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Quantum Hall States and Boson Triplet Condensate for Rotating Spin-1 Bosons

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We propose and analyze two series of clustered quantum Hall states for rotating systems of spin-1 bosons. The first series [labeled SU(4)k] includes the exact ground states of a model Hamiltonian at large angular momentum L, and also for N = 3k particles at L = N. The latter is a spin-singlet boson-triplet condensate. The second series, labeled SO(5)k, includes exact ground states at large L for different parameter values.

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In the study of the phenomenon of Bose-Einstein condensation (BEC) of cold atoms, the possibility of using bosons with internal degrees of freedom (spin 1 in particular) and of rotating the condensate has led to fascinating new possibilities for the states of matter that can be formed and observed. A multicomponent BEC can be realized by trapping higher spin atoms such as 87Rb [1] and 23Na [2] in optical traps, which do not affect the spin degeneracy, and spin-1 or “vector” BEC’s have been realized and studied [2,3]. In a model description, the two-body interactions of the spin-1 atoms are written as $\delta(r_i - r_j) [c_0 + c_2 S_i \cdot S_j]$, with $c_0 = (g_0 + 2g_2)/3$, $c_2 = (g_2 - g_0)/3$, and $g_2 = 4\pi\hbar^2a_2/M_b$ with $M_b$ the boson mass and $a_2$ the s-wave scattering length in the total spin S channel [4,5]. In the case of 23Na one has $c_2 > 0$ and, hence, “antiferromagnetic” spin correlations. The “polar” ground state [4,5] has $S = 0$; it supports many interesting collective excitations (see [6] for a review). The case of 87Rb has $c_2 < 0$, leading to a spin-polarized (ferromagnetic) ground state.

The effect of rotating a “scalar” BEC of spinless or spin-polarized atoms in an isotropic harmonic trap of frequency $\omega_0$ can be studied at mean field level (the Gross-Pitaevskii equation), and a vortex lattice is found at sufficiently high rotation frequency $\omega$, in agreement with experimental observations [7]. At still higher frequencies, quantum fluctuations become important and it has been argued that the vortex lattice is replaced by a sequence of distinct quantum fluids, which can be understood using a mapping to the fractional quantum Hall (qH) effect [8]. In this Letter, we explore the corresponding possibilities in a rotating system of spin-1 bosons.

In the regime $\omega_0 - \omega \ll \omega_0$ and very weak interaction, only the lowest-energy single-particle states in the trap are of interest. These have non-negative angular momentum $m = 0, 1, \ldots$ about the rotation axis, and no excitation of the motion along the axis, so the motion is effectively in the xy plane. The $x, y$ dependence of these wave functions is $\psi_m(z) \propto z^m e^{-|z|^2/2}$ [where $z = x + iy$, and the quantum length ($\hbar/M_b \omega_0)^{1/2}$ of the trap, and $\hbar$, have been set to 1]; they correspond to the lowest Landau level (LLL) [8].

We consider the interaction Hamiltonian for bosons $i = 1, \ldots, N$, restricted to the LLL,

$$H_{\text{int}} = \sum_{i<j} \delta(r_i - r_j) [c_0 + c_2 S_i \cdot S_j]$$

(1)

The ground state (GS) for the system rotating at frequency $\omega$ is found by minimizing $H_{\text{tot}} = H_{\text{int}} + (\omega_0 - \omega)L$ [8], where $L = \sum m_i$ is the total angular momentum. For $N$ particles in the LLL with zero center-of-mass angular momentum, the average “filling factor” of the occupied states can be defined as $\nu = N/\langle N \rangle$, where $N = 2L/N$ is the number of vortices there would be if if there were a Bose condensate (we assume here a uniform average occupation of states up to $m = N\nu$). The use of only the LLL states is physically reasonable when $\omega_0 - \omega \ll \omega_0/\langle N \rangle$ (so that particles in the LLL at $m = N\nu$ cannot lower their energy by moving to a non-LLL state with $m = 0$) and $\nu c_5 \ll \omega_0$, $S = 0, 2$ (so that perturbative corrections from mixing non-LLL states into the GS are negligible).

For the scalar case (in the LLL, without $c_2$), a study [9] using periodic boundary conditions found that the vortex lattice melts at a critical $\nu = \nu_c \sim 10$, that incompressible fluid states occur at $\nu = 1, 2, 3, \ldots < \nu_c$, and that the corresponding GS wave functions have large overlaps with the Read-Rezayi (RR) [10] series of fractional qH states.

Here, for the case of vector bosons, we first propose two series of spin-singlet generalizations of the RR states [labeled SU(4)k and SO(5)k, respectively, and $k = 1, 2, \ldots$] and explain their structure. In a thermodynamic limit $N \to \infty$ with $\nu$ fixed, they each represent incompressible liquid phases of the system, with $\nu = 3k/4$ for the SU(4)k states and $\nu = k$ for the SO(5)k series. Then we present results for the GS’s of $H_{\text{tot}}$ as a function of $L$, $c_0$, and $c_2$. We identify a region in the $c_0$-$c_2$ plane where the
proposed qH states with \( k = 1 \) give the exact GS’s at high values of \( L \). We demonstrate that the \( k = N/3 \) member of the SU(4)\(_k\) series is an exact eigenstate and, apparently, the GS of \( H_{\text{int}} \) with \( c_2 = 0 \), \( c_0 > 0 \), at angular momentum \( L = N \). In second quantization, this state, a spin-singlet boson-triplet condensate (bog HM), becomes the form

\[
|\text{BTC} \rangle = [e^{\mu_1 \mu_2 \mu_3} b_{\mu_1}^\dagger b_{\mu_2}^\dagger b_{\mu_3}^\dagger]^N/3|0\rangle, \tag{2}
\]

where \( b_{\mu_1}^\dagger \) is a creation operator for the single-particle state \( \psi_{\mu_1}(z) \) with spin \( \mu \), where \( \mu = x, y, z \). Finally, we show the GS’s of \( H_{\text{int}} \) as a function of \( \omega \).

We now explain our analysis, starting with the construction of spin-singlet qH states for spin-1 bosons. This construction is a generalization of similar constructions for scalar and spin-1/2 particles (either of which could be electrons or bosons) [10–12]. These papers all employed a correspondence [13] between qH states and specific conformal field theories (CFT’s). This connection allows one to obtain trial qH wave functions as chiral correlators in CFT’s that are associated to a Lie algebra \( G \) and integers \( k \geq 1 \) and \( M \geq 0 \), where \( M \) is even (odd) for bosons (fermions). These states generalize the pairing familiar from the theory of superconductivity to the formation of “clusters.” The resulting qH wave functions have a property that guarantees that they are exact zero-energy eigenstates of certain model Hamiltonians. For example, starting from \( G = \text{SU}(n + 1) \) and putting \( M = 0 \), one can find completely symmetric wave functions that obey

\[
\Psi(z_1, \ldots, z_N) \neq 0 \quad \text{for} \quad z_1 = \cdots = z_l \quad (l \leq k),
\]

\[
\Psi(z_1, \ldots, z_N) = 0 \quad \text{for} \quad z_1 = \cdots = z_{k+1}, \tag{3}
\]

independent of the spins of the particles involved. Hence, these are zero-energy eigenstates of a Hamiltonian with a repulsive \( k + 1 \)-body \( \delta \)-function interaction [10] and are the unique states of the lowest \( L \) with this property. The existence of such a Hamiltonian and the fact that it has a gap in its energy spectrum ensures that a corresponding incompressible liquid phase of matter exists (when \( N \to \infty \) at fixed \( k, M \)) over a range of interaction parameters, not just for the \( k + 1 \)-body model. The filling factors of these SU\((n + 1)\)_\(k\) states are

\[
\nu(n, k, M) = \frac{nk}{nkM + n + 1}. \tag{4}
\]

The parameter \( n \) corresponds to the number of components of an internal degree of freedom of the particles, and these models have U\((n)\) symmetry; the clusters contain \( nk \) particles. For \( n = 1 \) this construction gives the RR states, while for \( n = 2 \) it produces spin-singlet states for spin-1/2 particles [11]. Finally, clustered qH states admit excitations (quasiparticles) of effectively fractional “charge” (i.e., particle number). The simplest of these can be viewed as the result of the adiabatic insertion of a fraction of a quantum of magnetic flux (or vorticity), where the allowed fraction is \( 1/nk \) in the present cases. States with more than three quasiparticles at well-separated fixed positions display large degeneracies, which give rise to “non-Abelian statistics” [13] and are understood for \( n = 1, 2 \) [11,14].

Our first series of clustered spin-singlet qH states of spin-1 bosons consists of the SU\((4)\)_\(k\) states (\( n = 3 \) above), where we put \( M = 0 \). The filling factor is \( \nu = 3k/4 \). The CFT construction guarantees that, for \( N \) divisible by 3k, this state is a singlet under SU(3), and hence also under its SO(3) subgroup, the usual spin-rotation group. The wave function for \( N = 3kp \) particles can be written in terms of its components for particular spin states \( \mu = x, y, z \) specified for each particle (similar to [15]), as

\[
\Psi_k^{\text{SU}(4)} = S_{\text{groups}}[P_{\text{groups}}^{4, 2, 2, 1, 1, 1}], \tag{5}
\]

where

\[
\Psi^{2, 2, 2, 1, 1, 1}(z_{x_1}^1, \ldots, z_{x_p}^p; z_{y_1}^1, \ldots, z_{y_p}^p; z_{z_1}^1, \ldots, z_{z_p}^p) = \prod_{\mu=x,y,z} \prod_{i<j} \left( z_{i}^{\mu} - z_{j}^{\mu} \right)^2 \prod_{\mu' < \mu} \prod_{i,j} \left( z_{i}^{\mu'} - z_{j}^{\mu'} \right). \tag{6}
\]

In words, the operations \( P_{\text{groups}} \) and \( S_{\text{groups}} \) in Eq. (5) tell one to divide the \( 3kp \) bosons into \( k \) groups of \( 3p \) bosons each (\( p \) of each spin polarization), to write a factor \( \Psi^{2, 2, 2, 1, 1, 1} \) as in Eq. (6) for each group, and, finally, to symmetrize over all ways the particles can be divided over the \( k \) groups. For \( k = 1 \) there is a single group, and one finds a state of total degree \( L = 3p(2p - 1) \) that generalizes the Laughlin and Halperin states for scalar and spin-1/2 particles, respectively, and which is a zero-energy eigenstate of \( H_{\text{int}} \) for all \( c_0, c_2 \), and the unique GS when both \( g_0, g_2 > 0 \) since \( H_{\text{int}} \) is then positive. Putting instead \( k = N/3 \) gives \( k \) groups of three particles each, the resulting state being the \( L = N \) BTC, Eq. (2). The fundamental quasiparticles over the SU\((4)\), qH liquids have fractional charge equal to \( \pm 1/4 \), and spin 1, for all \( k \). We note, however, that such assignments, while meaningful for \( N \gg k \), may be meaningless when \( N \sim 3k \), where the quasiparticle size is comparable to the size of the fluid drop.

An important point is that for \( c_2 = 0 \), the Hamiltonian (1) has SU(3), not just SO(3), spin-rotation symmetry. Therefore, we expect the SU\((4)\)_\(k\) states to be relevant near this line when \( c_0 > 0 \), for sufficiently large \( L \).

Our second series of qH states for spin-1 bosons is obtained from a CFT with \( G = \text{SO}(5) \) and generalizes the construction in Ref. [12]. The GS’s possess a symmetry under an SO(3) subgroup of SO(5). The general construction, which involves spin-singlet clusters of \( k \) particles, gives states of filling factor \( \nu = k/(kM + 1) \). Putting \( M = 0 \), we obtain qH states with \( \nu = k \). For \( k = 1 \), the wave function with values in the spin space of \( N \) spin-1 particles can be written as

\[
\Psi_{k=1}^{\text{SO}(5)}(z_i) = \text{Pf} \left[ \frac{\rho_i \rho_j + a_i a_j + \tau_i \tau_j}{(z_i - z_j)} \right] \Psi_1^{k}(z_i). \tag{7}
\]

Here \( \Psi_1^{k}(z_i) = \prod_{i<j}(z_i - z_j) \) is a spin-independent
The Laughlin factor in all $N$ particle coordinates; the Pfaffian $\mathcal{P}$ is defined by $\mathcal{P}(M_{ij}) = \mathcal{A}(M_{12}M_{34}\ldots M_{N-1,N})$, where $M_{ij}$ are the elements of an antisymmetric matrix and $\mathcal{A}$ denotes the operation of antisymmetrization; $\rho_i$, $\sigma_j$, $\tau_j$ are basis vectors in the spin space for particle $i$, which correspond to $x$, $y$, $z$, and the product is the tensor product. For $k = 1$, $N$ must be even, and the cluster (or pairing) of particles is seen explicitly. Clearly, there must be $k = 1$ wave function for $N = 2q$ particles has total degree $L = 2q(q - 1)$.

The function (7) can be viewed as spin-singlet $p$-wave pairing of composite fermions of spin 1 (for a review, see Ref. [16]). There are also excited states with unpaired charge-neutral spin-1 fermions. We note that, in the state $|0\rangle$, the spin state is ferro, $S = N$, giving energy $c_0 + c_2 N(N - 1)/(2Nc_2)(2\pi)^{-3/2}$ [17]. At $c_0 < 0$, the spin states again form SU(3) multiplet. In the “repulsive” region II, the GS is in general not a common eigenstate of the $c_2$ and $c_3$ parts of the interaction, and the GS energy depends nonlinearly on the ratio $c_2/c_0$. The $L = N$ GS’s all have $S = 0$ or $S = 1$ (depending on the value of $N$ modulo 6, and on the ratio $c_2/c_0$). For $L > N$, larger values of $S < N$ do occur near the boundary at $c_2 = -c_0/2$. Turning to $L \gg N$ in region II, we have already pointed out that the SU(4)$_1$ state is the zero-energy GS for $N = 3p$ particles at $L = 3p(p - 1)$ when $g_0 > 0$, while the SO(5)$_1$ state is the zero-energy GS for $N = 2q$ particles at $L = 2q(q - 1)$ for $g_0 = 0$. (For $N = 3, 4$, these states occur at $L = N$). For even larger $L$, each of these model cases possesses many degenerate GS’s of zero energy. This implies that, within our model, the SU(4)$_1$ and SO(5)$_1$ GS’s are those found (for the parameters as stated) at the critical rotation frequency $\omega = \omega_0$, and so the lowest possible filling factor is $3/4$ in the region $g_0$, $g_2 > 0$, but is 1 when $g_0 = 0$.

At intermediate $L > N$ values in this region, we expect similar physics. Thus, for $c_2 = c_0/2 > 0$, the system can lower its energy by forming SO(3) singlet pairs of bosons. For $L < 2q(q - 1)$ ($N = 2q > 4$) we do not have exact eigenstates, but we expect that, similar to Ref. [9], for $\nu = k - 1$ less than some critical value $\nu'_k > 1$, the bulk of the fluid will be in the SO(5)$_1$ state. For $c_2 = 0$, $c_0 > 0$, the preferred behavior is singlet formation via triples of bosons of spin zero; each such three bosons must be in an antisymmetric orbital state as well. For $L < 3p(p + 1)$ ($N = 3p > 3$), we do not generally have exact eigenstates, but we expect the bulk of the fluid to be the SU(4)$_p$ states for $\nu = 3k/4$ less than another critical $\nu''_k > 3/4$.

We also found that the exact GS for $N = 6$, $L = 6$ at $c_2 = 0$ is the SU(4)$_2$ or BTC state (2). Prompted by this, we have proved analytically that for $N = 3p$ particles the SU(4)$_p$ state is an exact eigenstate of $H_{\text{int}}$ with $c_2 = 0$, with eigenvalue.

![FIG. 1. Overview of $c_0-c_2$ plane, with special regions and directions marked.](image)
member of the state is the GS and believe this is true for all \( N \). The form is independent of \( N \). We caution again that one should probably not think of the limit \( N = 3k \) as a qH state, since the drop is so small. We identified the state as an extreme member of the SU(4)\( _k \) qH series, but it might be more useful to view it as a boson-triplet analog of the BCS paired electron states, or alternatively as an SU(3) analog of a Skyrmion spin texture (see Ref. [18]).

The response of the spin-1 boson system to a rotation frequency \( \omega \) is found by minimizing \( H_{\omega k} \). In Fig. 3 we display the result for \( N = 6 \) particles. The BTC is at \( L = 6 \), and the SU(4)\( _1 \) state at \( L = 18 \). The degenerate spin multiplets listed at the steps at \( L < 6 \) each form a single SU(3) multiplet. The wave functions of these GS's at \( L \leq N \) are uniquely determined by their SU(3) and \( L \) quantum numbers.

The SO(5)\( _1 \) state found here will survive for some distance off \( g_0 = 0 \). For \( N \) large at fixed \( \nu \), there will be several phases within region II, and, in particular, a boundary between the SO(5)\( _1 \) and SU(4)\( _1 \) phases that approaches \( g_0 = 0 \) as \( \omega \to \omega_0 \).

Our constructions for \( c_2 = 0 \) directly generalize to an \( n \)-component rotating Bose gas with repulsive spin-independent \( \delta \)-function interactions, implying SU(\( n \)) symmetry. In particular, we expect a "boson \( n \)-plet condensate" at \( L = (n - 1)N/2 \), and SU(\( n + 1 \))\( _k \) states at sufficiently small \( \nu = nk/(n + 1) \).

To conclude, we have found several interesting states of matter, not containing vortices, in rotating spin-1 bosons.

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As this manuscript was being prepared, two e-prints appeared [19,20] that also address a rotating BEC. These papers point out the relevance of the SU(\( n + 1 \)) \( _1 \) states at large \( L \) but do not discuss the other qH states or the BTC state that we consider.

\[ E = \frac{11}{48} N(N - 3)(2\pi)^{-3/2} c_0. \quad (8) \]

FIG. 2. Density profiles versus \( x \) at \( y = 0 \) in the LLL model: BTC (solid line); nonrotating BEC (broken line). Vertical axis is in units of the particle number \( N = 3p \).

FIG. 3. GS values for \( L \) as a function of the rotation frequency \( \omega \) for \( N = 6 \) particles at \( c_2 = 0 \). [We put \( \omega_0 = 1 \) and \( c_0 = (2\pi)^{3/2}/4 \).] The state at \( L = 6 \) is the BTC, while the \( L = 18 \) state is the SU(4)\( _1 \) state. The spin values \( S \) are shown for each GS.