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Chapter 6

Reconstructing a random scenery in polynomial time

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Abstract

Let \( \xi \) be an i.i.d. uniformly colored scenery on \( \mathbb{Z} \). Let \( S \) be a recurrent random walk with bounded jumps on \( \mathbb{Z} \), independent of \( \xi \). Assume that \( S \) can reach every integer with positive probability. We prove that almost all sceneries can be reconstructed with high probability in polynomial time if the number of colors exceeds the number of possible single steps for the random walk.

1 Introduction and Result

In this article, we consider the scenery reconstruction problem under the assumption there are no errors in the observations. We make the same assumptions on scenery and random walk as in [3]: The scenery on \( \mathbb{Z} \) is i.i.d. uniformly colored with finitely many colors \( 1, \ldots, C \). The random walk is independent of the scenery, has i.i.d. increments with finite support, is recurrent, and can reach every integer with positive probability. We require that there is at least one color more than possible jumps for the random walk. The maximal jump lengths to the left and to the right of the random walk are assumed to be equal.

Löwe, Matzinger, and Merkl [2] proved that under these assumptions almost all sceneries can be almost surely reconstructed up to reflection and translation. Our main theorem refines their reconstruction result: We prove that for \( n \) large a finite piece of scenery of length \( l(n) = 10 \cdot 2^n + 1 \) around the origin can be reconstructed with high probability from the first \( 2n^7 + 2 \cdot 2^{12an} \) observations with a constant \( a > 0 \); thus the number of observations needed is polynomial in \( l(n) \).

In the following, we assume \( n_0 \in \mathbb{N}_0 \) and we set

\[
\begin{align*}
  n_1 &:= 2\lfloor \sqrt{n_0} \rfloor, \\
  n_2 &:= 2\lceil \sqrt{n_0} \rceil.
\end{align*}
\]

Formally, our result reads as follows:
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Theorem 1.1. For infinitely many \( n_0 \in \mathbb{N} \) there exists a measurable map \( A_{\text{initial}}^{n_0} : \mathbb{C}[0,n_0^{20} + n_2^7 + 2 \cdot 2^{12a n_2}] \rightarrow \mathbb{C}[{-5 \cdot 2^{n_2}, 5 \cdot 2^{n_2}}] \) such that

\[
\lim_{n_0 \to \infty} P([E_{\text{initial works}}^{n_0}]^c) = 0,
\]

where \( E_{\text{initial works}}^{n_0} := \)

\[
\{\xi|[-2^{n_2-1}, 2^{n_2-1}] \preceq A_{\text{initial}}^{n_0} (\chi|[0,n_0^{20} + n_2^7 + 2 \cdot 2^{12a n_2}]) \preceq \xi|[-10 \cdot 2^{n_2}, 10 \cdot 2^{n_2}]\}.
\]

The remainder of this article is organized as follows: In Section 2, we review a result from [2]. Section 3 contains the definition of the reconstruction algorithm \( A_{\text{initial}}^{n_0} \) and the proof of Theorem 1.1.

2 Review of a result of Löwe/Matzinger/Merkl

Unless otherwise stated, we use the same notation as in [3]. We work on the probability space

\[
\Omega := (\mathbb{C}^Z, \Omega_2), \quad P := \nu^\otimes \mathbb{Z} \otimes Q_0
\]

with the product \( \sigma \)-algebra generated by the canonical projections; here \( \nu \) denotes the uniform distribution on the set of colors \( \mathbb{C} = \{1, 2, \ldots, C\} \) and \( Q_0 \) the distribution of the random walk starting at the origin. We define the shift \( \Theta : \Omega \rightarrow \Omega \) by

\[
(\xi, S) \mapsto (\xi(\cdot + S(1)), S(\cdot + 1) - S(1)).
\]

Löwe, Matzinger, and Merkl [2] showed the existence of measurable maps \( A_{n_2} \) which do "partial reconstructions"; we will specify in which sense. We define the event that \( A_{n_2} \) reconstructs correctly a piece of scenery around the origin:

\[
E_{\text{recon}}^{n_0} := \{\xi|[-2^{n_2}, 2^{n_2}] \preceq A_{n_2} (\chi|[0, 2 \cdot 2^{12a n_2}]) \preceq \xi|[-9 \cdot 2^{n_2}, 9 \cdot 2^{n_2}]\}.
\]

For \( n \in \mathbb{N} \), we denote by \( E_B(n) \) the event that the first \( n + 1 \) observations are all equal to 1:

\[
E_B(n) := \{\chi_k = 1 \text{ for all } k \in [0,n]\}.
\]

We define the event that there is "a long block of ones close to the origin":

\[
\text{BigBlock} := \begin{cases} \text{There exists an integer interval } J_0 \subseteq [-2L n_0^{20}, 2L n_0^{20}] \text{ with } |J_0| \geq n_0^4 \\ \text{such that } \xi|J_0 = (1)_{J_0} \end{cases}.
\]

We denote by \( P_B \) the image of the conditional distribution \( P(\cdot | E_B(n_0^{20})) \) under the shift \( \Theta^{n_0^{20}} \), and we abbreviate \( \tilde{P} := P_B(\cdot | \text{BigBlock}) \). We set

\[
\varepsilon_1(n_0) := P_B ([E_{\text{recon}}^{n_0}]^c), \quad (2.1)
\]

and observe

\[
P_B ([E_{\text{recon}}^{n_0}]^c) \leq P_B ([E_{\text{recon}}^{n_0}]^c \cap \text{BigBlock}) + P_B ([\text{BigBlock}]^c) \\
\leq \tilde{P} ([E_{\text{recon}}^{n_0}]^c) + P_B ([\text{BigBlock}]^c). \quad (2.2)
\]
In [2], Löwe, Matzinger, and Merkl prove \( \lim_{n_0 \to \infty} \tilde{P} (|E^{n_0}_{\text{recon}}|^2) = 0 \). The second term on the right-hand side of (2.2) converges to 0 as \( n_0 \to \infty \) by Lemma 3.3 in [2]. Hence

\[
\lim_{n_0 \to \infty} \varepsilon_1 (n_0) = 0. \tag{2.3}
\]

Intuitively, this means the following: If the input for \( A_{n_2} \) begins with a long block of 1's, then \( A_{n_2} \) reconstructs with high probability a large piece of scenery around the origin.

3 Proof of Theorem 1.1

3.1 The algorithm \( A_{\text{initial}}^{n_0} \)

For \( k \in \mathbb{N}_0 \), let \( Z_k \) be the Bernoulli random variable taking values in \( \{0,1\} \) which is equal to one if and only if \( \chi (kn_2^6 + j) = 1 \) for all \( j \in [0,n_0^{-2}] \). We set

\[
\bar{k} := \min \{ k \geq 0 : Z_k = 1 \}. \tag{3.1}
\]

**Definition 3.1.** We define \( A_{\text{initial}}^{n_0} : C^{[0,n_0^{-2}+2 \cdot 2^{12n_2}] \to C^{[-5 \cdot 2^{n_2},5 \cdot 2^{n_2}]}} \) as follows: If \( \bar{k} \leq n_2 \), then we define

\[
A_{\text{initial}}^{n_0} (\chi | [0,n_0^{-2}+n_2^7+2 \cdot 2^{12n_2}]) := A_{n_2} (\chi | [\bar{k}n_2^6 + n_0^{-2},\bar{k}n_2^6 + n_0^{-2} + 2 \cdot 2^{12n_2}]).
\]

Otherwise we define \( A_{\text{initial}}^{n_0} (\chi | [0,n_0^{-2}+n_2^7+2 \cdot 2^{12n_2}]) := (1 | [-5 \cdot 2^{n_2},5 \cdot 2^{n_2}]) \), i.e. the output equals the piece of scenery which is constantly equal to 1.

For \( k \in \mathbb{N} \) we define the events

\[
E_{\text{no block}}^{n_0,k} := \left\{ \sum_{i=0}^{k-1} Z_i = 0 \right\},
\]

\[
E_{\text{rw apart}}^{n_0,k} := \left\{ |S (kn_2^5) - S (\bar{k}n_2^5)| > 2Ln_0^{-2} \text{ for all } i < k \right\}.
\]

Note that \( E_{\text{no block}}^{n_0,k} = \{ \bar{k} \geq k \} \).

**Lemma 3.1.** For \( n \in \mathbb{N}_0 \), let \( \mathcal{F}(n) \) be the \( \sigma \)-algebra generated by the whole scenery and the random walk up to time \( n \): \( \mathcal{F}(n) := \sigma (\xi, S(i); i \in [0,n]) \). There exists a constant \( c_3 > 0 \) such that for all \( n_0 \in \mathbb{N} \) and all \( k \leq n_2 \)

\[
P \left( \left| E_{\text{rw apart}}^{n_0,k} \right| \mathcal{F} (n_0^{-2} (k - 1) + n_0^{-2}) \right) \leq c_3 n_0^{-2} n_2^{-2}.
\]

**Proof.** By the local central limit theorem (see e.g. [1], Theorem (5.2), page 132), the probability that the random walk after \( n_0^{-2} - n_0^{-2} \) steps hits a set containing at most \( (4Ln_0^{-2} + 1) \cdot n_2 \) points is bounded by \( c_4 n_0^{-2} \cdot (4Ln_0^{-2} \cdot n_2) = c_3 n_0^{-2} n_2^{-2} \), here \( c_4 > 0 \) is a constant independent of \( n_0 \) and \( c_3 = 4Lc_4 \). The claim follows. \( \square \)
3.2 Some properties of $\bar{k}$

For $k \in \mathbb{N}_0$, let $\bar{F}(k)$ denote the tail probability:

$$\bar{F}(k) := P(\bar{k} > k).$$

We set

$$\varepsilon_2(n_0) := \sqrt{\max\left(\frac{1}{n_0}, \varepsilon_1(n_0)\right)}, \quad (3.3)$$

$$k_{n_0} := \min\{k : \bar{F}(k) \leq \varepsilon_2(n_0)\}. \quad (3.4)$$

**Lemma 3.2.** There exists $c_5$ such that $k_{n_0} \leq n_2$ for all $n_0 \geq c_5$.

**Proof.** For $k \in \mathbb{N}_0$, we define the Bernoulli random variable $\tilde{Z}_k$ which is equal to 1 if and only if $Z_k = 1$ or $E_{\text{rw}_o}^{n_0,k}$ does not hold. We have that

$$\left\{ \sum_{k=0}^{n_2} \tilde{Z}_k > 0 \right\} \cap \bigcap_{k=0}^{n_2} E_{\text{rw}_o}^{n_0,k} \subseteq \{ \bar{k} \leq n_2 \};$$

thus

$$\bar{F}(n_2) = P(\bar{k} > n_2) \leq \sum_{k=0}^{n_2} P\left(\left[E_{\text{rw}_o}^{n_0,k}\right]^c\right) + P\left(\left\{ \sum_{k=0}^{n_2} \tilde{Z}_k = 0 \right\}\right). \quad (3.5)$$

By Lemma 3.1,

$$\sum_{k=0}^{n_2} P\left(\left[E_{\text{rw}_o}^{n_0,k}\right]^c\right) \leq (n_2 + 1)c_3n_0^{20}n_2^{-2}. \quad (3.6)$$

We set $\mathcal{G}_k := \sigma(\tilde{Z}_i, S(n_0^2) - S(jn_0^2), \xi(S(n_0^2) + z); 0 \leq i < j \leq k + 1, z \in [-L_{n_0^2}, L_{n_0^2}])$. The sequence $(\tilde{Z}_k; k \geq 0)$ is adapted to the filtration $(\mathcal{G}_k; k \geq 0)$. Furthermore, by the definition of $\tilde{Z}_k$,

$$P(\tilde{Z}_k = 1|\mathcal{G}_{k-1}) = \frac{1}{E_{\text{rw}_o}^{n_0,k}} P(Z_k = 1|\mathcal{G}_{k-1}) + 1 \left[E_{\text{rw}_o}^{n_0,k}\right]^c \geq 1 \frac{1}{E_{\text{rw}_o}^{n_0,k}} P\left(E_{\mathcal{G}_{n_0^2}}^{n_0^2}\right);$$

here $1_A$ denotes the indicator function of the set $A$. For the last inequality we used that $Z_k$ is measurable with respect to $\sigma(\xi(S(kn_0^2) + z); z \in [-L_{n_0^2}, L_{n_0^2}])$. We abbreviate $p_{n_0} := P\left(E_{\mathcal{G}_{n_0^2}}^{n_0^2}\right)$. The preceding estimate yields

$$P\left(\sum_{k=0}^{n_2} \tilde{Z}_k = 0 \right) = P\left(\left\{ \sum_{k=0}^{n_2} \tilde{Z}_k = 0 \right\} \cap \bigcap_{k=0}^{n_2} E_{\text{rw}_o}^{n_0,k} \right)$$

$$= E\left[ 1 \left\{ \sum_{k=0}^{n_2-1} \tilde{Z}_k = 0 \right\} \cap \bigcap_{k=0}^{n_2} E_{\text{rw}_o}^{n_0,k} \right] P\left( Z_{n_2} = 0|\mathcal{G}_{n_2-1} \right)$$

$$\leq (1 - p_{n_0}) P\left(\left\{ \sum_{k=0}^{n_2-1} \tilde{Z}_k = 0 \right\} \cap \bigcap_{k=0}^{n_2-1} E_{\text{rw}_o}^{n_0,k} \right);$$
here $E$ denotes the expectation with respect to $P$. Using an induction argument, we conclude

$$P \left( \sum_{k=0}^{n_0^2} Z_k = 0 \right) \leq (1 - p_{n_0})^{n_0^2}. \quad (3.7)$$

In order to obtain a lower bound for $p_{n_0}$, we first note that $P (\xi(z) = 1 \forall z \in [-n_0^{11}, n_0^{11}]) = (1/C)^{2n_0^{11}+1}$. Furthermore, by Doob's inequality (see e.g. [1], page 250),

$$P \left( \max_{i \in [-n_0^{11}, n_0^{11}]} |S(i)| > n_0^{11} \right) \leq n_0^{-22} \text{Var}(S(n_0^{20})) = c_6 n_0^{-2}$$

with some constant $c_6 > 0$. Thus, we obtain for all $n_0$ sufficiently large

$$p_{n_0} = P (E_B(n_0^{20})) \geq C^{-2n_0^{11}-1} (1 - c_6 n_0^{-2}) \geq C^{-2n_0^{11}-2}. \quad (3.8)$$

It follows from (3.7) and (3.8) that

$$P \left( \sum_{k=0}^{n_0^2} Z_k = 0 \right) \leq \left( 1 - C^{-2n_0^{11}-2} \right)^{n_0^2} \leq \exp \left( -n_2 C^{-2n_0^{11}-2} \right).$$

Using the definition of $n_2$ (1.1), we see that the right-hand side is bounded above by $2^{-1} n_0^{-1/2}$ for all $n_0$ sufficiently large. Combining this with (3.5) and (3.6) yields for all $n_0$ sufficiently large

$$\bar{F}(n_2) \leq (n_2 + 1)c_3 n_0^{20} n_2^{-2} + 2^{-1} n_0^{-1/2} \leq n_0^{-1/2} \leq \varepsilon_2(n_0);$$

for the last inequality we used the definition of $\varepsilon_2(n_0)$ (3.3). The claim follows from the definition of $k_{n_0}$. \hfill \Box

We define

$$E_{\text{tw apart}}^{n_0} := \{ |S(i n_2^k) - S(k n_2^k)| > 2 Ln_0^{20} \text{ for all } i < k \},$$

$$E_{\text{ok}}^{n_0} := \{ k \leq k_{n_0} \} \cap E_{\text{tw apart}}^{n_0}.$$  

**Lemma 3.3.** The following holds:

$$\lim_{n_0 \to \infty} P (|E_{\text{ok}}^{n_0}|^c) = 0.$$  

**Proof.** We observe that

$$P (|E_{\text{ok}}^{n_0}|^c) = P (k > k_{n_0}) + P \left( \{ \bar{k} \leq k_{n_0} \} \cap [E_{\text{tw apart}}^{n_0}]^c \right).$$

By the definition of $\bar{F}$ and $k_{n_0}$, $P (k > k_{n_0}) = \bar{F}(k_{n_0}) \leq \varepsilon_2(n_0)$, which converges to 0 as $n_0 \to \infty$; recall the definition of $\varepsilon_2(n_0)$ and (2.3). Using Lemmas 3.1 and 3.2, we obtain for all $n_0$ sufficiently large

$$P \left( \{ \bar{k} \leq k_{n_0} \} \cap [E_{\text{tw apart}}^{n_0}]^c \right) \leq \sum_{i=0}^{k_{n_0}} P \left( [E_{\text{tw apart}}^{n_0}]^c \right) \leq c_3 n_0^{20} n_2^{-2} (n_2 + 1)$$

which converges to 0 as $n_0 \to \infty$ by the definition of $n_2$ (1.1). The claim follows. \hfill \Box
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Proof of Theorem 1.1. The following holds:

\[ P \left( [E^0_{\text{in works}}]^c \right) \leq P \left( [E^0_{\text{in works}}]^c \cap E^0_{ok} \right) + P \left( [E^0_{\text{in works}}]^c \right). \]

By Lemma 3.3, the second term on the right-hand side converges to 0 as \( n_0 \to \infty \). It suffices to prove that the same is true for the first term. We observe that

\[
P \left( [E^0_{\text{in works}}]^c \cap E^0_{ok} \right) = \sum_{k=0}^{k_{n_0}} P \left( [E^0_{\text{in works}}]^c \cap E^0_{ok} \mid \bar{k} = k \right) P \left( \bar{k} = k \right) \quad (3.9)
\]

By the definition of the shift \( \Theta \), we have for \( m \geq 0 \)

\[ \Theta^m(\xi, S) = (\xi(\cdot + S(m)), S(\cdot + m) - S(m)). \]

Consequently, we obtain for \( n_0 \) sufficiently large and \( k \leq k_{n_0} \)

\[ \Theta^{-kn_2^5-n_0^20} \left( E^0_{\text{recon}} \right) \cap \{ \bar{k} = k \} \subseteq E^0_{\text{in works}} \cap \{ \bar{k} = k \}; \]

here we used that if \( \bar{k} = k \), then the first observation which \( \mathcal{A}^0_{\text{initial}} \) uses is \( \chi(kn_2^5 + n_0^2) \) and for all \( n_0 \) sufficiently large,

\[ |S(kn_2^5+n_0^2)| \leq (kn_2^5+n_0^2) \leq (kn_2^5+n_0^2) L \leq (n_2^7+n_0^2) L \leq 2^{n_2^2-1} \]

because the random walker can jump in each step at most a distance of \( L \). Hence the reconstruction starts not at the origin, but at position \( S(kn_2^5+n_0^2) \) which has a distance \( \leq 2^{n_2^2-1} \) from the origin. This is why different pieces of scenery of \( \xi \) are concerned in the definitions of \( E^0_{\text{recon}} \) and \( E^0_{\text{in works}} \). Thus (3.9) yields

\[
P \left( [E^0_{\text{in works}}]^c \cap E^0_{ok} \right) \leq \sum_{k=0}^{k_{n_0}} P \left( \left[ \Theta^{-kn_2^5-n_0^20} \left( E^0_{\text{recon}} \right) \right]^c \cap E^0_{ok} \mid \bar{k} = k \right) P \left( \bar{k} = k \right)
\]

\[
\leq \sum_{k=0}^{k_{n_0}} P \left( \left[ \Theta^{-kn_2^5-n_0^20} \left( E^0_{\text{recon}} \right) \right]^c \mid \{ \bar{k} = k \} \cap E^0_{ok} \right) P \left( \bar{k} = k \right). \quad (3.10)
\]

Note that \( E^0_{\text{no block}} \cap \{ \bar{k} \geq k \} \) and \( E^0_{\text{no block}} = E^0_{\text{in works}} \cap \{ \bar{k} = k \} \). Consequently, for \( k \leq k_{n_0} \),

\[ \{ \bar{k} = k \} \cap E^0_{ok} = \Theta^{-kn_2^5}(E_B(n_0)) \cap E^0_{\text{no block}} \cap E^0_{\text{in works}}. \quad (3.11)\]

By the Markov property, we know that \( (S(t+kn_2^5)-S(kn_2^5); t \geq 0) \) is independent of \( (S(t); t \in [0, kn_2^5]) \). The event \( \Theta^{-kn_2^5}(E_B(n_0)) \) depends only on the increments \( (S(t+kn_2^5)-S(kn_2^5); t \geq 0) \) and the scenery \( (\xi(S(kn_2^5)+z); z \in [-Ln_2^50, Ln_2^50]) \) because the random walker can jump in each step at most a distance of \( L \). On the other hand, we have that \( E^0_{\text{no block}} \cap E^0_{\text{in works}} \) only depends on \( (S(t); t \in [0, kn_2^5]) \) and the scenery \( (\xi(S(kn_2^5)+z); z \notin [-Ln_2^50, Ln_2^50]) \). Since the scenery \( \xi \) is i.i.d., the event \( \Theta^{-kn_2^5}(E_B(n_0)) \) is independent of \( E^0_{\text{no block}} \cap E^0_{\text{in works}} \). For any events \( A, B, C \) with the property that \( A \) and \( B \) are independent and \( P(B) > 0 \) the following inequality holds:

\[ P(C \mid A \cap B) \leq \frac{P(C \mid A)}{P(B)}. \]
In our case this yields together with (3.11):

$$P \left( \left[ \Theta^{-kn_0^2 - n_0^{20}} \left( E_{\text{recon}}^{n_0} \right) \right] \{ \tilde{k} = k \} \cap E_{\text{ok}}^{n_0} \right)$$

$$= P \left( \left[ \Theta^{-kn_0^2 - n_0^{20}} \left( E_{\text{recon}}^{n_0} \right) \right] \{ \Theta^{-kn_0^2} (E_B(n_0)) \cap E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \} \left[ \Theta^{-kn_0^2} (E_B(n_0)) \right] \right)$$

$$\leq \frac{P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right)}{P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right)}$$

$$= \frac{P \left( \left[ \Theta^{-n_0^{20}} \left( E_{\text{recon}}^{n_0} \right) \right] \{ E_B(n_0) \} \right)}{P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right)} \leq \frac{\varepsilon_1(n_0)}{P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right)}; \quad (3.12)$$

for the second but last inequality we used that the shift $\Theta$ preserves the measure $P$ by Lemma 4.1 of [2]; for the last equality we used definition (2.1). Using the monotonicity of $\tilde{F}$ and the definition of $k_{n_0}$, we obtain for all $n_0$ sufficiently large and all $k \leq k_{n_0}$

$$P \left( E_{\text{no block}}^{n_0,k} \right) = P (k \geq k) = \tilde{F} (k - 1) \geq \tilde{F} (k_{n_0} - 1) \geq \varepsilon_2 (n_0).$$

Combining the last inequality with Lemma 3.1 and the fact $E_{\text{no block}}^{n_0,k} \in \mathcal{F}(t)$ for $t = n_0^2 (k - 1) + n_0^{20}$, we obtain

$$P \left[ P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right) \mathcal{F}(t) \right] \geq \left[ 1 - c_3 n_0^{-20} n_2^{-2} \right] P \left( E_{\text{no block}}^{n_0,k} \right) \geq \left[ 1 - c_3 n_0^{-20} n_2^{-2} \right] \varepsilon_2 (n_0).$$

For $n_0$ big enough we get that $\left( 1 - c_3 n_0^{20} n_2^{-2} \right) > 1/2$. In that case we conclude

$$P \left( E_{\text{no block}}^{n_0,k} \cap E_{\text{rw apart}}^{n_0,k} \right) \geq \varepsilon_2 (n_0) / 2.$$  

Combining the last estimate with (3.13), we obtain

$$P \left( \left[ \Theta^{-kn_0^2} \left( E_{\text{recon}}^{n_0} \right) \right] \{ \tilde{k} = k \} \cap E_{\text{ok}}^{n_0} \right) \leq \frac{2 \varepsilon_1 (n_0)}{\varepsilon_2 (n_0)}.$$

By the definition of $\varepsilon_2 (n_0)$ (3.3), we have $\varepsilon_2 (n_0) \geq \sqrt{\varepsilon_1 (n_0)}$. Thus,

$$P \left( \left[ \Theta^{-kn_0^2} \left( E_{\text{recon}}^{n_0} \right) \right] \{ \tilde{k} = k \} \cap E_{\text{ok}}^{n_0} \right) \leq 2 \sqrt{\varepsilon_1 (n_0)}.$$

Using (3.10) we get

$$P \left( E_{\text{ini works}}^{n_0,1} \cap E_{\text{ok}}^{n_0} \right) \leq 2 \sqrt{\varepsilon_1 (n_0)}.$$  

It follows from (2.3) that $\lim_{n_0 \to \infty} P \left( E_{\text{ini works}}^{n_0,1} \cap E_{\text{ok}}^{n_0} \right) = 0$. This completes the proof of Theorem 1.1.

References

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