Semi-classical dynamics of excess quantum noise

van Exter, M.P.; van Druten, N.J.; van der Lee, A.M.; Dutra, S.M.; Nienhuis, G.; Woerdman, J.P.

Publication date
2001

Published in
Physical Review A

PHYSICAL REVIEW A, VOLUME 63, 043801

Semiclassical dynamics of excess quantum noise

M. P. van Exter, N. J. van Druten,* A. M. van der Lee, S. M. Dutra, G. Nienhuis, and J. P. Woerdman
Leiden University, Huygens Laboratory, P.O. Box 9504, 2300 RA Leiden, The Netherlands
(Received 5 October 2000; published 5 March 2001)

A semiclassical theoretical framework is presented to describe the essential features of the excess quantum noise that occurs in systems with nonorthogonal eigenmodes. Excess noise is shown to be always spectrally colored, instead of white, so that the Petermann excess noise factor is best written as $K(\omega)$ instead of K. The consequences of this spectral coloring are analyzed for lasers, both below and above the lasing threshold.

DOI: 10.1103/PhysRevA.63.043801 PACS number(s): 42.50.Lc, 42.60.Da

I. INTRODUCTION

The fluctuation-dissipation theorem states that noise and damping go together and that systems that are open to the outside world will inevitably experience noise input from this outside world. This also applies to quantum-mechanical systems, where the fluctuation-dissipation theorem is linked to the discrete nature of the excitation or, equivalently, the commutation behavior of the creation and annihilation operators [1]. A well-studied example of quantum noise is the spontaneous-emission noise in lasers [2,3]. When the optical modes in the laser are orthogonal, this noise amounts effectively to ''one photon per mode.'' When the modes are nonorthogonal the amount of noise will be enhanced by the excess noise factor K, which expresses that there are effectively ''K noise photons'' in the lasing mode [3]. The reason for this noise enhancement is purely geometrical; the noise projection into each mode may be enhanced, but the integrated spontaneous-emission rate will not change, as the noise inputs in these nonorthogonal modes are correlated [4].

Experimentally the existence of excess quantum noise is well established. For lasers with nonorthogonal eigenmodes, several studies [5–11] have shown how the fundamental laser linewidth, which is due to laser phase diffusion, is enhanced by an excess noise factor K with respect to the so-called Schawlow-Townes limit. Similar noise enhancements have been observed in the laser intensity fluctuations [12]. Excess noise is quite universal. It has been observed for nonorthogonal longitudinal modes in cavities with large outcoupling ($K_{\mathrm{long}} \approx 7$) [5,6], for transverse modes in unstable cavities ($K_{\mathrm{trans}} \approx 500$) [12–15], or stable cavities with large diffractional outcoupling ($K \approx 13$) [7–9], and for polarization modes ($K_{\mathrm{pol}} \approx 60$) [10,11]. In all these cases the mode nonorthogonality is related to and enforced by ''anisotropy in the net losses,'' which can exist either in the longitudinal or transverse direction in real space, or in the polarization direction.

In this paper we will show that the geometric description of excess noise is too simple and that the excess noise factor rather acts as a frequency-dependent multiplier $K(\omega)$, i.e., that excess noise is spectrally colored. So far, this spectral coloring has been only briefly discussed in the literature, and mainly for the two-mode case [16–19]. Here we will give a systematic and general description using a semiclassical framework, where the field is treated classically and the quantum noise enters as a Langevin noise source. The reason that the spectral coloring is often overlooked is that one generally assumes one mode to dominate over all others, and thus neglects the dynamics of weak nonorthogonal side modes. It is the correlated dynamics in these side modes that projects into the measurement direction, and can thereby partially cancel the excess noise. These projections can be relatively strong as they correspond to a type of heterodyning and are therefore first order in the side-mode amplitude. Spectral coloring shows up in our treatment because we explictly take into account the time dependence of all side modes and thereby go beyond the standard geometric picture of excess noise [3].

From a physical point of view it is not the noise input that is enhanced, but rather the sensitivity of the system to specific noise inputs. In the geometric picture of excess noise this is not immediately obvious as one considers only the dynamics of the dominant mode and projects the input noise from its creation onto the adjoint of this mode. In this paper, we will keep track of all modal amplitudes and project onto the measurement direction only after the system evolution. The notion that the fluctuations in the system variables arise from the combined action of noise input and system dynamics is therefore an implicit part of our description. In Sec. II we will introduce a general mathematical framework for noise in nonorthogonal systems, which is based on a linearized description in an N-dimensional state space. In Sec. III we take several points of view to show that all excess noise is spectrally colored. In Sec. IV we discuss the implications for laser dynamics, while Sec. V contains a concluding discussion.

II. GENERAL FORMALISM FOR EXCESS NOISE

A. Introducing the problem

Although excess quantum noise has been studied only for lasers, it should occur in any noise-driven system with nonorthogonal eigenmodes. As a general problem we consider the noise-driven dynamics of a system with states that can be characterized by a state vector $|x\rangle$ in an N-dimensional state space. The noise-free evolution of $|x\rangle$ is taken to be time

*Present address: Delft Technical University, Department of Applied Physics, Lorentzweg 1, 2628 CJ Delft, The Netherlands.

independent and linear because the system dynamics is either linear or has been linearized around the steady state (see also Sec. IV D). The noise-driven dynamics of our system is given by

$$\frac{d}{dt}|x(t)\rangle = -i\mathcal{H}|x(t)\rangle + |f(t)\rangle, \tag{1}$$

where $-i\mathcal{H}$ is the (linearized) evolution operator, and where $|f(t)\rangle$ is a Langevin type, i.e., spectrally white, noise source. The time-correlation operator of this noise source is assumed to be given by

$$\overline{|f(t_1)\rangle\langle f(t_2)|} = 2\mathcal{D}\delta(t_1 - t_2), \tag{2}$$

where $\mathcal{D}$ is the Hermitian diffusion operator, which corresponds to a frequency-correlation operator and power spectral density operator of

$$\overline{|f(\omega_1)\rangle\langle f(\omega_2)|} = 4\pi\mathcal{D}\delta(\omega_1 - \omega_2), \tag{3a}$$

$$S_{|f\rangle\langle f|} = \overline{|f(\omega)\rangle\langle f(\omega)|} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \overline{|f(\omega)\rangle\langle f(\omega')|} = 2\mathcal{D}, \tag{3b}$$

where $\langle f| \equiv |f\rangle^\dagger$ and $|f(\omega)\rangle$ are the Hermitian conjugate and Fourier transform of $|f(t)\rangle$, respectively, where the single overline denotes ensemble (or time) averaging, and where the overline over frequency-dependent quantities should be interpreted as power spectral densities as in Eq. (3b). Note that our notation in terms of kets $|x\rangle$ and $|f\rangle$ and linear operators $\mathcal{H}$ and $\mathcal{D}$, is of course equivalent to a description that uses vectors $\vec{x}$ and $\vec{f}$ and matrices $\mathcal{H}$ and $\mathcal{D}$; the bra $\langle x| = |x\rangle^\dagger$ corresponds to the row vector $\vec{x}^\dagger$, which has consecutive elements that are the complex conjugates of the elements of the column vector $\vec{x}$.

In the following sections we will use Eq. (1) as a general and compact description of laser dynamics, in which case the variables $|x\rangle$ and $|f\rangle$ specify the (slowly varying component of the) intracavity optical field and noise field, respectively, in the state space of all optical modes. The intracavity optical field $|x\rangle$ can be specified to various levels of accuracy. For lasers with very lossy mirrors or apertures one should specify its complete spatial profile, thus opening the possibility to incorporate the longitudinal excess noise factor in the description. For lasers with high-reflecting mirrors the field changes per round trip are limited, so that it is sufficient to specify the transverse and polarization profile of the optical field in one specific transverse plane.

Equation (1) contains the essential ingredients for the generation and spectral coloring of excess noise. It gives a complete description of the field dynamics in a laser that operates sufficiently below threshold, where the laser acts as a regenerative amplifier of input noise, operating at fixed population inversion. It also gives a proper description for a laser operating above threshold, where the nonlinear process of optical saturation becomes important. However, this description is then incomplete, as Eq. (1) has to be supplemented with a description of the specific population dynamics. Still, the generic structure of Eq. (1) remains intact, and contains the essentials of excess noise, when we let the evolution matrix $\mathcal{H}$ depend on the atomic populations (see Secs. IV B–IV D for details).

There are two alternative descriptions for the optical field dynamics in the laser cavity. Instead of using the *evolution matrix* $-i\mathcal{H}$, we could have equally well worked with the *round-trip matrix* $\mathcal{M}$, which relates the intracavity field after consecutive roundtrips and which for high-reflecting mirrors is equal to $\mathcal{M} = \exp(-i\mathcal{H}T)$, T being the round-trip time. As a third alternative we could have used the *scattering matrix* $\mathcal{S}$, which relates the incoming field (read ‘‘vacuum fluctuations’’) to the outgoing field [20]. All three descriptions involve the same aspects of mode nonorthogonality and colored excess noise. We have chosen to describe our laser in a semiclassical way with the help of the evolution matrix $-i\mathcal{H}$ because (i) this matrix is directly linked to the time evolution, (ii) this description uses the same intracavity field that determines the optical saturation, in contrast to the external fields that appear in the scattering formalism, and (iii) a full quantum-mechanical description of optical saturation is too complicated anyhow.

The essentials of excess noise are contained in the (eigenmodes and eigenvalues of the) evolution matrix $-i\mathcal{H}$. If the matrix $\mathcal{H}$ is Hermitian, its eigenmodes form an orthogonal basis and the N-dimensional Eq. (1) can be solved trivially, because it separates to N simple one-dimensional problems, as the orthogonal projections of the noise sources $|f\rangle$ are uncorrelated. If the matrix $-i\mathcal{H}$ is non-Hermitian, as is generally the case for open or lossy systems, the eigenmodes can be nonorthogonal [21], in which case the solution is less trivial as the N-dimensional problem does not separate, and excess noise will develop. It is then convenient to separate the evolution matrix, via $-i\mathcal{H} = -i\mathcal{H}_0 - \mathcal{A}$, in a ‘‘dispersive,’’ i.e., energy-conserving part $i\mathcal{H}_0$, and an ‘‘absorptive,’’ i.e., energy-nonconserving part $-\mathcal{A} = -\mathcal{L} + \mathcal{G}$, where the submatrices $\mathcal{H}_0$, $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{G}$ are all Hermitian, and where the latter two characterize the loss and gain, respectively.

The diffusion matrix $\mathcal{D}$ plays no essential role in the generation process of excess noise, as it only specifies the strength and modal distribution of the Langevin-type noise input. The fluctuation-dissipation theorem imposes an (operator-type) relation between the diffusion matrix $\mathcal{D}$ and the energy-nonconserving term $-\mathcal{L}+\mathcal{G}$ in the evolution matrix $i\mathcal{H}$. Just as for the one-dimensional case [22,23], the semiclassical N-dimension fluctuation-dissipation theorem will depend on the chosen ordering of the quantum-mechanical operators. However, differences between the various semiclassical treatments, as derived from different operator orderings, can be neglected for lasers operating not too far above threshold, as they arise from ‘‘reflected vacuum fluctuations’’ [24] and give corrections of the order of shot-noise level. For symmetric operator ordering and complete population inversion the N-dimensional version of the fluctuation-dissipation theory reduces to the simple expression $2\mathcal{D} = \mathcal{L} + \mathcal{G}$, as can be deduced from calculations based on the concepts discussed in Ref. [25]. For normal operator ordering, where all noise can be attributed to spontaneous emission, the diffusion operator $\mathcal{D}$ can be calculated from the modal projections of a position-dependent noise source that is proportional to the local excited-state population $N_2(\vec{r})$.

In many practical lasers the diffusion operator $\mathcal{D}$ is very simple and approximately isotropic over the relevant part of state space, as both loss matrix $\mathcal{L}$ and gain matrix $\mathcal{G}$ have this property. Of course, small deviations from perfect isotropy are still needed to produce the mode coupling required for mode nonorthogonality, but these deviations can be and are in fact typically small. As the mode (non)orthogonality is determined by the relation between $\mathcal{H}_0$ and $-\mathcal{L}+\mathcal{G}$, it emphasizes small differences between $\mathcal{L}$ and $\mathcal{G}$, whereas the input noise $D\propto\mathcal{L}+\mathcal{G}$ is hardly sensitive to these differences.

B. Formal solution and eigenmodes

Starting from $|x(t=-\infty)\rangle=0$, the formal solution of Eq. (1), in either the time or frequency domain, is

$$|x(t)\rangle=\int_0^\infty d\tau\, e^{-i\mathcal{H}\tau}|f(t-\tau)\rangle, \tag{4a}$$

$$|x(\omega)\rangle=\frac{1}{i(\mathcal{H}-\omega)}|f(\omega)\rangle. \tag{4b}$$

The corresponding time-correlation matrix and power spectral density matrix are

$$\overline{|x(t_1)\rangle\langle x(t_2)|}=\Theta(t_1-t_2)e^{-i\mathcal{H}(t_1-t_2)}\mathcal{C}+\Theta(t_2-t_1)\mathcal{C}e^{i\mathcal{H}^\dagger(t_2-t_1)}, \tag{5a}$$

$$\overline{|x(\omega)\rangle\langle x(\omega)|}=(\mathcal{H}-\omega)^{-1}\,2\mathcal{D}\,(\mathcal{H}^\dagger-\omega)^{-1}, \tag{5b}$$

where $\Theta(t)$ is the Heavyside function, and where the Hermitian operator

$$\mathcal{C}=\int_0^\infty d\tau\, e^{-i\mathcal{H}\tau}2\mathcal{D}e^{i\mathcal{H}^\dagger\tau} \tag{6}$$

characterizes the mean fluctuations or the fluctuation probability distribution. The general Eqs. (5) can be found in at least one textbook [26]. It is therefore somewhat surprising that their implications to the existence of excess noise and its spectral coloring has been discussed only recently [16,19].

In the above rather abstract notation one operator, $\overline{|x(t_1)\rangle\langle x(t_2)|}$, describes all fluctuations and correlations in the system. In this notation the observation on a single variable, being a linear combination of the system's degrees of freedom, corresponds to a projection in state space onto the ''measurement direction'' $|m\rangle$. The time-correlation function of this measured variable is simply given by

$$\frac{\langle m\overline{|x(t_1)\rangle\langle x(t_2)|}m\rangle}{\langle m|m\rangle}, \tag{7}$$

while the corresponding power spectrum is obtained by a similar sandwich construction. When $|x\rangle$ denotes the optical field, these sandwich constructions are the time and frequency representations of the optical spectrum as measured after projection onto the $|m\rangle$ direction in state space, whereas the inner product $\langle x(t)|x(t)\rangle$ denotes the total intensity in all modes.

The possibly non-Hermitian character of the operator $\mathcal{H}$ can produce rich system dynamics. This is best appreciated by the introduction of the ''left and right'' eigenstates as

$$\mathcal{H}|u_i\rangle=(\omega_i-i\gamma_i)|u_i\rangle, \tag{8a}$$

$$\langle v_i|\mathcal{H}=(\omega_i-i\gamma_i)\langle v_i|, \tag{8b}$$

where each eigenvalue has been separated into a frequency ω_i and damping rate γ_i and where $|u_i\rangle$ and $|v_i\rangle$ are the corresponding eigenstate and its adjoint. By sandwiching operator $\mathcal{H}$ between adjoint modes and eigenmodes one can immediately show that the sets $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are bi-orthogonal when the eigenvalues are nondegenerate. Using the standard assumption that these eigenstates form a complete set, and normalizing them such that $\langle v_i|u_j\rangle=\delta_{ij}$, we can rewrite the unity operator and system operator as

$$\hat{I}=\sum_i |u_i\rangle\langle v_i|=\sum_i |v_i\rangle\langle u_i|, \tag{9a}$$

$$\mathcal{H}=\sum_i (\omega_i-i\gamma_i)\,|u_i\rangle\langle v_i|. \tag{9b}$$

For later convenience we will also normalize the eigenvectors via $\langle u_i|u_i\rangle=1$. This fixes the normalization of $\langle v_i|v_i\rangle \geqslant|\langle u_i|v_i\rangle|^2/\langle u_i|u_i\rangle=1$.

C. The standard result of nonorthogonality theory

With the eigenmodes introduced above we can give a more physical interpretation to the abstract result of Eqs. (5). For this we expand the state vector in the system's eigenmodes as

$$|x(t)\rangle=\sum_i a_i(t)|u_i\rangle. \tag{10}$$

Substitution into Eq. (1) and projection onto the adjoint modes $\langle v_j|$ yields the evolution of the expansion coefficients $a_i(t)=\langle v_i|x(t)\rangle$

$$\frac{d}{dt}a_i(t)=-(\gamma_i+i\omega_i)a_i(t)+\langle v_i|f(t)\rangle. \tag{11}$$

The various correlation functions can be found after formal solution, or directly by sandwiching Eqs. (5) between two adjoint modes. In both cases the power spectrum

$$\overline{a_i(\omega)a_j^*(\omega)}=\langle v_i\overline{|x(\omega)\rangle\langle x(\omega)|}v_j\rangle=\frac{\langle v_i|2\mathcal{D}|v_j\rangle}{[\gamma_i+i(\omega_i-\omega)][\gamma_j-i(\omega_j-\omega)]} \tag{12}$$

shows how the eigenmode amplitudes are driven by projections of the diffusion operator $\mathcal{D}$ onto the *adjoint* modes $|v_j\rangle$ instead of onto the eigenmodes $|u_j\rangle$. For a system with nonorthogonal modes the fluctuations in the expansion coefficients a_i can therefore be much larger than in an orthogonal system, as $\langle v_i|v_i\rangle \gg \langle u_i|u_i\rangle = 1$.

Two assumptions are needed to derive the standard result of excess noise, being a simple enhancement of the projected noise power by a geometric factor K_n. First, one has to assume that the diffusion operator $\mathcal{D}$ is sufficiently isotropic in state space, which for a laser means that one assumes that the spontaneous-emission noise contributes evenly to the various relevant eigenmodes. In this case, one can set $\mathcal{D} = D\hat{I}$ and rewrite $\langle v_n|\mathcal{D}|v_n\rangle = DK_{geo,n}$ [27], where

$$K_{geo,n} = \langle v_n|v_n\rangle \tag{13}$$

is the usual geometric excess noise factor of the nth eigenmode [3], which has been given an additional subscript geo to contrast it with the frequency-dependent excess noise factor $K_n(\omega)$ introduced in Sec. III. Second, one generally assumes that one specific eigenmode n dominates over all others, because it experiences considerably less damping than the others, i.e., $\gamma_i \gg \gamma_n \geq 0$. In this case, it seems reasonable to neglect the modal amplitudes of the other (weakly excited) eigenmodes, so that $|x(t)\rangle \approx a_n(t)|u_n\rangle$. The fluctuations in the dominant mode $|u_n\rangle$ are then fully determined by $a_n(t)$ and seeded only by the noise projection in the ''adjoint'' direction $|v_n\rangle$. Together, these two assumptions yield the standard excess noise factor $K_{geo,n}$ [3,28,29].

In the present paper we will retain the first assumption of almost isotropic diffusion, but relax the second assumption of modal dominance. We will show how even weakly excited eigenmodes can significantly alter the projected noise characteristics, basically because their amplitude fluctuations are correlated with the main mode [see Eq. (12)] so that effects already occur to first-order in side-mode amplitude. A second argument to explain the surprisingly strong influence of even relatively weak side modes is that their relatively fast dynamics ($\gamma_i \gg \gamma_n \geq 0$) enhances their contribution to the time-correlation function at short times, i.e., large frequencies. As a result of the side-mode dynamics, the amount of excess noise can differ from the geometric factor $K_{geo,n}$ and its strength will always depend on the observation frequency and bandwidth, i.e. excess noise is spectrally colored.

It is interesting to note that the dynamics of the modal amplitude a_n can be observed in a pure form, i.e., without admixture of other modal amplitudes, by direct projection in the adjoint direction $|v_n\rangle$. Moreover, such a projection exhibits no excess noise, as the projected fluctuations should be normalized by the same quantity $\langle v_n|v_n\rangle$ that generates $K_{geo,n}$ [see Eq. (7)]. This shows that excess noise is not a result of the ''adjoint projection'' by itself, which reduces the noise-driven dynamics of the full system to the simple one-dimensional problem of Eq. (11), but rather of the interpretation of $a_n(t)$ as the amplitude in the eigenmode direction $|u_n\rangle$ instead of the adjoint direction $|v_n\rangle$. As excess noise is only observed for measurements in some directions in state space, but not for others it is not unreasonable to state that ''all excess noise is a form of projection noise.'' Still, some projections appear naturally in the experiments (lasers, for instance, automatically choose the dominant eigenmode for laser action), whereas other projections need more experimental effort, like strong suppression of the lasing mode to admix sufficient power from weak side modes.

III. EXCESS NOISE IS SPECTRALLY COLORED

A. Derivation of colored excess noise

In the rest of the paper we will discuss some general aspects of measurements in the direction of eigenmode $|u_n\rangle$; such projections occur quite naturally as one generally measures specifically on the dominant (lasing) mode. As mentioned above, a projection onto the eigenmode $|u_n\rangle$ will not only give the amplitude a_n, but also a projected fraction of the coefficients a_i $(i \neq n)$ of the other nonorthogonal modes, as

$$\langle u_n|x\rangle = a_n + \sum_{i\neq n} a_i \langle u_n|u_i\rangle. \tag{14}$$

By combining Eqs. (12) and (14) the general power spectrum for a measurement in the $|u_n\rangle$ direction can be written as

$$\begin{aligned}
&\langle u_n|\overline{x(\omega)\rangle\langle x(\omega)}|u_n\rangle \\
&\quad= \sum_{i,j} \langle u_n|u_i\rangle\langle u_j|u_n\rangle\, \overline{a_i(\omega)a_j^*(\omega)} \\
&\quad= \frac{\langle v_n|2\mathcal{D}|v_n\rangle}{[(\omega_n-\omega)^2+\gamma_n^2]} \\
&\qquad+ \sum_{i\neq n} \mathrm{Re}\left[\frac{\langle u_n|u_i\rangle\,\langle v_i|4\mathcal{D}|v_n\rangle}{[\gamma_i+i(\omega_i-\omega)][\gamma_n-i(\omega_n-\omega)]}\right] \\
&\qquad+ \sum_{i\neq n}\sum_{j\neq n} \frac{\langle u_n|u_i\rangle\,\langle v_i|2\mathcal{D}|v_j\rangle\,\langle u_j|u_n\rangle}{[\gamma_i+i(\omega_i-\omega)][\gamma_j-i(\omega_j-\omega)]},
\end{aligned} \tag{15}$$

where Re denotes the real part. A direct projection of the general result Eq. (5) onto the eigenmode direction $|u_n\rangle$ yields the same result.

When the diffusion operator $\mathcal{D}$ is sufficiently isotropic in state space, so that $\mathcal{D} \approx D\hat{I}$, the first term in the right-most part of Eq. (15) reduces to the standard result for $|a_n(\omega)|^2$: a Lorentzian spectrum with a strength that is enhanced by a factor $K_{geo,n}$ due to excess noise. The two other terms are corrections to this result due to the projected contributions from other (nonorthogonal) eigenmodes. For the typical situation, where one modal amplitude a_n is much larger than all others, we have separated these corrections into terms that contain only one ''side-mode amplitude'' a_i $(i \neq n)$ and others that contain two side-mode amplitudes. The first-order term can be viewed upon as a kind of heterodyning between the dominant mode and the side modes. Its relative importance depends crucially on the frequency difference of these modes as compared to their damping rate difference. That first-order corrections exist at all already shows that it might be difficult to downplay the importance of the side modes. These corrections are the prime cause of the spectral coloring discussed below.

Although Eq. (15) gives a complete description of the projected power spectrum, it is often not the most convenient one, as the fluctuating modal amplitudes a_i can be strongly correlated. For a measurement in the eigenmode direction $|u_n\rangle$ it is often more convenient to remove part of these correlations, by combining the noise sources into two uncorrelated parts: one projection in the eigenmode direction $|u_n\rangle$ and a complementary projection in the orthogonal subspace of all adjoint modes $|v_i\rangle$ $(i\neq n)$. After substituting the expansion (10) in the right-hand side of Eq. (1), one obtains the following time evolution:

$$\frac{d}{dt}\langle u_n|x(t)\rangle = -(\gamma_n+i\omega_n)\langle u_n|x(t)\rangle + f_n(t)+g_n(t), \tag{16}$$

where $f_n(t)\equiv\langle u_n|f(t)\rangle$ is the usual spectrally white-noise source, which exhibits no excess noise, whereas the second noise source

$$\begin{aligned} g_n(t) &\equiv \sum_{i\neq n}[(\gamma_n-\gamma_i)+i(\omega_n-\omega_i)]\langle u_n|u_i\rangle a_i(t)\\ &= \sum_{i\neq n}[(\gamma_n-\gamma_i)+i(\omega_n-\omega_i)]\langle u_n|u_i\rangle\\ &\quad\times\int_0^\infty d\tau\langle v_i|f(t-\tau)\rangle e^{-(\gamma_i+i\omega_i)\tau} \end{aligned} \tag{17}$$

contains all excess noise effects.

Equation (17) demonstrates that the excess noise $g_n(t)$ may be viewed upon as a delayed response in the $|u_n\rangle$ direction from white noise that was originally projected in the adjoint directions $|v_i\rangle$ $(i\neq n)$. Such a delayed response is known to transform white noise into spectrally colored noise, where the spectral coloring is given by the Fourier transform of the memory kernel in Eq. (17). It generally appears when a system of many coupled degrees of freedom is reduced to a few or rather a single variable, where we note that a given spectral coloring can also be removed by the reverse process, i.e., by the introduction of new variables into a stochastic rate equation containing colored noise [30]. The noise source $g_n(t)$ is uncorrelated with $f_n(t)$ as it is built up from projections of $|f(t-\tau)\rangle$ onto all other $(i\neq n)$ adjoint, i.e., orthogonal, directions. Due to its delayed response the noise source $g_n(t)$ is spectrally colored by the dynamics of all other degrees of freedom. More specifically, its power spectrum is given by

$$\begin{aligned} \overline{|g_n(\omega)|^2} &= \sum_{i\neq n}\sum_{j\neq n}[(\gamma_i-\gamma_n)+i(\omega_i-\omega_n)][(\gamma_j-\gamma_n)-i(\omega_j-\omega_n)]\langle u_n|u_i\rangle\langle u_j|u_n\rangle\overline{a_i(\omega)a_j(\omega)^*}\\ &= \sum_{i\neq n}\sum_{j\neq n}\frac{[(\gamma_i-\gamma_n)+i(\omega_i-\omega_n)][(\gamma_j-\gamma_n)-i(\omega_j-\omega_n)]}{[\gamma_i+i(\omega_i-\omega)][\gamma_j-i(\omega_j-\omega)]}\langle u_n|u_i\rangle\langle v_i|2\mathcal{D}|v_j\rangle\langle u_j|u_n\rangle. \end{aligned} \tag{18}$$

Equation (18) is the key equation for the coloring of excess noise. It shows how excess noise is not only determined by the geometry of the eigenmodes, i.e., by the various inner products, but also by the system's dynamics, as described by the prefactor with the various eigenvalues. This equation *only* reduces to the standard result when we resort to the standard assumption [3] that one specific eigenmode n dominates over all others because it experiences considerably less damping than the others, i.e., when we take $\gamma_i\gg\gamma_n\geq 0$. In this case the prefactor with the various eigenvalues will be approximately unity around the observation frequency $\omega\approx\omega_n$. When we further assume the diffusion operator $\mathcal{D}$ to be isotropic in state space, the double summation in Eq. (18) simplifies to

$$\overline{|g_n(\omega\approx\omega_n)|^2} = 2D(K_{geo,n}-1), \tag{19}$$

where we have used the completeness relation of Eq. (9a) and the geometric excess noise factor $K_{geo,n}$ of Eq. (13).

For the general case, where the above "standard assumption" might not be valid, one should include the frequency-dependent prefactors in the summation. This can change both the magnitude and behavior of the excess noise factor. A convenient way to parametrize these changes is by introducing a frequency dependence, i.e., spectral coloring, to the excess noise factor via

$$\frac{\overline{|g_n(\omega)|^2}}{\overline{|f_n(\omega)|^2}} \equiv K_n(\omega)-1. \tag{20}$$

By rewriting the new excess noise factor as $1+[K_n(\omega)-1]$ we want to stress that the noise separates into a "normal" and an "excess" contribution, and that the latter is spectrally colored, as it originates from projections into a subspace orthogonal to the measurement direction $|u_n\rangle$, which take time to evolve into the measurement direction [see Eqs. (17) and (18)]. The new excess noise factor $K_n(\omega)$

can differ substantially from the geometric value $K_{geo,n}$ when side modes of reasonable power and nonorthogonality are present close to the dominant mode $|u_n\rangle$. Sufficient overlap between the eigenmodes is clearly a necessity to obtain excess noise [20]. Only when one mode dominates over all others, and when we choose our observation frequency close to the resonance of that mode, i.e., $\omega \approx 0$ (in the rotating basis where $\omega_n = 0$), do the two excess noise factors coincide $K_n(\omega \approx 0) = K_{geo,n}$. For frequencies much larger than the system's response, the colored excess noise will always disappear, i.e., $K_n(\omega \to \infty) = 1$, as $\overline{|g_n(\omega)|^2}$ decreases rapidly for $(\omega - \omega_n) \gg (\omega_i - \omega_n), (\gamma_i - \gamma_n)$ [see Eq. (18)]. Equivalently, one can say that the slope of the time-correlation function for a very short time scale [small $|t_1 - t_2|$ in Eq. (7)] does not yet notice the presence of excess noise, as it takes time for the excess noise to develop.

B. The Gramm-Schmidt normalized basis

A description in the eigenmode basis has the advantage that the dynamics of the modal amplitudes becomes simple, but the disadvantage that a measurement will generally sample a linear combination of these amplitudes so that first-order interference effects often show up. In Sec. III A we tried to remove some of these effects by separating the noise sources into two parts $f_n(t)$ and $g_n(t)$, being parallel and perpendicular to eigenmode $|u_n\rangle$. We will now go one step further by introducing a convenient orthogonal basis for our dynamic description. This orthogonal basis $|c_i\rangle$ can be created from the eigenmodes $|u_i\rangle$ by a Gramm-Schmidt orthogonalization procedure, where we take $|c_1\rangle = |u_1\rangle$ as the (dominant) mode under observation, $|c_2\rangle$ proportional to that part of $|u_2\rangle$ that is orthogonal to $|u_1\rangle$, i.e., $|c_2\rangle \propto |u_2\rangle - \langle u_1|u_2\rangle |u_1\rangle$, $|c_3\rangle$ proportional to that part of $|u_3\rangle$ that is orthogonal to both $|c_1\rangle$ and $|c_2\rangle$, and likewise for all other basis vectors $|c_i\rangle$, and where we normalize to $\langle c_i|c_j\rangle = \delta_{ij}$. When we decompose the state vector in this convenient orthogonal basis, via $|x(t)\rangle = \Sigma_i c_i(t)|c_i\rangle$, we find that the evolution of the coefficients $c_i(t)$ is described by the following triangular matrix equation [31]:

$$\frac{d}{dt}\begin{pmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ \cdots \end{pmatrix} = \begin{pmatrix} -(\gamma_1 + i\omega_1) & \kappa_{12} & \kappa_{13} & \cdots \\ 0 & -(\gamma_2 + i\omega_2) & \kappa_{23} & \cdots \\ 0 & 0 & -(\gamma_3 + i\omega_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \\ \cdots \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ \cdots \end{pmatrix}, \tag{21}$$

where the Langevin noise sources $h_i(t) = \langle c_i|f(t)\rangle$ are spectrally white and uncorrelated, as the $|c_i\rangle$ basis is orthogonal, and have equal strength when $\mathcal{D}$ is isotropic. Note how the elements $\langle c_i| - i\mathcal{H}|c_j\rangle$ of the evolution matrix are such that the eigenvalues of $-i\mathcal{H}$ appear as on-diagonal elements. The lower off-diagonal elements are all zero due to the Gramm-Schmidt procedure, which results in $\langle c_i|u_k\rangle = 0$ for $i > k$, but the upper off-diagonal elements are generally nonzero. Grangier and Poizat [18] have introduced the term ''loss-induced coupling'' to describe the role of these upper off-diagonal elements in the non-Hermitian character of the evolution. However, in the present semiclassical description it is better to drop the label ''loss-induced,'' as the Hermitian and anti-Hermitian parts of the evolution are on equal footing; the coupling can be removed both for the case of pure Hermitian and pure anti-Hermitian evolution, and even for some special case in between (see Sec. III C). Note that the coupling constants κ_{ij} can be easily rewritten in the eigenmode and adjoint basis with the help of Eq. (9b).

The top row of Eq. (21) describes the noise-driven dynamics of the amplitude of eigenmode $|u_1\rangle = |c_1\rangle$ via

$$\frac{d}{dt}c_1(t) = -(\gamma_1 + i\omega_1)c_1(t) + f_1(t) + \sum_{i \geq 2} \kappa_{1i}c_i(t), \tag{22}$$

where we recognize the earlier separation into two noise sources $f_1(t) = h_1(t)$ and $g_1(t)$ [see Eq. (16)]. The additional (excess) noise source $g_1(t)$ is again attributed to fluctuations in the orthogonal subspace that evolve into the measurement direction $|u_1\rangle$ due to the non-Hermitian character of $\mathcal{H}$. The evolution in this orthogonal subspace is independent of the amplitude c_1 in the measurement direction, as the first column of the evolution matrix in Eq. (21) contains only zero's, apart from it's upper element. As this coupling has a well-defined direction, the strength and spectral coloring of the excess noise source $g_1(t)$ is independent of the dynamics (read ''eigenvalues'') of the eigenmode $|u_1\rangle$.

C. Excess noise and the ''maximum emission principle''

Now that we have interpreted the generating mechanism of excess noise as the evolution of fluctuations out of an orthogonal subspace into the measurement direction, one may wonder why these first-order effects occur in systems with nonorthogonal eigenmodes but not in orthogonal systems. To explain this difference we will again separate the evolution operator as $-i\mathcal{H} = -i\mathcal{H}_0 - \mathcal{A}$, where the ''dispersion'' $\mathcal{H}_0$ and ''absorption'' $\mathcal{A} = \mathcal{L} - \mathcal{G}$ are both Hermitian operators. When these two operators commute, via $[\mathcal{H}_0, \mathcal{A}] = 0$, they will have a joint (orthogonal) eigenbasis and there is no excess noise; this is, for instance, the case when either the absorption or the dispersion are fully isotropic in state space. When these operators do not commute, excess noise will develop as the (orthogonal) eigenbases of $\mathcal{H}_0$ and $\mathcal{A}$ will differ and the eigenmodes $|u_i\rangle$ of the combined evolution ''look for a compromise between both evolutions'' via

$$-i\mathcal{H}_0|u_i\rangle = -i\omega_i|u_i\rangle + |b_i\rangle, \tag{23a}$$

$$-\mathcal{A}|u_i\rangle = -\gamma_i|u_i\rangle - |b_i\rangle, \tag{23b}$$

for some $|b_i\rangle \neq |0\rangle$. This strongly suggests that the size of $|b_i\rangle$ is a measure for the amount of excess noise.

The eigenmodes $\{|u_i\rangle\}$ of the combined evolution, which are now generally not orthogonal to each other, will differ from the eigenmodes of the ''absorption operator'' $\mathcal{A}$, so that the dominant eigenmode will generally differ from the state in state space that experiences maximum gain. One might therefore say that lasers with nonorthogonal eigenmodes violate the ''maximum emission principle,'' which can be expressed as the ''natural tendency of the laser to extract as much of its stored energy as possible'' [32]. Excess noise appears precisely in systems where the dominant eigenmode differs from the state of maximum gain (not an eigenmode), as small noise-induced deviations from the eigenmode can then already lead to *first-order* corrections in the experienced gain. In systems with orthogonal eigenmodes, where the dominant eigenmode does coincide with the state of maximum gain, similar deviations will produce only *second-order* effects and can thus generally be ignored.

The above argument brings us to the essence of excess noise and to the concept of ''injected-wave excitation'' [3,28], which is based on the notion that fluctuations originate from the combined result of noise injection and system evolution. From this point of view, one might wonder which type of injected wave produces the maximum system perturbance or in other words, ''in which direction $|m\rangle$ should we inject a signal of unity strength to have a maximum effect in the observation direction $|u_1\rangle$?'' The answer to this question depends not only on the mode nonorthogonality, but also on the various eigenvalues. When the mode under observation dominates over all others, with as limiting case $\gamma_1=0$, the best excitation is obtained for injection in the adjoint direction $|v_1\rangle$. This adjoint injection will produce a more efficient excitation of the low-loss eigenmode $|u_1\rangle$ than, e.g., direct injection will, basically because it contains a larger $|u_1\rangle$ component on top of a series of other modal components that decay much more rapidly. Equivalently, one might say that when the adjoint injection is added as a perturbation to the eigenmode it will give the largest first-order corrections to the experienced gain. When the mode under observation does not dominate completely, the situation is more complicated and the amount of excess noise will differ from its geometric value [see Eq. (18) and Fig. 1].

Figure 1 illustrates the concept of injected-wave excitation for a simple two-dimensional (real-valued) state space with two nonorthogonal eigenmodes $|u_1\rangle=\vec{u}_1$ and $|u_2\rangle=\vec{u}_2$, and two adjoint mode $|v_1\rangle=\vec{v}_1$ and $|v_2\rangle=\vec{v}_2$ (note that these modes satisfy the biorthogonality relation $\langle v_i|u_i\rangle=\delta_{ij}$). As mentioned above, the key idea is that the evolution after excitation by noise will generally depend on the noise ''direction'' in state space. Excitation in the direction of a pure eigenmode will result in a simple exponential decay along a straight line towards equilibrium. Excitation in other directions will lead to a more complicated evolution, as these state vectors decompose into both eigenmodes, which evolve via different eigenvalues. The adjoint direction plays a special role in this description. When the dominant eigenmode experiences (virtually) no damping, i.e., when $\gamma_1=0$, it can be optimally excited by injection into the $|v_1\rangle$ direction. The reason why this injection is more efficient than direct injection is that the state vector passes through a region of net gain, before it projects onto the eigenmode $|u_1\rangle$. When the weaker mode cannot be neglected, i.e., when the stronger eigenmode also experiences considerable damping, injection into the adjoint direction might not be the most efficient one (see below).

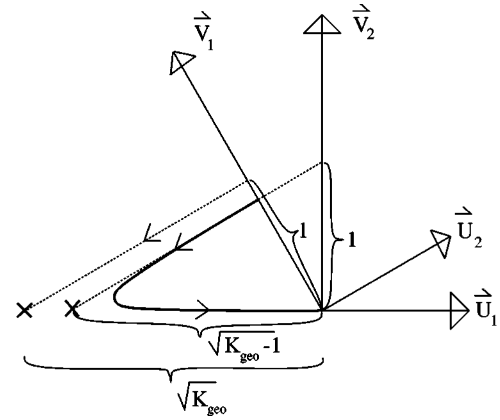

FIG. 1. Sketch of two nonorthogonal eigenvectors $\vec{u}_1,\vec{u}_2$, and their adjoint vectors $\vec{v}_1$ and $\vec{v}_2$. The dotted lines depict the evolution, after unity excitation in the adjoint directions $\vec{v}_1$ and $\vec{v}_2$, for the case $\gamma_2>\gamma_1=0$, where γ_i is the loss rate of mode *i*. The solid curve depicts the evolution, after unity excitation in the direction $\vec{v}_2$, for the case $\gamma_2>\gamma_1>0$.

Figure 1 also shows the time evolution after injection in the $|v_2\rangle$ direction, i.e., orthogonal to the eigenmode $|u_1\rangle$. The dotted curve again shows the evolution for the case $\gamma_1=0$, where the projected amplitude is $\sqrt{K_{geo}-1}$; the solid curve shows the evolution for the case $\gamma_2>\gamma_1>0$. These traces clearly demonstrate the two subtleties that are generally overlooked in the geometric description of excess noise. First, we note that the excess noise takes time to develop; it takes a certain time ($\approx 1/\gamma_2$) for the injected noise in the adjoint direction $|v_2\rangle$ to evolve and project onto the eigenmode direction $|u_1\rangle$. This finite response time produces a spectral coloring of excess noise. Second, we note that only for $\gamma_1=0$ will the final projected noise amplitude for orthogonal injection be equal to $\sqrt{K_{geo}-1}$. For $\gamma_1\neq 0$ the system evolution will follow a curve like the solid one in Fig. 1, which does not reach the maximum projected amplitude of $\sqrt{K_{geo}-1}$. Integrated over all possible noise directions this will make the actual low-frequency excess noise factor $K_1(\omega\approx 0)$ smaller than its geometric value $K_{geo,1}$.

IV. APPLICATION TO LASER DYNAMICS

A. Laser below threshold

The general formalism of excess noise can be easily applied to a laser, where $|x\rangle$ represents the optical field, if we neglect optical saturation and treat the laser as a regenerative amplifier of input noise, i.e., an amplifier with fixed gain. In the geometric picture of excess noise, where one only considers the evolution of the coefficient a_1 of the dominant eigenmode, one finds that the optical output is concentrated in the usual Lorentzian line shape, which in a system with nonorthogonal eigenmodes now contains $K_{geo,1}$ times as many photons as compared to the orthogonal system. This optical output will have the usual thermal or chaotic statistics, as we are dealing with a regenerative amplifier, so that the modal intensity fluctuations can be easily derived from the optical spectrum.

Inclusion of the other (weaker) modes can change this simple picture, if these modes are strong enough to carry any weight in the summation in Eq. (15), which describes the optical spectrum as projected onto the eigenmode $|u_n\rangle$. Side-mode contributions will also show up in the (time-dependent) laser output power, as projected in the eigenmode direction $|u_n\rangle$, which is given by

$$\langle u_n|x\rangle\langle x|u_n\rangle = |a_n|^2 + \sum_{i\neq n} 2\,\mathrm{Re}[\,a_i a_n^* \langle u_n|u_i\rangle] + \sum_{i\neq n}\sum_{j\neq n} a_i a_j^* \langle u_j|u_n\rangle\langle u_n|u_i\rangle. \quad (24)$$

If the $|u_n\rangle$ mode dominates and if there is sufficient spectral overlap between the eigenmodes, the single summation in the above expression behaves as a heterodyne term and will therefore be more important than the double summation, as the former is *first-order* in the side-mode amplitudes. It will thereby often dominate over the projected mode partition effects, i.e., the double summation, which are only *second-order* order in the side-mode amplitudes. Still, these first-order effects have hardly been discussed, probably because they only exist in nonorthogonal systems, whereas there is extensive literature on mode partition noise in orthogonal systems (see, for instance, Refs. [33,34] and references therein). In Ref. [31] and below, these first-order effects are shown to seriously hamper the generation of intensity-squeezed light in lasers with nonorthogonal eigenmodes.

B. Laser above threshold; the effect of optical saturation

The formalism presented above can also be used for lasers operating above the lasing threshold. As optical saturation becomes important the complex amplitude of the dominant mode should be separated into its (real-valued) amplitude and phase, which exhibit completely different dynamics. Simple results are only obtainable for lasers that are almost single mode, up to the degree that mode partition noise is irrelevant. The amplitude and phase dynamics of the dominant mode can then be straightforwardly obtained by linearization around steady state, but only for the case of isotropic saturation, where all ''atoms'' are equivalent and where a single (average) inversion is sufficient to characterize the gain. For class-A lasers, i.e., lasers in which the inversion dynamics is so fast that it can be eliminated adiabatically, this corresponds to the case of neutral coupling, where the ''self-saturation'' of the lasing mode equals the cross saturation that it forces upon the side modes, so that the net loss of the side modes is independent of output power [23]. In practical lasers, the saturation is often isotropic due to the fast diffusion of the inversion [35]. In the other case of anisotropic saturation, spatial, spectral, and/or polarization hole burning should be taken into account and the solution is generally much more complicated (see below).

As a starting point we use the general Eq. (1) for the noise-driven dynamics of the (slowly varying component of the) intracavity optical field $|x(t)\rangle$, and separate the evolution operator $-i\mathcal{H}$ in three parts (dispersion $\mathcal{H}_0$, loss $\mathcal{L}$, and gain $\mathcal{G}$) so that

$$\frac{d}{dt}|x(t)\rangle = \{-i\mathcal{H}_0 - \mathcal{L} + \mathcal{G}(N(t))\}|x(t)\rangle + |f(t)\rangle, \quad (25)$$

where only the gain operator $\mathcal{G}(N)$ depends on the inversion N. We consider an almost single-mode laser and use the normalized Gramm-Schmidt basis of Sec. III B to write

$$|x(t)\rangle = [A_0 + \Delta A(t)]e^{-i\phi(t)}|c_1\rangle + \sum_{i\geq 2} c_i(t)|c_i\rangle, \quad (26)$$

where A_0, $\Delta A(t)$, and $\phi(t)$ are the steady-state and time-dependent parts of the amplitude, and the phase of the dominant mode $|c_1\rangle$, respectively (all real-valued), and where $c_i(t)$ are the complex-valued amplitudes of the weak side modes $|c_i\rangle$. As we assumed mode $|c_1\rangle$ to dominate over the others, this mode must have relatively low losses and must practically maintain itself also in the absence of noise, so that the steady-state condition reads

$$\{-i\mathcal{H}_0 - \mathcal{L} + \mathcal{G}(N_0)\}|c_1\rangle \approx 0, \quad (27)$$

where N_0 is the threshold inversion.

With the above steady-state condition (27), the linearized laser rate equations can be derived by inserting Eq. (26) into Eq. (25), expanding the gain operator via $\mathcal{G}(N) = \mathcal{G}(N_0) + \partial\mathcal{G}/\partial N\,\Delta N$, and projecting onto the eigenmode $|c_1\rangle$. The real and imaginary parts of the resulting equation separate into

$$\frac{d}{dt}\Delta A(t) = \mathrm{Re}\left[\langle c_1|\frac{\partial\mathcal{G}}{\partial N}|c_1\rangle\right]A_0\Delta N(t) + \mathrm{Re}[\{f_1(t)+g_1(t)\}e^{i\phi}], \quad (28a)$$

$$\frac{d}{dt}\phi(t) = \mathrm{Im}\left[\langle c_1|\frac{\partial\mathcal{G}}{\partial N}|c_1\rangle\right]\Delta N(t) + \frac{1}{A_0}\mathrm{Im}[\{f_1(t)+g_1(t)\}e^{i\phi}], \quad (28b)$$

where Re and Im denote the real and imaginary parts. As f and g have random phases the real and imaginary parts of these noise sources are uncorrelated.

In each of these equations the three consecutive terms at the right-hand side are effective ''noise sources'' due to the inversion fluctuations, the spontaneous emission noise, and the fluctuating side-mode amplitudes. Of these three only the middle corresponds to spectrally white noise, whereas the other two contain the dynamics of the inversion and all side-mode amplitudes, respectively. The expressions for the (complex) noise sources

$$f_1(t)=\langle c_1|f(t)\rangle, \tag{29a}$$

$$g_1(t)=\sum_{i\geqslant 2}\langle c_1|-i\mathcal{H}|c_i\rangle\, c_i(t)=\sum_{i\geqslant 2}\kappa_{1i}\, c_i(t) \tag{29b}$$

are identical to their counterparts below threshold [see Eq. (22)]. Even the magnitude of $f_1(t)$ will be about the same below and above threshold, as the diffusion matrix $\mathcal{D}$ is determined by the loss and (saturated) gain, and thereby hardly depends on laser power. The magnitude and dynamics of the excess noise $g_1(t)$, however, might be different below and above threshold, but only when the optical saturation is anisotropic, i.e., when the presence of a strong lasing mode affects the strength and dynamics of the side-mode amplitudes $c_i(t)$.

The generating mechanism of excess noise is thus found to be the same below and above threshold; in both cases the excess noise originates from field fluctuations in states orthogonal to the lasing mode that project into this mode upon evolution. Also above threshold, the excess noise factor is best characterized by a frequency-dependent multiplier $K(\omega)>1$ that acts on the noise sources in both evolution equations for the laser amplitude variation and optical phase. However, above threshold the function $K(\omega)$ will only be identical to its below-threshold counterpart, when the optical saturation is isotropic; anistropic saturation can make $K(\omega)$ dependent on output power (see Ref. [17] for an example).

We will finish this section with a brief discussion of the case of anisotropic saturation, where the atoms are not equally saturated, to explore this more complicated problem. Depending on the type of anisotropy, we have to deal with spatial, spectral, and/or polarization hole burning. One way to do this is to separate the atomic inversion into different spatial, frequency, and/or spin classes. Apart from the average inversion $N(t)$, we might have to introduce a spin-difference inversion, as is used to describe the polarization dynamics in semiconductor vertical-cavity lasers [36] and HeXe gas lasers [10], or spatial Fourier components of a position-dependent inversion $N(\vec{r},t)$, as is used in the case of spatial hole burning [23]. The anisotropy of the optical saturation is now related to differences in the loss rates of the various inversion classes. If the extra inversion classes exhibit rapid decay, for instance due to fast spin flips or fast spatial diffusion, the saturation becomes almost isotropic and a treatment in terms of a single average inversion $N(t)$ might suffice. If the decay is not so rapid the populations in these extra inversion classes have to be included in the description. These populations can scatter light and thereby couple the various modes in a nonlinear way, i.e., with a power-dependent coupling strength, which leads among others to the appearance of four-wave-mixing peaks in the optical spectrum [37].

The saturation properties of a class-A laser with a position-dependent inversion $N(\vec{r},t)$ have been discussed at length in Ref. [23]. With the optical field expressed in the Gramm-Schmidt basis of Eq. (25), the position-dependent saturation, which scales with the local intensity $I(\vec{r})=|E(\vec{r})|^2$, is induced by two effects [23]: (i) the intensity profile $|E_1(\vec{r})|^2$ of the dominant mode can burn a spatial (or polarization) hole in the inversion distribution (hole burning due to weak side modes can generally be neglected), and (ii) the intensity profile $E_1(\vec{r})E_i^*(\vec{r})+\text{c.c.}$ that results from interference between the dominant mode and any side mode, can redistribute the inversion, leaving the spatial-average unaffected (as $\langle c_i|c_j\rangle=\delta_{ij}$). For class-A lasers these effects have been denoted as the hole-burning part and the population-pulsation part, respectively [23]. To keep track of the full position dependence of $N(\vec{r},t)$ we would have to introduce a large number of inversion reservoirs, i.e., Fourier components of $N(\vec{r},t)$, each of which could in principle have its own dynamics. Together with the average inversion N these determine the gain matrix G, which becomes quite complicated since a position-dependent inversion will not only produce isotropic gain, but will also scatter the optical field from one eigenmode to another. For simplicity we will therefore assume that the additional inversion reservoirs decay very rapidly, so that the buildup of any position dependence in $N(\vec{r},t)$ is heavily frustrated, and it is sufficient to work with a single, spatially averaged, inversion $N(t)$.

C. Phase fluctuations in lasers above threshold

The time evolution of the phase of the dominant eigenmode, as described by Eq. (28b), is rather simple as this optical phase has no intrinsic dynamics, i.e., it experiences no damping, and is driven only by a combination of three "noise sources." However, of the three noise sources only the second term, $\mathrm{Im}[f_1(t)e^{i\phi}]/A_0$, is spectrally white, while the other two are spectrally colored, by the dynamics of inversion and side-mode amplitudes, respectively. In general, the optical phase will therefore not perform a pure diffusion, and the optical spectrum of the emitted laser light can deviate significantly from a simple Lorentzian form. On a long-time scale, i.e., for small frequencies, all noise sources will be effective and the phase evolution will be approximately diffusive, with a diffusion rate that is enhanced by the usual geometric excess noise factor $K_{geo,1}$, possibly multiplied by an additional excess noise factor $(1+\alpha^2)$ (see below). However, on a very-short-time scale, i.e., short with respect to the fluctuations in N and c_i, the phase evolution is only partially diffusive and the diffusion rate reduces to its standard value, being $D_{\phi\phi}=D/|A_0|^2$ in our notation, as two of the three noise sources act only as a static frequency shift.

To demonstrate the effect of colored noise on the dynamics of the optical phase, we will first set $\mathrm{Im}[\langle c_1|\partial\mathcal{G}/\partial N|c_1\rangle]=0$ and consider only the excess noise due to side-mode dynamics, as contained in the noise source $g_1(t)$. We take as an example the two-mode case, which a.o. applies to the polarization dynamics in a laser [16,17]. For the two-mode case, the noise power spectrum $\overline{|g(\omega)|^2}$ is contained in a Lorentzian-shaped spectrum, centered around the relative frequency $\omega_2-\omega_1$, having a width that is given by the damping rate γ_2 of the side mode. For $\omega_2=\omega_1$ the time

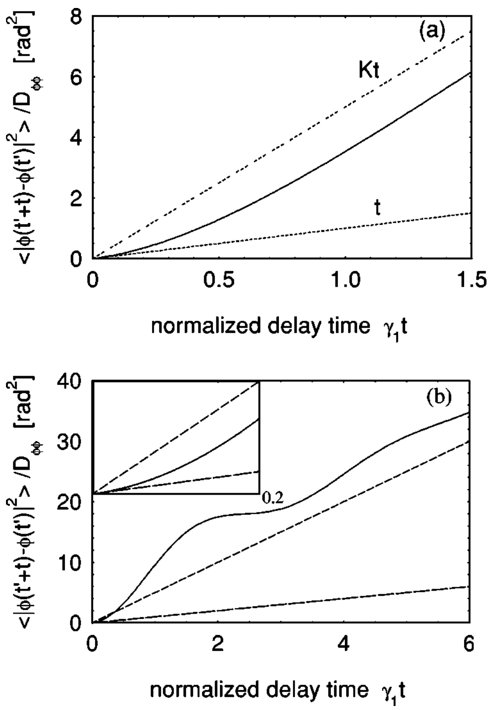

FIG. 2. Sketch of the time evolution of the laser phase in a two-mode system. In (a) the excess noise is generated by a weak side mode that has the same resonance frequency, but a larger damping rate than the main mode [(a) corresponds to $K=5$]. In (b) the excess noise is generated by the relaxation oscillation dynamics of the inversion, which can also affect the laser phase [(b) corresponds to $\alpha=2$ and $\gamma_{ro}/\omega_{ro}=0.2$]. In both figures, the slope is unaffected for small time delays, but increases at larger times by a factor K and $1+\alpha^2$, respectively. This demonstrates how excess noise develops only after a certain time delay.

evolution of the mean-square phase difference is easily found to be

$$\overline{|\phi(t'+t)-\phi(t')|^2} = [\,1+(K_{geo,1}-1)(1-e^{-\gamma_2|t|})\,]\,D_{\phi\phi}|t|, \quad (30)$$

where the overline denotes averaging over t'. The solid curve in Fig. 2(a) depicts this time evolution and thereby demonstrates one of the consequences of the spectral coloring of excess noise. For small times $|t|\ll 1/\gamma_2$ the phase evolves as if there is no excess noise, while for larger times the phase diffusion rate is enhanced by a factor $K_{geo,1}$; the excess noise takes time to develop and is noticeable only on a sufficiently long time scale. The nondiffusive short-time evolution will show up in the optical spectrum as deviations from a pure Lorentzian shape for frequencies $|\omega-\omega_1|\geqslant\gamma_2$. However, such spectral deviations will be experimentally accessible only when the diffusion rate $\mathcal{D}$ is not too small as compared to γ_2, i.e., when the nondiffusive evolution persists to large enough phase deviations (nondiffusive evolution up to angles of ≈ 1 rad will produce highly non-Lorentzian line shapes).

For the general N-mode case, the phase equation (28b) can still be solved separately from the amplitude and inversion dynamics, as long as $\mathrm{Im}[\langle c_1|\partial G/\partial N|c_1\rangle]=0$. In that case a simple Fourier transformation yields

$$\overline{|\phi(\omega)|^2} = \frac{\overline{|f(\omega)|^2}+\overline{|g(\omega)|^2}}{2A_0^2\omega^2} = K_n(\omega)\,\frac{D_{\phi\phi}}{\omega^2}, \quad (31)$$

where we recognize the generic structure of spectrally colored excess noise in the form of a frequency-dependent multiplier $K_n(\omega)$.

We will now discuss the physics of the first noise source in Eq. (28b), which contains a factor $\mathrm{Im}[\langle c_1|\partial\mathcal{G}/\partial N|c_1\rangle]$ and describes how the modal resonance frequency changes with population inversion. This factor is zero in lasers that operate on the center of a symmetric gain profile, but can be nonzero otherwise. In semiconductor lasers it plays an important role and is usually quantified by the so-called linewidth enhancement factor or Henry parameter α, being defined as the ratio of the imaginary and real part of $\partial\mathcal{G}/\partial N$ [38]. As the name indicates, a nonzero α leads to an enhancement of the laser linewidth, by a factor $(1+\alpha^2)$ as compared to lasers with $\alpha=0$. From the structure of Eq. (28b) it is clear that this enhancement is also spectrally colored, and can be interpreted as just another type of excess noise. Figure 2(b) depicts a typical time evolution of the mean-square phase difference, where we used Eq. (11a) from Ref. [39] with $\alpha=2$ and $\gamma_{ro}/\omega_{ro}=0.2$ for the ratio of the relaxation oscillation damping rate and frequency. For small t the phase evolution is unperturbed (see inset, which runs from 0 to 0.2 in normalized delay time), while the phase diffusion rate is enhanced by a factor $(1+\alpha^2)$ for large t, i.e., larger than the inverse relaxation oscillation frequency and damping rate, which sets the inversion dynamics. As a consequence of this nondiffusive phase evolution the optical spectrum is expected to deviate from a Lorentzian shape for frequency offsets comparable to the relaxation oscillation frequency, as has been observed experimentally [39].

For a laser with both $\alpha\neq 0$ and $K_{geo}>1$ the long-term phase diffusion rate will be enhanced by the product $K_{geo}(1+\alpha^2)$, but only if the inversion dynamics is much slower than the side-mode dynamics, so that the inversion fluctuations ΔN have sufficient time to get multiplied by the same excess factor K_{geo} that also enhances the intensity fluctuations. For the other extreme case, where the inversion dynamics is much faster than the side-mode dynamics, the spectral coloring of the noise $K(\omega)$ is such that the fast intensity and inversion fluctuations will hardly be enhanced. In this case the overall enhancement of the long-term phase diffusion is expected to be equal only to $K_{geo}+\alpha^2$ instead of $K_{geo}(1+\alpha^2)$.

D. Intensity fluctuations in lasers above threshold

We only consider the case of isotropic saturation, which applies a.o. to lasers with fast spatial diffusion of the inversion. In this case it is sufficient to work with a single average inversion $N(t)$, and Eq. (28a) already gives a proper description of the amplitude dynamics of the dominant mode, when supplemented by the rate equation for the inversion dynamics. For the case of isotropic saturation this equation reads

$$\frac{d}{dt}N(t)=\Lambda-\gamma_{\parallel}[1+P_{\mathrm{tot}}(t)]N(t)+f_N(t), \tag{32}$$

where Λ is the pump rate, $\gamma_{\parallel}$ is the inversion decay rate in the absence of light, P_{tot} is the total optical power, and f_N is a Langevin noise source. The important point to note is that, for the considered case of isotropic saturation, the optical saturation through stimulated emission is proportional only to the total power in all optical modes ($P_{\mathrm{tot}}=\langle x(t)|x(t)\rangle$ in our notation), and is insensitive to the modal power distribution. In the normalized Gramm-Schmidt basis of Sec. (III B) $P_{\mathrm{tot}}=\Sigma|c_i|^2$ can be linearized to $A_0^2+2A_0\Delta A_0$ when one mode dominates and when the second-order terms due to weak side modes are neglected. Linearization of Eq. (32) around the steady state and in combination with the earlier Eq. (28a) then gives the following description of the coupled intensity-inversion dynamics:

$$\frac{d}{dt}\Delta A_0(t)=g'A_0\Delta N(t)+\mathrm{Re}[\{f_1(t)+g_1(t)\}e^{i\phi}], \tag{33a}$$

$$\frac{d}{dt}\Delta N(t)=-M\gamma_{\parallel}\Delta N-\gamma_{\parallel}N_0 2A_0\Delta A_0(t)+f_N(t), \tag{33b}$$

where we have defined the gain derivative $g'=\mathrm{Re}[\langle c_1|\partial\mathcal{G}/\partial N|c_1\rangle]$, where N_0 is the steady-state inversion, and where M is the normalized pump parameter ($M=\Lambda/\Lambda_{\mathrm{th}}$ and the steady-state value of P_{tot} is $M-1$).

As the above equations are linear in $\Delta A_0(t)$ and $\Delta N(t)$, they can be easily solved in the Fourier domain to yield the usual damped relaxation oscillation resonance for the case of small $\gamma_{\parallel}\ll g'N_0$ (class-B laser), or the overdamped resonance for the case of large $\gamma_{\parallel}\gg g'N_0$ (class-A laser). However, the above equations are different from the usual ones, as the field equation (33a) contains an additional noise source $\mathrm{Re}[g_1(t)e^{i\phi}]$. This leads to an effective enhancement of the total Langevin noise by a frequency-dependent multiplier $K_n(\omega)$, and thereby confirms the heuristic approach used in Ref. [12] to explain the observed intensity noise in lasers with an unstable resonator, without yet considering the possibility of spectral coloring. Furthermore, it is important to note that the extra noise source shows up only in the field equation (33a), where it arises from the admixture of weak side-mode amplitudes into the dominant mode, but not in the inversion equation (33b), which contained only second-order contributions of the side modes that have been neglected. As the excess noise source $g_1(t)$ is not compensated by an anticorrelated noise source in the inversion equation, as is the case for $f(t)$, it can seriously hinder the generation of intensity-squeezed light, up to the point $K>1.5$, where intensity squeezing is thought to become impossible [31].

We will finish this section with an alternative description of excess intensity and inversion noise, now formulated directly in terms of the total optical power $P_{\mathrm{tot}}(t)$ and the inversion $N(t)$. To properly describe excess noise, we need a third variable (or set of variables) χ to specify the other degrees of freedom of the optical field, as was previously done with the side-mode amplitudes c_i. In these variables the coupled dynamics of the inversion and total power can be written as

$$\frac{d}{dt}P_{\mathrm{tot}}=G_{\mathrm{eff}}(N,\chi)P_{\mathrm{tot}}+f_P, \tag{34a}$$

$$\frac{d}{dt}N=\Lambda-\gamma_{\parallel}(1+P_{\mathrm{tot}})N+f_N, \tag{34b}$$

where G_{eff} is the net intensity gain (= gain minus loss), and where the ''stimulated-emission part'' of the noise sources f_P and f_N are anticorrelated. In this notation, excess noise will appear when the net intensity gain already depends to first order on χ, so that fluctuations around $\chi=0$ will produce an extra (excess) noise term $(\partial G/\partial\chi)\,\Delta\chi\,P_{\mathrm{tot}}$ in Eq. (34a), being equivalent to the term $\mathrm{Re}[g_1(t)e^{i\phi}]$ in Eq. (28a). The present formulation thus provides a convenient physical picture for the origin of excess noise; excess noise appears whenever the distribution of the optical power over the various degrees of freedom has a serious (read ''first-order'') effect on the net intensity gain. We note that Eq. (34b) does not contain such an excess noise term, as the optical saturation was assumed to be isotropic, i.e., to depend only on the total power P_{tot}, but not on the distribution of this power over the optical state space. This alternative description could a.o. be used to explain the occurence of excess noise in laser cavities with tilted end mirrors and nonuniform transverse loss [40]. The variable χ could then specify the shape, i.e., position, direction, and width, of the laser beam, to emphasize that the excess noise is generated by shape changes in the laser beam that lead to changes in the net gain, which, after a finite time, evolve into power and phase changes of the optical field.

V. CONCLUDING DISCUSSION

We finish with a few general remarks. Excess noise was found to originate from fluctuations in other, nonorthogonal, eigenmodes that project into the dominant (=lasing) mode. For a laser below threshold, with its simple linear dynamics, the excess noise can be canceled by observation in the adjoint direction, i.e., by admixture of the proper amount of correlated side-mode fluctuations. As the noise generating mechanism is the same, similar tricks are possible for lasers operating above the lasing threshold. Both the relation between correlated side-mode dynamics and excess noise, and their use in the suppression of power fluctuations, have recently been demonstrated in two experiments. For transverse modes (K_{trans}), Poizat *et al.* cut off part of the laser beam and thereby project correlated transverse modes into the measured intensity fluctuations [41]. For polarization-related excess noise (K_{pol}), van der Lee *et al.* used a rotatable polarizer to admix the weak polarization mode and change the excess intensity noise [10].

Spectral coloring of excess noise is thought to be universal in lasers, both below and above threshold. For polarization-related excess noise (K_{pol}) the existence of spectral coloring and its relation to the side-mode dynamics has been clearly demonstrated in the measured intensity noise [10]. For lasers with excess noise due to nonorthogonal transverse modes ($K_{\rm trans}$) spectral coloring has not yet been observed. From the above discussion we predict that this coloring will again be related to the side-mode dynamics, i.e., to the time it takes injected noise to evolve from the adjoint direction into the eigenmode direction. In the terminology of unstable resonators, this translates into the number of loss-free periods that the injected adjoint light experiences before it is projected onto the eigenmode [7].

Finally, it is interesting to consider which parts of the physical picture developed here remain valid in a full quantum treatment. Reference [19] presents a treatment where only a single Fox-Li mode, the lasing mode, is used in the quantum theory, with all the other modes being made orthogonal to it. A description in such an orthogonal basis has the advantage that one can use the standard quantum recipe and commutation relations between the modal amplitudes, but the disadvantage that the mentioned orthogonal modes are generally not eigenmodes of the system anymore [3]. To follow as closely as possible the semiclassical approach developed here, we think it is more convenient to keep the quantum treatment entirely in terms of biorthogonal Fox-Li modes as in Ref. [42]. Then the main change that is needed to carry the semiclassical picture developed here over to a quantum theory is a revision of the assumptions behind Eq. (9a). As Fox-Li modes are outgoing modes [43], they can only form a complete set for the space of outgoing fields [42]. Semiclassically, this is not a problem because there are no incoming components as the external field emanates from the cavity only. In a quantum theory, however, there is always the external vacuum field, part of which does not emanate from the cavity. This external vacuum field generates necessarily an incoming component of the cavity field [42] making an expansion in Fox-Li modes alone incomplete.

In summary, excess noise appears when one reduces the dynamics of an N-dimensional system with nonorthogonal eigenmodes, as in Eq. (1), into a one-dimensional description of one of the eigenmodes. A consequence of this reduction is that the original white noise source separates into two parts, one part that represents the direct noise injection and that remains spectrally white, and another part that represents the noise injection into other modes and that evolves into the measurement direction due to the non-Hermitian system evolution. As the ''time-delayed'' part is spectrally colored and reflects the dynamics of the other modes, the excess noise factor is best written as $K_n = 1 + [K_n(\omega) - 1]$. As a result of this spectral coloring, the amount of the excess noise was found to be not only related to the geometry, i.e., the nonorthogonality between the eigenmodes, but also to the system dynamics, i.e., The eigenvalues of the evolution matrix $-i\mathcal{H}$. For a laser with isotropic saturation, these results were shown to apply both below and above the lasing threshold.

ACKNOWLEDGMENTS

We acknowledge support of the ''Stichting voor Fundamenteel Onderzoek der Materie (FOM)'' and of the European Union in the the TMR network ERB4061 PL951021 (Microlasers and Cavity QED). The research of N.J. van Druten was made possible by the ''Koninklijke Nederlandse Akademie van Wetenschappen.''

[1] C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[2] K. Petermann, IEEE J. Quantum Electron. **15**, 566 (1979).
[3] A.E. Siegman, Phys. Rev. A **39**, 1253 (1989); **39**, 1264 (1989).
[4] H.A. Haus and S. Kawakami, IEEE J. Quantum Electron. **21**, 63 (1985).
[5] W.A. Hamel and J.P. Woerdman, Phys. Rev. Lett. **64**, 1506 (1990).
[6] S.J.M. Kuppens, M.P. van Exter, M. van Duin, and J.P. Woerdman, IEEE J. Quantum Electron. **31**, 1237 (1995).
[7] Å.M. Lindberg, M.A. van Eijkelenborg, K. Joosten, G. Nienhuis, and J.P. Woerdman, Phys. Rev. A **57**, 3036 (1998).
[8] Å.M. Lindberg, G.P. Karman, K. Joosten, G. Nienhuis, and J.P. Woerdman, Opt. Commun. **153**, 55 (1998).
[9] O. Emile, M. Brunel, F. Bretenaker, and A. LeFloch, Phys. Rev. A **57**, 4889 (1998).
[10] A.M. van der Lee, N.J. van Druten, A.L. Mieremet, M.A. van Eijkelenborg, Å.M. Lindberg, M.P. van Exter, and J.P. Woerdman, Phys. Rev. Lett. **79**, 4357 (1997).
[11] O. Emile, M. Brunel, A. LeFloch, and F. Bretenaker, Europhys. Lett. **43**, 153 (1998).
[12] M.A. van Eijkelenborg, M.P. van Exter, and J.P. Woerdman, Phys. Rev. A **57**, 571 (1998).
[13] M.A. van Eijkelenborg, Å.M. Lindberg, M.S. Thijssen, and J.P. Woerdman, Phys. Rev. Lett. **77**, 4314 (1996).
[14] M.A. van Eijkelenborg, Å.M. Lindberg, M.S. Thijssen, and J.P. Woerdman, Opt. Commun. **137**, 303 (1997).
[15] Y.-J. Cheng, C.G. Fanning, and A.E. Siegman, Phys. Rev. Lett. **77**, 627 (1996).
[16] A.M. van der Lee, N.J. van Druten, A.L. Mieremet, M.P. van Exter, and J.P. Woerdman, Phys. Rev. Lett. **81**, 5121 (1998).
[17] A.M. van der Lee, A.L. Mieremet, M.P. van Exter, N.J. van Druten, and J.P. Woerdman, Phys. Rev. A **61**, 033812 (2000).
[18] Ph. Grangier and J.-Ph. Poizat, Eur. Phys. J. D **1**, 97 (1998).
[19] Ph. Grangier and J.-Ph. Poizat, Eur. Phys. J. D **7**, 99 (1999).
[20] M. Patra, H. Schomerus, and C.W.J. Beenakker, Phys. Rev. A **61**, 023810 (2000).
[21] Strictly speaking, the necessary and sufficient condition for $\mathcal{H}$ to have orthogonal eigenmodes is $[\mathcal{H}_0, \mathcal{A}] = 0$, with $\mathcal{H}_0$ and $-i\mathcal{A}$ as Hermitian and anti-Hermitian parts of $\mathcal{H}$, which is weaker than the condition that $\mathcal{H}$ is Hermitian.
[22] W.H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).
[23] M. Sargent III, M.O. Scully, and W.E. Lamb, Jr., *Laser Physics* (Addison-Wesley, London, 1974).
[24] Y. Yamamoto and N. Imoto, IEEE J. Quantum Electron. **QE-22**, 2032 (1986).
[25] L. Davidovich, Rev. Mod. Phys. **68**, 127 (1996).

[26] P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer-Verlag, Berlin, 1990).
[27] If the ''isotropy assumption'' is violated the diffusion operator cannot be removed from $\langle v_i|\mathcal{D}|v_j\rangle$, so that the geometric expression for excess noise becomes somewhat more complicated [see a.o. Y. Champagne and N. McCarthy, IEEE J. Quantum Electron. **28**, 128 (1992)].
[28] G.H.C. New, J. Mod. Opt. **42**, 799 (1995).
[29] A.E. Siegman, Appl. Phys. B: Lasers Opt. **60**, 247 (1995).
[30] H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1996), Appendix A1.
[31] A.M. van der Lee, N.J. van Druten, M.P. van Exter, J.P. Woerdman, J.-Ph. Poizat, and Ph. Grangier, Phys. Rev. Lett. **85**, 4711 (2000).
[32] L.A. Westling, M.G. Raymer, and J.J. Snyder, J. Opt. Soc. Am. B **1**, 150 (1984).
[33] F. Marin, A. Bramati, E. Giacobino, T.C. Zhang, J.-Ph. Poizat, J.-F. Rouch, and P. Grangier, Phys. Rev. Lett. **75**, 4606 (1995).
[34] S. Lathi and Y. Yamamoto, Phys. Rev. A **59**, 819 (1999).
[35] S.M. Dutra, K. Joosten, G. Nienhuis, N.J. van Druten, A.M. van der Lee, M.P. van Exter, and J.P. Woerdman, Phys. Rev. A **59**, 4699 (1999).
[36] M. San Miguel, Q. Feng, and J.V. Moloney, Phys. Rev. A **52**, 1728 (1995).
[37] M.P. van Exter, A. Al-Remawi, and J.P. Woerdman, Phys. Rev. Lett. **80**, 4875 (1998).
[38] C.H. Henry, IEEE J. Quantum Electron. **QE-19**, 1391 (1983).
[39] M.P. van Exter, W.A. Hamel, J.P. Woerdman, and B.R.P. Zeijlmans, IEEE J. Quantum Electron. **28**, 1470 (1992).
[40] S. Longhi, Opt. Lett. **25**, 811 (2000).
[41] J.-Ph. Poizat, T. Chang, and P. Grangier, Phys. Rev. A **61**, 043807 (2000).
[42] S.M. Dutra and G. Nienhuis, Phys. Rev. A **62**, 063805 (2000).
[43] L. Ronchi, in *Laser Handbook*, edited by F. T. Arecchi and E. O. Schultz-DuBois (North-Holland, Amsterdam, 1972), Vol. 1, pp. 151–190.