Numerical methods for steady free surface flows
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Chapter 6
Interface Capturing

6.1 Introduction

Free-surface flows can be construed as a particular instance of two-fluid flow, in which the stresses exerted on the interface by one fluid are negligible on a reference scale that is appropriate for the other. If the objective is the numerical solution of a free-surface flow problem, then it can be attractive to adhere to the two-fluid-flow formulation. The interface behaves as a contact discontinuity and can be treated as such. This treatment of the interface is referred to as interface capturing. For examples of interface capturing see, for instance, Refs. [14,34,46].

The interface capturing approach requires that the employed numerical techniques remain robust and accurate in the presence of discontinuities. Godunov-type schemes [22] can be particularly useful in these circumstances. These schemes apply the (approximate) solution of an associated Riemann problem to determine the fluxes that are required in the numerical computation. This approach ensures robustness and accuracy at discontinuities. Godunov-type schemes can be suitably combined with finite volume methods and with discontinuous Galerkin finite element methods. For finite volume methods, the schemes can be implemented with higher-order limited interpolation methods, to achieve accuracy and secure monotonicity preservation in regions where large gradients occur (see, e.g., Refs. [61,65]). For discontinuous Galerkin methods, accuracy and monotonicity preservation is obtained by appropriate $hp$-adaptivity; see, e.g., Refs. [27,64].

A disadvantage of the method originally proposed by Godunov is that it requires the solution of an associated Riemann problem with each flux evaluation. In practice, many such evaluations are performed during an actual computation. Consequently, the method is notorious for its high computational costs. To avoid this problem, several approaches have been suggested to reduce the computational costs of the flux evaluations, by approximating the Riemann solution. Examples
of such approximate Riemann solvers are the flux vector splitting schemes (such as those by Van Leer [40] and Steger & Warming [62]) and the flux-difference splitting schemes (such as Roe's [53] and Osher's [49]).

A common objection to interface capturing is the occurrence of so-called \textit{pressure oscillations}. These pressure oscillations expose the loss of certain invariance properties of the continuum problem under discretization. Several correctives have been proposed to avoid pressure oscillations, e.g., (locally) nonconservative discretization methods [1, 32, 33, 56], correction methods [31] and the ghostfluid method [18]. For an overview of these correctives, and of their merits and deficiencies, see [2]. However, it is not evident that the loss of the aforementioned invariance properties is inevitable. In fact, since the invariance properties are inherent to the continuum equations, it should be possible to devise conservative numerical schemes that inherit these invariance properties.

In the present chapter we consider the interface capturing approach to solving two-fluid flow problems. We investigate an eminent flux-difference splitting scheme for the approximate solution of Riemann problems, viz., Osher's scheme, and we consider its application to two-fluid flows. Moreover, we examine the pressure oscillations that are commonly incurred by discrete approximations to two-fluid flow problems, and we set up a non-oscillatory conservative discretization.

The contents of this chapter are organized as follows: As a preliminary, Section 6.2 presents a general introduction to the Riemann problem. In Section 6.3 we examine Osher's approximate Riemann solver. Motivated by the fact that Osher's scheme suffers loss of accuracy in the presence of slow, strong shock waves, we propose a modified scheme. Section 6.4 presents the specifics for two-fluid flows. In Section 6.5 we examine pressure oscillations and we propose a non-oscillatory conservative discretization.

### 6.2 Riemann Problem

In this section we investigate the Riemann problem. To define the problem, we represent space and time by \( x \in (-\infty, \infty) \) and \( t \in [0, \infty) \), respectively, and we consider \textit{state variables} \( q := (q_1, \ldots, q_n) \) with \( q_k := q_k(x,t) \) and a continuously differentiable \textit{flux function} \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). The Riemann problem is defined by

\[
\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0, \quad (x, t) \in (-\infty, \infty) \times (0, \infty) \tag{6.1a}
\]

subject to the initial condition

\[
q(x, 0) = \begin{cases} 
q_L, & \text{if } x < 0, \\
q_R, & \text{if } x > 0.
\end{cases} \tag{6.1b}
\]

for certain constant left and right states, \( q_L \) and \( q_R \).
6.2.1 Weak Formulation

To allow discontinuous solutions a different setting of the Riemann problem is required. Classical solutions of (6.1) are differentiable. Discontinuous solutions can be taken into consideration if (6.1) is replaced by its weak formulation, viz., the variational problem: find \( q(x,t) \) such that

\[
\int_0^\infty \int_{-\infty}^{\infty} (q a_t + f(q) a_x) \, dx \, dt + \int_{-\infty}^0 q_L a(0,x) \, dx + \int_0^\infty q_R a(0,x) \, dx = 0 ,
\]

(6.2)

for all continuously differentiable \( a(x,t) \) with compact support in \((-\infty, \infty) \times [0, \infty)\). A discontinuous solution of (6.2) is called a weak solution.

The variational formulation (6.2) generally allows multiple solutions, because a classical solution can often be replaced by a weak solution. Therefore, the weak formulation (6.2) must be supplemented with an entropy condition to single out the physically correct solution. For entropy conditions we refer to [39,47,60].

6.2.2 Preliminaries

To facilitate the presentation, we first introduce some elementary concepts. The Jacobian of \( f(q) \) is denoted by \( A(q) := \partial_q f(q) \). Its eigenvalues are \( \lambda_k(q) \), with \( k = 1, 2, \ldots, n \), and are assumed to be indexed such that \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \). The corresponding eigenvectors are \( r_k(q) \).

The differential equation (6.1a) constitutes a hyperbolic system if the eigenvalues \( \lambda_k(q) \) are real and nonzero. The Jacobian-matrix \( A(q) \) can then be decomposed with respect to a basis of its eigenvectors:

\[
A(q) = R(q) \cdot \Lambda(q) \cdot R(q)^{-1} ,
\]

(6.3)

where \( \Lambda(q) := \text{diag}(\lambda_1(q), \ldots, \lambda_n(q)) \) and the matrix \( R(q) := (r_1(q), \ldots, r_n(q)) \) contains the eigenvectors.

From [38] we adopt the following classification of the eigenpairs: An eigenvalue \( \lambda_k(q) \) and an eigenvector \( r_k(q) \) are called genuinely nonlinear in \( \Omega \subseteq \mathbb{R}^n \) if

\[
\partial_q \lambda_k(q) \cdot r_k(q) \neq 0 , \quad \text{for all } q \in \Omega .
\]

(6.4)

Equation (6.4) implies that the eigenvalue is a strictly convex or concave function of \( q \) in the direction of the corresponding eigenvector. An eigenvalue \( \lambda_k(q) \) and an eigenvector \( r_k(q) \) are said to be linearly degenerate in \( \Omega \) if

\[
\partial_q \lambda_k(q) \cdot r_k(q) = 0 , \quad \text{for all } q \in \Omega .
\]

(6.5)

Equation (6.5) implies that \( \lambda_k(q) \) is constant in the direction of the corresponding eigenvector. The eigenvalues that are genuinely nonlinear are related to rarefaction waves and shock waves in the solution of the Riemann problem. The eigenvalues that are linearly degenerate correspond to contact discontinuities in the solution.
More complicated wave types can occur for eigenvalues that are neither genuinely nonlinear nor linearly degenerate. This occurs, for instance, in the case of the Buckley–Leverett equation; see, e.g., Refs. [42, 77].

With each of the eigenpairs \((\lambda_k(q), r_k(q))\) we associate two paths in state space: Firstly, the \(k\)-\textit{shock path}, which is defined by the set

\[
S_k(q_L) := \{ q \in \mathbb{R}^n : s(q, q_L)(q - q_L) = f(q) - f(q_L) \},
\]

where \(s(q, q_L)\) is referred to as the \textit{shock speed}. Secondly, the \(k\)-\textit{rarefaction path}, defined by the set

\[
R_k(q_L) := \{ q \in \mathbb{R}^n : q = h(\xi), \xi \in \mathbb{R} \},
\]

with \(h(\xi)\) the solution to the ordinary differential equation

\[
\begin{align*}
  h'(\xi) &= r_k(h(\xi)), & \xi \in \mathbb{R}, \\
  h(\xi_L) &= q_L,
\end{align*}
\]

for some \(\xi_L \in \mathbb{R}\).

To each \(k\)-rarefaction path corresponds a set of \textit{Riemann invariants}, i.e., functions which are invariant on \(R_k\). If \(r_k(q)\) denotes the \(k\)-th eigenvector of the Jacobian, then a \(k\)-Riemann invariant is any continuously differentiable function \(\psi_k : \mathbb{R}^n \to \mathbb{R}\) with the property

\[
\partial_q \psi_k(q) \cdot r_k(q) = 0,
\]

for all considered states. There are at most \(n - 1\) such \(k\)-Riemann invariants with linearly independent derivatives with respect to \(q\) in \(\mathbb{R}^n\). Note that for a linearly degenerate eigenpair \((\lambda_k(q), r_k(q))\) the eigenvalue \(\lambda_k(q)\) is a \(k\)-Riemann invariant.

### 6.2.3 Waves

The general solution to (6.1) consists of regions in the \((x, t)\)-plane where the solution is constant, separated by waves, in particular, rarefaction waves, contact discontinuities and shock waves. Before constructing the general solution, we first obtain the solution to (6.1) or (6.2) in the case that it contains only one of the aforementioned waves.

Classical and weak solutions of the Riemann problem can generally be written in the similarity form

\[
q(x, t) = h(x/t);
\]

see, e.g., Ref. [60]. To prove this, we note that if \(q(x, t)\) solves (6.1), then for all \(\zeta \in \mathbb{R}\), \(q_\zeta(x, t) := q(\zeta x, \zeta t)\) is also a solution. This is evident from

\[
\begin{align*}
  \frac{\partial q(\zeta x, \zeta t)}{\partial t} + \frac{\partial f(q(x, t))}{\partial x} &= \zeta [D_2 q(\zeta x, \zeta t) + A(q(\zeta x, \zeta t)) \cdot D_1 q(\zeta x, \zeta t)] = 0,
\end{align*}
\]
where $D_t$ denotes differentiation with respect to the $l^{th}$ function argument. If the solution is unique, then $q(x,t) = q_C(x,t)$ and, hence, $q(x,t)$ can be cast into the similarity form (6.10).

A classical $k$-rarefaction wave (or $k$-simple wave) solution of (6.1) exists if $\lambda_k(q)$ is a genuinely nonlinear eigenvalue, $\lambda_k(q_L) < \lambda_k(q_R)$ and $q_R$ is on the $k$-rarefaction path through $q_L$. For later reference, we note that this implies that the $k$-Riemann invariants are equal for $q_L$ and $q_R$, i.e., $\psi^m_k(q_L) = \psi^m_k(q_R)$ for $m = 1, \ldots, n$, $m \neq k$. To determine the rarefaction wave solution, we assume that the eigenvector in (6.8) is normalized such that

$$\partial_q \lambda_k(q) \cdot r_k(q) = 1, \quad \text{for all } q \in \mathbb{R}^n. \quad (6.12)$$

This assumption does not restrict generality. The solution of (6.8) is then the similarity solution in the simple wave region $\lambda_k(q_L) < x/t < \lambda_k(q_R)$; see, e.g., Refs. [39, 60]. For verification, we insert $q(x,t) = h(x/t)$ in (6.1a):

$$\frac{\partial h(x/t)}{\partial t} + \frac{\partial f(h(x/t))}{\partial x} = \frac{1}{t} \left( -h + A(h(x/t)) \right) \cdot D h(x/t) = 0, \quad (6.13)$$

where $I$ denotes the $\mathbb{R}^{n \times n}$ identity matrix and $D$ denotes differentiation with respect to the function argument. Equation (6.13) requires that $x/t = \lambda_k(h(x/t))$ and that $D h(x/t) = r_k(h(x/t))$. The latter trivially follows from (6.8), the former from (6.12). Hence, the Riemann solution in the case of a $k$-rarefaction wave reads

$$q(x,t) = \begin{cases} q_L, & \text{if } x/t < \lambda(q_L), \\ h(x/t), & \text{if } \lambda(q_L) < x/t < \lambda(q_R), \\ q_R, & \text{if } x/t > \lambda(q_R). \end{cases} \quad (6.14)$$

The states $q_L$ and $q_R$ are connected by an isolated $k$-contact discontinuity if $(\lambda_k(q), r_k(q))$ is a linearly degenerate eigenpair and $q_R$ is on the $k$-rarefaction path through $q_L$. By (6.5), $\lambda_k(q_R) = \lambda_k(q_L)$. The solution to the Riemann problem can be obtained from (6.14):

$$q(x,t) = \begin{cases} q_L, & \text{if } x/t < \lambda(q_L) = \lambda(q_R), \\ q_R, & \text{if } x/t > \lambda(q_L) = \lambda(q_R). \end{cases} \quad (6.15)$$

However, because (6.15) is discontinuous at $x/t = \lambda(q_L) = \lambda(q_R)$, it must be verified that (6.15) complies with the variational problem (6.2). Inserting (6.15) in (6.2) and using integration by parts and the compact support of $a$, it follows that (6.15) indeed satisfies (6.2).

A $k$-shock wave occurs if $\lambda_k(q)$ is a genuinely nonlinear eigenvalue, $\lambda_k(q_L) > \lambda_k(q_R)$ and $q_R$ is on the $k$-shock path through $q_L$. A solution of the form (6.14) is then necessarily multiple valued and must therefore be discarded. Instead, the weak solution reads

$$q(x,t) = \begin{cases} q_L, & \text{if } x/t < s(q_L; q_R), \\ q_R, & \text{if } x/t > s(q_L; q_R). \end{cases} \quad (6.16)$$
where \( s(q_L; q_R) \) denotes the shock speed, determined by the Rankine-Hugoniot relation
\[
s(q_L; q_R) (q_L - q_R) = f(q_L) - f(q_R).
\]

The shock relation (6.17) is in fact implied by the variational problem (6.2); see, for instance, Refs. [77]. Conversely, (6.17) ensures that (6.16) satisfies (6.2) for all appropriate \( a \) with support on the discontinuity. Clearly, (6.16) satisfies (6.2) for all \( a \) with support outside the discontinuity. Hence, (6.16) is a valid weak solution of (6.1).

### 6.2.4 General Solution

If the eigenvalues of the Jacobian are either genuinely nonlinear or linearly degenerate, then the general solution to the Riemann problem consists of \( n + 1 \) (possibly empty) regions \( \Omega_l \) in the \((x, t)\)-plane where the solution is constant, separated by rarefaction waves, contact discontinuities and shock waves. To construct the general solution, we define \( q_0 := q_L \), \( q_1 := q_R \) and \( q_{l/n}, \ l = 0, \ldots, n \) is the solution in \( \Omega_l \). If \( q_{l-1/n} \) is connected to \( q_{l/n} \) by a rarefaction wave, then \( h_l(x/t) \) denotes the similarity solution in the rarefaction wave region. Moreover, if \( q_{l-1/n} \) is connected to \( q_{l/n} \) by a shock wave, then \( s_l \) is the corresponding shock speed. The Riemann solution reads
\[
q(x, t) = \begin{cases} 
q_0, & \text{if } x/t < \sigma_0^+, \\
q_{l/n}, & \text{if } \sigma_l^- < x/t < \sigma_l^+, \\
h_l(x/t), & \text{if } \sigma_{l-1}^- < x/t < \sigma_l^-, \\
q_1, & \text{if } x/t > \sigma_n^- \end{cases},
\]

with
\[
\sigma_l^+ = \begin{cases} 
\lambda_{l+1}(q_{l/n}), & \text{if } \lambda_{l+1}(q_{l/n}) \leq \lambda_{l+1}(q_{(l+1)/n}), \\
s_{l+1}, & \text{otherwise}, 
\end{cases}
\]
\[
\sigma_l^- = \begin{cases} 
\lambda_l(q_{l/n}), & \text{if } \lambda_l(q_{l/n}) \geq \lambda_l(q_{(l-1)/n}), \\
s_l, & \text{otherwise}. 
\end{cases}
\]

We refer to the velocities \( \sigma_l^\pm \) as contact speeds. The general solution (6.18) is schematically depicted in Figure 6.1 on the facing page. The figure illustrates the contiguity of regions connected by shock waves and contact discontinuities (e.g., \( \Omega_{(l-1)/n} \) and \( \Omega_{l/n} \)) and the separation of regions connected by rarefaction waves (e.g., \( \Omega_{l/n} \) and \( \Omega_{(l+1)/n} \)).

As an important side note, we mention that existence of a solution to the Riemann problem is not evident. Depending on the properties of \( f \), if \( \|q_L - q_R\| \) is too large, then it can occur that a solution does not exist; see, for instance, Ref. [60].
6.3 Approximate Riemann Solution

The solution to the Riemann problem plays an important role in numerical methods for hyperbolic conservation laws: Following Godunov's approach [22], it can be used to determine the flux across the discontinuities in a piecewise continuous approximation to the solution of a hyperbolic problem.

The benefits of Godunov's method are widely appreciated. However, a disadvantage of the method is the high computational cost of solving the Riemann problem. It is therefore attractive to revert to approximate solution techniques.

In this section we investigate an eminent approximate Riemann solver, viz., Osher's scheme. We examine the approximate Riemann solution underlying Osher's scheme and, based on the results, we propose a modified scheme.

6.3.1 Osher's Scheme

We consider the Riemann solution in similarity form for given left and right states, \( h(x/t; q_L, q_R) \). The Godunov flux is defined by the flux at \( x = 0 \):

\[
\mathbf{f}(q_L, q_R) := \mathbf{f}(h(0; q_L, q_R)) ,
\]

The flux (6.19) is referred to as the Godunov flux, in view of its role in the numerical method proposed by Godunov.

In Osher's scheme [48, 49], the Godunov flux is approximated by

\[
\bar{\mathbf{f}}(q_L, q_R) := \frac{1}{2} \mathbf{f}(q_L) + \frac{1}{2} \mathbf{f}(q_R) - \frac{1}{2} \mathbf{d}(q_L, q_R) ,
\]

with

\[
\mathbf{d}(q_L, q_R) := \int_{q_L}^{q_R} \mathbf{A}(\mathbf{w}) \cdot d\mathbf{w} ,
\]
and
\[
|A(q)| := R(q) \cdot |A(q)| \cdot R(q)^{-1}, \quad (6.20c)
\]
\[
|A(q)| := \text{diag}(\lambda_1(q), \ldots, \lambda_n(q)). \quad (6.20d)
\]

The term (6.20b) contributes to the numerical dissipation. In general, upwind schemes can be cast into the canonical form (6.20a) (see, for instance, Ref. [73]), with \(d(q_L, q_R)\) depending on the specifics of a scheme. The integral (6.20b) characterizes Osher's scheme.

The integral (6.20b) is evaluated along a path in state space. To facilitate the description of the integration path, we represent it by
\[
\Gamma := \{ q(s) : 0 \leq s \leq 1, \; q(0) = q, \; q(1) = q \} . \quad (6.21)
\]

Osher proposed to compose the integration path of the \(k\)-rarefaction paths: The path (6.21) is separated into subpaths \(\Gamma_l, l = 1, 2, \ldots, n\). Each of these subpaths connects two states \(\bar{q}(l-1)/n\) and \(\bar{q}_l/n\). The initial and end state are defined by \(\bar{q}_0 := q_L\) and \(\bar{q}_1 := q_R\), respectively. Moreover, \(\Gamma_l\) is tangential to the eigenvector \(r_k(l)\), where \(k : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}\) is a bijective mapping. This implies that \(\Gamma_l\) is a section of the \(k(l)\)-rarefaction path, connecting \(\bar{q}_{(l-1)/n}\) to \(\bar{q}_{l/n}\). An illustration of the integration path is presented in Figure 6.2.

**Figure 6.2:** Osher’s integration path.

Usual choices for the ordering of the subpaths are the O-variant \(k(l) = n+1-l\) (cf. Ref. [49]) and the P-variant \(k(l) = l\) (cf. Ref. [28]). Note that the O-variant and the P-variant have mutually reversed orderings.

For the O- and P-variant orderings of the subpaths, if a linearly degenerate eigenvalue with algebraic multiplicity \(\mu > 1\) occurs, then this eigenvalue appears on successive subpaths. It is then possible to condense the corresponding subpaths. The integral (6.20b) can be rewritten as a sum of the contributions of the integral over each of the subpaths:

\[
d(q_L, q_R) = \sum_{l=1}^{n} d_l, \quad (6.22a)
\]

with
\[
d_l := \int_{\Gamma_l} |A(w(\xi))| \cdot r_{k(l)}(w(\xi)) \, d\xi . \quad (6.22b)
\]

From (6.20c)–(6.20d) it follows that
\[
d_l = \int_{\Gamma_l} \text{sign}(\lambda_{k(l)}(w(\xi))) \, A(w(\xi)) \cdot r_{k(l)}(w(\xi)) \, d\xi . \quad (6.23)
\]
If $\lambda_{k(l)}$ does not change sign along $\Gamma_l$, then the sub-integral (6.23) evaluates to
\[ d_l = \text{sign}(\lambda_{k(l)}) \left( f(\tilde{q}_{l/n}) - f(\tilde{q}_{(l-1)/n}) \right). \tag{6.24} \]

This implies that if successive eigenvalues have equal signs on their subpaths, then the sum of their contributions concatenates. In particular, if $\lambda_{k(l_0)}(q) = \lambda_{k(l_0+1)}(q) = \ldots = \lambda_{k(l_0+\mu-1)}(q)$ is a linearly degenerate eigenvalue with sign $S$ and algebraic multiplicity $\mu$, then
\[ \sum_{j=0}^{\mu-1} d_{l_0+j} = S \left( f(\tilde{q}_{(l_0+\mu-1)/n}) - f(\tilde{q}_{(l_0-1)/n}) \right). \tag{6.25} \]

The intermediate states $\tilde{q}_{(l_0+j)/n}$, $j = 0, 1, \ldots, \mu - 2$ are of no consequence and can be eliminated from the composite integration path a priori.

Due to the choice of the subpaths, the intermediate states can be conveniently determined by means of the Riemann invariants. Because the subpath $\Gamma_l$ is the $k(l)$-rarefaction path between $\tilde{q}_{(l-1)/n}$ and $\tilde{q}_{l/n}$,
\[ \psi_{k(l)}^m(\tilde{q}_{(l-1)/n}) = \psi_{k(l)}^m(\tilde{q}_{l/n}), \quad l, m = 1, 2, \ldots, n, \quad m \neq k(l); \tag{6.26} \]
cf. Section 6.2.3. If the $k$-Riemann invariants have linearly independent derivatives with respect to $q$, then, by the implicit function theorem, (6.26) is a solvable system of equations from which the intermediate states can be extracted. In many practical cases the intermediate states can then be solved explicitly from (6.26). Otherwise, it is necessary to determine the intermediate states by numerical approximation. Once the intermediate states have been obtained, the flux approximation follows by straightforward operations.

### 6.3.2 Accuracy

The flux obtained by means of the Osher scheme is based on an approximation to the solution of the Riemann problem. Because the approximate Riemann solution can again be written in similarity form, it is useful to introduce the notation
\[ \tilde{f}(q_L, q_R) = f(\tilde{h}(0; q_L, q_R)), \tag{6.27} \]
where $\tilde{h}(x/t; q_L, q_R)$ represents the approximate Riemann solution in similarity form.

To assess the accuracy of the approximate Riemann solution underlying Osher’s scheme, we examine its representation of rarefaction waves, contact discontinuities and shock waves. We consider only the $P$-variant ordering of the subpaths. However, by virtue of the fact that the $O$-variant and $P$-variant have mutually reversed orderings, it follows that
\[ \tilde{h}_O(x/t; q_L, q_R) = \tilde{h}_P(-x/t; q_R, q_L), \tag{6.28} \]
where the subscript distinguishes the variants. By (6.28), the results for the P-variant can be extended mutatis mutandis to the O-variant.

In section 6.3.1 it was emphasized that the subpaths in Osher's scheme are subsets of the \( k(l) \)-rarefaction paths. Referring to Section 6.2.4, it follows that the intermediate states in the approximate Riemann solution are connected by simple waves only. Clearly, this representation is correct for simple waves and contact discontinuities. However, shock waves in the actual solution are then replaced by so-called overturned simple waves; see Ref. [41].

To examine the appropriateness of the overturned-simple-wave representation of shocks. We consider a left state \( q_L \) and a right state \( q_R \), connected by a weak \( k \)-shock. The \( k \)-shock strength is defined by:

\[
\epsilon := \lambda_k(q_L) - \lambda_k(q_R) .
\]

In [60] it is proved that the change in the \( k \)-Riemann invariants across a \( k \)-shock with strength \( \epsilon \) is \( O(\epsilon^3) \) as \( \epsilon \rightarrow 0 \). This implies that a \( \tilde{q}_R \in R_k(q_L) \) exists such that \( \lambda_k(\tilde{q}_R) = \lambda_k(q_R) \) and \( \|\tilde{q}_R - q_R\| = O(\epsilon^3) \). To prove this, we note that \( \tilde{q}_R \in R_k(q_L) \) implies

\[
\psi^m_k(\tilde{q}_R) = \psi^m_k(q_L) , \quad m = 1, 2, \ldots, n , \quad m \neq k .
\]

The change in the \( k \)-Riemann invariants from \( q_L \) to \( q_R \) is only \( O(\epsilon^3) \). Therefore,

\[
\psi^m_k(\tilde{q}_R) = \psi^m_k(q_R) + O(\epsilon^3) , \quad m = 1, 2, \ldots, n , \quad m \neq k . \tag{6.31a}
\]

Equations (6.31) can be augmented with

\[
\lambda_k(\tilde{q}_R) = \lambda_k(q_R) , \tag{6.31b}
\]

to obtain \( n \) equations for \( \tilde{q}_R \). If the \( k \)-Riemann invariants have linearly independent derivatives with respect to \( q \), then \( \text{rank}(\partial_q \psi^1_k, \ldots, \partial_q \psi^n_k) = n - 1 \). Moreover, because \( \lambda_k \) is linearly degenerate, \( \partial_q \lambda_k \notin \text{span}(\partial_q \psi^1_k, \ldots, \partial_q \psi^n_k) \). Therefore, \( (\partial_q \psi^1_k, \ldots, \partial_q \psi^n_k, \partial_q \lambda_k) \) is nonsingular and, by the implicit function theorem, (6.31) is solvable. Taylor expansion of \( \psi^m_k(\tilde{q}_R) \) and \( \lambda_k(\tilde{q}_R) \) with center at \( q_R \) then yields that \( \|\tilde{q}_R - q_R\| = O(\epsilon^3) \).

Summarizing, we find that the error in the intermediate states in the simple-waves-only approximation of the Riemann solution is at most

\[
\sup_{i=1 \ldots n} \left( (\lambda_i(q_{(i-1)/n}) - \lambda_i(q_{l/n}))^3 , 0 \right) . \tag{6.32}
\]

Although the computed intermediate states are accurate, even in the presence of (weak) shocks, the flux approximation is not necessarily so. By (6.22)–(6.23), if \( \tilde{q}_R \in R_k(q_L) \) and \( \lambda_k(q_L) > 0 > \lambda_k(\tilde{q}_R) \), then

\[
d(q_L, \tilde{q}_R) = \int_{q_L}^{q^*} A(w(\xi)) \cdot r_k(l)(w(\xi)) \, d\xi - \int_{q^*}^{\tilde{q}_R} A(w(\xi)) \cdot r_k(l)(w(\xi)) \, d\xi
\]

\[
= 2f(q^*) - f(q_L) - f(\tilde{q}_R) , \tag{6.33}
\]
with \( q^* \in \mathcal{R}_k(q_L) \) such that \( \lambda_k(q^*) = 0 \). The Osher flux (6.20) then yields

\[
\tilde{f}(q_L, q_R) = f(q_L) + f(q_R) - f(q^*).
\]

(6.34)

In contrast, the Godunov flux corresponding to the \( k \)-shock is \( f(q_L) \) if \( s(q_L; q_R) > 0 \) and \( f(q_R) \) if \( s(q_L; q_R) < 0 \). Hence, ignoring terms of \( O(\epsilon^3) \), the error in the flux approximation is

\[
\tilde{f}(q_L, q_R) - f(q_L, q_R) = \begin{cases} 
   f(q_R) - f(q^*), & \text{if } s(q_L; q_R) > 0, \\
   f(q_L) - f(q^*), & \text{if } s(q_L; q_R) < 0.
\end{cases}
\]

(6.35)

To elaborate the error, we note that the states \( q_L, q_R \), and \( q^* \) are interrelated by the \( k \)-Riemann invariants in the following manner:

\[
\psi_k^m(q^*) = \psi_k^m(q_L) = \psi_k^m(q_R) + O(\epsilon^3), \quad m = 1, 2, \ldots, n, \quad m \neq k.
\]

(6.36)

Moreover, because \( q_L \) and \( q_R \) are connected by a shock with strength \( \epsilon \),

\[
\lambda_k(q_L) = O(\epsilon), \quad \lambda_k(q_R) = O(\epsilon),
\]

(6.37a)

and, by definition,

\[
\lambda_k(q^*) = 0.
\]

(6.37b)

From (6.36)–(6.37) it follows that

\[
\|q_{L/R} - q^*\| = O(\epsilon),
\]

(6.38)

where \( q_{L/R} \) is either \( q_L \) or \( q_R \). Taylor expansion of \( f(q_{L/R}) \) with center at \( q^* \) then yields

\[
\|f(q_{L/R}) - f(q^*)\| = O(\epsilon).
\]

(6.39)

Hence, by (6.35), the error in Osher’s flux approximation in the instance of a slow shock \( (\lambda_k(q_R) < 0 < \lambda_k(q_L)) \) with strength \( \epsilon \) is of \( O(\epsilon) \). This failure of Osher’s scheme is exemplified by means of the Burgers equation in [41].

### 6.3.3 Modified Scheme

To avoid the aforementioned deficiency of Osher’s scheme, we propose a modification of the scheme. The simple-waves-only approximation of the Riemann solution is maintained. However, the overturned-simple-wave representation of shocks in the approximate Riemann solution is replaced.

We propose to extract the intermediate states in the approximate Riemann solution from

\[
\psi_l^m(\bar{q}_{l/n}) = \psi_l^m(\bar{q}_{l/n}), \quad l, m = 1, 2, \ldots, n, \quad m \neq l,
\]

(6.40)

with \( \bar{q}_0 = q_L \) and \( \bar{q}_1 = q_R \). This is in fact equivalent to (6.26) with a presumed \( P \)-variant ordering of the subpaths.
The intermediate states from (6.40) are subsequently used to construct the approximate Riemann solution:

\[
\tilde{h}(x/t; \mathbf{q}_L, \mathbf{q}_R) = \begin{cases} 
\tilde{q}_0, & \text{if } x/t < \tilde{\sigma}_0^+, \\
\tilde{q}_{l/n}, & \text{if } \tilde{\sigma}_l^- < x/t < \tilde{\sigma}_l^+, \\
\mathbf{h}(x/t), & \text{if } \tilde{\sigma}_{l-1}^- < x/t < \tilde{\sigma}_l^-, \\
\tilde{q}_R, & \text{if } x/t > \tilde{\sigma}_n^-;
\end{cases}
\]

with the approximate contact speeds \(\tilde{\sigma}_l^\pm\),

\[
\tilde{\sigma}_l^+ = \begin{cases} 
\lambda_{l+1}(\tilde{q}_{l/n}), & \text{if } \lambda_{l+1}(\tilde{q}_{l/n}) \leq \lambda_{l+1}(\tilde{q}_{(l+1)/n}), \\
\tilde{s}_{l+1}, & \text{otherwise,}
\end{cases}
\]

\[
\tilde{\sigma}_l^- = \begin{cases} 
\lambda_l(\tilde{q}_{l/n}), & \text{if } \lambda_l(\tilde{q}_{l/n}) \geq \lambda_l(\tilde{q}_{(l-1)/n}), \\
\tilde{s}_l, & \text{otherwise},
\end{cases}
\]

and

\[
\tilde{s}_l = \frac{1}{2} \lambda_l(\tilde{q}_{(l-1)/n}) + \frac{1}{2} \lambda_l(\tilde{q}_{l/n}).
\]

In (6.41a), \(\mathbf{h}(x/t)\) represents the simple-wave solution between \(\tilde{q}_{(l-1)/n}\) and \(\tilde{q}_{l/n}\). Recall that \(\tilde{q}_{(l-1)/n}\) and \(\tilde{q}_{l/n}\) can indeed be connected by a simple wave due to the choice of the subpaths.

Comparison of the approximate Riemann solution (6.41) with the exact Riemann solution (6.18) shows that \(\tilde{s}_l\) approximates the shock speed. In Ref. [60] it is proved that if \(\mathbf{q}_R \in S_k(\mathbf{q}_L)\) and \(\epsilon := \lambda_k(\mathbf{q}_L) - \lambda_k(\mathbf{q}_R) > 0\), then the k-shock speed satisfies

\[
s(\mathbf{q}_L; \mathbf{q}_R) = \frac{1}{2} \lambda_k(\mathbf{q}_L) + \frac{1}{2} \lambda_k(\mathbf{q}_R) + O(\epsilon^2),
\]

as \(\epsilon \to 0\). Hence, the shock-speed approximation (6.41d) is \(O(\epsilon^2)\) accurate. However, because the objective is to obtain an approximation to the Godunov flux, it is the sign of the shock speed that is of primary interest. The sign of the shock speed is correctly predicted by (6.41d), provided that

\[
\left| \frac{1}{2} \lambda_l(\tilde{q}_{(l-1)/n}) + \frac{1}{2} \lambda_l(\tilde{q}_{l/n}) \right| \geq (\lambda_l(\tilde{q}_{(l-1)/n}) - \lambda_l(\tilde{q}_{l/n}))^2.
\]

Otherwise, the sign of the shock speed depends on the \(O(\epsilon^2)\) remainder and becomes ambiguous. This occurs for strong, slow shocks.

The approximation of the Godunov flux, corresponding to the approximate Riemann solution (6.41), is \(\tilde{f}(\mathbf{q}_L, \mathbf{q}_R) = f(\tilde{h}(0; \mathbf{q}_L, \mathbf{q}_R))\), with

\[
\tilde{h}(0; \mathbf{q}_L, \mathbf{q}_R) = \begin{cases} 
\tilde{q}_0, & \text{if } \tilde{\sigma}_0^+ > 0, \\
\tilde{q}_{l/n}, & \text{if } \tilde{\sigma}_l^- < 0 < \tilde{\sigma}_l^+, \\
\tilde{q}_l, & \text{if } \tilde{\sigma}_{l-1}^- < 0 < \tilde{\sigma}_l^-, \\
\tilde{q}_1, & \text{if } \sigma_n^- < 0,
\end{cases}
\]

where \(\tilde{q}^* \in \mathcal{R}_l(\tilde{q}_{(l-1)/n})\) such that \(\lambda_l(\tilde{q}^*) = 0\), in case of a centered rarefaction wave.
6.4 Two-Fluid Flow Application

Inviscid, compressible two-fluid flows can be modeled by hyperbolic conservation laws. The interface between the contiguous fluids then appears as a discontinuity in the solution. In this section we consider the application of the modified Osher’s scheme to two-fluid flows.

6.4.1 Two-Fluid Euler Equations

We consider the flow of two inviscid immiscible fluids. The fluids occupy a domain \( \mathcal{V} \subset \mathbb{R}^d \) (\( d = 2,3 \)). Positions in \( \mathcal{V} \) are identified by \( x := x_1 e_1 + \ldots + x_d e_d \). Time is denoted by \( t \geq 0 \). In both fluids, the flow is characterized by the state variables \( \rho(x,t), p(x,t) \) and \( \mathbf{v}(x,t) \), which represent density, pressure and velocity, respectively. The Cartesian components of the velocity are denoted by \( v_j(x,t) := e_j \cdot \mathbf{v}(x,t), j = 1,\ldots,d \).

The fluids are separated by an interface, which we identify by the level set
\[
S := \{ x \in \mathcal{V} : \theta(x,t) = 0 \} .
\]

The following kinematic condition ensures the immiscibility of the fluids:
\[
\theta_t + \mathbf{v} \cdot \nabla \theta = 0, \quad x \in \mathcal{V}, t > 0 .
\]

To distinguish the fluids, we arbitrarily designate one of the fluids as the primary fluid and the other as the secondary fluid. We assume that \( \theta(x,t) \) is positive in the primary fluid and negative in the secondary fluid.

In both fluids, the flow is governed by the Euler equations. Because our interest is in discontinuous solutions, we consider the weak formulation of the equations. To present the equations in weak formulation, we denote by \( \Omega \) any arbitrary bounded subset of \( \mathcal{V} \times [0,\infty) \), by \( \Theta \) its boundary and by \( (\mathbf{n}, \gamma) := n_1 e_1 + \ldots + n_d e_d + \gamma e_{d+1} \) the outward unit normal vector to \( \Theta \); see the illustration in Figure 6.3 on the following page. The weak formulation of the Euler equations reads
\[
\int_{\Theta} a (\rho \gamma + \rho \mathbf{v} \cdot \mathbf{n}) \, d\Theta - \int_{\Omega} \rho (a_t + \mathbf{v} \cdot \nabla a) \, d\Omega = 0 , \quad (6.47a)
\]
\[
\int_{\Theta} b (\rho v_j \gamma + \rho v_j \mathbf{v} \cdot \mathbf{n} + p n_j) \, d\Theta - \int_{\Omega} \rho v_j (b_t + \mathbf{v} \cdot \nabla b) + p \frac{\partial b}{\partial x_j} \, d\Omega = 0 , \quad (6.47b)
\]
with \( j = 1,\ldots,d \), for all subsets \( \Omega \) and all continuously differentiable functions \( a, b \).

The equations (6.47a) and (6.47b) express conservation of mass and conservation of momentum, respectively.

The kinematic condition (6.46) is not in conservation form. This renders (6.46)–(6.47) unsuitable for treatment by Godunov’s method. However, under the conditions imposed by (6.47a), we can replace (6.46) by
\[
\int_{\Theta} c (\rho g(\theta) \gamma + \rho g(\theta) \mathbf{v} \cdot \mathbf{n}) \, d\Theta - \int_{\Omega} \rho g(\theta) (c_t + \mathbf{v} \cdot \nabla c) \, d\Omega = 0 ,
\]
with \( c \neq 0 \).
for all subsets $\Omega$ and all continuously differentiable functions $c$, with $g : \mathbb{R} \to \mathbb{R}$ any strictly monotone continuously differentiable function. Equation (6.48) is in weak conservation form. To prove that (6.47a) and (6.48) imply (6.46), we note that the sum of (6.47a) and (6.48) yields:

$$\int_{\Omega} \rho \left( cg(\theta) + a \right) (\gamma + v \cdot n) \, d\Theta - \int_{\Omega} \rho (a_t + v \cdot \nabla a) + \rho g(\theta) (c_t + v \cdot \nabla c) \, d\Omega = 0, \quad (6.49)$$

which holds for all continuously differentiable $a, c$ and all subsets $\Omega$. If we choose $a = -cg(\theta)$, then the boundary integrals cancel and we obtain

$$\int_{\Omega} \rho \left( \left[ cg(\theta) \right]_t + v \cdot \nabla [cg(\theta)] - g(\theta) (c_t + v \cdot \nabla c) \right) \, d\Omega = 0. \quad (6.50)$$

Partial differentiation yields

$$\int_{\Omega} c \rho \left( (g(\theta)_t + v \cdot \nabla g(\theta) \right) \, d\Omega = 0, \quad (6.51)$$

or, equivalently,

$$\int_{\Omega} c \rho g(\theta) (\theta_t + v \cdot \nabla \theta) \, d\Omega = 0, \quad (6.52)$$

for all continuously differentiable $c$ and all subsets $\Omega$. Equation (6.52) implies (6.46) weakly. Therefore, (6.47a) and (6.48) imply (6.46) weakly.

To identify the associated Riemann problem, we collect (6.47)–(6.48) in:

$$\int_{\Omega} a \left( q_{j} \gamma + \sum_{i=1}^{d} f_{j}^{i}(q) n_{i} \right) \, d\Theta - \int_{\Omega} \left( q_{j} a_{t} + \sum_{i=1}^{d} f_{j}^{i}(q) \frac{\partial a}{\partial x_{i}} \right) \, d\Omega = 0, \quad (6.53a)$$
for \( d = 3 \). The case \( d = 2 \) can be treated as a special case of \( d = 3 \), with \( q_3 \) and \( f^3 \) set to 0, and will therefore not be considered separately. Integrating (6.53) by parts, we obtain

\[
\int_{\Omega} a \left( \frac{\partial q_i}{\partial t} + \sum_{i=1}^{d} \frac{\partial f_i(q)}{\partial x_i} \right) d\Omega = 0 ,
\]

which implies the following strong form of the equations:

\[
\frac{\partial q}{\partial t} + \frac{\partial f^1(q)}{\partial x_1} + \cdots + \frac{\partial f^d(q)}{\partial x_d} = 0 .
\]

To obtain the Riemann problem associated with the two-fluid Euler equations, we stipulate \( \partial f^i/\partial x_i = 0 \) for \( i = 2, \ldots, d \) and we prescribe discontinuous initial conditions conform (6.1b).

### 6.4.2 Equation of State

Closure of the two-fluid Euler equations requires an equation of state which interrelates \( p, \rho, \mathbf{v} \) and \( \theta \). We consider fluids that separately satisfy a barotropic equation of state:

\[
p := p_{p/s}(\rho) , \quad \text{or} \quad \rho := \rho_{p/s}(p) .
\]

with \( p_{p/s} : (0, \infty) \mapsto \mathbb{R} \) and \( \rho_{p/s} : \mathbb{R} \mapsto (0, \infty) \) mutually inverse functions, i.e.,

\[
p_{p/s}(\rho_{p/s}(p)) = p , \quad \text{and} \quad \rho_{p/s}(p_{p/s}(\rho)) = \rho ,
\]

for all \( p \in \mathbb{R} \) and all \( \rho \in (0, \infty) \). The subscripts \( p \) and \( s \) distinguish the primary and secondary fluid.

A relevant example of an equation of state conform (6.56) is Tait’s equation of state (see, e.g., [71]):

\[
p(\rho) = p_0 \left( 1 + \eta_1 \left( \frac{\rho}{\rho_0} \right)^{\eta_2} - \eta_1 \right) ,
\]

with \( p_0 \) an appropriate reference pressure, e.g., the atmospheric pressure, \( \rho_0 \) the corresponding density of the fluid and \( \eta_1 \) and \( \eta_2 \) fluid-specific constants. Equation (6.58) can be used to describe the behavior of air in homentropic flow and of
Table 6.1: Constants in Tait’s equation of state (6.58) for water and for air in homentropic flow.

<table>
<thead>
<tr>
<th></th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>water</td>
<td>3000</td>
<td>7</td>
</tr>
<tr>
<td>air</td>
<td>0</td>
<td>7/5</td>
</tr>
</tbody>
</table>

water. The corresponding constants are listed in Table 6.1. Appropriate constants for other fluids are given in [71].

In order to construct an equation of state for the two-fluid compound, we consider a domain $\Omega$ which is occupied by the two co-existing fluids. If the fluids are separated, then the primary and the secondary fluid occupy domains $\Omega_p$ and $\Omega_s$, respectively; see the illustration in Figure 6.4. Denoting by $|\Omega|$ the volume of $\Omega$, we define the primary *volume fraction* by

$$\alpha := \frac{|\Omega_p|}{|\Omega_p| + |\Omega_s|}. \quad (6.59)$$

If the pressure is constant in $\Omega$, then the compound density reads:

$$\rho(p, \alpha) := \alpha \rho_p(p) + (1 - \alpha) \rho_s(p). \quad (6.60)$$

Equation (6.60) specifies the equation of state for the two-fluid compound. The compound equation of state is illustrated in Figure 6.5 on the next page. An equation of state of the form $p(\rho, \theta)$ is obtained by introducing a mapping $\theta \mapsto \alpha(\theta)$.

**Figure 6.4:** Separation of the fluids in a volume $\Omega$. The primary and secondary fluid occupy $\Omega_p$ and $\Omega_s$, respectively.
6.4. Two-Fluid Flow Application

Figure 6.5: Compound equation of state for two barotropic fluids according to (6.60).

6.4.3 Riemann Invariants

We consider the flux function $f^1$ according to (6.53b) with an equation of state of the general form $p := p(\rho, \theta)$. To facilitate the derivation of the associated Riemann invariants, we introduce the notation:

\[
c_p(\rho, \theta) := \sqrt{\frac{\partial p(\rho, \theta)}{\partial \rho}},
\]

\[
c_\theta(\rho, \theta) := \sqrt{\frac{\partial p(\rho, \theta)}{\partial \theta}}.
\]

The Jacobian of the flux function then reads

\[
A(q) = \begin{pmatrix}
2q_1/q_5 & 0 & 0 & c_\theta^2/q_5 & -(q_1^2 + c_\theta^2 q_4)/q_5^2 + c_\rho^2 \\
q_2/q_5 & q_1/q_5 & 0 & 0 & -q_2 q_1/q_5^2 \\
q_3/q_5 & 0 & q_1/q_5 & 0 & -q_3 q_1/q_5^2 \\
q_4/q_5 & 0 & 0 & q_1/q_5 & -q_4 q_1/q_5^2 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

and its eigenvalues and the corresponding eigenvectors are

\[
\lambda_1 = q_1/q_5 - c_\rho, \quad \lambda_{2,3,4} = q_1/q_5, \quad \lambda_5 = q_1/q_5 + c_\rho.
\]
and
\[
\begin{pmatrix}
q_1/q_5 - c_\rho \\
q_2/q_5 \\
q_3/q_5 \\
q_4/q_5 \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}
\] (6.63b)
\[
\begin{pmatrix}
q_1 \\
0 \\
0 \\
-(c_\rho/c_\theta)^2 q_5^2 + q_4 \\
q_5
\end{pmatrix},
\begin{pmatrix}
q_1/q_5 + c_\rho \\
q_2/q_5 \\
q_3/q_5 \\
q_4/q_5 \\
1
\end{pmatrix}
\]

The eigenpairs \((\lambda_k, r_k)\) are genuinely nonlinear for \(k = 1, 5\) and linearly degenerate for \(k = 2, 3, 4\).

To derive the 1-Riemann invariants, we first solve the following system of ordinary differential equations:
\[
h'(\xi) = r_k(h(\xi)) , \quad \xi \in \mathbb{R} ,
\]
\[
h(0) = h^0 ,
\] (6.64a)
(6.64b)

for \(k = 1\). Trivially,
\[
h_5(\xi) = \xi + h_5^0 .
\] (6.65)

Hence, for \(j = 2, 3, 4\), \(h_j(\xi)\) is determined by
\[
h_j'(\xi) = h_j(\xi)/(\xi + h_5^0) , \quad h_j(0) = h_j^0 ,
\] (6.66)

which yields
\[
h_j(\xi) = (h_j^0/h_5^0) \xi + h_j^0 , \quad j = 2, 3, 4 .
\] (6.67)

Furthermore, \(h_1(\xi)\) is governed by
\[
h_1'(\xi) = h_1(\xi)/(\xi + h_5^0) - c_\rho(\xi) , \quad h_1(0) = h_1^0 ,
\] (6.68)

with \(c_\rho(\xi) := c_\rho(h_5(\xi), h_4(\xi)/h_5(\xi))\). The solution to (6.68) is
\[
h_1(\xi) = (\xi + h_5^0) \left( h_1^0/h_5^0 - \int_{h_5^0}^{\xi + h_5^0} \frac{c_\rho(\omega)}{\omega} \, d\omega \right) .
\] (6.69)

To obtain 1-Riemann invariants, we construct \(\xi\)-independent functions of \(h_j(\xi)\), \(j = 1, \ldots, 5\). By (6.65) and (6.67),
\[
\frac{h_j(\xi)}{h_5(\xi)} = \frac{(h_j^0/h_5^0) \xi + h_j^0}{\xi + h_5^0} = \frac{h_j^0}{h_5^0} , \quad j = 2, 3, 4 .
\] (6.70)
Therefore, \( q_j/q_5, j = 2, 3, 4 \) are 1-Riemann invariants. Moreover,

\[
\frac{h_1(\xi)}{h_5(\xi)} + \int_{h_5^0}^{h_5(\xi)} \frac{c_\rho(\omega)}{\omega} \, d\omega = \left( \frac{h_1^0}{h_5^0} - \int_{h_5^0}^{h_5(\xi)} \frac{c_\rho(\omega)}{\omega} \, d\omega \right) + \int_{h_5^0}^{h_5(\xi)} \frac{c_\rho(\omega)}{\omega} \, d\omega = \frac{h_1^0}{h_5^0} .
\]

(6.71)

Hence, for an arbitrary constant \( q_5^0 \in (0, \infty) , \)

\[
\frac{q_1}{q_5} + \int_{h_5^0}^{q_5^0} \frac{c_\rho(\omega)}{\omega} \, d\omega ,
\]

(6.72)

is also a 1-Riemann invariant.

Due to the simple structure of the eigenvectors \( r_k \) for \( k = 2, 3 \), the corresponding Riemann invariants can be determined by inspection. Moreover, by (6.63), the 5-Riemann invariants are identical to the 1-Riemann invariants, with \( c_\rho \) replaced by \(-c_\rho\).

To derive the 4-Riemann invariants, we solve (6.64) for \( k = 4 \). Obviously,

\[
\frac{h_1(\xi)}{h_5^0} e^\xi , \quad \frac{h_2(\xi)}{h_5^0} , \quad \frac{h_3(\xi)}{h_5^0} , \quad \frac{h_4(\xi)}{h_5^0} = \frac{h_5^0}{h_5^0} e^\xi .
\]

(6.73)

To facilitate the determination of \( h_4(\xi) \), we recall that \( c_\rho \) and \( c_\theta \) are defined by Eq. (6.61) and that \( h_5 \) and \( h_4 \) are defined by \( \rho \) and \( \rho \theta \), respectively. Therefore,

\[
\frac{h_4(\xi)}{h_5^0} = \frac{c_\rho^2}{c_\theta^2} (h_5(\xi))^2 + h_4(\xi) \Rightarrow \frac{\partial \rho}{\partial \theta} \left( \theta \frac{d\rho}{d\xi} + \rho \frac{d\theta}{d\xi} \right) = -\frac{\partial \rho}{\partial \rho} \rho^2 + \frac{\partial \rho}{\partial \theta} \rho \theta .
\]

(6.74)

From (6.73) it follows that \( \rho = \rho'(\xi) \) and, hence,

\[
\frac{\partial \rho}{\partial \theta} \frac{d\rho}{d\xi} + \frac{\partial \rho}{\partial \rho} \frac{d\rho}{d\xi} = \frac{d\rho}{d\xi} = 0 .
\]

(6.75)

This implies that \( h_4(\xi) \) is implicitly given by

\[
p(h_5(\xi), h_4(\xi)/h_5(\xi)) = p(h_5^0, h_4^0/h_5^0) .
\]

(6.76)

Summarizing, by (6.73)–(6.76), 4-Riemann invariants are:

\[
q_1/q_5 , \quad q_2 , \quad q_3 , \quad p .
\]

(6.77)

In conclusion, we can associate the following Riemann invariants with the flux function \( f^1 \) according to (6.53b), with an equation of state of the form \( p := p(\rho, \theta) \):

\[
\psi_1^2 = v_1 + \psi(\rho, \theta) , \quad \psi_1^3 = v_2 , \quad \psi_1^4 = v_3 , \quad \psi_1^5 = \theta , \quad (6.78a)
\]
\[
\psi_1^2 = v_1 , \quad \psi_1^3 = \rho v_3 , \quad \psi_1^4 = \rho \theta , \quad \psi_1^5 = \rho \theta , \quad (6.78b)
\]
\[
\psi_1^2 = v_1 , \quad \psi_1^3 = \rho v_3 , \quad \psi_1^4 = \rho \theta , \quad \psi_1^5 = \rho \theta , \quad (6.78c)
\]
\[
\psi_1^2 = v_1 , \quad \psi_1^3 = \rho v_3 , \quad \psi_1^4 = \rho v_3 , \quad \psi_1^5 = \rho v_3 , \quad (6.78d)
\]
\[
\psi_1^1 = v_1 - \psi(\rho, \theta) , \quad \psi_1^2 = v_2 , \quad \psi_1^3 = v_3 , \quad \psi_1^5 = \theta , \quad (6.78e)
\]
with
\[ \Psi(\rho, \theta) := \int_{\rho^0}^{\rho} \frac{c_p(\omega, \theta)}{\omega} \, d\omega, \quad (6.78f) \]
and \( \rho^0 \) an arbitrary constant in \((0, \infty)\).

### 6.4.4 Approximate Two-Fluid Solution

The Riemann invariants (6.78) can be used to extract from (6.40) the intermediate states in the simple-waves-only approximation to the solution of the two-fluid Riemann problem. Because the linearly degenerate eigenvalue \( v_1 \) has algebraic multiplicity \( m = 3 \), only two intermediate states have to be distinguished; see Sect. 6.3.1. These intermediate states are connected by a contact discontinuity. The eigenvalues \( v_1 \pm \Psi \) are genuinely nonlinear. This implies that simple waves connect the intermediate states to the left and right states. In particular, the structure of the approximate Riemann solution is as follows: The left state \( \tilde{q}_0 := q_L \) is connected to the intermediate state \( \tilde{q}_{1/3} \) by a simple wave. The intermediate states \( \tilde{q}_{1/3} \) and \( \tilde{q}_{2/3} \) are connected by a contact discontinuity. A simple wave connects the intermediate state \( \tilde{q}_{2/3} \) to the right state \( \tilde{q}_1 := q_R \). The approximate solution of the two-fluid Riemann problem is illustrated in Figure 6.6.

![Figure 6.6: Approximate solution of the two-fluid Riemann problem. Simple waves (shaded) connect the left and right states to the intermediate states. The intermediate states are connected by a contact discontinuity (dashed).](image)

It is important to note that \( \theta \) is a \( k \)-Riemann invariant for \( k = 1, 5 \). Hence, a fluid transition can only occur across the contact discontinuity. Moreover, both \( v_1 \) and \( p \) are \( k \)-Riemann invariants for \( k = 2, 3, 4 \). Hence, \( v_1 \) and \( p \) are continuous across any contact discontinuity. In particular, \( v_1 \) and \( p \) are continuous across the interface. This implies that the kinematic and dynamic interface conditions are fulfilled. The interface between the fluids can therefore be construed as a particular instance of a contact discontinuity.
From (6.40) and (6.78) it follows that

\[
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_{1/3}
=
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_0
, \\
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_{2/3}
=
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_1
, \\
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_{1/3}
=
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_0
, \\
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_{2/3}
=
\begin{pmatrix}
\tilde{v}_2 \\
\tilde{v}_3
\end{pmatrix}
_1
,
\]

(6.79a)

and

\[
(\tilde{v}_1)_{1/3} = (\tilde{v}_1)_{2/3} := (\tilde{v}_1)_{1/2}.
\]

(6.79b)

Furthermore, \((\tilde{v}_1)_{1/2}, \tilde{p}_{1/3}\) and \(\tilde{p}_{2/3}\) are determined by

\[
(\tilde{v}_1)_{1/2} + \int_{\tilde{p}_0}^{\tilde{p}_{1/3}} \frac{c_\rho(\rho, \theta_0)}{\rho} \, d\rho = (\tilde{v}_1)_{0},
\]

(6.80a)

\[
(\tilde{v}_1)_{1/2} - \int_{\tilde{p}_1}^{\tilde{p}_{2/3}} \frac{c_\rho(\rho, \theta_1)}{\rho} \, d\rho = (\tilde{v}_1)_{1},
\]

(6.80b)

\[
p(\tilde{p}_{1/3}, \tilde{\theta}_0) = p(\tilde{p}_{2/3}, \tilde{\theta}_1).
\]

(6.80c)

In some cases the intermediate states can be explicitly extracted from (6.79)–(6.80). Otherwise, the intermediate states have to be determined by numerical approximation.

If the equation of state is of the form \(\rho := \rho(p, \theta)\), or can be cast in this form, then the conditions for the intermediate states can be formulated in a convenient form. To derive this formulation, we note that

\[
\int_{\rho_a}^{\rho_b} \frac{c_\rho(\rho, \theta)}{\rho} \, d\rho = \int_{\rho_a}^{\rho_b} \frac{1}{\rho} \sqrt{\frac{\partial \rho(p, \theta)}{\partial \rho}} \, d\rho = \int_{\rho_a}^{\rho_b} \frac{1}{\rho(p, \theta)} \sqrt{\frac{\partial \rho(p, \theta)}{\partial p}} \, d\rho,
\]

(6.81)

for certain integration limits \(\rho_a\) and \(\rho_b\). The latter identity follows from the transformation \(\rho := \rho(p, \theta)\). From (6.80)–(6.81) it follows that

\[
\int_{\tilde{p}_0}^{\tilde{p}_{1/2}} \frac{1}{\rho(p, \tilde{\theta}_0)} \sqrt{\frac{\partial \rho(p, \tilde{\theta}_0)}{\partial p}} \, d\rho + \int_{\tilde{p}_1}^{\tilde{p}_{1/2}} \frac{1}{\rho(p, \tilde{\theta}_1)} \sqrt{\frac{\partial \rho(p, \tilde{\theta}_1)}{\partial p}} \, d\rho = (\tilde{v}_1)_{0} - (\tilde{v}_1)_{1},
\]

(6.82)

with \(\tilde{p}_{1/2} := p(\tilde{p}_{1/3}, \tilde{\theta}_0) = p(\tilde{p}_{2/3}, \tilde{\theta}_1)\), conform (6.80c). Equation (6.82) presents a concise condition for the intermediate pressure \(\tilde{p}_{1/2}\).

It is important to note that (6.82) is well suited to treatment by numerical approximation techniques. In particular, the derivatives of the integrals with respect to \(\tilde{p}_{1/2}\), which are required in Newton’s method, are simply the integrands evaluated at \(\tilde{p}_{1/2}\). Moreover, for a given approximation to \(\tilde{p}_{1/2}\), the integrals can be evaluated by a numerical integration method (see, e.g., Ref. [30]).

Once the intermediate pressure has been extracted from (6.82), the intermediate densities follow immediately from the equation of state:

\[
\tilde{\rho}_{1/3} = \rho(\tilde{p}_{1/2}, \theta_0), \quad \text{and} \quad \tilde{\rho}_{2/3} = \rho(\tilde{p}_{1/2}, \tilde{\theta}_1).
\]

(6.83)

We recall that once the intermediate states have been obtained, the modified Osher flux follows by straightforward computation; cf. Section 6.3.3.
6.5 Pressure Oscillations

A common objection to interface capturing is the occurrence of pressure oscillations. These pressure oscillations expose the loss of certain invariance properties of the continuum problem under discretization. Below, we exemplify the pressure oscillations and we derive a pressure-invariance condition for discrete approximations to two-fluid flow problems. Furthermore, we construct a non-oscillatory conservative discretization for two-fluid flow problems.

6.5.1 Exemplification

To exemplify the pressure oscillations that are generally incurred by conservative discretizations of two-fluid flow problems, we consider

\[ \dot{q} + (f(q))_x = 0, \quad x \in \mathcal{L}, t \geq 0, \quad (6.84a) \]

with

\[ q = \begin{pmatrix} \rho v \\ \rho \alpha \\ \rho \end{pmatrix}, \quad \text{and} \quad f(q) = \begin{pmatrix} \rho v^2 + p(\rho, \alpha) \\ \rho \alpha v \\ \rho v \end{pmatrix}, \quad (6.84b) \]

or its weak formulation, if appropriate. The considered equation of state is

\[ \rho(p, \alpha) = \alpha \rho_p(p) + (1 - \alpha) \rho_s(p), \quad (6.84c) \]

with \( \rho_p(p) \) and \( \rho_s(p) \) given equations of state for the primary and secondary fluid. The interval \( \mathcal{L} \) is subdivided into intervals \((x_j, x_{j+1})\) and (6.84) is supplemented with the initial conditions

\[ q(x, 0) = q_0, \quad v(x, 0) = V, \quad \alpha(x, 0) = \alpha_0, \quad \text{if } x \in (x_j, x_{j+1}), \quad (6.85a) \]

with \( V \) an arbitrary positive constant and \( q_0 \) and \( \alpha_0 \) constants such that

\[ \rho_j^0 = \alpha_j^0 \rho_p(P) + (1 - \alpha_j^0) \rho_s(P), \quad (6.85b) \]

for some constant \( P \). The equations (6.84)–(6.85) represent a two-fluid flow in which the velocity \( v \) is uniform and in which the density \( \rho \) and the (primary) volume fraction \( \alpha \) are such that the pressure \( p \) is uniform as well.

The solution to (6.84)–(6.85) reads

\[ q(x, t) = q(x - Vt, 0). \quad (6.86) \]

The pressure \( p(x, t) \) corresponding to (6.86) follows from the equation of state:

\[ \rho(x, t) = \alpha(x, t) \rho_p(p(x, t)) + (1 - \alpha(x, t)) \rho_s(p(x, t)). \quad (6.87) \]

By (6.86)–(6.87),

\[ p(x - Vt, 0) = \alpha(x - Vt, 0) \rho_p(p(x, t)) + (1 - \alpha(x - Vt, 0)) \rho_s(p(x, t)), \quad (6.88) \]
6.5. Pressure Oscillations

which implies that \( p(x, t) = P \), i.e., the pressure is invariant. Hence, if the initial velocity and pressure are uniform, then the pressure is invariant under (6.84).

Next, we consider the first order forward Euler discretization to (6.84)–(6.85) on the grid \( \{(x_j, t_n) : j \in \mathbb{Z}, n \in \mathbb{N}\} \), with \( t_0 = 0 \) and \( t_n < t_{n+1} \):

\[
\frac{q_j^{n+1} - q_j^n}{t_{n+1} - t_n} + \frac{f(q_j^n, q_j^{n+1}) - f(q_j^n, q_j^n)}{x_{j+1} - x_j} = 0, \quad n = 0, 1, \ldots, \quad (6.89)
\]

with the initial conditions

\[
q_j^0 = \begin{pmatrix} \rho_j^0V \\ \rho_j^0 \alpha_j^0 \\ \rho_j^0 \end{pmatrix}, \quad (6.90)
\]

conform (6.85). The grid function \( q_j^n \) approximates \( q(x, t_n) \) according to (6.86) in the interval \( (x_j, x_{j+1}) \).

The states \( q_j^0 \) and \( q_j^0 \) are connected by a contact discontinuity with velocity \( V \). The corresponding Godunov flux reads:

\[
f(q_j^0, q_{j+1}^0) = V \begin{pmatrix} \rho_j^0V \\ \rho_j^0 \alpha_j^0 \\ \rho_j^0 \end{pmatrix} + \begin{pmatrix} P \\ 0 \\ 0 \end{pmatrix}. \quad (6.91)
\]

Because shocks are absent, expression (6.91) is also valid for the (modified) Osher scheme. By (6.89),

\[
q_j^1 = q_j^0 - C(q_j^0 - q_{j-1}^0), \quad (6.92a)
\]

with

\[
C := V(t_1 - t_0)/(x_{j+1} - x_j), \quad (6.92b)
\]

the CFL-number. In conjunction with (6.85b), equation (6.92) implies

\[
\rho_j^1 = \alpha_j^* \rho_j(P) + (1 - \alpha_j^*) \rho_s(P), \quad (6.93a)
\]

with

\[
\alpha_j^* := \alpha_j^0 - C(\alpha_j^0 - \alpha_{j-1}^0). \quad (6.93b)
\]

From (6.93) it follows that a necessary and sufficient condition for pressure invariance of the discrete approximation is \( \alpha_j^1 = \alpha_j^0 \). However, from (6.85b) and (6.92) we obtain

\[
\alpha_j^1 = \frac{((1 - C)(\alpha_j^0)^2 + C(\alpha_j^{0-1})^2) \rho_p + ((1 - C)\alpha_j^0(1 - \alpha_j^0) + C\alpha_j^{0-1}(1 - \alpha_j^{0-1})) \rho_s}{(\alpha_j^0 - C(\alpha_j^0 - \alpha_{j-1}^0)) \rho_p + (1 - (\alpha_j^0 - C(\alpha_j^0 - \alpha_{j-1}^0))) \rho_s}, \quad (6.94)
\]

with \( \rho_{p/s} := \rho_{p/s}(P) \). In general, \( \alpha_j^1 \neq \alpha_j^* \) and, hence, the discrete approximation (6.89) lacks the pressure-invariance property of the continuum equations (6.84).
Specific exceptions are:

\[ C = 0 \quad \Rightarrow \quad q_j^1 = q_j^0 , \quad (6.95a) \]

\[ C = 1 \quad \Rightarrow \quad q_j^1 = q_{j-1}^0 , \quad (6.95b) \]

\[ \alpha_j^0 = \alpha_{j-1}^0 \quad \Rightarrow \quad q_j^0 = q_{j-1}^0 , \quad (6.95c) \]

\[ \rho_p = \rho_s \quad \Rightarrow \quad \text{single-fluid flow}. \quad (6.95d) \]

Exceptions (6.95a)-(6.95c) are trivial. Exception (6.95d) confirms that the loss of the pressure-invariance property is special for two-fluid flows.

It is noteworthy that if \((\rho \alpha)_t + (\rho \alpha v)_x = 0\) in (6.84) is replaced by

\[ \alpha_t + v \alpha_x = 0 , \quad x \in \mathcal{L}, t \geq 0 , \quad (6.96) \]

then, subject to the initial conditions (6.90), the first order forward Euler discretization yields

\[ \alpha_j^1 = \alpha_j^0 - C(\alpha_j^0 - \alpha_{j-1}^0) . \quad (6.97) \]

Hence, \(\alpha_j^1 = \alpha_j^*\), and pressure invariance is ensured. Equation (6.96) is, however, in non-conservation form. Hence, the pressure invariance is in this case accomplished at the expense of the conservation form of the equations.

### 6.5.2 Pressure-Invariance Condition

The implications of the above exemplification are restricted: The analysis does not imply that pressure oscillations are inherent to conservative discretizations of two-fluid flow problems. It merely implies that discrete approximations to two-fluid flow problems do not obviously inherit the pressure-invariance property of the continuum equations.

To avoid pressure oscillations, it is necessary that a discrete approximation to a two-fluid flow problem complies with a **pressure-invariance condition**. We formulate this pressure-invariance condition for discrete approximations of

\[ q_t + (f(q))_x = 0 , \quad x \in \mathcal{L}, t \geq 0 , \quad (6.98a) \]

or its weak formulation, with

\[ q = \begin{pmatrix} \rho v \\ \rho \beta(\theta) \end{pmatrix} , \quad f(q) = \begin{pmatrix} \rho v^2 + p(\rho, \theta) \\ \rho \beta(\theta) v \end{pmatrix} , \quad (6.98b) \]

and \(p(\rho, \theta)\) a given equation of state, e.g., \(p(\rho, \theta) = p(\rho, \alpha(\theta))\), with \(p(\rho, \alpha)\) implicitly defined by (6.84c). We do not yet attach a specific connotation to \(\beta\). For instance, \(\beta\) can be a continuously differentiable strictly monotone function, so that \(\theta_t + v \theta_x = 0\) is implied (cf. §6.4.1), or \(\beta\) can be the primary volume fraction \(\alpha\). Note that \(\theta\) only serves as an intermediary between \(\beta\) and \(p\). Hence, it is not necessary that \(\theta\) appears explicitly in the formulation.
6.5. Pressure Oscillations

We consider a discretization of (6.98) on a grid \( \{(x_j, t_n) : j \in \mathbb{Z}, n \in \mathbb{N}\} \), with \( t_0 = 0 \) and \( t_n < t_{n+1} \). We denote by \( q^n_j \) the discrete approximation to \( q(x, t_n) \) in the interval \( (x_j, x_{j+1}) \). In particular, \( q^n_j \) is the discrete representation of the initial data. The discretization of (6.98) is characterized by the mapping \( \{q^n_j\} \mapsto \{q^{n+1}_j\} \). The pressure-invariance condition for discretizations \( \{q^n_j\} \mapsto \{q^{n+1}_j\} \) of (6.98) reads: If \( v^n_j = V \), with \( V \) a constant, and \( \rho^n_j \) and \( \theta^n_j \) satisfy

\[
p(\rho^n_j, \theta^n_j) = P ,
\]
for some constant \( P \), then

\[
p(\rho^{n+1}_j, \theta^{n+1}_j) = P .
\]

For instance, for an equation of state of the form (6.84c), the pressure-invariance condition reads: If \( v^n_j = V \) and

\[
\rho^n_j = \alpha(\theta^n_j) \rho_p(P) + (1 - \alpha(\theta^n_j)) \rho_s(P) ,
\]
then

\[
\rho^{n+1}_j = \alpha(\theta^{n+1}_j) \rho_p(P) + (1 - \alpha(\theta^{n+1}_j)) \rho_s(P) .
\]

6.5.3 A Non-Oscillatory Conservative Scheme

In order to construct a pressure-invariant conservative discretization for two-fluid flow problems, we define the partial primary and secondary densities by:

\[
\rho'_p := \alpha \rho_p , \quad \text{and} \quad \rho'_s := (1 - \alpha) \rho_s ,
\]
where \( \alpha \) is the primary volume fraction conform (6.59) and \( \rho_p \) and \( \rho_s \) denote the densities of the primary and secondary fluids, respectively. In one spatial dimension, conservation of mass of the primary and secondary fluid stipulates

\[
\rho'_p + (\rho'_p v)_x = 0 , \quad (6.102a)
\]
\[
\rho'_s + (\rho'_s v)_x = 0 . \quad (6.102b)
\]

The compound density is defined by

\[
\rho := \alpha \rho_p + (1 - \alpha) \rho_s = \rho'_p + \rho'_s . \quad (6.103)
\]
Hence, if \( \beta \) denotes the primary mass fraction,

\[
\beta := \rho'_p / \rho , \quad (6.104)
\]
then conservation of mass, of both the primary and the secondary fluid, and of momentum is described by:

\[
q_t + (f(q))_x = 0 , \quad x \in \mathcal{L}, t \geq 0 , \quad (6.105a)
\]
or its weak formulation, with

\[
\mathbf{q} = \begin{pmatrix} \rho v \\ \rho \beta \\ \rho \end{pmatrix}, \quad \mathbf{f}(\mathbf{q}) = \begin{pmatrix} \rho v^2 + p(\rho, \beta) \\ \rho \beta v \\ \rho v \end{pmatrix}.
\] (6.105b)

Moreover, if the primary and secondary fluid are equipped with barotropic equations of state \( \rho_p(p) \) and the compound equation of state is given by (6.84c), then \( p(\rho, \beta) \) in (6.105) is implicitly given by

\[
\rho \beta = \alpha \rho_p(p),
\] (6.106a)

\[
\rho - \rho \beta = (1 - \alpha) \rho_s(p),
\] (6.106b)

or, by eliminating \( \alpha \),

\[
\frac{1}{\rho} = \frac{1 - \beta}{\rho_s(p)} + \frac{\beta}{\rho_p(p)}.
\] (6.107)

We consider the first order forward Euler discretization of (6.105)–(6.106) on a grid \( \{ (x_j, t_n) : j \in \mathbb{Z}, n \in \mathbb{N} \} \), conform (6.89). If \( v_j^n = V \), with \( V \) a positive constant, and

\[
(\rho \beta)_j^n = \alpha_j^n \rho_p(P),
\] (6.108a)

\[
\rho_j^n - (\rho \beta)_j^n = (1 - \alpha_j^n) \rho_s(P),
\] (6.108b)

for all \( j \in \mathbb{Z} \), with \( P \) a constant, then the first order forward Euler discretization yields

\[
\rho_j^{n+1} = \rho_j^n - C(\rho_j^n - \rho_j^{n-1}),
\] (6.109a)

\[
(\rho \beta)_j^{n+1} = (\rho \beta)_j^n - C((\rho \beta)_j^n - (\rho \beta)_j^{n-1}),
\] (6.109b)

with \( C \) according to (6.92b). From (6.108)–(6.109) it follows that

\[
(\rho \beta)_j^{n+1} = \alpha_j^{n+1} \rho_p(P),
\] (6.110a)

\[
\rho_j^{n+1} - (\rho \beta)_j^{n+1} = (1 - \alpha_j^{n+1}) \rho_s(P),
\] (6.110b)

with

\[
\alpha_j^{n+1} := \alpha_j^n - C(\alpha_j^n - \alpha_j^{n-1}).
\] (6.110c)

Therefore, \( p(\rho_j^{n+1}, \beta_j^{n+1}) = P \), which implies that the first order forward Euler discretization of (6.105)–(6.106) complies with the pressure-invariance condition.