solution [30].
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