A term structure model of interest rates and forward premia: an alternative monetary approach
Daal, W.H.

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Chapter 4

The Two-Country General Equilibrium: A Simplified Case

In this chapter we consider the general equilibrium of the monetary-production economy, where the dynamics of production and money supply are governed by a geometric Brownian motion. These i.i.d. (i.e. independent and identically distributed) processes characterize the dynamics of the equilibrium quantities in the two-country world economy. This simple i.i.d. economy (i.e. an economy where the production and money supply processes follow a geometric Brownian motion) is a convenient analytical tool to understand the interaction between the nominal and real quantities and its impact on the equilibrium foreign exchange rate and interest rates. The equilibrium in this simple monetary-production economy has some important distinctive properties. For example it determines endogenously and simultaneously the demand for money, the consumption-production plan, the price of money and its dynamics, the foreign exchange rate, the rate of depreciation, the risk premia, and the market clearing interest rates. More importantly, it allows for an interaction between the real economy, the financial markets, and the monetary economy of the two countries.

In line with several general equilibrium models in financial economics we assume in this chapter, for analytical convenience, that the production process in equation (3.3) is
driven by a geometric Brownian motion [see Stulz (1986), Foresi (1990), Bakshi and Chen (1996)]. This assumption implies that the drift and diffusion of the real rate of return in production are constants. The i.i.d. production process in country $i$, for $i \in \{h, f\}$, is governed by the following stochastic differential equation:

$$\frac{d\eta_i(t)}{\eta_i(t)} = \alpha_{\eta_i} dt + S_{\eta_i} dw_{\eta_i}(t), \quad (4.1)$$

where $\alpha_{\eta_i}$ is a constant scalar denoting the instantaneous expected real rate of return on the productive assets in country $i$ and $S_{\eta_i} dw_{\eta_i}(t) = \sigma_{\eta_i} dw_{\eta_i}(t) + \sigma_{\eta_i} dw_{\eta_i}(t)$ is a constant scalar that represents the diffusion term of this return. The variance of the real rate of return on investment in $i$-th production process is given by $S_{\eta_i,\eta_i} = \sigma_{\eta_i}^2 + \sigma_{\eta_i}^2$ and the covariance between the rate of return on productive investments in both countries is given by $S_{\eta_i,\eta_j} = \sigma_{\eta_i} \sigma_{\eta_j}$, for $i \neq j$.

The money supply process in country $i$, for $i \in \{h, f\}$, as formulated in equation (3.4), is governed by the following i.i.d. process in this simplified economy:

$$\frac{dM^*_i(t)}{M^*_i(t)} = \mu^*_{m_i} dt + S_{m_i} dw_{m_i}(t), \quad (4.2)$$

where $\mu^*_{m_i} = \mu_{m_i} + \gamma_i \alpha_{\eta_i}$ is a constant that denotes the expected rate of money growth in country $i$. The expected instantaneous rate of growth of money is determined by the expected autonomous monetary rate of growth and the monetary response to the expected production growth, $\gamma_i \alpha_{\eta_i}$. The diffusion term of money growth is captured by $S_{m_i} dw_{m_i}(t) = \sigma_{m_i} dw_{m_i}(t) + \gamma_i S_{\eta_i} dw_{\eta_i}(t)$, with $\gamma_i S_{\eta_i} dw_{\eta_i}(t)$ representing the monetary response to production uncertainty. The instantaneous variance of the rate of money growth in country $i$ is a constant scalar given by $S^2_{m_i} = \sigma^2_{m_i} + \gamma_i^2 \left( \sigma^2_{\eta_i} + \sigma^2_{\eta_i} \right)$. This modelling approach allows for a plausible correlation structure in this two-country world economy. The money supply processes in both countries are not perfectly correlated with the production process, as the autonomous monetary component of the diffusion term allow for the monetary disturbances to move independent of the production process. In
addition the money supply processes in the two countries are (not perfectly) correlated with each other, \( S_{m_i,m_j} = \gamma_i \sigma_{m_i} \gamma_j \sigma_{m_j} \), for \( i \neq j \). As will be shown below, these assumptions for the money supply and production processes imply that the real and nominal interest rates, and the velocity of money are constant in this i.i.d. setting.

In this economy the representative agents use an admissible feedback control vector defined as \( \varphi_i = \begin{bmatrix} c_i & m_{a_i} & a_i & b_i & f_i \end{bmatrix} \) to solve

\[
\max_{\varphi_i} E_i \int_t^\infty e^{-\rho s} U \left( c_{ih}, c_{if}, m_{da_i}, m_{df_i}, s \right) ds,
\]

subject to their dynamic budget constraint (3.18). To solve the agent’s optimization problem we apply standard stochastic dynamic programming technique as in Merton (1969) to obtain the continuous-time Bellman fundamental optimality equation (see Appendix A.1),

\[
0 = \max_{\varphi} \left\{ U \left( c_{ih}, c_{if}, m_{da_i}, m_{df_i}, t \right) + J_t + J_{X\mu_x} + J_{Y\mu_y} 
+ J_{W_i} \left[ a'_i (\alpha - 1 r_i) W_i + b'_i (\beta - 1 r_i) W_i 
+ f'_i (\zeta - 1 r_i) W_i + r_i W_i - c_{ih} - c^*_f - R_h m_{da_i} - R_f m^*_f \right] 
+ \frac{1}{2} J_{W_i, W_i} \left[ a'_i S_{m_i} a_i + b'_i S_{BB} b_i + f'_i S_{FF} f_i + 2b'_i S_{B_i a_i} 
+ 2a'_i S_{B_f} f_i + 2b'_i S_{B_f} f_i \right] W^2_i 
+ \frac{1}{2} J_{X X} S_{x x} 
+ \frac{1}{2} J_{Y Y} S_{y y} + [a'_i S_{n_x} + b'_i S_{B_x} + f'_i S_{F_x}] J_{W_i, X} W_i 
+ [a'_i S_{n_y} + b'_i S_{B_y} + f'_i S_{F_y}] J_{W_i, Y} W_i \right) \right.,
\]

(4.3)

where \( X(t) = \begin{bmatrix} X_h(t) & X_f(t) \end{bmatrix} \). The subscripts on the \( J \) term denote the corresponding partial derivatives of the agent’s value function \( J(W_i, X, Y, t) \) and the subscripts on the \( S \)-term denote a variance-covariance matrix of the variables involved. For example

\[
S_{m} = \begin{bmatrix} S_{n_i}^2 & S_{n_i} S_{n_j} \\ S_{n_j} S_{n_i} & S_{n_j}^2 \end{bmatrix} \quad \text{or} \quad S_{zz} = \begin{bmatrix} \sigma_{x x}^2 X_h(t) & 0 \\
0 & \sigma_{y}^2 X_f(t) \end{bmatrix}.
\]

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In this economy we ruled out the possibility of negative productive investments in both capital goods and negative transaction and portfolio demand for money balances. In addition the agent’s consumption, must be non-negative. Standard Kuhn-Tucker optimization technique results in the following first order conditions:

\[ L_{c_i} = U_{c_i} - J_{w_i} \leq 0 \]
\[ c_i L_{c_i} = 0 \] \hspace{1cm} (4.4)
\[ L_{m_{d_i}} = U_{m_{d_i}} - J_{w_i} R_i(t) \leq 0 \]
\[ m_{d_i} L_{m_{d_i}} = 0 \] \hspace{1cm} (4.5)
\[ L_{a_i} = (\alpha - \mathbf{1} r_i(t)) W_i J_{w_i} + (S_{\eta y} a_i + S_{\eta B} b_i + S_{f \eta f_i}) W_i^2 J_{w_i} \]
\[ + S_{\eta x} J_{w_i} J_{w_i} W_i + S_{\eta y} J_{w_i} W_i \leq 0 \]
\[ a_i L_{a_i} = 0 \] \hspace{1cm} (4.6)
\[ L_{b_i} = (\beta - \mathbf{1} r_i(t)) W_i J_{w_i} + (S_{BB} b_i + S_{\eta B} a_i + S_{BB} f_i) W_i^2 J_{w_i} \]
\[ + S_{\eta x} J_{w_i} J_{w_i} W_i + S_{\eta y} J_{w_i} W_i \leq 0 \]
\[ b_i L_{b_i} = 0 \] \hspace{1cm} (4.7)
\[ L_{f_i} = (\zeta - \mathbf{1} r_i(t)) W_i J_{w_i} + (S_{ff} f_i + S_{\eta f} a_i + S_{BB} b_i) W_i^2 J_{w_i} \]
\[ + S_{\eta x} J_{w_i} J_{w_i} W_i + S_{\eta y} J_{w_i} W_i = 0 . \] \hspace{1cm} (4.8)

Given the first order conditions of the representative agents, we define the equilibrium as a set \((R_i, r_i, e_{ij}, \pi_i, F; a_i, b_i, m_{d_i}, c_i)\). This equilibrium set must satisfy the following market clearing conditions: \(a_i'1 + b_i'1 = 1, B_i = 0, f_i = 0, M_i^d = M_i^* = M_i, \) and \(\eta_i = c_i^* + I_i^*, \) where \(I_i^*\) represents the amount re-invested in production. These conditions imply, among others, that the default-free nominal bonds and contingent claims in this economy are in zero net supply in equilibrium. The market clearing condition on the goods market, \(\eta_i\), implies that in equilibrium the total amount of the \(i\)-th good produced must be equal to the sum of the amount consumed, \(c_i^*\), during that period and the amount re-invested in the \(i\)-th production technology, \(I_i^*\). The total consumption of the \(i\)-th good
and the total re-investment in the \(i\)-th production process can be defined as \(c^*_i = c^*_i + c^*_{ji}\), and \(I^*_i = I^*_{ii} + I^*_{ji}\), respectively. Let \(\zeta_i^g\) denote the fraction of the \(i\)-th production allocated to the \(i\)-th consumption good. Then we have \(c^*_i = \zeta_i^g \eta_i\) and \(c^*_{ji} = \zeta_j^g \eta_i\), where \(\zeta_i^g + \zeta_j^g = \zeta_i^g\). The market clearing condition on the money markets entails that the total demand for the \(i\)-th money must be equal to the its supply. Since the sum of the demand for the \(i\)-th money by both agents must satisfy \(M^d_i = M^d_{ii} + M^d_{ji}\), we have that \(M^d_{ii} = \zeta^m_{ii} M_i\) and \(M^d_{ji} = \zeta^m_{ji} M_i\), where \(1 = \zeta^m_{ii} + \zeta^m_{ji}\).

### 4.1 Money demand, spot foreign exchange rate, and the price of money in equilibrium

In this section we derive the marginal rate of substitutions and the equilibrium conditions for the following variables: the demand for real cash balances, the spot foreign exchange, the rate of depreciation, the price of money, and the rate of inflation. We show that the dynamics for these variables are govern by a geometric Brownian motion, which is a direct result of the assumed production and money supply process. This section gives us, even in this simple economy, a good insight in the dynamics of these variables in an equilibrium characterized by uncertainty.

**Proposition 1.** In equilibrium, the marginal rate of substitution of the foreign good for the domestic good, of money balances for goods, and of foreign money for domestic money in this two-country world economy are given, respectively, by:

\[
\frac{U_{ch_f}(c_h(t), m_{dh}(t), t)}{U_{ch_h}(c_h(t), m_{dh}(t), t)} = \epsilon_{hf} (X, Y, t),
\]

\[(4.9)\]

\[
\frac{U_{mh_f}(c_h(t), m_{dh}(t), t)}{U_{ch_h}(c_h(t), m_{dh}(t), t)} = R_i(t) \quad i \in \{h, f\},
\]

\[(4.10)\]

\[
\frac{U_{mh_f}(c_h(t), m_{dh}(t), t)}{U_{ch_h}(c_h(t), m_{dh}(t), t)} = \frac{\epsilon_{hf}(t) R_f(t)}{R_h(t)}.
\]

\[(4.11)\]

**Proof:** See Appendix A.2.
Equation (4.9) shows that in equilibrium, the marginal utility of consuming one additional unit of the foreign good by the domestic agent is equal to its marginal cost in terms of the local currency, the spot foreign exchange rate. An increase in the spot exchange rate increases the marginal utility of consuming an additional unit of the foreign good. Therefore, to maintain the same level of satisfaction, the number of units of the local good, \( c_{hh}(t) \), that has to be sacrificed in exchange for an extra unit of the foreign good, \( c_{hf}(t) \), increases. The intuitive explanation therefore is that the foreign good has become more expensive in terms of the domestic good. Note that the marginal rate of substitution of the foreign representative agent for the good produced in the home country \( h \) is equal to the inverse of the foreign exchange rate level.

\[
\frac{U_{cfh}(c_{f}(t), m_{d}(t), t)}{U_{cf}(c_{f}(t), m_{d}(t), t)} = \frac{1}{\epsilon_{hf}(t)}. \tag{4.12}
\]

Equation (4.10) shows that in equilibrium the marginal rate of substitution between consumption of the \( i \)-th good and the \( i \)-th real cash holdings is equal to the nominal interest rate in country \( i \), \( R_i(t) \). This result is consistent with the results in the existing literature, e.g., LeRoy (1984), Svensson (1985b), Foresi (1990), Bakshi and Chen (1996), and Basak and Gallmeyer (1999), which states that in equilibrium, the marginal benefit of holding one additional unit of the \( i \)-th currency balance in terms of the good is equal to its marginal cost, i.e. the foregone nominal interest rate, \( R_i(t) \). The inflation risk does not affect the marginal rate of substitution of goods for money directly because the investor can use the nominal bond to hedge against the inflation risk of real cash holdings. As argued by LeRoy (1984) this implies that the nominal price of the instantaneously riskless nominal bond in this economy captures all the users cost of holding real balances. Thus, beside the measure of liquidity services provided by money, the nominal interest rate also represents a measure of the utility gain from holding real balances in this economy. Or stated differently the nominal interest rate is the relative price of real balances in terms of the consumption-investment good in equilibrium.
The marginal rate of substitution of foreign money for domestic money as shown by equation (4.11) is equal to the relative opportunity cost of the two monies. This means that the marginal utility of one additional unit of foreign money relative to domestic money is proportional to its relative marginal cost expressed in domestic currency. The relative marginal cost of holding foreign money is measured by the opportunity cost of holding foreign money denominated in domestic currency, i.e. $\epsilon_{hf}(t)R_f(t)$, divided by the opportunity cost of holding local money, $R_h(t)$. An increase in either the spot exchange rate or the foreign nominal interest rate relative to the domestic nominal interest rate increases the marginal utility of one additional unit of foreign cash balances.

**Theorem 1** In equilibrium, the demand for real money balances, the relative real demand for money balances, the spot foreign exchange level, and the price of the monies in this two-country economy are given, respectively, by:

$$\epsilon_{hf}(t) = \frac{\eta^n_{hh} \theta_{hf} \eta_h(t)}{\eta^n_{hf} \theta_{hh} \eta_f(t)},$$

$$M_{hi}(t)\pi_i(t) = \frac{\delta_{hi} c^*_h(t)}{\theta_{hi} R_i(t)} \quad i \in \{h, f\},$$

$$m^*_h(t)_{hh}(t) = \frac{\delta_{hi} \epsilon_{hf}(t) R_f(t)}{\delta_{hf} R_h(t)},$$

$$\pi_i(t) = \frac{\eta^n_{ji} \delta_{ji} \eta_i(t)}{\eta^n_{ji} \theta_{ji} M_i(t)} \left[ \frac{1}{\rho + \mu^*_m - \sigma^2_{\eta_m} - \gamma_i^2 \left( \sigma^2_{\eta_m} + \sigma^2_{\eta_n} \right)} \right] \quad i \in \{h, f\}.$$

**Proof:** See Appendix A.3.

Equation (4.13) shows that the exchange rate level is determined by the domestic and foreign production processes as in Nielsen and Saá-Raquejo (1993). An increase in the domestic production level relative to the foreign production level increases the spot price of the currency, i.e. the spot rate depreciates. In addition, the level of the exchange rate is determined by the relative preferences between the domestically produced and the imported good, $\theta_{hf}/\theta_{hh}$, and the relative share of the goods consumed by the local agent, $\eta^n_{hh}/\eta^n_{hf}$. By expressing equation (4.13) as $d\epsilon_{hf}/\epsilon_{hf} = (d\eta_h/\eta_h)/(d\eta_f/\eta_f)$ it can be
observed that the exchange rate in this economy functions as an adjustment mechanism on the world equity markets. It restores the equilibrium between the real rate of return of domestic and foreign production, expressed in the domestic currency. An increase in the real rate of return of domestic production relative to that of the foreign production, ceteris paribus, induces rational agents to shift away from foreign capital investment towards investment in the domestic production technology. An increase in the spot foreign exchange rate increases the rate of return on foreign equity investments when expressed in the domestic currency. An intuitive explanation herefore is that an increase in the spot exchange rate reduces the value of the domestic currency relative to the value of the foreign currency. As a result the representative investor is indifferent between investing in the local or the foreign production technology.

To obtain more insight into how both real rates of return on production affect the exchange rate dynamics we apply Itô's lemma on equation (4.13) and use equation (4.1) to obtain the following stochastic differential equation for the instantaneous rate of depreciation of the domestic currency:

\[
\frac{d\varepsilon_{hf}(t)}{\varepsilon_{hf}(t)} = \mu_t dt + S_{\varepsilon_{hf}} dw_t (t),
\]

where

\[
\mu_t = \alpha_{h} - \alpha_{f} + \sigma_{\varepsilon_{hf}}^2 + \left(\sigma_{\varepsilon_{hf}} - \sigma_{\varepsilon_{hf}}\right) \sigma_{\varepsilon_{hf}},
\]

\[
S_{\varepsilon_{hf}} dw_t (t) = \sigma_{\varepsilon_{hf}} dw_{\varepsilon_{h}} (t) - \sigma_{\varepsilon_{hf}} dw_{\varepsilon_{f}} (t) + \left(\sigma_{\varepsilon_{hf}} - \sigma_{\varepsilon_{hf}}\right) dw_y (t).
\]

Equation (4.17) shows that in this i.i.d. economy the equilibrium rate of depreciation of the currency is governed by a geometric Brownian motion. The constant expected instantaneous rate of depreciation of the spot exchange rate, \(\mu_t\), is determined by the difference between the real rate of return of domestic production and that of foreign production. The expected rate of depreciation of the currency is also affected by the
difference of the impact of the common source of uncertainty on domestic production relative to foreign production. The drift term of the depreciation rate is increasing in the uncertainty with respect to the real rate of return of foreign production, as measured by \( \sigma_{\eta_{x,f}}^2 \). In this setup, where the money supply process and the production process follow geometric Brownian motions, the exchange rate volatility, as measured by the variance of the rate of depreciation, is constant. It is given by
\[
S_{\epsilon_{n,f}}^2 = \sigma_{\eta_{x,h}}^2 + \sigma_{\eta_{x,f}}^2 + \left( \sigma_{\eta_{y,h}} - \sigma_{\eta_{y,f}} \right)^2.
\]

In the following chapter we allow for stochastic volatility in the rate of depreciation, by relaxing the assumption of an i.i.d. production process in each country.

Based on the equilibrium solution for the rate of depreciation of the spot exchange rate we can determine the real rate of return on foreign equity investment, expressed in domestic currency. By using the equilibrium expression for the exchange rate dynamics, equation (4.17), and the conjecture in equation (3.7), we can obtain the endogenous expression for the real rate of return on foreign productive investment expressed in domestic currency as,
\[
d\left[ \tilde{\eta}_j(t) \right] \eta_j(t) = \alpha_{\tilde{\eta},i} dt + S_{\tilde{\eta},i} d\eta_j(t), \quad \text{for } i \neq j,
\] (4.18)

where
\[
\tilde{\eta}_j(t) = \epsilon_{ij}(t) \eta_j(t),
\]
\( \alpha_{\tilde{\eta},i} = \alpha_{\eta,i} \), and
\( S_{\tilde{\eta},i} d\eta_j(t) = \sigma_{\eta_{y,i}} d\eta_y(t) + \sigma_{\eta_{x,i}} d\eta_x(t) \).

Note that investment in foreign production technology has the same real rate of return as domestic productive investment when expressed in domestic currency. This is a direct result of the equilibrium condition on the goods/equity market, whereby the spot exchange rate adjusts instantaneously to restore the equilibrium between the two real rate of returns.

In equation (4.14), the demand of the domestic representative agent for the i-th real currency balances is increasing in the i-th real consumption and decreasing in the nominal interest rate, \( R_i(t) \). Intuitively, in equilibrium an increase in real consumption must be
accompanied by an increase in real money holdings for transaction purposes. The inverse relationship between the real demand for money and the nominal interest rate can be explained by the portfolio considerations of the investor. The rationale for the risk averse investor to hold real balances is that it reduces the riskiness of an asset portfolio and facilitates transactions. The opportunity cost involved in this is the expected return forgone by not holding bonds. An increase in the rate of return on bonds due to an increase in the nominal interest rate, entails higher opportunity cost of holding money. This results in a shift away from cash balances, as an object of portfolio choice, in favor of interest-bearing nominal assets and real assets. Let us examine the relation between the real demand for both monies in this economy.

By using equation (4.14) and the market clearing conditions, \( c_i^* = c_{hi}^* + c_{fi}^* \), \( c_i^* = \xi_i \eta_i \), \( M_{hi}^d = \zeta_{hi}^n M_i \), and \( M_{fi}^d = \zeta_{fi}^n M_i \), we obtain the equilibrium velocity of the \( i \)-th money in this economy,\(^1\)

\[
\nu_i(t) = \frac{\eta_i(t)}{m_i(t)} = \varphi_{\nu_i} R_i(t) \quad i \in \{h, f\}.
\]

where

\[
\varphi_{\nu_i} = \left( \frac{\theta_{hi} \zeta_{hi}^m}{\delta_{hi} \zeta_{hi}^m} + \frac{\theta_{fi} \zeta_{fi}^m}{\delta_{fi} \zeta_{fi}^m} \right) / \xi_i^7.
\]

Note that the velocity of the \( i \)-th money in country \( i \) at time \( t \) can be defined as \( \nu_{hi}(t) = \varphi_{\nu_{hi}} R_i(t) \), where \( \varphi_{\nu_{hi}} = \left( \frac{\theta_{hi} \zeta_{hi}^m}{\delta_{hi} \zeta_{hi}^m} \right) / \xi_{hi}^7 \). The same holds for the velocity of the \( j \)-th money in the foreign country \( f \). The equilibrium velocity of money in this two-country economy is a function of the opportunity cost of holding that particular money. As mentioned above an increase in the nominal interest rate results in a shift away from cash balances, as an object of portfolio choice, in favor of interest-bearing nominal assets.

\(^1\)Note that the demand of the foreign representative agent for the \( i \)-th real currency balances is given by:

\[
M_{fi}(t) \pi_i(t) = \frac{\delta_{fi}}{\theta_{fi}} \frac{\zeta_{fi}(t)^*}{R_i(t)} \quad i \in \{h, f\}.
\]
and real assets. This decline in the demand for real balances results in an increase of the velocity of money. Thus, as indicated by Bakshi and Chen (1996), the velocity of money co-moves with the nominal interest rate in this type of economies. Since in this i.i.d. setting the nominal interest rate is constant, as shown below, the velocity of money is also constant.

Equation (4.15) implies that the relative real demand for both monies in equilibrium is proportional to the inverse of the relative opportunity cost of both monies, expressed in the domestic currency. If there is a positive shock, which increases production in the domestic economy, the exchange rate will increase as we can observe from equation (4.13). This depreciation of the domestic currency, ceteris paribus, increases the opportunity cost of holding foreign money, expressed in domestic currency, relative to the opportunity cost of holding domestic money. As a result the real demand for domestic money for transaction purposes will increase relative to the demand for foreign money. The increased demand for domestic money relative to that for foreign money is consistent with the increased production of the domestic good relative to the foreign good, and as such the equilibrium condition on both markets is maintained through the exchange rate. If we allow for the other variables to adjust, it can be observed below that the domestic nominal interest rates and the price of domestic money increase. Thus, the increased opportunity cost of holding foreign money expressed in domestic currency is partly offset by the increased opportunity cost of holding domestic money. The resulting increased real demand for domestic money relative to the foreign money is consistent with the increased price of money and the increased domestic production. If we rewrite equation (4.15) as

$$\frac{\pi_h(t)M_{hh}(t)}{\pi_f(t)M_{hf}(t)} = \frac{\delta_h \epsilon_{hf}(t) R_f(t)}{\delta_h R_h(t)},$$  \hspace{1cm} (4.20)

it can be noted that the increased relative real demand for domestic money can be decomposed in an increase in the nominal money stock and the resulting increased price of money (i.e. a decrease in the price level), as will be shown below.
The endogenous price of domestic money in equation (4.16) in this two-country set-up is consistent with the general price level in the closed economy in theorem 3 of Bakshi and Chen (1996). A monetary expansion decreases the local price of money, whereas an increase in domestic production increases the price of money. When expressed in terms of the price level of the local good, \( P_i(t) \), that is the inverse of the price of domestic money \( \pi_i(t) \), monetary expansion leads to an increase in the price level of the good. An increase in the domestic production level of the good decreases the price level of that particular good in the economy. Note that the separability of the representative agent's preferences over the two monies does not lead to the local price level being detached from real influences in this economy. In addition, the price level is also affected by the international uncertainty both directly, through \( \gamma^2 \sigma_{\eta_i}^2 \), and indirectly through both the supply of domestic money and domestic production. The policy implication of equation (4.16) is that, ceteris paribus, an accommodating monetary policy requires that the rate of growth of money is equal to that of the real rate of return on production. The price of money remains constant at such a pace. In order to allow for a countercyclical monetary policy, money supply must increase less than production. As a result the price of domestic money (the price level of domestic good) increases (decreases). This is consistent with economic theory, that is contracting money supply decreases the general price level in the case of a positive production shock. An inflationary monetary policy is obtained when the money supply is allowed to increase more than the rate of growth of production. An increase in the expenditure share on domestically produced good, \( \theta_{ji} \), implying that the agent's preferences is shifting away from money holdings and/or from the foreign good, decreases the price of domestic money \( i \) (that is increases the price level of that particular good).

Applying Itô's lemma on equation (4.16) results in the following endogenous dynamics for the price of the money in country \( i \):

\[
\frac{d\pi_i(t)}{\pi_i(t)} = \mu_i dt + S_{\pi_i} dw(t) \quad i \in \{h,f\}, \quad (4.21)
\]
where,

\[
\begin{align*}
\mu_{x_i} &= -\mu_m + (1 - \gamma_i) \left( \alpha_{\eta_i} - \gamma_i \left( \sigma_{\eta_{ni}}^2 + \sigma_{\eta_{xi}}^2 \right) \right) + \sigma_{m_i}^2, \\
S_{x_i} dw(t) &= -\sigma_m dw_m(t) + (1 - \gamma_i) \left( \sigma_{\eta_{yi}} dw_y(t) + \sigma_{\eta_{xi}} dw_x(t) \right).
\end{align*}
\]

Equation (4.21) provides more insight into how the domestic money supply process and the production process affects the dynamics of the local price of money. Note that the expected domestic inflation rate in terms of the price level of the good, \( P_i(t) \), can be written as

\[
\mu_{p_i} = \mu_m - (1 - \gamma_i) \left[ \alpha_{\eta_i} - \left( \sigma_{\eta_{ni}}^2 + \sigma_{\eta_{xi}}^2 \right) \right].
\]  

(4.22)

The expected rate of inflation in country \( i \) in terms of the general price level is, ceteris paribus, increasing in the expected rate of growth of the domestic money supply, \( \mu_m \). Rational investors expect that a positive autonomous money growth results in an increased demand for the good in the economy. The increased demand will lead to higher prices. Rational investors in this economy anticipate on this developments, by incorporating the price increases in their expectation. The impact of the expected rate of growth of production, \( \alpha_{\eta_i} \), and production uncertainty (as measured by its variance \( \sigma_{\eta_{ni}}^2 + \sigma_{\eta_{xi}}^2 \)) depends on the value of the monetary response parameter. There exist three possibilities, namely, accommodating monetary response (\( \gamma_i = 1 \)), contracting money supply (\( \gamma_i < 1 \)), and inflationary monetary growth (\( \gamma_i > 1 \)).

When the value of the monetary response parameter is equal to one, \( \gamma_i = 1 \), the long term mean of the rate of inflation is equal to the expected rate of growth of the domestic money supply, \( \mu_m \). The explanation therefore is that money and output have opposing affect on the expected long term rate of inflation. Therefore, when \( \gamma_i = 1 \) the impact of production on the long term rate of inflation is neutralized by the monetary growth, since money responds fully accommodating to output dynamics in this case. When there is a partial response of money on the production dynamics, i.e. \( \gamma_i < 1 \), the expected rate of inflation in country \( i \) in terms of the general price level is, ceteris paribus, decreasing.
in the expected rate of growth of production. Note that in this case there is a surplus on the goods market due to the increased production and a contracting money supply. Rational investors expect, due to the surplus on the goods market, that prices of that good will fall. The reverse holds for the case of an inflationary monetary response, that is $\gamma_i > 1$. The resulting surplus demand for goods, due a faster growing money supply relative to production, causes prices to increase.

Contrary to "standard" price theory that does not take uncertainty into account, rational investors in this economy incorporate the uncertainties underlying the production process in their expectations with regard to the inflation rate. Both the variance of the change in production process and the covariance of the rate of growth of money supply with that of production are incorporated in the expected inflation. An increase in the variance of production growth, will induce the rational investors to adjust their expectation regarding the rate of inflation upwards. The rationale therefore is that the larger variability in production growth increases the variability in the rate of inflation. Note that the expected rate of inflation is also affected by the variability of the production due to the international uncertainty. The covariance between the rate of growth of money supply and that of output has a negative impact on the expected rate of inflation, which ultimately depends on the feedback parameters. As in the case of the expected production growth it depends on the value of monetary response parameter whether the uncertainties underlying the economy have an impact on the expected domestic inflation and whether this effect is increasing or decreasing.

The domestic inflation risk, as measured by the variance of the rate of inflation, $S_{\pi_i,\pi_i} = \sigma_{\pi_i}^2 + (1 - \gamma_i)^2 \left( \sigma_{\pi_y}^2 + \sigma_{\pi_x}^2 \right)$, is fueled by both production and monetary shocks. Note that the affect of production uncertainty on the inflation risk depends on the value of the monetary response parameter. When monetary response is accommodating ($\gamma_i = 1$) the inflation risk is driven only by the autonomous monetary shocks. In all other case ($\gamma_i \neq 1$) it is increasing in the production variability. Equation (4.21) implies that the covariance between production and domestic rate of inflation,
\[ \sigma_{\eta P_t} = - (1 - \gamma_t) \left( \sigma_{\eta_u}^2 + \sigma_{\eta_e}^2 \right), \]
is determined by the technology shocks and the feedback parameters. The rate of return of domestic production is positively correlated with the domestic price of goods when the monetary response to output shocks is inflationary, that is for \( \gamma_i > 1 \). Otherwise, there is a negative correlation between the production process and inflation. In addition observe that the covariance of the domestic rate of inflation with the foreign inflation rate is determined as follows:

\[ S_{P_tP_f} = (1 - \gamma_h) (1 - \gamma_f) \sigma_{\eta_h} \sigma_{\eta_f}. \]

This covariance is non-zero unless monetary dynamics is fully accommodating to production growth in either country, that is \( \gamma_h = 1 \) or \( \gamma_f = 1 \). The rate of inflation in the two-country world economy co-varies positively when the real sources of inflation risk are larger than the monetary sources in each country (\( \gamma_h < 1 \) and \( \gamma_f < 1 \)) or when the monetary sources in each country are larger (\( \gamma_h > 1 \) and \( \gamma_f > 1 \)). This can be interpreted as the correlation between inflation in the two countries is positive when the monetary policy in both countries with respect to the rate of growth of production are consistent with each other. When the monetary sources of inflation risk in one country is larger than the output risk, while in the other country the reverse is true, the two inflation processes are negatively correlated (\( \gamma_h > 1 \) and \( \gamma_f < 1 \) or \( \gamma_h < 1 \) and \( \gamma_f > 1 \)).

### 4.2 The asset pricing formula

To show that in this two-country monetary-production economy all factor risk are priced we derived the equilibrium asset pricing formula.

**Proposition 2.** The asset pricing formula, based on the general equilibrium conditions in a two-country monetary-production economy with separable logarithmic prefer-
ences, is given by:

\[ E[R_q(t)] = r_i(t) + \sum_j \beta_{q,j}^X (E[R_{X_j}(t)] - r_i(t)) + \beta_q^Y (E[R_Y] - r_i(t)) \]

for \( q = 1, \ldots, N + 5 \).

(4.23)

where,

- \( R_q(t) = \) the real rate of return on security \( q \),
- \( R_{X_j}(t) = \) the rate of change in the local state variable in country \( j \), \( X_j(t) \),
- \( R_Y(t) = \) the rate of change in the international state variable, \( Y(t) \),
- \( \beta_{q,j}^X = \frac{\text{cov}(R_q(t), R_{X_j}(t))}{\text{var}(R_{X_j}(t))} \) and \( \beta_q^Y = \frac{\text{cov}(R_q(t), R_Y(t))}{\text{var}(R_Y(t))} \) for \( j \in \{h, f\} \).

Proof: See Appendix A.4.

From equation (4.23) we can observe that each asset's risk is completely specified by the instantaneous covariance of its return with the changes in the state variables. In equilibrium, the risk-averse investor is compensated in terms of expected return for bearing both monetary and production risk. Therefore, the asset pricing formula in equation (4.23) implies that all sources of systematic risk in this two-country world-economy are priced. This result generalizes the asset pricing models in the existing log utility based monetary models, e.g. Stulz (1986), Foresi (1990), and Bakshi and Chen (1996). In these models, the separability of preferences, which implies that \( U_{cm} = 0 \), allows only one source of risk, i.e., production risk, to be priced in equilibrium. In Bakshi and Chen (1996), both production and monetary risk are priced only in the general formulation of the preferences. Only one of the sources of risk would remained priced if they apply their log utility specification to their asset pricing model. As a result, in the case of log utility preferences, these models reduce to the standard consumption-based capital asset pricing model developed by Breeden (1979).

The financial assets in this two-country monetary-production economy are priced to yield a risk premium that depends on the hedging capabilities of that asset against the
real and monetary uncertainties in the economy. Since, for instance, the real domestic assets\(^2\) can be used to hedge against unfavorable developments in the foreign state and/or monetary uncertainty, its real risk premium is determined by its covariance with the changes in both the domestic and international state variables and by the factor risk premiums on \(R_{X_h}\) and \(R_Y\). Thus, the covariance between the return on the domestic productive asset and the foreign state variable is zero in this economy, which implies that 
\[
\beta_{X_f}^{\eta_h} = \frac{\text{cov}(R_{X_h}(t), R_{X_f}(t))}{\text{var}(R_{X_h}(t))} = 0.
\]
This, however, does not imply that the risk premium of real assets is not affected by foreign (and monetary) quantities in this economy. To illustrate the effects of monetary and foreign quantities, we use the first order condition (4.6 and 4.7) and our asset pricing model to obtain an explicit expression for the risk premium on real domestic asset. First, note that we can write the factor risk premium on \(R_{X_h}\) and \(R_Y\) as, respectively,
\[
(E[R_{X_h}] - r_h) = \left[2a_h^* + (1 - \gamma_h)b_{hh}^* + b_{hf}^*\right] \sigma_{x_h} \sigma_{x_h} \sqrt{X_h(t)}
\]
and
\[
(E[R_Y] - r_h) = \left[(2a_h^* + b_{hh}^* (1 - \gamma_h) + b_{hf}^*) \sigma_{y_h} - \gamma_f \sigma_{y_f} b_{hf}^*\right] \sigma_Y \sqrt{Y(t)},
\]
where \(a_h^*, b_{hh}^*,\) and \(b_{hf}^*\) are the optimal demand for real asset, domestic nominal bond, and foreign nominal bond, respectively (see below for the solutions). The covariance between the return on the real asset with the changes in the domestic and international state variables, \(\beta_{X_h}^{\eta_x}\) and \(\beta_{X_h}^{\eta_y}\), can be written as 
\[
\beta_{X_h}^{\eta_x} = \frac{\sigma_{x_h} \sqrt{X_h}}{\sigma_{x_h} \sqrt{X_h}} \quad \text{and} \quad \beta_{X_h}^{\eta_y} = \frac{\sigma_{y_i}^2}{\sigma_y \sqrt{Y}}.
\]
By substituting these expressions and the equations (4.24) and (4.25) in equation (4.23), we can write the expected risk premium the domestic agent obtains when investing in the

\(^2\)Note the foreign real asset expressed in domestic currency has the same risk exposure as the domestic real asset. This is a direct result of our model, where the exchange rate adjust instantaneously to equate the rate of returns on the two real assets.
domestic real asset as

$$\alpha_h - r_h = \sigma_{\eta_{x,h}}^2 + \sigma_{\eta_{v,h}}^2 - \rho \xi_{rp_1},$$

(4.26)

where

$$\xi_{rp_1} = \frac{S_{m_h}\sigma_{m_h} - S_{m_h}m_f + S_{m_f}S_{m_h}m_h}{S_{m_f}^2 S_{m_h}m_h - S_{m_h}^2 m_f}.$$  

As can be observed from equation (4.26) the equilibrium expected real rate of excess return on the productive assets depends on both production and monetary variability. The production variability, $\sigma_{\eta_{x,h}}^2$ and $\sigma_{\eta_{v,h}}^2$, has a direct positive effect on the expected risk premium on domestic productive investments, because the risk averse investor requires an increase in the risk premium on real assets to compensate for an increase in the instantaneous variance of the productive assets. Indirectly there is a decreasing effect through the covariance of the real rate of return of domestic production with the domestic money supply process ($S_{m_h}$) or, say, with the inflationary uncertainty. Both domestic and foreign monetary sources of uncertainty is noticeable through the impact of the term, $\xi_{rp_1}$. The variability of the domestic money supply is captured by $\sigma_{m_h}^2$ and $S_{m_h}^2$, whereas the foreign sources of monetary uncertainty enter through the variability of the foreign money supply process, $S_{m_f}^2$, and the covariance with domestic money supply, $S_{m_h}m_f$. The explanation therefore is that the monetary sources of uncertainty affect the portfolio demand for nominal assets and, consequently, the risk premium on these assets. In order to ensure that the representative investor maintains its stock of wealth invested in the domestic production technology, the risk premium on this asset must adjust. The risk premium on real assets is negatively affected by a positive money supply shock. This is because the risk-averse investor holding the real asset requires a lower risk compensation for the monetary shock because of the hedging capabilities of the real asset against this source of uncertainty in the economy. Notice that the real risk premium on domestic real asset is not affected by the foreign production variability, because the exchange rate accounts for it.
From equation (4.23) it can be noted that the equilibrium expected real rate of excess return on the domestic nominal asset, that is the difference between the expected instantaneous real rate of return on this security and the riskless real interest rate, has non-zero betas with respect to the local and common state variable. In contrast, the real risk premium on the foreign nominal asset, expressed in domestic currency, has non-zero betas with respect to all three state variables involved. By using the market clearing conditions for nominal and real assets in our economy and the first order conditions (4.6 and 4.7), we can alternatively express the real risk premium on the domestic and foreign nominal assets (expressed in domestic currency), respectively, as follows:

\[ \beta_h - r_h = (1 - \xi_{rp_1}) \rho + S_{\eta_h}^2 - S_{\eta_h m_h} \]  
\[ (4.27) \]

and

\[ \tilde{\beta}_f - r_h = \rho \xi_{rp_2} + S_{\eta_h}^2 - S_{\eta_h m_f} , \]  
\[ (4.28) \]

where

\[ \beta_h = R_h + \mu_{\eta_h} , \]
\[ \tilde{\beta}_f = \beta_f + \mu_{\eta_x} + S_{\eta_x h_f} , \]
\[ S_{\eta_x h_f} = (1 - \gamma_f ) \left( \frac{\sigma_{\eta_x h} - \sigma_{\eta_x f}}{\sigma_{\eta_x f} - \sigma_{\eta_x h}} \right) \]
\[ \xi_{rp_2} = \frac{(S_{m_h m_f} - S_{\eta_h m_h}) (S_{m_f}^2 - S_{m_h m_f}) + S_{m_f}^2 (S_{m_h}^2 - S_{m_h m_f})}{S_{m_f}^2 S_{m_h}^2 - S_{m_h m_f}^2} . \]

This representation of the real risk premium on the domestic nominal asset in equation (4.27) can be considered as the extended Fisher equation. According to the standard Fisher theory the interest rate differential, or stated differently the expected real rate of return on nominal assets in excess of the real risk-free interest rate, should be equal to zero. Equation (4.27) differs from the standard Fisher identity, because the different sources of uncertainty are priced in this two-country world economy. The monetary uncertainty affects the risk premium on the nominal assets through the term \( \xi_{rp_1} \), for
\( i = 1, 2 \), and therefore the same arguments apply as for the risk premium on the real assets. The real risk premium on the nominal bonds is increasing in the production shocks, except when production is almost perfectly correlated with changes in the money supply, that is when \( \gamma_i = 1 \).

### 4.3 The equilibrium interest rates and fundamental valuation equation

**Theorem 2** The equilibrium interest rates and fundamental valuation equation in this two-country monetary production economy are as follows:

(a) The equilibrium nominal and real interest rate are given respectively by:

\[
R_i(t) = \rho + \mu_m + \gamma_i \alpha_n - \sigma^2_{m_i} - \gamma_i^2 \left( \sigma^2_{\eta_n} + \sigma^2_{\eta_{n_i}} \right) \quad i \in \{ h, f \} \tag{4.29}
\]

and

\[
\tau_i = \alpha_i - \sigma^2_{\eta_{r,i}} - \sigma^2_{\eta_{y,i}} + \rho \xi_{rp_i} \quad i \in \{ h, f \}. \tag{4.30}
\]

(b) In equilibrium, the fundamental valuation equation for any contingent claim in this two-country economy, \( F^k(X_h(t), X_f(t), Y(t), t, T) \), is represented by:

\[
0 = \frac{1}{2} \sum_{i=1}^{n_f} \sigma^2_{x_i, F^k X_i} + \frac{1}{2} \sigma^2_{y, Y} + [\mu_{x_h}(X_h, t) - \lambda_{x_h}X_h(t)] F^k_{X_h}
+ [\mu_{x_f}(X_f, t) - \lambda_{x_f}X_f(t)] F^k_{X_f} + [\mu_y(Y, t) - \lambda_yY(t)] F^k_Y
+ F^k_i - \tau_i(t) F^k, \tag{4.31}
\]

where,

\[
\lambda_y Y = \left( S_{\psi h} a^*_h + S_{\psi f} a^*_f + S_{\psi B_h} b^*_h + S_{\psi B_f} b^*_f \right),
\]

\[\text{\footnotesize{The subscripts on } F \text{ represent the corresponding partial derivatives.}}\]
\[
\lambda z_h X_h = \left( S_{z_h \eta_h} a^{*h} + S_{z_h \eta_h} b^{*h} \right)
\]
\[
\lambda z_f X_f = \left( S_{z_f \eta_f} a^{*f} + S_{z_f \eta_f} b^{*f} \right).
\]

**Proof: See Appendix A.5.**

Inspection of equations (4.29) and (4.30) shows that the equilibrium nominal and real interest rates, respectively, depend on both real and monetary variables in this two-country monetary-production economy. This an extension of existing monetary equilibrium models of interest rates [e.g. Stulz (1986), Foresi (1990), and Bakshi and Chen (1996, 1997)]. The nominal interest rate in country \(i\) is increasing in the expected autonomous monetary growth rate in that country and decreasing in the autonomous component of the variance of the local money supply. A higher expected rate of growth of the \(i\)-th money leads to an excess demand for the corresponding good, which results in an increase in the price level of that good. To restore equilibrium the nominal interest rates in country \(i\) must increase. The rationale for this is that, as shown in Proposition 1, the increased domestic nominal interest rate has a negative affect on the demand for that particular real cash balances. Note, that the nominal interest rate is positively affected by the time preference parameter.

The response of the equilibrium domestic nominal interest rate to the real sources of uncertainty is largely determined by the monetary feedback parameter, \(\gamma_i\). We can note that if monetary response to production shocks is zero, the nominal interest rate is completely determined by autonomous monetary dynamics and the subjective discount factor. The mean and the instantaneous variance of the rate of return of production has, respectively, a positive and negative impact on the nominal interest rate, when the monetary feedback parameter is non-zero. In particular, an increase in the production variability results in an increase in the variability of domestic money supply, and, therefor, increased inflationary uncertainty in these countries. As stated by general economic theory, higher inflation risk induces a substitution away from financial assets and into physical capital. This has a decreasing effect on the demand for real money balances. In
order to counteract this effect and restore the money market equilibrium, the opportunity
cost of holding that particular money, \( R_t \), must fall. Note from equation (4.16) that
the price level in terms of the price level co-varies with the nominal interest rates in
this economy. Even in this simple i.i.d. monetary-production economy, with constant
nominal interest rates, we can observe that the nominal interest rates in both countries
are driven by a common source of risk. In the next chapter, we will show that the nominal
interest rates in both countries are correlated in the stochastic case.

By using the equilibrium solutions for the nominal interest rates in this two-country
economy we can endogenously determine the real rate of return on nominal bonds. From
equations (3.8), (4.21), and (4.29), we obtain the real rate of return on the domestic bond
as
\[
\frac{dB_h}{B_h} = \beta_h dt + \sigma_{\eta,h} dw_{\eta,h}(t),
\]
where
\[
\beta_h = \rho + \alpha_{\eta,h} - \gamma_h \left( \sigma_{\eta,h}^2 + \sigma_{\eta,h}^2 \right).
\]
Equation (4.32) shows that the expected real rate of return on the domestic nominal
bond increases with the time preference parameter and the expected real rate of return
on productive investment (recall that the rate of return on the foreign equity investment
is equal to the domestic rate of return, when expressed in domestic currency). The
covariance of the domestic money supply process with the local production process has
a negative impact on the expected real rate of return on these bonds. The rationale
therefore is that the covariability of money with output has an increasing impact on the
price level. As can be noted the real rate of return on the domestic nominal bond has
the same risk exposure as the price of money.

Next, we can determine the real rate of return on foreign nominal bond expressed in
domestic currency endogenously by using the equilibrium solutions for the dynamics of
the price of money, the dynamics of the exchange rate and the nominal interest rate in
equations (3.9), (4.17), (4.21), and (4.29),

\[ \frac{d\bar{B}_j(t)}{\bar{B}_j(t)} = \tilde{\beta}_f dt + S_{\bar{B}_j} dw_{\bar{B}_j}(t), \]  

(4.33)

where

\[ \bar{B}_j(t) = \epsilon_{ij}(t)B_j(t), \]

\[ \tilde{\beta}_f = \rho + \alpha_{n,h} - \gamma_f \sigma_{n_f,} \sigma_{n_h}, \]

\[ S_{\bar{B}_j} dw(t) = \left( \sigma_{n_h} - \gamma_f \sigma_{n_f} \right) dw_y(t) - \gamma_f \sigma_{n_f} dw_{x_f}(t) + \sigma_{n_x,h} dw_{x_h}(t) - \sigma_{m_f} dw_{m_f}(t). \]

We can observe from equation (4.33) that the expected real rate of return on the foreign nominal bond (\( \bar{\beta}_f \)), expressed in domestic currency, is also positively affected by the time preference parameter and the expected real rate of return on domestic productive investment. The difference with the expected real rate of return on domestic nominal bond is that the covariability of domestic output with the foreign money supply is allowed to affect the real rate of return. An increase in this covariance decreases the expected rate of return expressed in domestic currency. In contrast, the domestic monetary sources of risk do not affect the expected real rate of return on the foreign bond, expressed in domestic currency. This can be observed by writing the diffusion term of the real rate of return of foreign bond expressed in domestic currency as \( S_{\bar{B}_j} dw_{\bar{B}_j}(t) = S_{n_h} dw(t) - S_{m_f} dw(t). \) Since, the exchange rate process functions as an arbitrage mechanism, the real sources of foreign uncertainty do not enter into this rate of return, when expressed in domestic currency.

The effects of production and monetary shocks are less straightforward in the case of the real interest rate in equation (4.30). In line with the results of CIR (1985b) in a one-country framework, the real interest rates in this two-country economy is increasing in the expected instantaneous rate of return on the productive assets. Equation (4.30) shows that the real rate of interest in this two-country monetary-production economy accounts
for all sources of risk, namely both real and monetary sources. A production shock in the economy, as measured by the variance of its real rate of return $S_{\eta,\eta} = \sigma_{\eta_x,\eta}^2 + \sigma_{\eta_y,\eta}^2$, has a decreasing effect on the level of the real interest rate in that particular country. The rationale for this effect is that increased uncertainty in the real economy due to technology shocks induces a tendency to shift away from that particular real asset toward the local nominal asset, which in turn implies that part of the capital stock is no longer invested in the production process of that particular country. To counteract this tendency, the real interest rate in that country must fall in order to increase the risk premium on that real asset and, subsequently, to ensure that the agent invests the whole stock of capital in that production technology. Note that the real rate of return of foreign productive investment do not affect the rate of return on domestic production directly, as the exchange rate dynamics accounts for this affect.

Further analysis shows that the instantaneous covariance of the real rate of return of domestic production with the money supply process in each country, which ultimately depends on the monetary feedback parameters, $\gamma_i$, could mitigate the decreasing impact of the production variability on the real interest rate in each country. This effect is captured through the term $\xi_{\rho p}$, which contains all the sources of monetary, c.q. inflation, risk. As can be inferred from equation (4.30), the impact of domestic monetary uncertainty on the real interest rate depends on the covariance between changes in production and domestic money supply. In the case of a small covariance, a monetary shock results in an increase in the real interest rate, and, subsequently, in a commensurate decrease of the risk premium on real assets. This effect can be attributed to the fact that an autonomous domestic monetary shock induces a tendency to substitute the local nominal assets for productive assets in either country. This tendency is reinforced by the hedging capabilities of the real assets against the monetary risk in each country. Conversely, a monetary shock has an decreasing effect on the real interest rate in country $i$ when the covariability with the output process is near perfect. This effect can be explained by the fact that an autonomous shock also increases the risk exposure of the productive asset.
when there is a positive covariance. Notice that in the absence of a demand for domestic nominal assets for portfolio purposes, the real interest rates will only be driven by real factors, even in the presence of monetary endogeneity.

Let us briefly consider the optimal portfolio choice in equilibrium. The optimal demand of the representative agent of home country for real assets, \( a_{h}^* = \begin{bmatrix} a_{hh}^* & a_{hf}^* \end{bmatrix} \), and for the nominal assets, \( b_{h}^* = \begin{bmatrix} b_{hh}^* & b_{hf}^* \end{bmatrix} \), is given by, respectively (see Appendix A.5 for derivation):

\[
\begin{align*}
a_{hh}^* &= a_{hf}^* = \frac{1}{2} \left[ 1 + \rho \frac{2S_{mhf} - S_{mhf}^2 - S_{mh}^2}{S_{mhf}^2 - S_{mh}^2} \right] \\
b_{hh}^* &= b_{hf}^* = \rho \frac{S_{mhf} - S_{mhf}^2}{S_{mhf}^2 - S_{mh}^2} \\
b_{hf}^* &= \rho \frac{S_{mhf}^2 - S_{mhf}^2 S_{mh}^2 - S_{mhf}^2}{S_{mhf}^2 - S_{mh}^2}.
\end{align*}
\] (4.34)

(4.35)

(4.36)

From the derivations above it can be seen how the optimal portfolio choices are determined in this economy. Note that the optimal portfolio demand of home representative agent for the \( j \)-th real asset, \( a_{h,j}^* \), is not affected by the expected instantaneous real rate of return on this particular asset. This is because in equilibrium the expected rate of returns on all securities in this economy is equally affected by the expected real rate of return on the productive investment [see equations (4.18), (4.32), and (4.33)]. Note that this effect is independent of the monetary response parameter. Thus a change in the expected real rate of return on productive investment does not alter the portfolio choice of rational investors. In addition, the equilibrium portfolio demand for securities is not affected by the production uncertainty, as measured by its variance, \( \sigma_{r}^2 = \sigma_{r,s}^2 + \sigma_{r,s}^2 \). The rationale therefore is that, as mentioned above, the real interest rates incorporate this effect and adjusts negatively towards productive uncertainty, such that the risk premium on all securities is compensated for this uncertainty. Equation (4.34) shows that the equilibrium demand of the home representative agent for productive investment in the home country does not differ from its demand for foreign productive investment, since
the spot exchange rate adjusts instantaneously to equate these returns when expressed in the domestic currency. Demand for both real assets in the economy increases with an increase in the monetary uncertainty, both at home and abroad. The intuitive explanation therefore is that the real assets can be used to hedge against the inflationary risk in both countries in these circumstances. This increase in demand for the productive assets, due to the increased inflationary uncertainty, is mitigated, if the time preference parameter is relatively small. The optimal portfolio demand for both nominal assets, $b_{hh}^*$ and $b_{hf}^*$, is decreasing in the nominal uncertainties in both countries. Demand for local nominal asset decreases relative to the demand for foreign nominal assets when the local monetary uncertainty increases relative to foreign monetary uncertainty.

With regard to the PDE in equation (4.31), the boundary conditions are provided by the contractual provisions of the contingent claim and are particular for each type of claim. The regularity conditions, under which existence and uniqueness of a solution for the PDE is established, have been documented extensively in the literature [e.g. CIR (1985a) and Duffie (1992)].

As in CIR (1985a), the fundamental valuation equation in (4.31) implies that the equilibrium expected return on any contingent claim in this economy is determined by the covariance between the changes in the state variables and percentage changes in optimally invested wealth. In conformity with the asset pricing model, the valuation equation in (4.31) reflects the fact that all the sources of risk are priced in equilibrium. In particular, the terms $\lambda_x X_i$ and $\lambda_i Y$ represent the market price of risk of changes in the level of domestic and international uncertainty in country $i$, respectively. Notice that the factor risk premiums depend on the covariance of the production process and the inflation process with the respective state variables. The effect of inflation in country $i$ on the factor risk premium depends on the domestic monetary response parameter, $\gamma_i$, and the demand for domestic nominal assets for portfolio considerations. For example, a countercyclical response ($\gamma_i < 1$) results in a lower covariance between domestic rate of inflation and changes in the domestic and common state variables and, consequently, in
a lower market price of risk of changes in the domestic and common uncertainties.

In addition, equation (4.31) implies that the equilibrium expected return on any contingent claim is uniquely determined by the equilibrium expected rates of return on the productive asset and the nominal asset. Thus, the payout structure of any contingent claim in our two-country world economy can be replicated by using the assets in the basis. This result is consistent with the redundancy property of contingent claims in a complete market. Notice that the expected excess return of the contingent claim in equation (4.31) is taken with respect to the real interest rate and that the valuation equation is developed in real terms. In the following chapter, we examine and solve the fundamental valuation equation in nominal terms.

Summarizing, it can be said that in this chapter we have derived in equilibrium, the price of money, the rate of inflation, the spot foreign exchange rate, and the rate of depreciation of the spot exchange rate in an i.i.d. monetary-production economy. In addition, we obtained the equilibrium nominal and real interest rates. In this chapter we have used a simplified framework, that is i.i.d. production and money supply processes, to obtain an inside in the equilibrium mechanism of the two-country monetary-production economy. However, in such a setup, it is inevitable that the resulting equilibrium interest rates and exchange rate volatility are constants. Since, it is not possible to account for the foreign exchange forward premium puzzle in this framework, we allow for stochastic interest rates and volatilities in the next chapter.