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Fendley, P.; Schoutens, K.; de Boer, J.

Publication date
2003

Published in
Physical Review Letters

Lattice Models with $\mathcal{N} = 2$ Supersymmetry

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(Received 1 November 2002; published 25 March 2003)

We introduce lattice models with explicit $\mathcal{N} = 2$ supersymmetry. In these interacting models, the supersymmetry generators $Q^{\pm}$ yield the Hamiltonian $H = \{Q^+, Q^-\}$ on any graph. The degrees of freedom can be described as either fermions with hard cores, or as quantum dimers; the Hamiltonian of our simplest model contains a hopping term and a repulsive potential. We analyze these models using conformal field theory, the Bethe ansatz, and cohomology. The simplest model provides a manifestly supersymmetric lattice regulator for the supersymmetric point of the massless $(1 + 1)$-dimensional Thirring (Luttinger) model. Generalizations include a quantum monomer-dimer model on a two-leg ladder.

DOI: 10.1103/PhysRevLett.90.120402 PACS numbers: 05.30.–d, 05.50.+q, 11.30.Pb

Supersymmetry is an exceptionally powerful theoretical tool. It can allow exact computations in field theory and string theory, even when interactions are strong. However, since only certain quantities can be computed nonperturbatively, a lattice formulation of supersymmetric field theories would be useful. Efforts to achieve this aim have concentrated mainly on taking a known field theory and discretizing it [1]. Since the supersymmetry algebra involves the Poincaré algebra, putting it on a lattice automatically breaks at least some of the supersymmetries. The difficulty is in ensuring that the supersymmetry is maintained in the continuum limit.

In this Letter, we approach the problem in a different way. We introduce simple lattice models with $\mathcal{N} = 2$ supersymmetry, describing interacting fermions and quantum monomer-dimer systems. We show that the continuum limit of the simplest of these models is a well-known $(1 + 1)$-dimensional quantum field theory with $\mathcal{N} = (2, 2)$ superconformal symmetry. We also note the close connections with models of current interest in the study of strongly correlated electrons, and show how supersymmetry can be useful in studying their properties. There are two key questions we will attempt to answer. The first is: What properties can be computed exactly using the supersymmetry? The second is: Which (if any) field theories describe the models in the continuum limit?

Our definition of $\mathcal{N} = 2$ supersymmetry is that the Hamiltonian H is built from two nilpotent fermionic generators denoted Q^+ and $Q^- = (Q^+)^\dagger$ [2]. It is

$$H = \{Q^+, Q^-\}. \tag{1}$$

The fact that Q^+ and Q^- commute with H follows from the nilpotency $(Q^+)^2 = (Q^-)^2 = 0$. Models with $\mathcal{N} = 2$ supersymmetry also have a fermion-number symmetry generated by F with $[F, Q^{\pm}] = \pm Q^{\pm}$. We shall show how, in at least some cases, this lattice supersymmetry extends to a space-time super(conformal) symmetry in the field theory describing the continuum limit. Lattice models with a symmetry involving fermionic generators, such as the t-J model at $J = \pm 2t$, are often called "supersymmetric" in the condensed-matter literature, but do not have a Hamiltonian of the form of Eq. (1).

The models we introduce can be defined on any lattice (or actually, any graph) in any dimension. The simplest model involves a single species of fermion c_i, placed at any site i of the lattice. The fermion obeys the usual anticommutator $\{c_i, c_j^\dagger\} = \delta_{ij}$, and the operator $F = \sum_i c_i^\dagger c_i$ counts the number of fermions. We impose the restriction that the fermions have hard cores, meaning that fermions are not allowed on neighboring sites. A hard-core fermion is created by $c_i^\dagger \mathcal{P}_{\langle i \rangle}$, where the projection operator $\mathcal{P}_{\langle i \rangle}$ requires all sites neighboring i to be empty:

$$\mathcal{P}_{\langle i \rangle} = \prod_{j \text{ next to } i} (1 - c_j^\dagger c_j). \tag{2}$$

The supersymmetry operators are defined by

$$Q^+ = \sum_i c_i^\dagger \mathcal{P}_{\langle i \rangle}, \qquad Q^- = \sum_i c_i \mathcal{P}_{\langle i \rangle}. \tag{3}$$

It is easy to verify that $(Q^+)^2 = (Q^-)^2 = 0$. The Hamiltonian is therefore

$$H = \sum_i \sum_{j \text{ next to } i} \mathcal{P}_{\langle i \rangle} c_i^\dagger c_j \mathcal{P}_{\langle j \rangle} + \sum_i \mathcal{P}_{\langle i \rangle}. \tag{4}$$

The first term in the Hamiltonian allows fermions to hop to neighboring sites, with the projectors maintaining the hard-core repulsion. The second term favors having more fermions, as long as they are more than two sites from each other. Thus, one can view it as a repulsive potential for fermions, in addition to the hard core.

The supersymmetry is very useful for finding the $E = 0$ ground states of a theory [2]. All eigenvalues E of the Hamiltonian Eq. (1) satisfy $E \geq 0$, and the eigenstates form either singlet or doublet representations of the supersymmetry algebra. All singlets must have $E = 0$, and all states $|g\rangle$ with $E = 0$ are singlets: $Q^+|g\rangle = Q^-|g\rangle = 0$. Eigenstates $|s\rangle$ with $Q^-|s\rangle = 0$ and $E > 0$ give rise to a doublet $(|s\rangle, Q^+|s\rangle)$ of eigenstates with the same eigenvalue E. All $E > 0$ eigenstates of H can be decomposed into doublets; i.e., the four-dimensional representation $(|s'\rangle, Q^-|s'\rangle, Q^+|s'\rangle, Q^+Q^-|s'\rangle)$ is reducible. Let

$$|s\rangle \equiv |s'\rangle - \frac{1}{E_s} Q^+ Q^- |s'\rangle,$$

where $E_s > 0$ is defined by $H|s'\rangle = E_s|s'\rangle$. Then $Q^-|s\rangle = 0$, and $(|s\rangle, Q^+|s\rangle)$ and $(Q^-|s'\rangle, Q^+Q^-|s'\rangle)$ form two irreducible doublets. These simple properties of the states make computing the Witten index [2],

$$W = \mathrm{tr}[(-1)^F e^{-\beta H}], \tag{5}$$

possible. Because the two states in a doublet have the same energy, their contribution to W cancels, leaving the trace only over $E = 0$ ground states. W is thus independent of β, and can thus be found by evaluating (5) in the $\beta \to 0$ limit, where all states contribute with weight $(-1)^F$. For example, for the hard-core fermions on a cube, one finds $W = 1 - 8 + 16 - 8 + 2 = 3$, so that in this case there are at least three ground states.

For another example, consider the six-site chain with periodic boundary conditions. There is one state $|0\rangle$ with $f = 0$, and six states $c_i^\dagger|0\rangle$ with $f = 1$, while, because of the hard cores, there are nine states with $f = 2$, and two with $f = 3$. This means $W = 1 - 6 + 9 - 2 = 2$. We can find these two ground states by grouping the other states into doublets. The vacuum obeys $Q^-|0\rangle = 0$ and $Q^+|0\rangle = \sum_{i=1}^6 c_i^\dagger|0\rangle$, so $(|0\rangle, Q^+|0\rangle)$ form a doublet. The remaining five states with $f = 1$ are all annihilated by Q^- but not Q^+, so there are five doublets with $(f, f+1) = (1, 2)$. The states with $f = 3$ are both annihilated by Q^+ but not Q^-, giving two doublets with $(f, f+1) = (2, 3)$. This accounts for all the states save two with $f = 2$. These two states cannot form a doublet, because they have the same fermion number. They therefore must be singlets, and so these states with $f = 2$ are the two $E = 0$ ground states. With a little more work, one finds that they have eigenvalues $\exp(\pm i\pi/3)$ under translation by one site.

Finding the ground states for supersymmetric hard-core fermions on a general graph poses a fascinating combinatorial problem. The powerful tool of cohomology theory allows us to compute the number of ground states at any fermion number. The supersymmetry generator Q^+ satisfies $(Q^+)^2 = 0$, and the $E = 0$ ground states are precisely the states $|s\rangle$ that satisfy $Q^+|s\rangle = 0$ and that cannot be written in the form $|s\rangle = Q^+|s'\rangle$. Those states form what is called the cohomology of the operator Q^+, and for its computation a variety of techniques are available; we apply the "spectral sequence" technique here as follows. We split the lattice in two sublattices, with corresponding fermion-number operators F_1 and F_2, so that $F = F_1 + F_2$. Q^+ can also be split as $Q_1^+ + Q_2^+$, so that Q_i^+ increases F_i by one. The two operators Q_1^+ and Q_2^+ are nilpotent and anticommute. The first step in the spectral sequence is to compute the cohomology of Q_1^+. Q_2^+ becomes an operator acting on this cohomology, and the second step is to compute the cohomology of Q_2^+ on this subspace. Often the process terminates here, and the result is the cohomology of Q^+. In general, the procedure continues for a finite number of additional steps (see, e.g., [3] for details). A similar procedure exists for a decomposition of a lattice into n sublattices, and, in particular, when every sublattice consists of a single site. Applying this to the N-site periodic chain, with F_1 consisting of every third site, we find ground states solely at fermion number $f = \mathrm{int}[(N+1)/3]$. For a chain with $N = 3p$ with p integer, we find two ground states [so $W = 2(-1)^p$], while for $N = 3p \pm 1$, there is a single ground state [so $W = (-1)^p$].

We can see heuristically why a ground state for the chain has $f = \mathrm{int}[(N+1)/3]$. The potential term in (6) alone is minimized by the state with a fermion on every third site; adding any more fermions forces fermions to be two sites away and raises the energy. The hopping term alone also discourages fermions from being only two sites away, because it has negative eigenvalues when fermions can hop to an adjacent site and back again, and the hard cores prevent this if there is another fermion two sites away. The state with a fermion on every third site,

$$\ldots\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\circ\ldots$$

resembles a Néel state for a Heisenberg antiferromagnet. It is not an eigenstate of the Hamiltonian: The full ground state is disordered. However, like the Néel state, we expect this state to be a part of the ground state. Our derivations of $f_{\mathrm{GS}} = \mathrm{int}[(N+1)/3]$ confirm this intuition. This heuristic picture also gives the fermion numbers of the low-lying excited states. The excitations include defects (domain walls) in the Néel-like state, such as

$$\ldots\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\ldots$$

The fermion number of this configuration is just one higher than that of the Néel-like state, and it has three identical defects. Since defects can be moved arbitrarily far apart with no change in the potential, it is natural to treat each defect as a quasiparticle with charge $1/3$. The existence of fractional charge in $1+1$ dimensions is an old story; this was first discovered in field theory [4].

The cohomology technique is applicable to more general models. Quantum dimer models are defined by placing the hard-core fermions $c_i^\dagger$ on the links instead of the sites of a lattice. The product in $\mathcal{P}_{\langle i \rangle}$ is then over links which meet the link i, so that the projectors prohibit overlapping dimers. On the chain with $3p$ sites, the ground states have dimer number $N_d = p$, which is 2/3 of the value for close-packed dimers. For dimers on a two-leg ladder with $R + 1$ rungs, we found that the number N_{GS} of ground states grows quickly with R, and that not all ground states have the same dimer number, leading to (partial) cancellations in the Witten index W. For example, for $R = 7$, one finds five $E = 0$ ground states with $N_d = 5, 5, 5, 5, 6$, so $W = -3$. Our results, asymptotically precise for R large, are $N_{GS} \sim (1.395)^R$ and $W \sim (1.356)^R$. It will be interesting to see if supersymmetry can be of use in other quantum dimer models [5].

We now analyze the one-dimensional case in more detail. The Hamiltonian Eq. (4) is on an N-site chain:

$$H = \sum_{i=1}^{N} [P_{i-1}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) P_{i+2} + P_{i-1} P_{i+1}], \quad (6)$$

where $P_i \equiv 1 - c_i^\dagger c_i$ is the projector on a single site. We take periodic boundary conditions, so indices are defined mod N. The translation operator T commutes with both H and F, and its eigenvalues t satisfying $t^N = 1$ characterize the eigenstates of H. The Hamiltonian Eq. (6) resembles a lattice version of the Thirring model, a $(1 + 1)$-dimensional field theory with a four-fermion interaction term. Below we will make the connection precise.

We originally obtained the space of states for this model by applying a systematic "finitization" procedure [6] to the chiral spectrum of a specific two-dimensional $\mathcal{N} = (2, 2)$ superconformal field theory (SCFT) with central charge $c = 1$. This construction follows two steps. In the first step, the full chiral Hilbert space of the SCFT is written in a "quasiparticle" basis, with the fundamental quasiparticles forming a supersymmetry doublet with charges $(1/3, -2/3)$. In the second step, the momenta of the individual quasiparticles are constrained to a maximum value in the order of N, corresponding to a discretization of the space direction of the SCFT with spacing of the order $1/N$. This then leads to a truncated or "finitized" partition sum $Q_N(q, w)$, which is closely related to the partition sum $Z_N(q, w) = \mathrm{Tr}(q^{[N/(2\pi i)] \log T} w^{(N-3F)})$, which keeps track of the eigenvalues of the translation operator T and the fermion-number F of the lattice model Eq. (6). This entire construction, to be detailed elsewhere, respects the supersymmetry. This gives a rationale for the existence of supersymmetry generators on this space of states, and it provides an independent way of determining the quantum numbers t and f of the supersymmetric ground states. We remark that an analogous construction based on the spinon basis of the simplest SU(2)-invariant CFT naturally leads to the space of states of the Heisenberg and Haldane-Shastry models for N spin-1/2 degrees of freedom. Clearly, many generalizations are possible.

We have discussed how the space of states in the model with Hamiltonian (6) arises from the finitization of the $c = 1$ conformal field theory. Moreover, the lattice model and the SCFT have the same Witten index $W = 2$. We now give the results of a Bethe ansatz computation [7]. which makes the correspondence precise.

An eigenstate of H with f fermions is of the form

$$\phi^{(f)} = \sum_{\{i_k\}} \varphi(i_1, i_2, \ldots, i_f) c_{i_1}^\dagger c_{i_2}^\dagger \ldots c_{i_f}^\dagger |0\rangle, \quad (7)$$

where we order $1 \le i_1 < i_2 - 1 < i_3 - 2 \ldots$. Bethe's ansatz for these eigenstates is [8]

$$\varphi(i_1, i_2, \ldots, i_f) = \sum_P A_P \mu_{P1}^{i_1 - 1} \mu_{P2}^{i_2 - 1} \ldots \mu_{Pf}^{i_f - 1}, \quad (8)$$

for some numbers $\{\mu_1, \ldots, \mu_f\}$ and A_P; the sum is over permutations P of the set $(1, 2, \ldots, f)$. By construction, the translation operator T has eigenvalue $t = \prod_{k=1}^f \mu_k^{-1}$. For a generic model, this ansatz does not work, but in the Heisenberg and other integrable models, the miracle is that it does. Here, the ansatz gives eigenstates of energy

$$E = N - 2f + \sum_{k=1}^{f} \left[\mu_i + \frac{1}{\mu_i} \right], \quad (9)$$

when the μ_k obey the Bethe equations

$$t^{-1}(\mu_k)^{N-f} = \prod_{j=1}^{f} \frac{\mu_k \mu_j + 1 - \mu_k}{\mu_k \mu_j + 1 - \mu_j}, \quad (10)$$

for all $k = 1, \ldots, f$. These are very similar to the Bethe equations for the antiferromagnetic XXZ spin chain at $\Delta = \pm 1/2$ [9]. The only difference in (10) is in the left-hand-side, which in the XXZ case reads $(\mu_k)^N \tau$, where $\tau \neq 1$ corresponds to twisted boundary conditions.

The supersymmetry doublets appear naturally within the Bethe ansatz: If $(\mu_1, \ldots, \mu_f)$ satisfies the Bethe equations, then the set $(1, \mu_1, \ldots, \mu_f)$ also satisfies them. It is straightforward to check that the states associated with these two sets are related by $\phi^{(f+1)} = Q^+ \phi^{(f)}$, and that both sets have the same energy E.

The Bethe equations are f coupled polynomial equations of order N. They cannot be solved in closed form and, to make further progress, one usually needs to take N large. In our case, however, the supersymmetry allows us to derive more results from the Bethe ansatz for finite N. Precisely, we define w_k in terms of μ_k as $w_k = (\mu_k - q)/(q\mu_k - 1)$, where $q \equiv \exp(-i\pi/3)$. Then Baxter's $\mathcal{Q}$ function [9] $\mathcal{Q}(w) \equiv \prod_{i=1}^f (w - w_k)$ has zeros at $w = w_k$. Defining $\mathcal{R}(w) \equiv \mathcal{Q}(w)(1 + w)^{N-f}$, we find that, for the w_k giving the ground state,

$$\mathcal{R}(q^{-2} w) = t q^{-N} \mathcal{R}(w) + t^{-1} q^N \mathcal{R}(q^2 w). \quad (11)$$

We derive an explicit expression for $\mathcal{R}(w)$ in the sequel [7], but from (11) directly we can rederive f and t for the ground state(s). When $N = 3p$ with p an integer, there are nontrivial solutions to (11) only when $f = N/3$ and $t = (-1)^N \exp(\pm i\pi/3)$. For $N \neq 3p$, one has only a single solution with $f = \mathrm{int}[(N + 1)/3]$, and $t = (-1)^{N-1}$.

We now discuss the field theory describing the continuum limit of (6). When N is large, one can rewrite the Bethe equations in terms of densities of roots [8], and then derive integral equations (known as thermodynamic Bethe ansatz equations) yielding the free energy. Our model has the same thermodynamic equations as the XXZ chain at $\Delta = 1/2$, so the two models coincide in the continuum limit. The continuum limit of the XXZ chain is described by the massless Thirring model [10], or, equivalently, a free massless boson Φ with action [11]

$$S = \frac{2g}{\pi} \int dxdt[(\partial_t \Phi)^2 - (\partial_x \Phi)^2].$$

The continuum limit of the $\Delta = 1/2$ model has $g = 2/3$; this is the simplest field theory with $\mathcal{N} = (2, 2)$ superconformal symmetry [11]. The $(2, 2)$ means that there are two left and two right-moving supersymmetries: In the continuum limit, the fermion decomposes into left- and right-moving components over the Fermi sea. The boson also can be decoupled into left and right pieces, so that $\Phi = \Phi_L + \Phi_R$, while its dual $\tilde{\Phi} = g(\Phi_L - \Phi_R)$. The states of the field theory are given by the vertex operators $V_{m,n} = \exp(im\Phi + in\tilde{\Phi})$, of conformal dimensions $h_{L,R} = (m \pm gn)^2/(4g)$. The four components of the Dirac fermion in the Thirring model are $V_{\pm 1, \pm 1/2}$, while the supersymmetry generators are $Q_L^{\pm} = V_{\pm 1, \pm 3/2}$ and $Q_R^{\pm} = V_{\pm 1, \mp 3/2}$. In a finite size $\mathcal{L}$, the lowest-energy state is in the Neveu-Schwarz sector, where the Thirring fermion has antiperiodic boundary conditions. This state has $E_{\mathrm{NS}} = -\pi/(6\mathcal{L})$. The lowest-energy states in the Ramond (periodic boundary conditions) sector are given by $|\pm\rangle_R = V_{0,\pm 1/2}|0\rangle_{\mathrm{NS}}$; both have energy zero. States in this conformal field theory can be built up by operating with the "spinons" $V_{\pm 1/3, \pm 1/2}$.

Comparing this superconformal field theory with the lattice model on $N = 3p$ sites, we identify our two $E = 0$ ground states with the two Ramond vacua; all other states of the lattice model are in the Ramond sector as well. The U(1) quantum number m corresponds to $f - N/3$ (the fermion number relative to the ground state). The spinons here have charge $\pm 1/3$, so it is natural to identify these with the fractionally charged excitations in the lattice model.

We close by noting several generalizations of our construction. Here the space of states is made up solely of fermions on which the supersymmetry acts nonlinearly, but in the sequel [7] we will study linear realizations as well. Other supersymmetric models arise by including more projectors in (3), or by including several species of fermions. Consider a two-species model, with fermions labeled by $+$ and $-$, and with the conditions that (i) a single site may not be occupied by two particles and (ii) same-type particles may not occupy nearest neighbor sites. On a periodic chain with $4n$ sites, the Witten index of this model turns out to be $W = 3$, and we have strong indications that the continuum limit of this theory is the second model (at $c = 3/2$) of the series of $\mathcal{N} = 2$ minimal superconformal field theories. A typical ground-state pattern is

$$\ldots \circ + \circ - \circ + \circ - \circ + \circ - \circ \ldots$$

and one recognizes the possibility of domain walls of charge $\pm 1/2$ ($\circ + - \circ$) and neutral defects ($+ \circ +$). The $+/-$ pattern indicates an Ising substructure in the model, in accord with the fact that the $c = 3/2$ $\mathcal{N} = 2$ minimal model can be written in terms of a Majorana fermion and a free boson.

It is a pleasure to thank Bernard Nienhuis, Erik Verlinde, and Alexander Schrijver for discussions. This work was supported in part by the foundations FOM and NWO of the Netherlands, by NSF Grant No. DMR-0104799, as well as support by DOE OJI.

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[1] For work using the Hamiltonian formulation, see S. Elitzur and A. Schwimmer, Nucl. Phys. **B226**, 109 (1983); S. Cecotti and L. Girardello, Nucl. Phys. **B226**, 417 (1983); S. Catterall and S. Karamov, Phys. Rev. D **65**, 094501 (2002). For some recent developments, see D. B. Kaplan, E. Katz, and M. Unsal, arXiv:hep-lat/0206019; for an overview see A. Feo, arXiv:hep-lat/0210015.
[2] E. Witten, Nucl. Phys. **B202**, 253 (1982).
[3] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics Vol. 82 (Springer-Verlag, New York, 1982).
[4] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976); J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[5] R. Moessner, S. L. Sondhi, and E. Fradkin, Phys. Rev. B **65**, 024504 (2002), and references therein.
[6] K. Schoutens, Phys. Rev. Lett. **79**, 2608 (1997); P. Bouwknegt and K. Schoutens, Nucl. Phys. **B547**, 501 (1999).
[7] P. Fendley, B. Nienhuis, and K. Schoutens (to be published).
[8] H. Bethe, Z. Phys. **71**, 205 (1931).
[9] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982), and references therein.
[10] H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981).
[11] D. Friedan and S. H. Shenker, in *Conformal Invariance and Applications to Statistical Mechanics* (World Scientific, Singapore, 1988), p. 578.