Essays on Political and Experimental Economics
Sadiraj, V.

Citation for published version (APA):
Sadiraj, V. (2002). Essays on Political and Experimental Economics

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 5

On the Size of the Winning Set

5.1 Introduction

In Chapter 3 interest groups were introduced in a spatial voting model of electoral competition between political parties. The function of interest groups in that model is three-fold: (i) to coordinate voting behavior; (ii) to transmit information about voter preferences (or the electoral landscape) to the political parties; and (iii) to influence the policy outcome via conditions on polling. Chapter 3 analyzed the model by means of stochastic simulations whereas Chapter 4 developed a mean-field approximation analysis. Both types of analysis indicate that, in the presence of interest groups, the probability with which the challenger wins an election as well as the separation between political parties platforms increases, while convergence to the center of the issue space is inhibited. These findings are mainly driven by a new property of the electoral landscape, which is that the winning set (i.e., the set of policy platforms that will defeat the incumbent) seems to increase in the presence of interest groups. In this chapter we focus our attention on a deeper investigation of this property. We construct slightly modified versions of the stochastic models introduced in Chapter 3, Sections 3.2 and 3.3. For the modified versions, we present a general result that says that the winning set of the positions around the center of the space indeed increases in the presence of interest groups.

The setup of this chapter is as follows. Section 5.2 introduces the modified versions of the stochastic models and the main result. The proof of this main result is given in
5.2 Main result

Let policy platforms be represented as points in an issue space. We take this issue space to be continuous and equal to $\mathcal{X} = B(O, K) \subset \mathbb{R}^2$, that is, political parties can choose issues on two dimensions in a disc centered at the origin and with radius $K > 0$. There is a continuum of voters where each voter is characterised by an ideal point $x \in \mathcal{X}$ and strengths $s_1$ and $s_2$, which are independently distributed according to some distribution on $\mathcal{S}$. We assume that $\underline{s} = \inf \mathcal{S} > 0$. Without loss of generality we take $\overline{s} = \sup \mathcal{S} = 1$. The utility of voters with respect to a certain policy outcome $y \in \mathcal{X}$ is given by the negative of the (weighted) Euclidean distance between this policy outcome and their ideal points; i.e., the utility for a voter $j$ of policy outcome $y$ is given by

\begin{equation}
U_j(y) = -s_{j1} (x_{j1} - y_1)^2 - s_{j2} (x_{j2} - y_2)^2.
\end{equation}

We assume that voters ideal points are uniformly distributed over $\mathcal{X}$.

The political system works as follows. There are two political parties and in each period there is an election. The party that wins the election in the current period will be the incumbent for the next period and will not change his policy platform. The challenger can choose any policy platform. The objective of the parties is to get elected in the next period, i.e., they view policy as a means to winning.

Given the initial configuration of voters an electoral landscape can be constructed as follows. There are two political parties entering the election, the incumbent and the challenger. Each voter votes for the political candidate yielding him the highest utility as given by (5.1) if there are no interest groups. Then for each position the height of the electoral landscape is determined as the fraction of voters voting for the challenger, if it would select that position. That is, we can define $h(z \mid y)$ as the fraction of the votes the challenger gets if the incumbent is at platform $y$ and if the challenger selects platform $z$. For every policy position $z$ with $h(z \mid y) > (\leq) \frac{1}{2}$, the challenger wins (loses) the election. For a policy position with $h(z \mid y) = \frac{1}{2}$, the challenger wins with probability $\frac{1}{2}$. For a given incumbent's platform $y$, we are interested in the platforms for the challenger so that
the challenger will win the election. A platform $z$ is said to (strictly) defeat the platform $y$ in expectation if and only if

$$E(h(z \mid y)) > \frac{1}{2},$$

that is, if the expected height of the electoral landscape at $z$, given the incumbent's platform $y$ is strictly larger than $\frac{1}{2}$. The winning set is defined as

$$W(y) = \left\{ z \in X \mid E(h(z \mid y)) > \frac{1}{2} \right\}$$

and gives the set of platforms that is expected to defeat the incumbent's platform $y$. Notice that $y \notin W(y)$, since $E(h(y \mid y)) = \frac{1}{2}$ for all $y$. By $W(y; s_1, s_2)$ we define the winning set in the specific situation where all voters have strength $s_1$ on issue 1 and strength $s_2$ on issue 2.

The next step is to incorporate interest groups in the model. We follow the modelling approach presented in Chapter 3. An interest group is interested in the policy outcome with respect to only one of the two issues, say the first issue. Now suppose that an interest group exists at a certain position $k$ on the first issue. This interest group induces its members, who take the same position on the first issue, to vote for that political candidate who is closest to the interest group on the first issue. That is, if $\tilde{y}$ and $\tilde{y}$ are the policy platforms of the two political parties, then an interest group member votes for the first party according to the following decision rule

$$\text{vote for } \tilde{y} \text{ if } \begin{cases} |\tilde{y}_1 - x_1| < |\tilde{y}_1 - x_1| \\ |\tilde{y}_1 - x_1| = |\tilde{y}_1 - x_1| \text{ and } |\tilde{y}_2 - x_2| < |\tilde{y}_2 - x_2|. \end{cases}$$

Furthermore, we will assume that the voter votes with probability $\frac{1}{2}$ for policy platform $\tilde{y}$ if $u_j(\tilde{y}) = u_j(\tilde{y})$ and $|\tilde{y} - x| = |\tilde{y} - x|$. A similar decision rule holds for members of interest groups on the other issue. Therefore, the introduction of interest groups may change voter preferences from weighted Euclidean distance to lexicographic preferences.

Finally, we have to determine which voters will become a member and how voters will decide which interest group to join. We are going to investigate the situation where all voters join an interest group.\footnote{Remember that the parameters used in the simulations reported in Chapter 3 were such that all} Each voter is a potential member of two interest groups.
Which interest group they join will depend upon the incumbent’s position. We assume that voters are more inclined to join an interest group if the present policy position on that issue is farther away from their own position on that issue. The idea is that voters tend to join an interest group on an issue that they are most dissatisfied with under the current government policy. Hence, given the incumbent policy platform \( y \) voter \( j \) will prefer the interest group on issue 1 to the one on issue 2 if

\[
 s_{j1} (x_{j1} - y_{j1})^2 > s_{j2} (x_{j2} - y_{j2})^2 ,
\]

and similarly for the interest groups on issue 2. Thus, the presence of interest groups may change the electoral landscape.

We will now focus on the difference between the basic model of spatial competition and the model with interest groups. We are particularly interested in the sizes of the winning sets for the two different models. Let us denote by \( W^I (y) \) the winning set in the presence of interest groups given the incumbent’s platform \( y \), and let \( |W (y)| \) and \( |W^I (y)| \) indicate the size of the winning set for the model without and with interest groups, respectively.

Our main result is

**Theorem 5.1** Let \( s > 0 \) be given. Furthermore, let

\[
\varphi (s) = \min \left\{ \frac{2 (1 + s^{-1})^{\frac{3}{2}}}{(1 + (1 + s^{-1})^3)}, \left( 1 + 4 \left( \sqrt{s} + \frac{1}{\sqrt{s}} \right) \right)^{-\frac{1}{2}} \right\} .
\]

Then, for all \( y \in B(O, \varphi (s) K) \setminus \{O\} \) the winning set increases in the presence of interest groups.

This result shows that for all incumbent positions \( y \) in a prespecified disc within the issue space, the winning set increases in the presence of interest groups. Note that this disc is shrinking as \( s \) approaches 0 and that the ray of this disc goes to \( \frac{1}{2} K \) as \( s \) approaches 1. For an intermediate value of \( s = \frac{1}{2} \), we have \( \varphi (\frac{1}{2}) = (1 + 6\sqrt{2})^{-\frac{1}{2}} \approx 0.325 \).

voters with one of the strengths unequal to zero and with ideal positions different from the incumbent platform join at least one interest group (see footnote 8, Chapter 3).

2To derive this condition apply the formula in Chapter 3, footnote 8, with the expected value for \( n^e_m \) (which is the same on both issues).
5.3. PROOF OF THE THEOREM

5.3 Proof of the Theorem

In this section we will prove the theorem from the previous section. It will be convenient, for a given $v \in \mathbb{R}^2$ and $a, b \in \mathbb{R}_+$, to define

$$E_{a,b}(v, R_0) = \left\{ z \in \mathbb{R}^2 : a(x_1 - v_1)^2 + b(x_2 - v_2)^2 \leq R_0^2 \right\},$$

that is, $E_{a,b}(v, R_0)$ corresponds to the set of points located inside the ellipse centered at the point $v$ and with "radius" $R_{a,b}(v, R_0) = \left( \frac{R_0}{\sqrt{a}}, \frac{R_0}{\sqrt{b}} \right)$. The area contained in $E_{a,b}(v, R_0)$ equals $\pi \frac{R_0^2}{\sqrt{ab}}$. Furthermore, let $E_{a,b}(v, w)$, where $w \in \mathbb{R}^2$, be the set of points inside the ellipse centered at $v$ that goes through the point $w$, that is, where $R_{a,b}^2 = \|v - w\|^2_{a,b} \equiv a(w_1 - v_1)^2 + b(w_2 - v_2)^2$.

Let us first consider the basic model. We have the following result.

**Proposition 5.1** Assume voters' ideal positions are independently (across issues and across voters) drawn from the uniform distribution on $X$ and voters' strengths are independently drawn from a discrete distribution on $S$ and are uncorrected with the ideal positions. Denote by $y = (y_1, y_2)$ the platform of the incumbent. We then have:

1. $W(O) = \emptyset$;

2. $W(y; s_1, s_2) = E_{s_1, s_2}(O, y)$;

3. $\forall y \in X \setminus \{O\}, W(y) \subset E_{1,1}(O, y) \cup E_{1,1}(O, y)$ and

$$|W(y)| \leq \pi \|y\|^2 \left( \sqrt{\frac{1}{2}} + \frac{1}{\sqrt{2}} \right).$$

**Proof.**

1 By symmetry, any line through the origin $O$ divides the issue space $X$ in two subspaces which are equally large. The uniform distribution then implies that we have the same measure of voters on either side of such a line.\(^3\) Therefore the origin cor-

\(^3\)Note that this is valid even if the space $X = \{-k, \ldots, k\} \times \{-k, \ldots, k\}$ and voters are uniformly distributed in $X$. In this case, any line going through $z$ would separate the space into two disjoint subspaces, such that the subspace that prefers $C$ to $z$ contain at least half of positions. Furthermore either the vertical or the horizontal (both probably) would generate a subspace with more than half of the positions and which contain voters that prefer $C$ to $z$. Therefore in expectations a point $z \neq C$ can not defeat $C$. 

responsible for the position of the generalized median voter and no position will defeat it. Hence \( W(O) = \emptyset \).

2 Without loss of generality we assume \( y_1 \geq y_2 > 0 \). Consider an arbitrary policy position \( z = (z_1, z_2) \), with \( z_1 \leq y_1 \) and \( z_2 \leq y_2 \). We want to determine under which conditions \( z \in W(y) \). Let us first determine all positions \( x^* = (x_1^*, x_2^*) \) such that voters with the considered strength profile \((s_1, s_2)\) are indifferent between \( z \) and \( y \). These positions \( x^* \) have to satisfy

\[
 s_1 (y_1 - x_1^*)^2 + s_2 (y_2 - x_2^*)^2 = s_1 (x_1^* - z_1)^2 + s_2 (x_2^* - z_2)^2. 
\]

Solving for \( x_2^* \) is straightforward and gives

\[
 x_2^* = \frac{1}{2} \frac{(s_1 y_1^2 + s_2 y_2^2) - (s_1 z_1^2 + s_2 z_2^2) - 2 s_1 (y_1 - z_1) x_1^*}{s_2 (y_2 - z_2)}. \tag{5.2} \]

Let us denote the line defined by (5.2) as \( l_{s_1,s_2}^* \). Clearly, \( l_{s_1,s_2}^* \) separates the issue space into two subspaces and all voters in the subspace below and to the left of \( l_{s_1,s_2}^* \) vote for the challenger, if it selects position \( z \). Now suppose \( l_{s_1,s_2}^* \) cuts the \((x_1 = 0)\) axis at some value \( x_2 > 0 \). Then we can draw a line that goes through \( O \) and that lies parallel to \( l_{s_1,s_2}^* \). This line through \( O \) divides \( X \) in two equal subspaces, implying that the subspace lying below and to the left of \( l_{s_1,s_2}^* \) will be larger than the subspace above and to the right of \( l_{s_1,s_2}^* \). Therefore, the challenger will win at \( z \) if \( l_{s_1,s_2}^* \) cuts \((x_1 = 0)\) axis at some value \( x_2 > 0 \). This condition reduces to

\[
 (s_1 y_1^2 + s_2 y_2^2) - (s_1 z_1^2 + s_2 z_2^2) > 0, \]

which defines the ellipse given in the proposition. In a similar fashion the same condition can be derived for positions \( z \) with \( z_1 \leq y_1 \) and \( z_2 > y_2 \) and for positions \( z \) with \( z_1 > y_1 \) and \( z_2 \leq y_2 \). Clearly, positions \( z \) with \( z_1 \geq y_1 \) and \( z_2 \geq y_2 \) never defeat the incumbent. Finally, let us, for the sake of completeness consider cases where exactly one of the strengths is 0. It is easily seen that \( W(y; 0, s_2) = \{ z : |z_2| < |y_2| \} \) and \( W(y; s_1, 0) = \{ z : |z_1| < |y_1| \} \).

\footnote{This applies to the discrete case as well since the bigger subspace contains at least one position more which is the center \( C \).}
5.3. PROOF OF THE THEOREM

3 First, we show that \( W(y) \subset E_{1,1}(O, y) \cup E_{a,1}(O, y) \). It should be clear that it is sufficient to show that \( \cup_{s_1,s_2} E_{s_1,s_2}(O, y) \subset E_{1,1}(O, y) \cup E_{a,1}(O, y) \).

Let \( x \in \cup_{s_1,s_2} E_{s_1,s_2}(O, y) \) which implies that there exist \( s_1 \) and \( s_2 \) such that \( x \in E_{s_1,s_2}(O, y) \), that is, \( x \) satisfies \( s_1 x_1^2 + s_2 x_2^2 \leq s_1 y_1^2 + s_2 y_2^2 \). Hence

\[
(x_1^2 - y_1^2) \leq \frac{s_2}{s_1} (y_2^2 - x_2^2).
\]

Note that \( s \leq \frac{s_2}{s_1} \leq \frac{1}{s} \), for all \( s_1, s_2 \in S \). If \( |x_2| \leq |y_2| \), then

\[
(x_1^2 - y_1^2) \leq \frac{s_2}{s_1} (y_2^2 - x_2^2) \leq \frac{1}{s} (y_2^2 - x_2^2) \iff s x_1^2 + x_2^2 \leq s y_1^2 + y_2^2
\]

and \( x \in E_{a,1}(O, y) \). On the other hand, if \( |x_2| > |y_2| \) we have

\[
(x_1^2 - y_1^2) \leq \frac{s_2}{s_1} (y_2^2 - x_2^2) \leq s (y_2^2 - x_2^2) < 0
\]

and \( x \in E_{1,1}(O, y) \). This proves that \( W(y) \subset E_{1,1}(O, y) \cup E_{a,1}(O, y) \).

Second, we calculate the area of \( E_{1,1}(O, y) \cup E_{a,1}(O, y) \). Using

\[
\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + c
\]

we have (after some tedious but straightforward computations)

\[
S' = \frac{1}{4} |E_{1,1}(O, y) \cup E_{a,1}(O, y)|
\]

\[
= \int_{y_1}^{y_1} \sqrt{\frac{y_1^2 + s y_2^2 - x^2}{s}}\,dx + \int_{y_1}^{y_1} \frac{\sqrt{(s y_1^2 + y_2^2)/s}}{s} \sqrt{s y_1^2 + y_2^2 - s x^2}\,dx
\]

\[
= \frac{1}{\sqrt{s}} \left( \frac{y_1}{2} \sqrt{\frac{y_1^2 + s y_2^2 - y_1^2}{s}} - \frac{y_1^2 + s y_2^2}{2} \arcsin \frac{y_1}{\sqrt{y_1^2 + s y_2^2}} \right) + \sqrt{s} \left( \frac{sy_1^2 + y_2^2}{2s} \arcsin \frac{1}{s} \right)
\]

\[
- \sqrt{s} \left( \frac{y_1}{2} \frac{\sqrt{sy_1^2 + y_2^2}}{s} - y_1^2 + \frac{sy_1^2 + y_2^2}{2s} \frac{y_1}{\sqrt{sy_1^2 + y_2^2}} \right)
\]

\[
= \frac{y_1^2 \sqrt{s}}{2} \left( \left( 1 + \frac{1}{s} \frac{y_2^2}{s y_1^2} \right) \left( \frac{\pi}{2} - \arcsin \frac{1}{\sqrt{1 + \frac{1}{s} \frac{y_2^2}{s y_1^2}}} \right) + \frac{1}{s} \frac{y_2^2}{y_1^2} \arcsin \frac{1}{\sqrt{1 + \frac{1}{s} \frac{y_2^2}{s y_1^2}}} \right)
\]

\[\text{Note that } ||y|| \leq K \sqrt{s} \text{ implies } E_{1,1}(O, y) \cup E_{a,1}(O, y) \subset \mathcal{X} = B(O, K). \text{ Indeed, } \forall x \in E_{a,1}(O, y), \text{ one has}
\]

\[
z_1^2 + z_2^2 \leq \frac{1}{s} z_1^2 + z_2^2 = \frac{1}{s} (z_1^2 + s z_2^2) \leq \frac{1}{s} (y_1^2 + s y_2^2) \leq \frac{1}{s} ||y||^2 \leq K
\]

Hence, \( x \in B(O, K) \). The same way it can be shown that, \( E_{a,1}(O, y) \subset B(O, K) \).
Using $\arccos x = \frac{\pi}{2} - \arcsin x$, and $\arcsin x$, $\arccos x \leq \frac{\pi}{2}$, we find

$$S' \leq \frac{1}{2} y_1^2 \sqrt{\delta} \left(1 + \frac{1}{\delta} \frac{y_1^2}{y_1^2} + \frac{1}{\delta} \frac{y_2^2}{y_2^2}\right) \frac{\pi}{2} = \frac{\pi}{4} \left(\sqrt{\delta} + \frac{1}{\sqrt{\delta}}\right) \|y\|^2.$$  

and hence we have

$$|W(y)| \leq |E_{1,4}(O, y) \cup E_{2,1}(O, y)| \leq \pi \left(\sqrt{\delta} + \frac{1}{\sqrt{\delta}}\right) \|y\|^2.$$

Now that we have derived an upper bound for the size of the winning set in the basic model, let us turn to the model with interest groups. We will derive a lower bound for the size of the winning set in the model with interest groups in a number of steps.

We have assumed that all voters that are not located at the incumbent position get organized into interest groups. The results are driven by the fact that preferences of interest group members change from weighted Euclidean distance to the distance in only one issue. Consider all voters with strengths $s_1, s_2 \in S$. The following definition will be helpful for our analysis.

**Definition 5.1** Define $l_-$ and $l_+$ by

$$l_+ = \left\{ x : (-\sqrt{s_1}, \sqrt{s_2}) (x_1 - y_1, x_2 - y_2)^T = 0 \right\},$$

$$l_- = \left\{ x : (\sqrt{s_1}, \sqrt{s_2}) (x_1 - y_1, x_2 - y_2)^T = 0 \right\}.$$

These lines divide the issue space $X = B(O, K)$ in the following four subspaces:

1. $I = \{ x \in X : x_2 < \min \{ l_-, l_+ \}(x_1) \}$;

2. $II = \{ x \in X : x_2 > \max \{ l_-, l_+ \}(x_1) \}$;

3. $III = \{ x \in X \setminus (I \cup II \cup l_- \cup l_+) : x_1 > y_1 \}$; and

4. $IV = \{ x \in X \setminus (I \cup II \cup l_- \cup l_+) : x_1 < y_1 \}$.

Clearly, $X = I \cup II \cup III \cup IV \cup l_- \cup (l_+ \setminus \{y\})$.

The next step is to determine for each of the subspaces defined above, what happens to voters with ideal positions in such a subspace. From now on we will, without loss of generality, assume $y_1 \geq 0$ and $y_2 \geq 0$. 
5.3. PROOF OF THE THEOREM

Lemma 5.1 Consider voter \( j \) with ideal position \( x^j \).

i) If \( x^j \in I \cup II \) then voter \( j \) derives the highest utility from the group on the second issue.

ii) If \( x^j \in III \cup IV \) then voter \( j \) derives the highest utility from the group on the first issue.

Proof. We will prove that if a voter has an ideal position in \( I \) then she would prefer the group on the second issue to the group on the first one. The result for voters from subspaces \( II, III \) and \( IV \) can be established in the same way. Consider voter \( j \) with ideal position \( x^j \in I \). Note that \( \min\{l_-, l_+\} < y_2 \) implies that

\[
|x_2^j - y_2| = -(x_2^j - y_2).
\]

We distinguish the following cases:

• \( x_1^j \leq y_1 \). We then have

(a) \( |x_1^j - y_1| = y_1 - x_1^j \), and

(b) \( \min\{l_+, l_-\} = l_+ \). Therefore \( x_2^j < y_2 + \sqrt{\frac{s_1}{s_2}} (x_1^j - y_1) \) and hence

\[-(x_2^j - y_2) > \sqrt{\frac{s_1}{s_2}} (y_1 - x_1^j). \]

Substituting (a) and (5.3) at the above inequality,

\[
\sqrt{s_2} |x_2^j - y_2| > \sqrt{s_1} |x_1^j - y_1| \]

which is a necessary and sufficient condition for voter \( j \) to prefer the interest group on the second issue over the interest group on the first issue.

• \( x_1^j > y_1 \). Note that,

(c) \( |x_1^j - y_1| = x_1^j - y_1 \),

(d) \( \min\{l_+, l_-\} = l_- \), and by a similar argument as above one obtains

\[
\sqrt{s_2} |x_2^j - y_2| > \sqrt{s_1} |x_1^j - y_1| \]

which is a necessary and sufficient condition for a voter \( j \) to prefer the group on the second issue \( x_1^j = y_1 \).
Before we turn to our main results we still need a set of definitions.

**Definition 5.2** Denote by $ll_-$ and $ll_+$ the lines passing through the origin $O$, and lying parallel to $l_-$ and $l_+$, respectively, that is

$$ll_+ = \left\{ x : (-\sqrt{s_1}, \sqrt{s_2}) (x_1, x_2)^T = 0 \right\},$$

$$ll_- = \left\{ x : (\sqrt{s_1}, \sqrt{s_2}) (x_1, x_2)^T = 0 \right\}.$$

Let

$$S_-(y; s_1, s_2) = \{ x \in \mathbb{R}^2 | \min(ll_-, l_-)(x_1) \leq x_2 \leq \max(ll_-, l_-)(x_1) \} \cap X,$$

$$S_+(y; s_1, s_2) = \{ x \in \mathbb{R}^2 | \min(ll_+, l_+)(x_1) \leq x_2 \leq \max(ll_+, l_+)(x_1) \} \cap X.$$

Using $S_-(y; s_1, s_2)$ and $S_+(y; s_1, s_2)$ we define

$$A_-(y; s_1, s_2) = \{ z : z_1 < y_1, z_2 < y_2 \} \cap E_{s_1, s_2} (y, 4\sqrt{s_1 s_2} |S_-(s, y)|) \cap X,$$

$$A_+(y; s_1, s_2) = \{ z : z_1 < y_1, z_2 \geq y_2 \} \cap E_{s_1, s_2} (y, 4\sqrt{s_1 s_2} |S_+(s, y)|) \cap X.$$

Finally $A(y; s_1, s_2) = A_-(y; s_1, s_2) \cup A_+(y; s_1, s_2)$.

We are now ready to prove our main results. First, in Proposition 5.2 we show that, given the incumbent’s position $y$ and a strength profile $s_1$ and $s_2$, the winning set with interest groups contains $A(y; s_1, s_2)$. Proposition 5.3 then looks at the intersection of all these sets over different strength profiles, in order to find a lower bound for $|W^I(y)|$. Finally, by comparing this lower bound with the upper bound for $|W(y)|$, that was found in Proposition 5.1, Theorem 5.1 is proven.

**Proposition 5.2** In the presence of interest groups any element of $A(y; s_1, s_2)$ is supported by more than half of the voters with strength profile $(s_1, s_2)$.

**Proof.** Consider a position $z \in A_-(y; s_1, s_2)$. Denote by $(m_1, m_2) = \left( \frac{y_1 + z_1}{2}, \frac{y_2 + z_2}{2} \right)$ the midpoint of the line connecting $z$ with the incumbent’s position $y$. Now draw a vertical and a horizontal line through $m$ and consider the intersections of these lines with $l_+$ and
5.3. PROOF OF THE THEOREM

Denote these intersections by \( i_* \), where \( * \in \{+, -\} \) and \( i \in \{1, 2\} \). Hence \( 1_+ (1_-) \) is the intersection between the vertical line through \( m_1 \) and \( 1_+ (1_-) \) and \( 2_+ (2_-) \) is the intersection between the horizontal line through \( m_2 \) and \( 1_+ (1_-) \). Notice that \( l_- \) separates \( \mathcal{X} \) in two disjoint subspaces. Let \( l_-(O) \) denote the subspace that contains \( O \) and consider the subspace \( (l_- (O) - (1_-1_2, 2_2)) \cap \mathcal{X} \). From Lemma 5.1 we know that all voters in \( I \) join the interest group on the second issue and all voters in \( IV \) join the interest group on the first issue. For the voters in \( I \cap (l_- (O) - (1_-1_2, 2_2)) \cap \mathcal{X} \) we have \( |x^2_1 - z_2| < |x^2_1 - y_2| \) and for all voters in \( IV \cap (l_- (O) - (1_-1_2, 2_2)) \cap \mathcal{X} \) we have \( |x^2_1 - z_1| < |x^2_1 - y_1| \). This proves that the challenger at position \( z \) gets votes from voters with ideal positions in \( (l_- (O) - (1_-1_2, 2_2)) \cap \mathcal{X} \). Hence it gets more than half of the votes if

\[ |S_-| > |(1_-1_2, 2_2) \cap \mathcal{X}| \]

since \( |S_-| = |l_- (O)| - \frac{1}{2} |\mathcal{X}| \).

First, note that

\[
|(1_-1_2, 2_2) \cap \mathcal{X}| \leq |\triangle 1_-1_2, y| + |\triangle 2_2, y| \\
= \sqrt{\frac{s_1}{s_2}} (y_1 - m_1)^2 + \sqrt{\frac{s_2}{s_1}} (y_2 - m_2)^2 \\
= \frac{1}{4} \sqrt{\frac{s_1}{s_2}} (y_1 - z_1)^2 + \frac{1}{4} \sqrt{\frac{s_2}{s_1}} (y_2 - z_2)^2 \leq |S_-| 
\]

The first inequality follows from the fact that some parts of the triangles \( \triangle i_-i, y \), \( i = 1, 2 \) might not be in \( \mathcal{X} \), the first equality follows from computation of the surface of the two triangles, the second equality follows from the definition of the midpoint \((m_1, m_2)\) and the final step follows from the fact that \( z \in A_- (y; s_1, s_2) \). A similar reasoning holds for \( A_+ (y; s_1, s_2) \). Thus, it is shown that all positions \( z \in A (y; s_1, s_2) \) are expected to defeat the incumbent. ■

This proposition tells us that the area of \( S_- (y; s_1, s_2) \) (or \( S_- (y; s_1, s_2) \)) defines part of an ellipse which has the property that all positions within this part of the ellipse defeat the incumbent for the given strengths.

A set \( A (y; s_1, s_2) \) can be constructed for all strength profiles. Clearly, a policy position which lies in the intersection \( \cap_{s_1, s_2} A (y; s_1, s_2) \) of these sets has the property that voters
with all strength profiles will vote for this policy position. The area of this intersection therefore gives a lower bound for $|W^+(y)|$. The next proposition deals with this intersection and a lower bound for its area, for the case with $y_1 > 0$ and $y_2 > 0$. For the other configurations a similar result can be shown.

**Proposition 5.3** Consider $y_1 > 0$ and $y_2 > 0$. Let $R_y = 2\sqrt{S_-(y)\sqrt{2}}$, where $S_-(y) = \inf_{s_1,s_2 \in S} |S_- (y; s_1, s_2)|$.

1. $B = B(y, R_y) \cap \{ z \in X : z_1 < y_1, z_2 < y_2 \} \subset \cap_{s_1,s_2} A(y; s_1, s_2)$.

2. $S_-(y) \geq \sqrt{\frac{4}{1+2}} \left( K + \sqrt{K^2 - \| y \|^2} \right) (|y_1| + |y_2|)$.

**Proof.**

1. First, we show that

$$B \subset \cap_{s_1,s_2} A(y; s_1, s_2)$$

Indeed, let $x \in B$ i.e. $x_1 < y_1$ and

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 \leq R^2 = 4S_-(y)\sqrt{2}$$

Thus, $\forall s \in S$

$$\frac{1}{4} \sqrt{\frac{s_1}{s_2}} (x_1 - y_1)^2 + \frac{1}{4} \sqrt{\frac{s_2}{s_1}} (x_2 - y_2)^2 \leq \frac{1}{4} \sqrt{\frac{1}{s}} ((x_1 - y_1)^2 + (x_2 - y_2)^2)$$

$$= \frac{1}{4} \sqrt{\frac{4}{s}} S_-(y)\sqrt{2}$$

$$\leq |S_- (y; s_1, s_2)|$$

and therefore, $x \in A_- (y; s_1, s_2) \subset A(y; s_1, s_2)$.

2. The next step consists of deriving a lower bound for the radius of $B$. First, let $d(O, l_+) \ (d(O, l_-))$ correspond to euclidean distance between the origin and that point on the line $l_+ \ (l_-)$, that is closest to the origin, i.e.

$$d(O, l_+) = \min \{ \| x \| | x \in l_+ \}.$$
5.3. PROOF OF THE THEOREM

For the case we are considering \((y_1 > 0 \text{ and } y_2 > 0)\), we denote \(h(y; s_1, s_2) = d(O, l_\text{--})\) (which means that we are focusing on \(A_\text{--} (y; s_1, s_2)\)). For all \(s \in \mathcal{S}\), one has

\[
|S_\text{--} (y; s_1, s_2)| = 2 \int_0^{h(y; s_1, s_2)} \sqrt{K^2 - z^2} \, dz = \left[ z\sqrt{K^2 - z^2} + K^2 \arcsin \frac{z}{K} \right]_0^{h(y; s_1, s_2)} = h (y; s_1, s_2) \sqrt{K^2 - h (y; s_1, s_2)^2} + K^2 \arcsin \frac{h(y; s_1, s_2)}{K}.
\]

By definition we have \(h(y; s_1, s_2) \leq \|y\| \leq K\). Furthermore, \(\arcsin x > x\) for all \(x \in (0, 1]\). Using these properties we find

\[
|S_\text{--} (y; s_1, s_2)| \geq h(y) \left( \sqrt{K^2 - \|y\|^2} + K \right),
\]

where \(h(y) = \inf_{s_1, s_2 \in \mathcal{S}} h(y; s_1, s_2)\). Furthermore,

\[
h (y; s_1, s_2) = \sqrt{s_1 |y_1| + \sqrt{s_2} |y_2|} \geq \frac{\sqrt{s_2}}{\sqrt{1 + s_2}} (|y_1| + |y_2|)
\]

implies \(h(y) \geq \frac{\sqrt{s} (|y_1| + |y_2|)}{\sqrt{1 + s}}\), and hence

\[
|S_\text{--} (y; s_1, s_2)| \geq \sqrt{\frac{s}{1 + s}} \left( \sqrt{K^2 - \|y\|^2} + K \right) (|y_1| + |y_2|).
\]

We now have determined an upper bound for the number of winning positions in the absence of interest groups (Proposition 5.1) and a lower bound for the number of winning positions in the presence of interest groups (Proposition 5.3). Our main theorem then follows from comparing these two bounds.

**Proof of Theorem 5.1.** Let \(\mathcal{S}\) be given. From Proposition 5.1 we know that

\[
|W(y)| \leq \pi \|y\|^2 \left( \sqrt{s} + \frac{1}{\sqrt{s}} \right).
\]

From Proposition 5.3 we know that all positions in \(B (y, 2\sqrt{s_1 s_2 |S_\text{--}|})\), with \(z_1 < y_1\) and \(z_2 < y_2\) are contained in \(W^f(y)\). Furthermore we found that

\[
|S_\text{--} (y; s_1, s_2)| \geq S_\text{--} (y) \geq \sqrt{\frac{s}{1 + s}} \left( K + \sqrt{K^2 - \|y\|^2} \right) (|y_1| + |y_2|).
\]
If this area lies in $\mathcal{X}$ we know that it presents a lower bound for $|W^I(y)|$. However, it does not necessarily lie in $\mathcal{X}$. Let us first compute the two points that have a coordinate equal to a coordinate of $y$ but that lie on the border of $\mathcal{X}$. These points are $(y_1, \sqrt{K^2 - y_1^2})$ and $(\sqrt{K^2 - y_2^2}, y_2)$. Now consider the following ball

$$B \left( y, \min \left\{ |y_1| + \sqrt{K^2 - y_2^2}, |y_2| + \sqrt{K^2 - y_1^2} \right\} \right).$$

One fourth of this ball lies in $\mathcal{X}$ completely. Therefore there are two possibilities. This ball is contained in $A_-$ or it contains $A_-$. So we have to take the minimum of the two lower bounds as a lower bound for $|W^I(y)|$. First consider the first case. We then have

$$|W^I(y)| \geq \frac{1}{4} \pi R_y^2$$

$$= \pi S_-(y) \sqrt{s}$$

$$\geq \pi \frac{s}{\sqrt{1 + s}} \left( K + \sqrt{K^2 - \|y\|^2} \right) (|y_1| + |y_2|)$$

and

$$|W(y)| \leq \pi \|y\|^2 \left( \sqrt{s} + \frac{1}{\sqrt{s}} \right).$$

So we find that, in the presence of interest groups, the size of the winning set is expected to increase for all $y$ satisfying $|W^I(y)| \geq |W(y)|$ or

$$\left( K + \sqrt{K^2 - \|y\|^2} \right) (|y_1| + |y_2|) \geq \left( \frac{1 + \frac{s}{K}}{s} \right)^{\frac{3}{2}} \|y\|^2.$$

Using $|y_1| + |y_2| \geq \|y\|$, and rewriting, we find that for all $y$ satisfying

$$\|y\| \leq \frac{2 \left( \frac{1 + s}{s} \right)^{\frac{3}{2}}}{(1 + (1 + s)^{\frac{3}{2}})} K,$$

our property holds. Defining $D_1 = \frac{2 \left( \frac{1 + s}{s} \right)^{\frac{3}{2}}}{(1 + (1 + s)^{\frac{3}{2}})} K$, the property holds for all $y \in B(0, D_1)$.

Now consider the second case with

$$|W^I(y)| \geq \frac{\pi}{4} \left( \min \left\{ |y_1| + \sqrt{K^2 - y_2^2}, |y_2| + \sqrt{K^2 - y_1^2} \right\} \right)^2.$$

Suppose, without loss of generality, that $|y_1| + \sqrt{K^2 - y_2^2} \leq |y_2| + \sqrt{K^2 - y_1^2}$. We then obtain

$$\frac{\pi}{4} \left( |y_1| + \sqrt{K^2 - y_2^2} \right)^2 \geq \pi \|y\|^2 \left( \sqrt{s} + \frac{1}{\sqrt{s}} \right).$$
Again, we derive a condition on $\|y\|^2$. Using $0 \leq |y_2|^2 \leq \|y\|^2$, we get

$$|y_1|^2 + K^2 - y_2^2 + 2|y_1|\sqrt{K^2 - y_2^2} \geq K^2 - \|y\|^2 \geq 4\|y\|^2\left(\sqrt{\frac{1}{K}} + \frac{1}{\sqrt{K}}\right)$$

or

$$\|y\|^2 \leq \frac{K^2}{1 + 4\left(\sqrt{\frac{1}{K}} + \frac{1}{\sqrt{K}}\right)}$$

That is, for all $y \in B(0, D_2)$, with $D_2 \equiv \frac{K}{\sqrt{1+4\left(\sqrt{\frac{1}{K}} + \frac{1}{\sqrt{K}}\right)}}$. This concludes the proof of Theorem 5.1. \[\square\]