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MATHEMATICAL LEVEL RAISING THROUGH COLLABORATIVE INVESTIGATIONS WITH THE COMPUTER

ABSTRACT. Investigations with the computer can have different functions in the mathematical learning process, such as to let students explore a subject domain, to guide the process of reinvention, or to give them the opportunity to apply what they have learned. Which function has most effect on mathematical level raising? We investigated that question in the context of developing learning materials for 16-year-old students in the domain of probability theory, consisting of computer simulations based on a gambling game and investigation tasks about these games. We compared the difference in level raising between three versions of the learning materials: investigations with the computer *before*, *during* or *after* the learning of a mathematical concept. It was shown that there was no significant difference in the final mathematical level that students attained in the three conditions (the product). However, there were differences in the level on which students approached the investigation tasks (the process). Furthermore, we found evidence of new categories in the students' answers, lying between the perceptual and conceptual levels, which may give important insight into the process of level raising.

KEY WORDS: collaborative learning, computer simulation, investigations, mathematical level raising, probability theory

INTRODUCTION

In the past decade, technological developments have lead to some important new tools for mathematics education. The Dutch research program 'Mathematics and ICT' has investigated possibilities to use these tools in the learning of mathematics by means of guided reinvention, with the aim of mathematical level raising. In one of the projects a computer game was developed (Pijls, 2000, 2001; Pijls, Dekker et al., 2000, 2001) which laid the foundation for several simulations that have been used to teach the subject of binomially- distributed probabilities and counting routes (see description below) to 16-year-old students of higher pre-vocational education. Earlier research made clear that the students in this type of education were not always motivated to perform investigations with the help of the computer (PRINT, 1998; Prent, 1999). We ascribed this to the fact that many computer simulations on probability theory were rather abstract. That is why we tried to develop simulations in a way that is very accessible for students, offering them many opportunities to explore the

subject on a concrete level. In a field experiment, we wanted to investigate how students can learn most from working with the simulations, that is: how can they attain level raising in the domain of binomially distributed probabilities? In this paper we compare three versions of the learning materials and find out which version enhanced level raising the most. We will look at both the product (have students attained the conceptual level at the end of the experimental lessons?) and the process (how do students attain level raising during the lessons?). This study is part of a developmental project and the results are being used to improve the learning materials for future research and development.

MATHEMATICAL LEVEL RAISING

When using the term ‘level raising’ we refer to the level theory of Van Hiele (1986). Van Hiele distinguishes several levels in the learning of a mathematical concept and gives examples of it in the field of geometry. On the first level students deal with mathematical objects in their physical appearance, such as a rhombus. On the second level students deal with properties of the objects such as the fact that it has four equal sides.

In our research we define level raising as follows. In the different stages of their learning, students approach a problem in various ways. Before they learn a concept, they approach it at a perceptual level. They see it ‘as it is’ and do not recognize properties of a mathematical concept in it. After they have acquired a certain mathematical concept, students will recognize aspects of that concept in a realistic problem and they can use these to solve the problem. Thus, they now treat the problem at another level: level raising has taken place, from the perceptual to the conceptual level. Level raising is thus defined for a given concept.

Figure 1. Mathematical level raising.

Level raising, as interpreted above, can be linked to the idea of *conceptual change* (Pressley and McCormick, 1995). Both level raising and

conceptual change deal with the acquisition of a certain concept. The difference, however, is that with conceptual change it is supposed that students start with misconceptions, which have to 'be transformed' into correct scientific concepts, whereas with level raising there is not necessarily a misconception at the perceptual level. For instance, regarding the concept 'probability' (defined as 'favorable amount of probabilities/total amount of probabilities'), at the perceptual level students can correctly estimate probabilities without using numbers, so this estimation is not necessarily a 'misconception'. On the other hand, in some cases mathematical level raising does deal with misconceptions at the perceptual level, so level raising may involve conceptual change.

COLLABORATIVE INVESTIGATION TASKS WITH THE COMPUTER

A number of studies have shown that the interplay between building knowledge and carrying out investigations by students is twofold. Studies in the field of discovery learning show that discovery activity only makes sense if students have a sufficiently extensive knowledge base at their disposal (see for instance Tuovinen and Sweller, 1998). On the other hand, other studies have shown that carrying out investigations can be useful for students to build up their knowledge base. For example, Freudenthal speaks of explorations at the visual (which we call perceptual) level as an essential aspect of the process of reinvention; and reflection on one's own activities at the perceptual level may evoke level raising (Freudenthal, 1973). The assumption that carrying out investigations contributes to the building of knowledge is in favor of using investigation tasks *before* or *during* the learning of a mathematical concept, whereas the idea that students have to know enough about a certain subject before they can profit from investigation tasks argues for performing such tasks only *after* learning a concept. In the first two cases, the use of a computer simulation could allow students to make explorations at the perceptual level, in the third case it would enable them to apply and restructure what they learned at the conceptual level. This last case was the most commonly used in an earlier research project, on which the current project is based (PRINT, 1998; Prent, 1999).

Not only the use of investigation tasks with the computer may lead students to reflection, working together with peers may contribute to this as well (see for example, Dekker, 1991; Webb, 1991). It has been shown that giving explanations about one's own work has a positive influence on learning. The same holds for showing, justifying and reconstructing one's

own work. All those activities are evidently related to reflection, either because they give rise to reflection (showing one's work) or because they are the result of it (justifying and reconstructing). Dekker and Elshout-Mohr (1998) developed a process model in which they describe these key activities for two students who are working on the same mathematical task but on different levels. Regulating activities (activities that evoke key activities, such as to ask to show one's work, to ask for explanation, to criticize), and mental activities (activities that are assumed to take place 'in the heads of the students', such as to become conscious of one's own work, to become conscious of the work of the other) are also mentioned in the model (Dekker and Elshout-Mohr, 1998).

RESEARCH QUESTIONS

In the previous section, we mentioned that in current classroom practice, students frequently carry out investigation tasks with the computer *after* they have learned a certain concept. However, educational theories suggest that carrying out investigations with the computer *before* or *during* the learning of a mathematical concept could be instructive too. Therefore, we carried out a field experiment to find out which of the three approaches enhances mathematical level raising the most. The three functions of collaborative investigation tasks with the computer are presented in the three conditions chosen for the experiment. In the condition BEFORE, students perform investigation tasks with the computer before studying a chapter from a regular textbook. In the condition DURING, they perform the same investigation tasks with the computer, and a set of paper-and-pencil tasks connected with the computer simulation instead of studying the regular textbook. Furthermore, in the DURING condition the students have to 'reinvent' the theory, that is, there are no 'rules' given in the teaching materials. Instead, there are exercises in which students have to formulate and summarize what they have discovered. In order to reduce the cognitive load while students are working with both the computer simulation and written materials (see Sweller and Chandler, 1994), the exercises are formulated as much as possible in the context of the games of the simulation, using screenshots of the software. In the condition AFTER students perform the investigation tasks with the computer after having studied the tasks using their regular textbook. Figure 2 gives an overview of the lessons in the three conditions.

The research questions concern firstly the mathematical levels attained by the students, and secondly the process of mathematical level raising in the three conditions:

Figure 2. The lessons in the three conditions.

- (1) *Are there differences in attained mathematical level between the following three conditions?*
 - a. BEFORE students work on collaborative investigation tasks with the computer before their regular textbook chapter;
 - b. DURING students work on collaborative investigation tasks with the computer without a textbook, but with textbook-like questions in the context of the computer simulations;
 - c. AFTER students work on collaborative investigation tasks with the computer after their regular textbook chapter.
- (2) *Do students in the different conditions approach the collaborative investigation tasks with the computer in the following ways?*
 - a. do students in BEFORE approach the tasks at the perceptual level?
 - b. do students in DURING approach the tasks initially at the perceptual level, and at the end at the conceptual level?
 - c. do students in AFTER approach the tasks at the conceptual level?

Concerning question (1) we had the following expectations. The BEFORE students will be able to explore the domain at the perceptual level with the computer simulations. This may lead to level raising if they are able to link these experiences to the theory they learn in the textbook chapter. We expect the DURING students to profit the most from the possibilities of the learning materials since the investigation tasks are highly embedded in the learning process. We expect the AFTER students to apply what they have learned in the investigation tasks. If they did not yet reach the conceptual level, working with the investigation tasks might help them to achieve level raising. Question (1) will be answered by analyzing the results of a post-test on the topic of binomially distributed probabilities.

For the second question we expect the students to approach the tasks as expressed in (2a), (2b) and (2c). This question will be answered by analyzing the students' answers in the investigation tasks.

METHOD

The experiment took place at a Montessori school in Amsterdam, with three classes (67 students) of 16-year-old students of higher pre-vocational education. The students in this school were used to working self-reliantly, while the teachers had a coaching role. The teachers attended the experimental lessons too, giving minimal help when students asked for it. The researcher carried out a major part of the teacher's role. This was because the learning materials were still in a developmental stage and the researcher knew most about them. The experiment consisted of 10 lessons, each lasting 45 minutes.

One month before the lessons started, the students completed in 90 minutes two pre-tests: pre-test 1 on prior knowledge of probability theory, and pretest 2 on the topic of binomially distributed probabilities and counting routes. Pretest 1 was a test of topics covered in preceding lessons on combinatorics and probability theory, and it consisted of 10 items. The researcher and the teacher constructed it and the reliability proved to be sufficient (Cronbach's $\alpha = 0.6550$). Pretest 2 consisted of 11 items on binomially distributed probabilities and counting routes. Those items were comparable to the items of the post-test. The reliability of this pre-test was not sufficient (Cronbach's $\alpha = 0.5739$), probably because the students could not yet answer the questions of this test. That is why we have left it out of our analysis.

After the 10 lessons of the instructional sequence the students completed a 45 minute post-test. It consisted of 13 items, 11 of which were very much comparable with pre-test 2. The items were constructed in such a way that they could not be answered correctly unless students had reached the conceptual level. The reliability was sufficient (Cronbach's $\alpha = 0.7010$) and the inter coder reliability was 95%.

Each class was divided into three condition groups that were equal in results on pre-test 1, by listing the students from low to high results and assigning them one by one to a different group. In each condition group the students could choose semi-heterogeneous pairs. This was organized as follows: each condition group was divided into two groups, A and B, on the basis of pre-test 1 (on prior knowledge). Group A consisted of students with a high or low score and group B of students with an average score. Then students could choose pairs themselves, so long as an A-student worked together with a B-student.

The three condition groups of each class worked together in a large room with computers. Each group had an own tester that carried out the teacher role, i.e., starting the lessons with a short introduction, handing out the learning materials and giving minimal help if the students asked for it. The regular class teacher also moved among the three groups to give minimal help. The students worked in pairs on the investigation tasks with the computer: they shared one computer and one task booklet and produced one answer. The tasks from the traditional textbook were done individually, that is, each student produced his or her own answer although they discussed their answers with each other. The BEFORE students first had four lessons on investigation tasks and then six lessons on their regular textbook chapter. The DURING students had ten lessons with the experimental learning materials. The AFTER students started with six lessons on their textbook chapter and then four lessons on investigation tasks.

We collected from all students their (written) answers from the learning materials, both answers from the investigation tasks (made in pairs) and answers to the textbook questions (made individually). Furthermore, we collected for all students log files created while they were working with the computer. Audio recordings of nine pairs (three per condition, one from each class) were made during all ten lessons. For these recordings, we chose pairs that contained at least one student that had made extensive elaborations (drawings, diagrams) in pre-test 1, because we expected that their written products might give rise to discussion by the pair. The log files and audiotapes were used as background information in case the answers of the students were unclear, and to get more information about the level of answer of the students.

THE LEARNING MATERIALS

The Topic of Routes and Probabilities

In this section, we give an outline of the topic ‘Routes and Probabilities’, for which we developed the learning materials. This is part of the Dutch pre-examination curriculum for 16 year-old students of higher pre-vocational education, which mainly involves study of applied mathematics. The central topic is the calculation of binomially distributed probabilities, as in example 1. The students are taught to do this with the help of counting routes in a grid, according to a given procedure.

Example 1

In the grid of Figure 3 you have a probability of 45% to go downwards and a probability of 55% to go upwards. You start in point A. What is the probability that you end up in point B?

Figure 3. Grid.

One can answer this question by starting in point A with 100% and dividing this at each point. Another possibility is to count all possible routes from point A to point B and multiply this by the probability of a route from A to B. We will call the second procedure ‘counting routes in a grid’. Students are taught to do this with help of the procedure ‘adding in end points’, as shown in Figure 4.

Figure 4. The procedure ‘adding in end points’.

Repeating this procedure leads to the structure of Pascal’s triangle. However, students are not taught the formula for binomial coefficients, so they have to keep in mind the counting procedure and/or the numbers in Pascal’s triangle. Moreover, they have to be able to apply the counting of routes and the calculation of binomially distributed probabilities for solving word problems as in example 2.

Example 2

We have a box with three red and four white balls. How many patterns of colors can you get by putting these balls in a row?

This is an example of a ‘counting word problem’. Using their prior knowledge, students may try to draw a ‘probability tree’ to solve this problem,

or try to systematically write down all strings having three R's and four W's, like RRRWWWW, RRWRWWW. However, this is a rather complex problem and the two ways of solving it just described are difficult to carry out correctly. That is why students are taught to solve this problem another way, by drawing a three-by-four grid and counting all possible routes from one angle to the opposite angle as in Figure 5, or to solve it by using Pascal's triangle.

Figure 5. Counting routes from one angle to the opposite angle in a three-by-four grid.

The regular Dutch textbooks deal with this subject in the following order. Students have prior knowledge about systematic counting and probability trees. The chapter 'Routes and Probabilities' of 'Moderne Wiskunde' [Modern Mathematics] (Boer et al., 1998), the mathematics textbook that was used in our experiment, starts with the subject of counting routes in a grid, followed by the use of counting routes to solve 'counting word problems' and Pascal's triangle. After that, follows the calculation of probabilities in a grid, like in example 1, and the use of calculating probabilities in a grid to solve 'probability word problems'. As we can see, the subjects covered by this chapter become increasingly complex.

For students, difficulties arise at several points. First, it can be difficult for them to calculate routes in a grid with a special shape, for instance a grid with a 'hole' in it. Second, they find it difficult to see that Pascal's triangle and the grid are in fact the same, in other words, that it is the structure of the numbers that counts and not the orientation. Third, the calculation of probabilities in a grid is problematic for them.

The Software

The software 'Whoopy Trainer' (Pijls, 2001) consists of several games that have a game board with a grid structure. Games were chosen as the subject of the software because students of this age are used to playing

many computer games. We expected that counting routes in a grid will become more meaningful for them when they play games with this same structure. One of the games is TIC-TAC, as shown in Figure 6.

Figure 6. Computer screen of the game TIC-TAC.

The game is played as follows: at the bottom left of the game board one can start a little ball moving. This ball ‘chooses’ a route to one of the nine boxes by going upwards or to the right. One then earns the number of points indicated on the box the ball gets into. The probability to go to the right and the probability to go upwards are both 50%. The route of a certain ball is shown step by step. In every round, a player can play five balls.

The essential ideas behind this game are

1. to focus on the relationship between the *routes* to a certain box and the *probability* of getting there;
2. to focus on the fact that every route towards a certain box consists of the same number of steps upwards and the same number of steps to the right.

We explicitly did not add the possibility to let the computer count the balls ending up in a certain box in this part of the learning material. The reason for this is that we want to give students the opportunity to problematize the counting of routes and to develop their own strategies.

The software provides opportunities for teachers and researchers to design other games, that vary in orientation (i.e., does a ball go from left to right or from the top downwards), lay-out (a grid with lines or boxes at end points), different shapes (so called ‘grids with holes’) and probabilities (in the game TIC-TAC the probability is 50–50, but it can be asymmetric

too). Furthermore one can change the number of points in the boxes, the number of balls per round and whether play proceed by rounds, or just one or 100 balls at a time.

Teachers can also use the software to design an environment for students, in which they can determine which variables of the game the students can manipulate. Thus, students can design their own games as well as playing the games provided.

Investigation Tasks

In Figure 7 we give an overview of the collaborative investigation tasks that we developed to be used with the software. There are five tasks that concern the analysis and playing of a game, and in the final investigation task students have to design a new game themselves. The aim of the first task is to raise the question of counting routes. The aim of the second task is to make clear that this game has the same underlying structure; it is made smaller so that students have the opportunity to develop a counting strategy. The third task lets students explore so-called 'grids with a hole'. The fourth task lets them experiment with probabilities. The idea is that in order to answer the question students should calculate all probabilities. The aim of the fifth task is to let students experience an asymmetric distribution. In the last investigation task students can apply what they have learned about grid structures, routes, and probabilities.

The Experimental DURING Tasks

In the conditions BEFORE and AFTER students use the mathematics textbook and the collaborative investigation tasks with the computer. In the experimental DURING condition, the investigation tasks take about ten lessons. Between the investigation tasks, students work on paper and pencil tasks based on the context of the computer simulation. The idea of these tasks is to give students the opportunity to make a start with level raising. Here is an example of that. After working on task 1 (see Figure 7) the two students of a pair both get a picture of the game board and they are asked to draw several routes from the starting point to the '100-box'. The second step is to describe the routes they drew in terms of 'Right' and 'Above'. Then they are asked to compare the routes they have described. The next step is to ask them whether they have found all possible routes from the start to the 100 box. The aim of these tasks is to problematize the counting of routes and to make clear that counting routes is the same as writing down all strings with three R's and five A's. The fact that the students have worked with the computer simulation, where they could explore at perceptual level, should give them an opportunity to reflect on this with help of the paper and pencil tasks.

Task	Computer screen
<p>1 Play the game TIC TAC several times and try to answer the following questions with the help of the results:</p> <p>a) How many points do you get on average in one round of five balls?</p> <p>b) What is the probability that a ball comes in the 100 points box?</p> <p>c) Can you discover a relation between the probability to come into a certain box and all the possible routes to that box?</p>
<p>2 a) Play the game Plinko several times. Which box is the best to click on? Explain clearly why you think so. If you have different opinions between you, please indicate this.</p> <p>b) Determine the probability to come into the 100 box from the second box from the left.</p>
<p>3 Play the game TIC-TAC NEW several times. What is the probability to come into the 100 point box using this game board?</p>
<p>4 With the game GALTON, you can distribute 100 points in total. The purpose of the game is to do this in such a way that you gain as many points as possible when you play the game.</p> <p>a) Which point distribution gives you most points on average?</p> <p>b) Determine for each box on the bottom row the probability that a ball will come there.</p> <p>c) If you wanted to calculate which score you get on average with a certain point distribution of the game, how can you use the question above to do so?</p>
<p>5 a) Play the game GALTON NEW several times. Try five point distributions. Record your average score.</p> <p>b) Which point distribution gives you the most points? Do you have an explanation for that?</p> <p>c) Determine by experimenting with all the boxes the probabilities to come there. Indicate how you determined these probabilities.</p>
<p>6 In the past lessons you have analyzed several games. Now you will design a new game yourselves. (. .) Make a clear explanation (with calculations) of how you have to play the game if you want to gain as many points as possible. (. .)</p>

Figure 7. Overview of the collaborative investigation tasks.

LEVEL RAISING IN THE DOMAIN OF ROUTES AND PROBABILITIES

We will clarify the operationalization of level raising in the domain of 'Routes and Probabilities' by means of task 1b (see Figure 7) of the learning materials. In task 1a, the students have to calculate the average number of points in one round of the game. In task 1b, they have to discover what the probability is that a ball will get into the 100 points box. Possible answers to this question were:

- (1) There are nine boxes to reach, so the probability is $1/9$.
- (2) By experimenting we find that out of the 100 balls, 14 of them get into the 100 points, so the probability is 14%.
- (3) The probability to finish in the middle is bigger than the probability to finish at either end.
- (4) There are 28 routes to point C, out of the 256 routes from the starting point, so the probability is $28/256 \approx 10,94\%$.

Answer (1) is what we call an answer at the perceptual level. In this case it is a misconception, because the students apply prior knowledge in the wrong way. The students are aware of the fact that 'probability = favorable possibilities/total number of possibilities' but they have not yet learned to count routes. A first step towards level raising can be experimenting, like in answer (2). When the students see, by experimenting, that the probability to get to each box is not equal for every box, they have the opportunity to find out that there are a different number of routes leading to each box. The notion that there is a higher probability to end up in the middle than at the ends, as in answer (3), can be understood as a starting point for the process of level raising too. The next step is to find a way to count the routes. 'By hand' this soon becomes too complex. That is why the method of 'adding the points' is taught. With this method students can calculate the exact number of routes to a certain box and so they will move on to answer (4), which is the right answer at the conceptual level.

RESULTS

The First Research Question: Are There Differences in Attained Mathematical Level between the Three Conditions?

This question is answered by analyzing the results of the post-test on knowledge of the topic 'Routes and Probabilities'. The test consisted of 13 items, the maximum score was 32 and the minimum score 0. Eight missing values (students who did not participate in the pre-tests or the

post-test) were excluded. Table I shows the mean values of the scores of the three conditions.

TABLE I
Mean scores and standard deviation for the post-test

	Post test Mean	N	Post test Standard deviation
Bofore	12.14	21	5.96
During	12.25	20	6.47
After	10.68	19	5.77
Total	11.72	60	6.02

All the means are not very high compared to the maximum score of 32. The standard deviations are high in all conditions and very comparable. It is clear that the mean scores of this test do not differ substantially. This was confirmed by an covariance analysis with pre-test 1 as a covariate, that showed no significant difference between the three means. ($F = 0.577$, $df = 2$, $p > 0.05$). When we consider the average final learning results in terms of mathematical level raising, it seems to make no difference whether students worked with those tasks either before or after their traditional textbook, or whether they worked with the condition in which textbook-like tasks were integrated with the simulation. In all the conditions, there were some students who did not attain the conceptual level.

The Second Question: Do Students in the Three Different Conditions Approach the Collaborative Investigation Tasks with the Computer in the Following Ways?

- a. *do students in BEFORE approach the tasks at the perceptual level?*
- b. *do students in DURING approach the tasks initially at the perceptual level, and at the end at the conceptual level?*
- c. *do students in AFTER approach the tasks at the conceptual level?*

A first review and analysis of the written answers of the investigation tasks of all pairs showed that for some tasks there was a clear difference between the answers of the three conditions and for other questions there appeared no such difference. The first analysis also made clear that the students' answers could not always be categorized as 'perceptual' or 'conceptual', but that some answers 'fell in between'. These answers were not correct, but we could see that students had learned something. We categorized

these answers as ‘*start level raising*’ or ‘*semi level raising*’, which we will explain below.

Figure 8. Four categories of students' answers.

Figure 8 shows the four categories of students' answers. With ‘perceptual level’, we mean that students are playing the game without reflection, or that they apply prior knowledge in an incorrect way (misconceptions). With ‘start level raising’, we mean that students come up with an answer that makes a start in the process of level raising. Often these were their own constructions. So ‘start level raising’ denotes a development from the perceptual level. With ‘semi level raising’, we mean that students wrongly apply the concepts they have been taught. So ‘semi level raising’ means that students have not attained the conceptual level. There is a difference in quality between ‘start level raising’ and ‘semi level raising’, since ‘start level raising’ means that students try to build up new concepts from their own ideas and experiences, while ‘semi level raising’ indicates that students are not able to link new learned concepts to their own ideas and experiences. Finally, on the conceptual level students can apply the concept in the correct way; thus conceptual change has taken place. In Table II we give as an example the different categories of answers for task 1b.

The analysis of the students' answers was carried out as follows. For each task, we classified the four levels of answers. Then, for each task (see Figure 7), we determined the level(s) of the answer of each pair of students. The dominant level(s) of answers for each task (i.e., the levels that occurred for more than 25% of the pairs) are illustrated in Table III.

Table III makes clear that some tasks show differences in level of answers between the conditions and other tasks show similarities. Before analyzing the pattern of answers in each condition, we first checked whether all tasks could be answered at different levels. It might be that

TABLE II
Categories of answers for investigation task 1b

Task 1b	<i>What is the probability that a ball comes in the 100 points box?</i>
Perceptual level	1/9; experiments
Start level raising	the probability is not equal for each box, so we have to count routes; own counting strategy
Semi level raising	counting routes wrongly, or linking it wrongly to probabilities; probability = 28
Conceptual level	counting routes correctly, 28/128

the nature of a certain task explains the differences or similarities in the levels of answers.

Tasks 1a, 2a, 4a and 5a were answered at perceptual level by most of the students in all conditions. These tasks are ‘starters’. They ask students to play a certain game a few times and to draw conclusions about how to win it. Although they could be answered at the conceptual level, these tasks in fact ask for an answer at the perceptual level.

Tasks 1b, 2b, and 4b show the most differentiation in the levels of the answers. All of these tasks ask students to calculate probabilities in a grid, so these are important questions from a conceptual point of view. These tasks give opportunities for level raising.

Task 1c shows homogeneity in answers across all conditions, apart from the fact that ‘semi level raising’ does not dominate in BEFORE. This was a so-called ‘yes/no’ question: answers at perceptual level are a simple yes or no without explanation. The aim of this task was to focus students on the fact that the calculation of probabilities has to do with counting routes. The fact that students do not formulate answers at the conceptual level might be because of the fact that the concept ‘total number of routes’ is not mentioned in the question.

The results of task 3 make clear that for students it is very difficult to count routes in a grid with a different shape. BEFORE students realized that there are fewer routes to the 100-box, however DURING students did not. Most of the AFTER students did not manage to count routes.

The answers to task 4c could not be categorized in levels, so we left it out of our analysis. The aim of the task was to show students how to calculate the expected value when you know the probability to come into a certain box and the amount of points you can win for every box. The students’ answers gave little information about the level at which they answered the question. This could be to do with the fact that the question

TABLE III

Dominant level of answer per condition per task. (Key: percep = perceptual level, start = 'start level raising', semi = semi level raising, concep = conceptual level)

Task	Before	During	After
1a	percep (100%)	percep (100%)	percep (100%)
1b	percep (60%) start (40%)	percep (60%) start (40%)	semi (36%) concep (36%)
1c	percep (50%) start (40%)	percep (33%) start (44%)	percep (27%) start (27%) semi (27%)
2a	percep (60%) start (40%)	percep (80%)	percep (80%)
2b	percep (60%)	start (40%)	concep (60%)
3	start (80%)	percep (70%) start (30%)	semi (55%)
4a	percep (90%)	percep (100%)	percep (91%)
4b	percep (86%)	concep (50%)	percep (36%) semi (27%) concep (27%)
4c	—	—	—
5a	percep (100%)	percep (86%)	percep (100%)
5b	percep (30%) start (70%)	percep (43%) unclear (57%)	start (73%)
5c	percep (100%)	percep (86%)	percep (91%)
Final task	percep (90%)	percep (38%) start (63%)	percep (50%) start (50%)

is rather 'closed', i.e., there is no room for the students' own constructions. Besides, it is very difficult from the conceptual point of view. The concept 'expected value' was introduced to give a motivation for the calculation of

probabilities for each box, but this might have been too advanced for the students to understand.

In task 5, the conceptual level and ‘semi level raising’ do not dominate. This can be related to the formulation of the task, especially task 5c, that explicitly asks students to determine the probability by experimenting, in other words, it asks for an answer at perceptual level. The task does not aim at level raising and is in fact not a good investigation task. However, it is interesting that two pairs in AFTER did answer that they did not only experiment, but that they also calculated the probability. In their calculation, they did not take into account the asymmetric probabilities.

In the final task, most students answered at perceptual level, or made a start with level raising. The aim of this task was to let students apply what they learned about probabilities by developing their own game and reflecting on it. This did not happen, which might have to do with the formulation of the task.

We will now summarize the results per condition in order to answer the second research question. Tasks 1a, 2a, 4a, 5a and 5c will be left out of the analysis, because they explicitly ask for an answer at the perceptual level, as we just explained. The students in the condition BEFORE indeed answered the investigation tasks at perceptual level and they made a start with level raising. It is interesting that they made a start in level raising in task 1b but not in 2b and 4b. This might have to do with the fact that they attempted task 1b at the beginning of the lessons and that at that time the question was new to them. We expected that students would make a start with level raising by working on the final task, but there they stayed at the perceptual level. It might be that they lacked tasks to evoked level raising, that is, that they had not made any exercises causing them to reflect on the mathematical structures they had been exploring. The students in DURING started at perceptual level and ended up at conceptual level, but not for task 5 and the final task. The students in AFTER answered some tasks at the conceptual level, but it was also clear that many students did not answer at conceptual level, but that semi level raising had occurred, that is that they were not able to apply the concept they learned in the correct way.

‘Start Level Raising’ and ‘Semi Level Raising’

In this section we will illustrate the levels of answers in students’ dialogues. In Table IV, we give an example of start level raising in a dialogue of two students, Susan and Peter, in the BEFORE condition. They are working on task 1b for the game TIC-TAC, and the question was ‘What is the probability that a ball ends up in the 100 points box?’

TABLE IV
Dialogue between Susan and Peter

	Statement	Level of answer
1	S: There are nine possibilities	<i>Perceptual</i>
2	P: (...) But the chance that it will come here is probably bigger than that it will come here	<i>Start level raising</i>
3	S: O, why?	
4	P: O, no, that is, yes, of course, because the chance that it will go like this is bigger than that it will go up all the time.	<i>Start level raising</i>
5	S: Ok, but the computer is also doing something, ok	<i>Perceptual</i>
6	P: Comput . . . , yes, but it is about theories	<i>Conceptual</i>
7	S: Yes, that's true . . .	<i>Conceptual</i>
8	P: I mean, there are . . .	<i>Start level raising</i>
9	S: But how can you calculate that?	<i>Conceptual</i>
10	P: For instance, that one can go like that and like that, all kind of routes, but that one only has one route ...	<i>Start level raising</i>
11	S: Yes, but look, with this one you have . . . yes . . . yes	<i>Start level raising</i>
12	P: O no, that one cannot go like that, well that is very difficult (...)	<i>Start level raising</i>
13	S: Different possibilities, then you have to . . . all those routes, you know	<i>Start level raising</i>

In line 1, Susan counts the total number of possibilities. Maybe she has in mind the idea 'probability = favorable possibilities/total number of possibilities', which was prior knowledge and an answer at perceptual level in this context. In line 2, Peter discusses this answer by expressing his idea that the probability is not equal for all boxes. By asking 'why' in line 3, Susan gives him the opportunity to refine his idea. In line 5, Susan criticizes the idea that one could calculate those probabilities, because the computer has a random influence, but Peter thinks that they have to find an answer at the conceptual level (line 6). Susan is convinced (line 7) and realizes that this is not easy to calculate (line 9). Peter wants to refine his idea (line 10), Susan gets involved in this and she finally realizes that they have to count routes in order to calculate the probability. It is interesting to see that Susan started with an answer at the perceptual level and that at the end of the discussion she has made a start with level raising. Peter,

however, started at a higher level and was stimulated to refine his thinking by Susan's questions.

In Table V, we give an example of semi level raising in a dialogue between two students in the AFTER condition, Anouk and Mara. They are also working on task 1b.

TABLE V
Dialogue between Anouk and Mara

	Statement	<i>Level of answer</i>
1	A 1, 3, 6, 10, 15, 21, 21 plus 7 is 28 chances	<i>Conceptual</i>
2	A 28, how do you say, it is 1 at the 28 or something, 28, no, there can be more chances, isn't it?	<i>Semi level raising</i>
3	M Just 28	<i>Semi level raising</i>
4	A Ok	<i>Semi level raising</i>

In line 1, Anouk counts routes in the correct way, which is an answer at the conceptual level with respect to the counting of routes. In line 2, she tries to link the number of routes towards the 100-box to the probability to end up in that box and she mixes up 'number of routes' with 'probability'. We categorize this answer as 'semi level raising' because they seem to approach the problem at the conceptual level, but they are not able to apply the concepts in the right way. In lines 3 and 4, they agree to accept this answer.

CONCLUSION AND DISCUSSION

Having compared three conditions of use of the learning materials, what can we conclude about the use of collaborative investigation tasks with the computer for the learning of the mathematical concepts involved? When we consider the average final results in terms of mathematical level raising, it seems to make no difference whether students worked with the tasks either before or after their traditional textbook, or whether they worked with tasks in which textbook-like tasks had been integrated with the computer-based activities. In all conditions, many students did *not* attain the *conceptual level* for all concepts. When looking at the levels of answers to specific investigation tasks, however, we found some significant differences. Students that worked with the textbook-like tasks integrated into the collaborative investigation tasks (the DURING condition) made

a start with level raising more frequently (using their own constructions) than students who worked with the investigation tasks before they worked with their traditional textbooks. It may be that the textbook-like (paper and pencil) tasks helped students to elaborate their experiences with the computer simulation. Compared to the students who worked with the investigation tasks with the computer after they worked with the textbook, the DURING students showed less 'semi level raising', that is that they showed less use of a concept without being able to integrate it in their experiences at the perceptual level. It is therefore possible that integration of textbook-like exercises and investigation tasks in the context of the computer simulation encouraged students to try to build up a concept with their own constructions, by reflection on their experiences at the perceptual level. One result of this is that they were less likely to 'learn a trick but not know how to apply it'. We value this as an important conclusion, since this last phenomenon is a well-known problem throughout mathematics education.

We will make some critical remarks on our study and give some outlines for further research. A point of critique could be that the students in the three conditions worked with the same formulation of the investigation tasks. In order to make the learning materials most effective in each of the three conditions it might have been better to adapt the instructional text of the tasks to the state of prior knowledge, which was different for the three conditions.

Another point of critique for all conditions is that students do not get the opportunity to invent the procedure of counting routes ('*adding in end points*') with the help of their own constructions. In the next version of the learning materials, we will try to give opportunity for this with the help of smaller models of the game board in the computer simulation. The work of Kafai et al. (1998) may give ideas on how to improve the final task, where students develop a game by themselves. In further research we will continue with the DURING approach for using the learning materials, and special attention will be given to the calculation of probabilities.

The categories 'start level raising' and 'semi level raising' may help to identify steps in the process of mathematical level raising. 'Start level raising' is in fact the first step in the process of guided reinvention, where students try to build up mathematical concepts from their own experiences. 'Semi level raising' is well known as a problem of 'transfer' in mathematics learning when students know how to apply knowledge in the same context that they learnt it, but are not able to link it properly to prior knowledge, or to apply it in new contexts. Although 'start level raising' and 'semi level raising' both indicate that a student has not attained the conceptual

level, we value the ‘start’ case more highly, since the knowledge is better founded. Such concepts appear to be useful in future research projects to further investigate the process of level raising.

So far, we have paid little attention to the role of the teacher, who gave minimal help when the students were working in pairs. In further research, we will focus on the quality of the teacher help.

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