Invariants of curves and Jacobians in positive characteristic
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1 Introduction

1.1 Foreword

The main theme of this thesis is the study of some phenomena which occur in the world of curves over algebraically closed fields of positive characteristic, and which do not have counterparts in the case of characteristic zero. Let us mention two examples of such phenomena. The first example is given by non-singular projective curves in positive characteristic having a non-zero exact regular differential form. This is the property of having a singular Hasse-Witt matrix. We know this phenomenon cannot occur over a characteristic zero field, since any non-constant rational function over a projective non-singular curve will always have poles and its differential will have poles as well.

The other example is existence of curves, over an algebraically closed field $k$ of characteristic $p$, whose jacobian variety $J$ has a trivial group of $p$-torsion points, $J_{p\text{-tors}}(k) = \{0\}$. This is the property of having “$p$-rank equal to zero”. One should compare this with the fact that in characteristic zero, or more generally when $p \not| n$, the cardinality of the group of $n$-torsion points $J_{n\text{-tors}}$ is $n^{2g}$, with $g = \dim J$.

Another common feature of the kind of geometrical properties which we consider in this thesis is that they are satisfied only by special (families of) curves. More precisely, they are satisfied only on proper closed subsets of the moduli space $M_g \otimes k$, the algebraic variety parametrizing the isomorphism classes of all curves of genus $g$ over $k$. For instance, on a generic curve $X$ of genus $g$, corresponding to a general point of $M_g \otimes k$, every non-zero regular differential form is non-exact, and the cardinality of $J_{p\text{-tors}}(k)$ is $p^g > 1$.

The interest in geometrical properties of this doubly special kind, i.e. typical of the positive characteristic world and, even there, verified in special cases, arises from two main motivations. First of all, one hopes that the families of curves mentioned above may eventually help to understand the geometry of $M_g$ itself better. The reason to be so optimistic lies in the fact that the definition of a geometrical property usually gives rise not only to a set of distinguished subvarieties of $M_g$, but also in many cases to a stratification of it. This is indeed the case for those properties analyzed in this thesis.

The second motivation, which we find, to our taste, not less attractive than the previous one, is that very often it happens that the study of very special objects in a mathematical theory helps to clarify the scope and the character of the theory itself. Moreover these special, or extremal, objects often have extra interesting features, like a lot of symmetries or beautiful arithmetical properties, or are used for the study of any other object of a
given category for their universal properties. Consider for example the role of the circle in euclidean geometry or that of projective spaces in algebraic geometry.

A guiding example in the context of this thesis may be provided by the hermitian curve $X^{p+1} + Y^{p+1} = Z^{p+1}$, defined over $\mathbb{F}_p$, and isomorphic to the Fermat curve of degree $p+1$ over $\mathbb{F}_{p^2}$. It is a well known curve much studied by many authors, and together with its generalizations, it finds applications in coding theory. The curve above is extremal according to both the properties mentioned before: every regular differential form on it is exact and it has $p-$rank zero. In the plane model given above, every point over $\overline{\mathbb{F}}_p$ is an inflection point. This might be thought of as a "pathological" phenomenon, when compared with the intuition coming from real or complex algebraic geometry. But the given curve also has the nice properties of achieving the maximum number of points over $\mathbb{F}_{p^2}$ and the maximum order of automorphism group, relatively to the set of all curves of the same genus.

Our feeling is that objects like the hermitian curve, far from being "pathological", indeed reveal much of the subtleties and beauty of the characteristic $p$ geometry of curves.

Finally let us draw the reader's attention to the following analogy between the hermitian curve and an old friend from high school: let us use the notation $\bar{x} = x^p$ for $x \in \mathbb{F}_{p^2}$. Note that $\bar{x}$ is indeed a conjugation in $\mathbb{F}_{p^2}$. Then an affine model for the hermitian curve is $\bar{x}x + \bar{y}y = 1$, and its "real points", i.e. the points over $\mathbb{F}_p$, satisfy the equation: $x^2+y^2 = 1$.

### 1.2 Historical overview of the basic concepts

The concept of an abstract algebraic curve over an arbitrary field has its origins in the end of nineteenth century, mainly in the work of Dedekind and Weber, who were among the first to observe that most of the theory of function fields in one variable could be developed in purely algebraic terms. However, we owe to F.K. Schmidt in the late 20’s and the 30’s of the twentieth century a systematic treatment of the theory of algebraic curves in arbitrary characteristic including a proof for the Riemann-Roch theorem for curves in positive characteristic. Among the concepts introduced by F.K. Schmidt, we will mention the general definition of the zeta function associated to a complete absolutely irreducible smooth curve $X$, defined over a finite field $\mathbb{F}_q$, obtained by extending previous work of E. Artin in the hyperelliptic case: let $N_m$ the number of points of $X$ rational over $\mathbb{F}_{q^m}$, then

$$Z(t) = \exp(\sum_{m=1}^{\infty} N_m \frac{t^m}{m}).$$

(1)
Moreover he proved the rational expression for $Z(t)$:

$$Z(t) = \frac{P_{2g}(t)}{(1-t)(1-qt)}$$

with $P_{2g}(t)$ a polynomial of degree $2g$ in $\mathbb{Q}[t]$, $g$ being the genus of $X$, and the functional equation for $Z(t)$:

$$Z(1/qt) = q^{1-2g}Z(t).$$

In the language developed by F.K. Schmidt it was possible to state the famous Riemann hypothesis for curves over finite fields. It states that

$$|\alpha| = q^{1/2}, \quad \text{for any root } \alpha \text{ of } P_{2g}(t).$$

In his unpublished Ph.D. thesis, E. Artin had already produced a lot of evidence for this conjecture in the hyperelliptic case. F.K. Schmidt did not prove the Riemann hypothesis, but we like to recall that he defined and studied the Weierstrass points for curves in positive characteristic, and his ideas on this subject have been revived in the mid 80’s by K-O Stöhr and F. Voloch in [36], leading to an elementary proof and strengthening of the Riemann hypothesis for curves over finite fields. Weierstrass points in positive characteristic are still an important tool for the study of curves with maximal number of points over a finite field.

The first important result on the Riemann hypothesis is due to H. Hasse who in 1932 proved the Riemann hypothesis for curves of genus 1. An important tool in Hasse’s work is the notion of the Hasse-Witt matrix, which he associated, together with E. Witt, to a curve in positive characteristic. In a modern language, it describes the $p$-linear operator $F^*: H^1(O_X) \to H^1(O_X)$, with $F$ the absolute Frobenius morphism of a curve $X$.

We owe to Hasse the definition and the study of the class of supersingular elliptic curves and the first results on the endomorphism rings of elliptic curves in arbitrary characteristic. An elliptic curve $E$ over an algebraically closed field $k$ with char($k$) = $p$ is said to be supersingular if the kernel of the multiplication by $p$ has cardinality $\#E[p](k) = 1$. A study of the $p$-linear operator $F^*$ defined above allowed H. Hasse to find an explicit polynomial whose roots are the $j$—invariants of the supersingular elliptic curves. In 1941 Max Deuring was able to count the isomorphism classes of supersingular elliptic curves, via a characterization of their endomorphism algebras. Let us state part of Hasse’s and Deuring’s results as follows.
Theorem 1.1. Let $E$ be an elliptic curve over $k = \overline{k}$. The following facts are equivalent:

1) $E$ is supersingular;
2) $F^* = 0$;
3) $\text{End}(E)$ is non-commutative.

Then $\text{End}(E) \otimes \mathbb{Q} \cong D_{\infty,p}$, with $D_{\infty,p}$ the unique quaternion division algebra ramified at $\infty$ and $p$, and the following formula holds (Deuring's mass formula):

$$\sum_{[E]} \frac{1}{\# \text{Aut}(E)} = \frac{p - 1}{24},$$

where the sum is over the isomorphism classes $[E]$ of supersingular elliptic curves over $k$.

The proof of the general Riemann hypothesis for curves was found by A. Weil in 1940, using the theory of correspondences for curves. An equivalent formulation of the Riemann hypothesis is the following upper bound for the number $N$ of rational points of $X$ over $\mathbb{F}_q$:

$$|N - q - 1| \leq 2gq^{1/2},$$

known as the Hasse-Weil bound. To Weil we owe the conjectural generalization of the properties of zeta functions stated above to arbitrary smooth projective varieties in positive characteristic, and the suggestion that the rationality and the functional equation should follow from the existence of a cohomological theory for such varieties with coefficients in characteristic 0 and properties analogous to the singular cohomology of complex varieties (Weil cohomologies). He gave evidence for his conjectures by the analysis of special cases, like the Fermat varieties $X_0^n + \cdots + X_n^n = 0$ and, especially, the case of abelian varieties. A. Weil was indeed the builder of the theory of abelian varieties over fields in arbitrary characteristic, cf. his memoir [40]. To an abelian variety $A$ over $k$, he associated the inverse system of the $t^n$ torsion subgroups $A_{t^n}$, denoted by $T_t(A)$, where $l \neq p = \text{char}(k)$. $T_t(A)$ is a $\mathbb{Z}_t$ module, now known as the Tate module of $A$. There is a canonical injection $T_t : \text{Hom}_k(A, B) \hookrightarrow \text{Hom}_{\mathbb{Z}_t}(T_t(A), T_t(B))$.

If $A$ is defined over a finite field $k$ and $F$ is the Frobenius over $k$, then the characteristic polynomial $P_A(t)$ of the endomorphism $T_t(F)$ has integer coefficients and if $A$ is the jacobian variety of a curve $X$, then this characteristic polynomial is equal to $t^{2g} P_{2g}(1/t)$, with $P_{2g}$ the polynomial introduced above.

Two decades later than the memoir of Weil, in his paper [38], J. Tate showed that for $A$ and $B$ abelian varieties defined over a finite field $k$, the image of the map $T_t$ introduced
above is the part of $\text{Hom}_{\mathbb{Z}}(T_i(A), T_i(B))$ invariant under the natural action of the absolute Galois group $\text{Gal}(\bar{k}, k)$. As a consequence of this, Tate also showed that $P_2(t)$ is a complete $k$-isogeny invariant of the abelian variety, that is, two abelian varieties $A$ and $B$ are $k$-isogenous if and only if $P_A(t) = P_B(t)$.

In the late 50's and early 60's many results were proved about the structure of $p$-subgroups of abelian varieties in characteristic $p$, thanks to the work of many people, among which we will mention P. Cartier, J. Dieudonné, A. Grothendieck, Yu. Manin, T. Oda, F. Oort, J.-P. Serre, J. Tate. Since the multiplication by $p$ is inseparable, the concept of group schemes, rather than group varieties, was crucial. The following analogue of the Tate group $T_i(A)$ was much studied: the group $A[p^\infty]$, defined as the union, for all $n$'s, of the group scheme kernels $A_{p^n}$ of the multiplications by $p^n$. It has a much richer structure and it contains more delicate information about the abelian variety $A$ than $T_i(A)$. For example, $A[p^\infty]$ has the structure of a $p$-divisible formal group scheme (see [5] for an introduction).

The main tool for studying finite commutative or $p$-divisible formal group schemes is the theory of Dieudonné modules. For example, one instance of this theory establishes an anti-equivalence of categories between the category of finite commutative group schemes over a perfect field $k$ of positive characteristic, and the category of modules of finite length over the non-commutative ring $W(k)[F, V]$, with $W(k)$ the ring of infinite Witt vectors over $k$, subject to the relations $FV = VF = p$, $Fa = a^pF$, $Va^p = VA$, with $a \in W(k)$ and $\sigma$ the Frobenius operator over $W(k)$.

A theorem of Oda in 1969 (see [22]) states that, for an abelian variety $A$ over a perfect field $k$, the first De Rham cohomology group $H^1_{DR}(A, k)$ is canonically isomorphic with the Dieudonné module of the group scheme $A_p$. Moreover, let $V : A^{(p)} \to A$ be the Verschiebung operator, i.e. the dual of $F' : A^t \to (A^t)^{(p)}$, the relative Frobenius operator for the dual abelian variety $A^t$. Then one can derive from Oda's results that the subgroup scheme $\ker(F) \cap \ker(V) \subset A$ has Dieudonné module equal to coker $F^* : H^1(O_A) \to H^1(O_A)$.

In general, from finite subgroup schemes of $A[p^\infty]$, or from their associated Dieudonné modules, it is possible to extract isomorphism invariants for $A$ (cf. [23]). The most basic one is obtained in the following way. Let $\alpha_p$ be the kernel of Frobenius in $G_\alpha$, the additive group scheme over $k$.

**Definition 1.1.** The number

$$a(A) = \dim_k \text{Hom}_{k, \text{gr.sch}}(\alpha_p, A),$$
is called the a-number of the abelian variety $A$.

We remark that the notion of the a-number has been recently extended to a very broad class of algebraic varieties by G. van der Geer and T. Katsura, see [10]. A much more manageable formula for $a(A)$ is the following:

$$a(A) = \dim(\ker F^* : H^1(O_A) \to H^1(O_A)).$$

It was proved by F. Oort in [25] that the a-numbers of abelian varieties of dimension $g$ may assume any value $0 \leq a \leq g$.

On the other hand, in the case of jacobian varieties, there are restrictions on the possible a-numbers, and one of the problems studied in this thesis is indeed to find such restrictions.

In his article [18], Yu. Manin classified the isogeny classes of $p$-divisible formal group schemes $G$ defined over an algebraically closed field, and he applied his results to the groups $A[p^\infty]$. The outcome was a very simple finite set of isogeny invariants of abelian varieties in positive characteristic. More precisely, he found that any $G$ is isogenous to a direct product of known iso-simple objects $G_{m,n}$, characterized by the couples of integers $(m, n)$ such that $m, n \geq 0$, $\gcd(m, n) = 1$, and with $\dim(G_{m,n}) = m$ and $\operatorname{rank}(G_{m,n}) = m + n$. We will not describe further these iso-simple objects, but we will just mention that $G_{m,n} = G_{n,m}$, where the notation $G^t$ denotes the Serre-dual of a $p$-divisible $G$, and that $G_{1,0} = G_m[p^\infty]$, with $G_m$ the multiplicative group over $k$. In the case $G = A[p^\infty]$, if $G_{m,n}$ appears in the isogeny decomposition of $G$, then also $G_{n,m}$ appears, and with the same multiplicity. This condition comes from the existence of polarizations for $A$, hence of isogenies between $A$ and its dual abelian variety $A^t$. Manin conjectured that this condition is necessary and sufficient for $G$ to be equal to a $A[p^\infty]$. This was proved later by means of Honda-Tate theory, but see also [27] for a different proof. The set of couples $(m, n)$ appearing in the decomposition of $G$, is usually encoded in the Newton polygon $\operatorname{NP}(G)$ of $G$. It is a lower convex polygon in $\mathbb{R}^2_0$ starting at $(0,0)$, and having slopes $\lambda_i$ equal to the ratios $m/(m + n)$, each repeated $m + n$ times, and arranged in non-decreasing order. For $G = A[p^\infty]$, the corresponding Newton polygon $\operatorname{NP}(A)$ is an isogeny invariant of $A$ and it is symmetric, that is, if $\lambda$ is a slope, then also $1 - \lambda$ is a slope. Hence, in particular, its endpoint is $(2g, g)$, with $g = \dim(A)$. The highest possible such Newton polygon has all slopes equal to $1/2$, (supersingular case), and the lowest possible is the graph of the function $f(x) = 0$ for $x \in [0, g]$, and $f(x) = x - g$ for $x \in [g, 2g]$ (ordinary case).

If $A$ is defined over a finite field $\mathbb{F}_q$, there is a tight relation between $\operatorname{NP}(A)$ and the
characteristic polynomial of Frobenius

\[ P_A(t) = t^{2g} + c_1 t^{2g-1} + \cdots + c_{2g-1} t + c_{2g}, \]

namely:

\[ \lambda_i = v_p(c_i)/v_p(q), \]

with \( v_p \) the \( p \)-adic valuation.

In just one case the Newton polygon characterizes a unique isogeny class, and this is the supersingular case. The following result is due to Oort, see [24].

**Theorem 1.2.** Let \( A \) be an abelian variety over an algebraically closed field \( k \) of positive characteristic. The following facts are equivalent:

1) the Newton polygon has all slopes equal to 1/2;

2) \( A \) is isogenous to a product of supersingular elliptic curves.

When one of the conditions above is satisfied, \( A \) is said supersingular.

Supersingular abelian varieties are known to exist for any dimension (cf. above), but it is not clear if the same fact holds for jacobian varieties. A curve is said to be supersingular if its jacobian variety is. In their article [11], G. van der Geer and M. van der Vlugt, showed the existence of supersingular curves of any genus in characteristic 2. To study the existence problem for supersingular curves in \( \text{char} \geq 3 \) is another objective of this thesis.

Since the late 80's, considerable work has been done on the moduli spaces \( \mathcal{A}_g \otimes \mathbb{F}_p \) of principally polarized abelian varieties in characteristic \( p \), using isogeny or isomorphism invariants of the kind described above. This research field has been developed mainly by the Dutch school of algebraic geometry, of which we will mention A. J. de Jong, G. van der Geer, B. Moonen, F. Oort, with contribution from the Swedish mathematician T. Ekedahl. Two different stratifications of the moduli space of principally polarized abelian varieties have been defined. The first one is based on Newton polygons. By work of Grothendieck and Katz, Newton polygons define a stratification, cf. [15]. The strata are given by the isomorphism classes of abelian varieties with a given Newton polygon, hence each stratum is invariant under isogeny. A survey on the topic of Newton polygon stratifications can be found in [28]. The supersingular stratum, i.e. the one associated to the supersingular Newton polygon, is the smallest of such strata and it has dimension \( [g^2/4] \), cf. [17]. For \( g \geq 9 \) a non-empty intersection between the supersingular stratum and Torelli locus, i.e. the locus of jacobian varieties in \( \mathcal{A}_g \), cannot follow from dimension
consideration, since the sum of the codimensions of those two sets exceeds \( \dim A_g \). A natural, but likely a difficult, open problem is to understand the intersection between the Newton polygon strata and the Torelli locus. The first important results in this direction have very recently been obtained by C. Faber and G. van der Geer, who proved that the locus of stable curves over an algebraically closed field \( k \) of \( \text{char} = p \) having \( p \)-rank at most \( f \) is pure of codimension \( g - f \) in \( \overline{M}_g \otimes k \), cf. their preprint [7].

The second stratification of \( A_g \otimes \mathbb{F}_p \) is called the Ekedahl-Oort stratification. Its strata are characterized by the isomorphism types of the finite group schemes \( A[p] \), for \( A \) a principally polarized abelian variety of dimension \( g \) over \( \bar{k} \) (cf [29]). The basic observation for this construction is that only a finite number of isomorphism types of such \( A[p] \) are possible. The smallest of the Ekedahl-Oort strata is the one consisting of the isomorphism classes of abelian varieties with \( a \)-number\(= g \). This stratum is zero dimensional, and the formula for its class in the Chow ring of \( A_g \otimes \mathbb{F}_p \) can be considered as a generalization of Deuring's mass formula 2. A formula of this class is a particular case of general formulas for the classes of Ekedahl-Oort strata, obtained by G. van der Geer and T. Ekedahl in [9].

In the same paper the Ekedahl-Oort strata are defined using the fact, mentioned above, that the de Rham cohomology groups \( H_{dR}^1(A, k) \), together with their \( p \)-linear operators \( F, V \) give the Dieudonné modules of the group schemes \( A[p] \). Also here an interesting open problem is to understand the intersections of the various strata of this stratification with the Torelli locus.

\[ \text{1.3 New results} \]

As already mentioned above, this thesis focuses on two different numerical invariants for a curve over an algebraically closed field of positive characteristic. The first invariant is the \( a \)-number of the jacobian of the given curve, see definition 1.1 above. It is not an isogeny invariant. The second one is the Newton polygon associated to the jacobian, which is an isogeny invariant. In this last case we focus our attention in particular on the supersingular case, i.e. the case when the Newton polygon is the highest possible.

In chapter 2 we report our study of the restrictions on the possible \( a \)-numbers of jacobians. The starting point is the article [6] of T. Ekedahl, which deals with the notion of superspecial curves and abelian varieties. An abelian variety \( A \) over an algebraically closed field \( k \) of positive characteristic is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves \( E_1 \times \cdots \times E_g \), as unpolarized abelian variety. A curve is said to be superspecial if its jacobian variety is. A result in [6] is that a curve
$X$ can be *superspecial* only if its genus $g(X)$ satisfies the inequality $g(X) \leq p(p - 1)/2$. Now, it is well known that an abelian variety is superspecial if and only if $\alpha(A) = \dim A$, i.e. its $\alpha$-number is the maximum possible, cf. for example [26], or [20], theorem 4.1. Equivalently, the rank of the Hasse-Witt matrix of a superspecial abelian variety is 0. This fact makes one suspect that for curves a bound on the genus depending on the rank of their Hasse-Witt matrix should exist. Another evidence in favour of this conjecture was a remark of Gerard van der Geer, who showed to us how to find such a bound in the case of char= 2, as a consequence of Clifford's theorem. In our approach, we use the Cartier operator acting on the space of regular differential on a curve $X$, which is dual to the Hasse-Witt operator by the Serre duality, see the introduction of chapter 2 for definitions. Then we observe that if the rank of the Cartier operator, equivalently the rank of the Hasse-Witt operator, is $\leq m$, then the curve $X$ has the following very special property: for general points $P_1, \cdots, P_m$ on $X$ there exists a rational function $f$ on $X$ having poles only on $P_1, \cdots, P_m$, and with multiplicity equal to $p$ over these points. If $g(X)$ is greater than $(m + 1)p$, this phenomenon can never happen in characteristic 0. Indeed a curve with such a property is non-classical, in the sense of Stöhr-Voloch, see [36]. Now, when a curve possesses a very special line bundle, i.e. a line bundle with unexpected number of sections relatively to its degree, it is sometimes possible to obtain a bound on the genus depending on the degree of such a line bundle. One encounters such a situation in classical algebraic geometry in Castelnuovo's proof for a bound of the genus of a non-degenerate irreducible curve of degree $d$ in $\mathbb{P}^r$. By suitable refinements of Castelnuovo techniques, and by applying them in our char= p context, we get the following results.

**Theorem 1.3.** Let $X$ be an hyperelliptic curve over an algebraically closed field in characteristic $p > 0$, and suppose that the Hasse-Witt operator $F^*: H^1(O_X) \to H^1(O_X)$ has rank $m$. Then

$$g(X) < (p + 1)/2 + mp$$

In the general case we prove the following result.

**Theorem 1.4.** Let $X$ be non-hyperelliptic, and suppose that $\text{rk } F^* = m$. Then

$$g(X) \leq (m + 1)p(p - 1)/2 + pm.$$  

We also consider the case when $F^*$ is nilpotent, with nilpotency order equal to $r$. This case is meaningful, because it is well known that $F^*$ is nilpotent if and only if the jacobian variety $J$ of the curve $X$ has $p$-rank= 0. We obtain the following result.
Theorem 1.5. In the notations above, if \((F^*)^r = 0\) for some \(r \geq 1\), then
\[ g(X) \leq p^r(p^r - 1)/2. \]

This bound is reached by the hermitian curves:
\[ y^{p^r} - y = x^{p^r+1}. \]

The last two theorems give back Ekedahl's bound \(g(X) \leq p(p - 1)/2\), for a curve \(X\) with \(F^* = 0\), as a particular case. A proof Ekedahl's bound by methods analogous to ours was given independently also by M. Baker in his paper [1]. Our approach to such bounds provides a clear geometrical interpretation of the effect of the degeneration of the operator \(F^*\) on the geometry of curves in positive characteristic.

In chapter 3 we start to study the problem of the existence of supersingular curves in arbitrary positive characteristic. The material contained in this chapter is not completely original, since it is our elaboration of ideas of N. O. Nygaard and T. Ekedahl, leading to some criteria of supersingularity for curves and abelian varieties. In the first section of chapter 3 we reformulate Nygaard's necessary and sufficient conditions for the supersingularity of curves or abelian varieties. We obtain conditions expressed in terms of the cohomology with Witt vectors coefficients defined by Serre, cf. [33]. Our motivation for such a formulation is that Witt vectors cohomology is computable in the Zariski site, hence we believe it easier to compute in concrete cases, for example for curves given by explicit equations in projective spaces. In the second section of chapter 3, we apply and extend ideas of T. Ekedahl, to get a sufficient condition for the supersingularity of curves admitting a group action of special kind. The precise result is the following.

Theorem 1.6. Let \(X\) be a curve defined over a number field \(K\), with an action of a finite group \(G\) of exponent \(n\), such that all irreducible representations of \(G\) occur at most once in \(H^1_{DR}(X,\mathbb{Q})\). If \(p^r \equiv -1 \mod n\) for some \(r\), and \(X\) has good reduction in char \(= p\), then \(X \mod p\) is supersingular.

This criterion can be applied to give a new proof of the supersingularity of the Fermat curves \(x^{q+1} + y^{q+1} + z^{q+1} = 0\), with \(q\) a power of the characteristic, which was already known since the article [14], where it was obtained with a different method. Indeed, we prove that any abelian covering \(X\) of \(\mathbb{P}^1\) ramified over three points satisfies the assumptions on the group action on \(H^1_{DR}(X)\) required by the theorem above, see theorem 3.5. But, potentially, the criterion stated above may provide also other examples of supersingular curves, as it is shown by an example at the end of chapter 3.
In chapter 4 we construct supersingular curves in many genera, producing some tables of genera of supersingular curves in characteristic equal to some small prime numbers \( p \geq 3 \). More precisely, we compute quotients of the Fermat curves with respect to many classes of subgroups of their (very big) automorphism group, which is isomorphic to \( \text{PGU}(3, q^2) \). In the course of the computations we extend some results of [8], where similar computations were performed for quotients of the Hermitian curves \( Y^{q+1} = X^qZ + XZ^q \), isomorphic to the Fermat curve as above over \( \mathbb{F}_q \). We implemented in a computer program the general formulas for the genera of the supersingular curves so constructed and we obtained some interesting numerical results. For instance we get any \( g \leq 100 \) except maybe \( g = 59 \) as the genus of a supersingular curve in char=3, any \( g \leq 100 \) with no exceptions if char= 5 and similarly good results for other small primes. At the end of the chapter we observe that analogous and indeed much simpler constructions are capable to produce explicit examples of \( p \)-rank 0 curves of any genus in char= 3 and char= 5.

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