Quantum Hall states and boson triplet condensate for rotating spin-1 bosons

Reijnders, J.W.; van Lankvelt, F.J.M.; Schoutens, K.; Read, N.

Publication date
2002

Published in
Physical Review Letters

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Quantum Hall States and Boson Triplet Condensate for Rotating Spin-1 Bosons

J. W. Reijnders, F. J. M. van Lankvelt, and K. Schoutens

Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
Department of Physics, University of Virginia, P.O. Box 400714, Charlottesville, Virginia 22904-4714

N. Read
Department of Physics, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120
(Received 13 March 2002; published 30 August 2002)

We propose and analyze two series of clustered quantum Hall states for rotating systems of spin-1 bosons. The first series [labeled SU(4)_k] includes the exact ground states of a model Hamiltonian at large angular momentum L, and also for N = 3k particles at L = N. The latter is a spin-singlet boson-triplet condensate. The second series, labeled SO(5)_k, includes exact ground states at large L for different parameter values.

DOI: 10.1103/PhysRevLett.89.120401 PACS numbers: 05.30.Jp, 73.43.Cd

In the study of the phenomenon of Bose-Einstein condensation (BEC) of cold atoms, the possibility of using bosons with internal degrees of freedom (spin 1 in particular) and of rotating the condensate has led to fascinating new possibilities for the states of matter that can be formed and observed. A multicomponent BEC can be realized by trapping higher spin atoms such as ^{87}\text{Rb} [1] and ^{23}\text{Na} [2] in optical traps, which do not affect the spin degeneracy, and spin-1 or “vector” BEC’s have been realized and studied [2,3]. In a model description, the two-body interactions of spin-1 or “vector” BEC’s has been realized and studied [2,3]. In a model description, the two-body interactions of spin-1 or “vector” BEC’s has been realized and studied [2,3].

In a model description, the two-body interactions of spin-1 or “vector” BEC’s has been realized and studied [2,3]. In a model description, the two-body interactions of spin-1 or “vector” BEC’s has been realized and studied [2,3]. In a model description, the two-body interactions of spin-1 or “vector” BEC’s has been realized and studied [2,3].

In the case of ^{23}\text{Na} one has c_2 > 0 and, hence, “antiferromagnetic” spin correlations. The “polar” ground state [4,5] has (S) = 0; it supports many interesting collective excitations (see [6] for a review). The case of ^{87}\text{Rb} has c_2 < 0, leading to a spin-polarized (ferromagnetic) ground state.

The effect of rotating a “scalar” BEC of spinless or spin-polarized atoms in an isotropic harmonic trap of frequency \( \omega_0 \) can be studied at mean field level (the Gross-Pitaevskii equation), and a vortex lattice is found at sufficiently high rotation frequency \( \omega \), in agreement with experimental observations [7]. At still higher frequencies, quantum fluctuations become important and it has been argued that the vortex lattice is replaced by a sequence of distinct quantum fluids, which can be understood using a mapping to the fractional quantum Hall (qH) effect [8]. In this Letter, we explore the corresponding possibilities in a rotating system of spin-1 bosons.

In the regime \( \omega_0 - \omega \ll \omega_0 \) and very weak interaction, only the lowest-energy single-particle states in the trap are of interest. These have non-negative angular momentum \( m = 0, 1, \ldots \) about the rotation axis, and no excitation of the motion along the axis, so the motion is effectively in the \( xy \) plane. The magnetic quantum number dependence of these wave functions is \( \psi_m(z) \propto z^m e^{-|z|^2/2} \) [where \( z = x + i y \) and the quantum length \( (\hbar/M_b \omega_0)^{1/2} \) of the trap, and \( \hbar \), have been set to 1; these correspond to the lowest Landau level (LLL) [8].

We consider the interaction Hamiltonian for bosons \( i, j = 1, \ldots, N \), restricted to the LLL,

\[
H_{\text{int}} = \sum_{i<j} \delta(r_i - r_j) [(c_0 + c_2 S_i \cdot S_j) \otimes \rho_i \otimes \rho_j] \tag{1}
\]

The ground state (GS) for the system rotating at frequency \( \omega \) is found by minimizing \( H_{\text{tot}} = H_{\text{int}} + (\omega_0 - \omega) L \) [8], where \( L = \sum_i m_i \) is the total angular momentum. For \( N \) particles in the LLL with zero center-of-mass angular momentum, the average “filling factor” of the occupied states can be defined as \( \nu = N/(N_V + 1) \), where \( N_V = 2L/N \) is the number of vortices there would be if there were a Bose condensate (we assume here a uniform average occupation of states up to \( m = N_V \)). The use of only the LLL states is physically reasonable when \( \omega_0 - \omega \ll \omega_0/N_V \) (so that particles in the LLL with \( m = N_V \) cannot lower their energy by moving to a non-LLL state with \( m = 0 \)) and \( \nu c_2 \ll \omega_0 \), \( S = 0, 2 \) (so that perturbative corrections from mixing non-LLL states into the GS are negligible).

For the scalar case (in the LLL, without \( c_2 \)), a study [9] using periodic boundary conditions found that the vortex lattice melts at a critical \( \nu = \nu_c \sim 10 \), that incompressible fluid states occur at \( \nu = \frac{1}{2}, \frac{3}{2}, 2, \ldots \ll \nu_c \), and that the corresponding GS wave functions have large overlaps with the Read-Rezayi (RR) [10] series of fractional qH states.

Here, for the case of vector bosons, we first propose two series of spin-singlet generalizations of the RR states [labeled SU(4)_k and SO(5)_k, respectively, and \( k = 1, 2, \ldots \)] and explain their structure. In a thermodynamic limit \( N \to \infty \) with \( \nu \) fixed, they each represent incompressible liquid phases of the system, with \( \nu = 3k/4 \) for the SU(4)_k states and \( \nu = k \) for the SO(5)_k series. Then we present results for the GS’s of \( H_{\text{tot}} \) as a function of \( L, c_0 \), and \( c_2 \). We identify a region in the \( c_0-c_2 \) plane where the
proposed qH states with \( k = 1 \) give the exact GS’s at high values of \( L \). We demonstrate that the \( k = N/3 \) member of the SU(4) series is an exact eigenstate and, apparently, the GS of \( H_{\text{tot}} \) with \( c_2 = 0, c_0 > 0 \), at angular momentum \( L = N \). In second quantization, this state, a spin-singlet boson-triplet condensate (borders, the form

\[
|\text{BTC} \rangle = [e^{\mu_1 \mu_2 \mu_3} b^\dagger_{1 \mu_1} b^\dagger_{1 \mu_2} b^\dagger_{2 \mu_3}]^{N/3}|0\rangle,
\]

where \( b^\dagger_{1 \mu} \) is a creation operator for the single-particle state \( \phi_{\mu}(\zeta) \) with spin \( \mu \), where \( \mu = x, y, z \). Finally, we show the GS’s of \( H_{\text{tot}} \) as a function of \( \omega \).

We now explain our analysis, starting with the construction of spin-singlet qH states for spin-1 bosons. This construction is a generalization of similar constructions for scalar and spin-1/2 particles (either of which could be electrons or bosons) [10–12]. These papers all employed a correspondence [13] between qH states and specific conformal field theories (CFT’s). This connection allows one to obtain trial qH wave functions as chiral correlators in CFT’s that are associated to a Lie algebra \( G \) and integers \( k \geq 1 \) and \( M \geq 0 \), where \( M \) is even (odd) for bosons (fermions). These states generalize the pairing familiar from the theory of superconductivity to the formation of “clusters.” The resulting qH wave functions have a property that guarantees that they are exact zero-energy eigenstates of certain model Hamiltonians. For example, starting from \( G = U(n + 1) \) and putting \( M = 0 \), one can find completely symmetric wave functions that obey

\[
\Psi(z_1, \ldots, z_N) \neq 0 \quad \text{for} \quad z_1 = \cdots = z_l \quad (l \leq k),
\]

\[
\Psi(z_1, \ldots, z_N) = 0 \quad \text{for} \quad z_1 = \cdots = z_{k+1},
\]

independent of the spins of the particles involved. Hence, these are zero-energy eigenstates of a Hamiltonian with a repulsive \( k + 1 \)-body \( \delta \)-function interaction [10] and are the unique states of the lowest \( L \) with this property. The existence of such a Hamiltonian and the fact that it has a gap in its energy spectrum ensures that a corresponding incompressible liquid phase of matter exists (when \( N \to \infty \) at fixed \( k, M \)) over a range of interaction parameters, not just for the \( k + 1 \)-body model. The filling factors of these SU\((n + 1)\) states are

\[
\nu(n, k, M) = \frac{nk}{nkM + n + 1}.
\]

The parameter \( n \) corresponds to the number of components of an internal degree of freedom of the particles, and these models have U\((n)\) symmetry; the clusters contain \( nk \) particles. For \( n = 1 \) this construction gives the RR states, while for \( n = 2 \) it produces spin-singlet states for spin-1/2 particles [11]. Finally, clustered qH states admit excitations (quasiparticles) of effectively fractional “charge” (i.e., particle number). The simplest of these can be viewed as the result of the adiabatic insertion of a fraction of a quantum of magnetic flux (or vorticity), where the allowed fraction is \( 1/nk \) in the present cases. States with more than three quasiparticles at well-separated fixed positions display large degeneracies, which give rise to “non-Abelian statistics” [13] and are understood for \( n = 1, 2 [11,14] \).

Our first series of clustered spin-singlet qH states of spin-1 bosons consists of the SU(4) states (\( n = 3 \) above), where we put \( M = 0 \). The filling factor is \( \nu = 3k/4 \). The CFT construction guarantees that, for \( N \) divisible by 3k, this state is a singlet under SU(3), and hence also under its SO(3) subgroup, the usual spin-rotation group. The wave function for \( N = 3kp \) particles can be written in terms of its components for particular spin states \( \mu = x, y, z \) specified for each particle (similar to [15]), as

\[
\Psi_{SU(4)}^k = S_{\text{groups}}[P_{\text{groups}} \Psi_{2,2,1.1.1}],
\]

where

\[
\Psi_{2,2,1.1.1} = \prod_{\mu=x,y,z} \prod_{i<j} \left( z^\mu_i - z^\mu_j \right)^2 \prod_{\mu<0, i,j} \left( z^\mu_i - z^\mu_j \right).
\]

In words, the operations \( P_{\text{groups}} \) and \( S_{\text{groups}} \) in Eq. (5) tell one to divide the \( 3kp \) bosons into \( k \) groups of \( 3p \) bosons each (\( p \) of each spin polarization), to write a factor \( \Psi_{2,2,1.1.1} \) as in Eq. (6) for each group, and, finally, to symmetrize over all ways the particles can be divided over the \( k \) groups. For \( k = 1 \) there is a single group, and one finds a state of total degree \( L = 3p(2p - 1) \) that generalizes the Laughlin and Halperin states for scalar and spin-1/2 particles, respectively, and which is a zero-energy eigenstate of \( H_{\text{tot}} \) for all \( c_0, c_2 \), and the unique GS when both \( g_0, g_2 \) are greater than zero. Here \( H_{\text{tot}} \) is then positive. Putting instead \( k = N/3 \) gives \( k \) groups of three particles each, the resulting state being the \( L = N \) BTC, Eq. (2). The fundamental quasiparticles over the SU(4), qH liquids have fractional charge equal to \( \pm 1/4 \), and spin 1, for all \( k \). We note, however, that such assignments, while meaningful for \( N \gg k \), may be meaningless when \( N \sim 3k \), where the quasiparticle size is comparable to the size of the fluid drop.

An important point is that for \( c_2 = 0 \), the Hamiltonian (1) has SU(3), not just SO(3), spin-rotation symmetry. Therefore, we expect the SU(4) states to be relevant near this line when \( c_0 > 0 \), for sufficiently large \( L \).

Our second series of qH states for spin-1 bosons is obtained from a CFT with \( G = SO(5) \) and generalizes the construction in Ref. [12]. The GS’s possess a symmetry under an SO(3) subgroup of SO(5). The general construction, which involves spin-singlet clusters of \( 2k \) particles, gives states of filling factor \( \nu = k/(kM + 1) \). Putting \( M = 0 \), we obtain qH states with \( \nu = k \). For \( k = 1 \), the wave function with values in the spin space of \( N \) spin-1 particles can be written as

\[
\Psi_{k=1}^{SO(5)}(z_i) = \text{Pf} \left[ \sigma_i^j + \frac{\tau_i \tau_j}{(z_i - z_j)} \right] \Psi_{k=1}^1(z_i).
\]

Here \( \Psi_{k=1}^1(z_i) = \prod_{i<j} (z_i - z_j) \) is a spin-independent
Laughlin factor in all $N$ particle coordinates; the Pfaffian 
$\text{Pf}$ is defined by $\text{Pf}(M_{ij}) = \mathcal{A}(M_{12}M_{34} \ldots M_{N-1,N})$, 
where $M_{ij}$ are the elements of an antisymmetric matrix 
and $\mathcal{A}$ denotes the operation of antisymmetrization; 
$\rho_i, \sigma_j, \tau_j$ are basis vectors in the spin space for particle $i$, 
which correspond to $x, y, z$, and the product is the tensor 
product. For $k = 1$, $N$ must be even, and the clustering (or pairing) 
of particles is seen explicitly. Clearly, there must 
be even (but otherwise arbitrary) numbers of particles in 
each of the three spin states $\rho, \sigma, \tau$. The $k = 1$ wave 
function for $N = 2q$ particles has total degree $L = 2q(q-1)$. 
The function (7) can be viewed as spin-singlet $p$-wave 
pairing of composite fermions of spin 1 (for a review, see 
Ref. [16]). There are also excited states with unpaired 
charge-neutral spin-1 fermions. We note that, in the state 
(7), two bosons are found at the same point only if the total 
spin of the pair is zero, not if it is 2, and hence the state is an 
exact zero-energy eigenstate, of lowest $L$, for a $\delta$-function 
interaction that includes a projection onto spin 2. In our 
parametrization, that is, $c_2 = c_0/2$ or $g_0 = 0$, and if also 
c_0 > 0, then $H_{\text{int}}$ is positive, so our $k = 1$ state is the GS at 
this $L$. For general $k$, the $\text{SO}(5)_k$ wave function can be 
written as a CFT chiral correlator and is an exact zero-
energy eigenstate of a certain $k + 1$-body $\delta$-function 
interaction (details will appear elsewhere). In general, the 
quasiparticles over the $\text{SO}(5)_k$ qH state have charge $\pm 1/2$ and 
spin $1/2$, thus displaying a fractionalization of both charge and 
spin.

We next report on our study (numerical and analytic) of 
the GS phase diagram of the model $H_{\text{int}}$. The numerical 
work is restricted to small values of $N$ (up to 12), but it 
indicates many features which we believe to hold for 
general values of $N$. To guide our discussion we have 
displayed in Fig. 1 various special directions (rays) and 
regions in the $c_0$-$c_2$ plane.

For $L = 0$, the GS has total spin $S = N$ for $c_2 < 0$ (ferro 
regime) and $S = 0$ (1) for $N$ even (respectively, odd) for 
c_2 > 0 (antiferro regime). For $c_2 = 0$, there is a single 
SU(3) multiplet of spin states, decomposing into unique 
SO(3) multiplets of each spin $S = N, N-2, \ldots$.

As $L$ increases, these two phases survive in part of the 
phase diagram, as compact drops of fluid, with the center of 
mass (CM) carrying all the angular momentum. Meanwhile, the positive $c_0$ axis gradually opens into a 
region that contains other phases. By the time $L$ is $\geq N$, 
the $c_0$-$c_2$ plane contains the three regions labeled Ia, Ib, 
and II in Fig. 1. The GS’s in regions Ia and Ib are similar to 
the GS in the “attractive” regime in the scalar case [8]. The 
orbital part of the GS wave function is of the form $\Psi(z_i) \propto \bar{z}_i^L$, 
with $\bar{z}_i = \sum_i z_i/N$ the CM coordinate. In region Ia 
$\left(c_0 < 0, c_2 > 0\right)$, the spin state is the same spin singlet as 
for the $L = 0$ GS, and the GS energy becomes $\left[c_0(N-N-1)/2 - Nc_2\right](2\pi)^{-3/2}$ [17]. In region Ib 
$\left(c_2 < 0, c_0 < -c_2\right)$, the spin state is ferro, $S = N$, giving energy $(c_0 + c_2)/2N(1-1/2)\pi^{-3/2}/2 < 0$ [8]. At 
c_2 = 0, $c_0 < 0$, the spin states again form the SU(3) multiplet. In the 
“repulsive” region II, the GS is in general not a common 
eigenstate of the $c_0$ and $c_2$ parts of the interaction, and the 
GS energy depends nonlinearly on the ratio $c_2/c_0$. The 
$L = N$ GS’s all have $S = 0$ or $S = 1$ (depending on the 
value of $N$ modulo 6, and on the ratio $c_2/c_0$). For $L > N$, 
larger values of $S < N$ do occur near the boundary at $c_2 = -c_0$.

Turning to $L \gg N$ in region II, we have already pointed 
out that the SU(4)$_4$ state is the zero-energy GS for $N = 3p$ 
particles at $L = 3p(2p-1)$ when $g_0$ and $g_2 > 0$, while 
the SO(5)$_k$ state is the zero-energy GS for $N = 2q$ particles 
at $L = 2q(q-1)$ for $g_0 = 0$. (For $N = 3, 4$, these states 
occur at $L = N$. For even larger $L$, each of these model 
cases possesses many degenerate GS’s of zero energy. This 
implies that, within our model, the SU(4)$_4$ and SO(5)$_k$ GS’s 
are those found (for the parameters as stated) at the critical 
rotation frequency $\omega = \omega_0$, and so the lowest possible 
filling factor is $3/4$ in the region $g_0$, $g_2 > 0$, but is 1 
when $g_0 = 0$.

At intermediate $L > N$ values in this region, we expect 
similar physics. Thus, for $c_2 = c_0/2 > 0$, ($g_0 = 0$), the 
system can lower its energy by forming SO(3) singlet 
pairs of bosons. For $L < 2q(q-1)$ ($N = 2q > 4$) we do not 
have exact eigenstates, but we expect that, similar to 
Ref. [9], for $\nu = k$ less than some critical value $\nu'_c > 1$, 
the bulk of the fluid will be in the SO(5)$_k$ state. For $c_2 = 0$, 
c_2 > 0, the preferred behavior is singlet formation via 
triples of bosons of spin zero; each such three bosons 
must be in an antisymmetric orbital state as well. For $L < 3p(2p-1)$ ($N = 3p > 3$), we do not generally have 
even eigenstates, but we expect the bulk of the fluid 
to be the SU(4)$_4$ states for $\nu = 3k/4$ less than another 
critical $\nu''_c > 3/4$.

We also found that the exact GS for $N = 6$, $L = 6$ at 
c_2 = 0 is the SU(4)$_2$ or BCT state (2). Prompted by this, 
we have proved analytically that for $N = 3p$ particles the 
SU(4)$_p$ state is an exact eigenstate of $H_{\text{int}}$ with $c_2 = 0$, 
with eigenvalue
We have numerically checked that for \( N = 3, 6, 9, 12 \) this state is the GS and believe this is true for all \( N = 3p \).

The BTC state in Eq. (2) has a clear experimental signature, that is, the density profile given in Fig. 2, whose form is independent of \( N \). We caution again that one should probably not think of the limit \( N = 3k \) as a qH state, since the drop is so small. We identified the state as an extreme member of the SU(4) \( k \) qH series, but it might be more useful to view it as a boson-triplet analog of the BCS paired electron states, or alternatively as an SU(3) analog of a Skyrmion spin texture (see Ref. [18]).

The response of the spin-1 boson system to a rotation frequency \( \omega \) is found by minimizing \( H_{\text{tot}} \). In Fig. 3 we display the result for \( N = 6 \) particles. The BTC is at \( L = 6 \), and the SU(4) \( l \) state is at \( L = 18 \). The degenerate spin multiplets listed at the steps at \( L < 6 \) each form a single SU(3) multiplet. The wave functions of these GS’s at \( L \leq N \) are uniquely determined by their SU(3) and \( L \) quantum numbers.

The SO(3) \( 1 \) state found here will survive for some distance off \( g_0 = 0 \). For \( N \) large at fixed \( \nu \), there will be several phases within region II, and, in particular, a boundary between the SO(3) \( 1 \) and SU(4) \( 1 \) phases that approaches \( g_0 = 0 \) as \( \omega \to 0 \).

As this manuscript was being prepared, two e-prints appeared [19,20] that also address a rotating BEC. These papers point out the relevance of the SU(\( n + 1 \)) \( 1 \) states at large \( L \) but do not discuss the other qH states or the BTC state that we consider.

[18] T. Mizushima et al., cond-mat/0203242.