The Foreground-Background Queue
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Comparing maximum queue lengths

In this chapter we compare the maximum queue lengths in two FB queues with different service times with the same mean. In the first queue the service times are deterministic, in the second queue the customers either leave the queue immediately, or have service time $1 + c$ for some $c > 0$. We show that the maximum queue length in the busy period is stochastically smaller in the second queue. This shows that a heavier tail of the service-time distribution does not necessarily lead to worse behaviour of the system.

4.1 Introduction

Asmussen [4], page 336, reads “a common folklore states that ‘adding variation decreases performance’”. On the other hand, Yashkov [69] claims that in the stationary FB queue “$EV$ decreases with an increase in the dispersion of $F(x)$, and conversely
increases as the dispersion of $F(x)$ decreases." Here $V$ denotes the sojourn time in the stationary queue. In this chapter and the next two we examine the effect of a higher variability in the service times, to be defined below, on the queue length process.

The stationary queue length in the $M/G/1$ queue under the PS queue depends only on the service-time distribution through its mean. This shows that more variation in the service-time distribution does not decrease performance. Some recent studies have indicated that more variability may even lead to better behaviour of certain stochastic systems. Ridder et al. [49] show that in an inventory system a larger variability in the demand may result in lower costs. Litjens and Boucherie [34] have done simulations that seem to indicate that in an $M/G/1$ PS queue with varying service capacity, more variation in the service times results in a smaller expected sojourn time in the queue. Kelly [30] describes a computer network that is much more stable when the variance of the packet sizes is larger.

In this thesis the variability of the service times is measured in the following way.

**Definition 4.1 (Convex order)** A random variable $X$ is smaller than a random variable $Y$ in the convex order, notation $X \leq_{cx} Y$, if $EX$ and $EY$ exist and

$$Eh(X) \leq Eh(Y)$$  \hspace{1cm} (4.1)

for all convex functions $h$ for which the expectations $Eh(X)$ and $Eh(Y)$ exist.

From the definition it follows that $EX = EY$, since $h(x) = x$ and $h(x) = -x$ are convex functions. If the second moments of $X$ and $Y$ exist as well, then the convexity of $h(x) = x^2$ implies $\text{Var}(X) \leq \text{Var}(Y)$. Therefore, if $X \leq_{cx} Y$, then $X$ is said to be *more variable* than $Y$.

Shaked and Shanthikumar [60] show that if $X$ and $Y$ have densities $f$ and $g$, then a sufficient condition for $X \leq_{cx} Y$ is that $f - g$ changes sign twice, from minus to plus to minus, or is zero everywhere. For conditions that lead to convex ordering, and characterisations, we refer to Shaked and Shanthikumar [60].

In this and the next two chapters we study the effect of an increase in the variability of the service-time distribution. In Chapter 6 we treat the effect on the mean queue length in the FB queue. In Chapter 5 we treat the effect on the maximum queue length in queues with several other disciplines. In this chapter we consider the effect of a larger variability on the behaviour of the maximum queue length in a busy period. The maximum queue length is a fair way to measure the performance of queues, since two queues with the same arrival rate and expected service time have asymptotically the same number of busy periods.
Furthermore, understanding the distribution of the maximum queue length is important when designing the buffer size for finite buffer (M/G/1/n) systems. These systems are good models for telecommunications channels. In this chapter we compare the distributions of the maximum queue length for two FB queues with the same arrival rate and expected service time for two special service-time distributions, namely the simplest pair of convexly ordered service times.

The main result in this chapter is that for such convexly ordered service times, the convexly larger service time yields a stochastically shorter maximum queue length in the busy period. Hence for the maximum queue length the FB discipline apparently takes more advantage from the presence of more small customers than it suffers from the presence of more large customers. In Chapter 5 it is shown that a similar result holds in the queue operating under the LIFO preemptive resume discipline.

4.2 Notation and the theorem

In this chapter we compare the queue lengths of two queues. One queue has constant service times equal to 1; the service times in the other queue have the same expectation, but may assume two different values. Denote the first queue by $Q_1$ and the second queue by $Q_2$. Both queues have i.i.d. $\exp(\lambda)$ distributed interarrival times for some $\lambda \in (0, 1)$. The service discipline is FB.

In $Q_2$ the service times are $0+$ and $1+c$ with probabilities $c/(1+c)$ and $1/(1+c)$ respectively, for some $c > 0$. Here a customer with service time $0+$ is understood as a customer that enters the queue but leaves immediately. We call such a customer a *null-customer*. If a null-customer arrives at time $t$ say, and the queue length $Q(s)$ satisfies $\lim_{s \uparrow t} Q(s) = n$, then $Q(t) = n + 1$ and $\lim_{s \uparrow t} Q(s) = n$ with probability one. It is clear that the service times in $Q_2$ are convexly larger than those in $Q_1$.

Let both queues be empty at time 0-. At time 0 in both queues a customer arrives. These customers start a period of time in which the server is continuously busy. These periods are called busy periods and we denote their lengths by $T_1$ and $T_2$ respectively. The periods between two busy periods, the idle periods, have strictly positive length. The queue length is the number of customers present in the queue, including the customer(s) being served. Let the maximum queue lengths in these busy periods $[0, T_1]$ and $[0, T_2]$ be $M_1$ and $M_2$ respectively. Recall that $\leq_{st}$ denotes the usual stochastic order relation, see Definition 2.6.

The main result of this chapter is the following.

**Theorem 4.2** The maximum queue lengths $M_1$ and $M_2$ over the first busy period
in the two queues introduced above satisfy

\[ M_1 \geq_{st} M_2. \quad (4.2) \]

In case of constant service times the FB discipline is the worst (work-conserving) discipline one could use: under this discipline all customers in a busy period stay in the queue till the end of the busy period. This observation however compares queues with different disciplines. It does not help us when we compare two FB queues with different service-time distributions, of which one is deterministic.

The organisation of the chapter and the proof of Theorem 4.2 is as follows. In Section 4.3 we introduce the last sojourn time in the busy period in the M/D/1 FB queue Q₁. This random variable plays a key role in the proof of Theorem 4.2. In Section 4.4 a transform of the last sojourn time is obtained. In Section 4.5 we analyse the queue Q₂ and show that the inequality \( P(M_1 = n) \geq P(M_2 = n) \) holds if the transform of the last sojourn time satisfies a certain inequality. By means of rather complicated calculations it is shown in Section 4.6 that this inequality indeed holds. Finally, in Section 4.7 the proof of Theorem 4.2 is given. We conclude the chapter with a result for random walks conditioned to stay positive.

## 4.3 The last of the sojourn times

First we consider the M/D/1 queue Q₁ with constant service times equal to 1. We give the probabilities \( P(M_1 = n) \) for \( n \in \mathbb{N} \) and introduce the last sojourn time in the busy period, which is an important random variable in this chapter.

Let the interarrival times \( A_1, A_2, \ldots \) be independent and \( \exp(\lambda) \) distributed. Define

\[ Y_n = n - (A_1 + \cdots + A_n), \quad n \geq 1. \quad (4.3) \]

As soon as \( Y_n \) becomes negative, the (first) busy period is over. Indeed, \( A_1 + \cdots + A_n \) is the arrival time of the \((n + 1)\)st customer. As soon as this arrival time is later than the time needed to complete the service of the first \( n \) customers, which is \( n \), the busy period is finished. Hence the length \( \tau \) of the first busy period is given by

\[ \tau = \min\{n : Y_n < 0\}. \quad (4.4) \]

Within an M/D/1 FB busy period no customers leave. Since the received amount of service of the customer being served is never more than that of the other customers,
all customers served in one busy period have to stay until the end and the maximum queue length is attained during the time interval \([T, \tau]\) for some stochastic time \(T\).

Since every customer has a service time equal to 1, the random variable \(\tau\) is also equal to the number of customers in the busy period, and in the M/D/1 FB queue also equal to the maximum queue length. Borel [9] and Prabhu [45] give the distribution of \(\tau\):

**Proposition 4.3 (Borel)** The distribution of the length \(\tau\) of an M/D/1 busy period with arrival rate \(\lambda\) and service times equal to 1 satisfies

\[
p_n = P(\tau = n) = \lambda^{n-1} e^{-\lambda n} \frac{n^{n-1}}{n!}.
\]

(4.5)

Recalling (4.3) we find that for \(\tau \geq 2\)

\[
Y_{\tau -1} + 1 = \tau - 1 - (A_1 + \cdots + A_{\tau -1}) + 1 = \tau - (A_1 + \cdots + A_{\tau -1}).
\]

(4.6)

Define \(Y_0 = 0\). Since \(\tau\) is the length of the first busy period, by (4.6) the random variable

\[
D := Y_{\tau -1} + 1
\]

is the difference between the arrival time of the \(\tau\)'th and last customer in this busy period \((A_1 + \cdots A_{\tau -1})\) and the end of the busy period. Hence \(D\) is the length of the period during which the maximum queue length (of size \(\tau\)) is attained.

### 4.4 A transform of the last sojourn time

In the proof of the main result in this chapter, Theorem 4.2, a crucial role is played by the transform \(\mathcal{L}_n(s)\) of \(D\), defined as follows:

\[
\mathcal{L}_n(s) = E[\exp(-sD) | \tau = n]. \quad s \geq 0.
\]

(4.8)

In this section we first find an expression for \(\mathcal{L}_n\). Surprisingly, it does not depend on the arrival rate \(\lambda\). Then we derive a relation between \(\mathcal{L}_n\) and \(\mathcal{L}_{n+1}\) that is needed later on.

**Proposition 4.4** The transform \(\mathcal{L}_n(s)\) satisfies

\[
\mathcal{L}_n(s) = \frac{1}{p_n(-s)} \left(1 - \sum_{m=1}^{n-1} p_m(-s)\right),
\]

(4.9)

where \(p_m(x) = \frac{m^{m-1}}{m!} x^{m-1} e^{-xm}\).
Note that \( p_m(x) \) is negative for \( x < 0 \) and \( m \) even, and that \( p_n = p_n(\lambda) \) in (4.5). The proposition is proved with the use of the following two technical lemmas.

**Lemma 4.5** Let \( q_1(y) = 1 \) and \( q_{n+1}(y) = \int_0^{1+y} q_n(z)dz \) for \( n \geq 1 \) and \( y \geq 0 \). Then

\[
q_n(y) = \frac{(y + n)^{n-1}}{(n-1)!} - \frac{(y + n)^{n-2}}{(n-2)!} \quad n \geq 2. \tag{4.10}
\]

**Proof** By induction. Suppose that (4.10) holds for \( n \). Then

\[
q_{n+1}(y) = \int_0^{1+y} q_n(z)dz = \int_0^{1+y} \left( \frac{(z + n)^{n-1}}{(n-1)!} - \frac{(z + n)^{n-2}}{(n-2)!} \right)dz
\]

\[
= \frac{(z + n)^n}{n!} - \frac{(z + n)^{n-1}}{(n-1)!} \bigg|_{z=0}^{z=1+y}
\]

\[
= \frac{(y + n + 1)^n}{n!} - \frac{(y + n + 1)^{n-1}}{(n-1)!} - \frac{n^n}{n!} + \frac{n^{n-1}}{(n-1)!}
\]

and the induction is completed. \( \square \)

**Lemma 4.6** Let \( r_1(y) = e^{-sy}, s > 0 \) and \( r_{n+1}(y) = \int_0^{1+y} r_n(z)dz \) for \( n \geq 1 \). Then

\[
r_{n+1}(y) = s^{-1} \sum_{k=0}^{n-1} q_{n-k}(y)(-se^s)^{-k} + (-se^s)^{-n}e^{-sy}. \tag{4.11}
\]

where \( q_k(y) \) is as in Lemma 4.5.

**Proof** By induction. Suppose that \( r_{n+1}(y) \) is as in (4.11). Then

\[
r_{n+2}(y) = \int_0^{1+y} \left( s^{-1} \sum_{k=0}^{n-1} q_{n-k}(z)(-se^s)^{-k} + (-se^s)^{-n}e^{-sz} \right)dz
\]

\[
= s^{-1} \sum_{k=0}^{n-1} (-se^s)^{-k} \int_0^{1+y} q_{n-k}(z)dz + (-se^s)^{-n} \int_0^{1+y} e^{-sz}dz
\]

\[
= s^{-1} \sum_{k=0}^{n-1} q_{n+1-k}(y)(-se^s)^{-k} + (-se^s)^{-n}(s^{-1} - s^{-1}e^{-s(1+y)})
\]

\[
= s^{-1} \sum_{k=0}^{n} q_{n+1-k}(y)(-se^s)^{-k} + (-se^s)^{-n}e^{-sy}.
\]
since \( q_1(y) = 1 \).

**Proof of Proposition 4.4** The joint density of the interarrival times \( A_1, \ldots, A_n \) is

\[
 f_{(A_1, \ldots, A_n)}(x_1, \ldots, x_n) = \lambda^n e^{-\lambda(x_1 + \cdots + x_n)}. \quad (x_1, \ldots, x_n) \in (0, \infty)^n. 
\]

For \( \tau = n \) we need that \( A_1 < 1, A_2 < 2 - A_1, \) etc. Hence

\[
 L_n(s) = E \left[ \exp \left( -s(\tau - (A_1 + \cdots + A_{\tau-1})) \right) \right] \mid \tau = n 
\]

\[
 = \frac{1}{P(\tau = n)} \int_0^1 \cdots \int_0^1 e^{-s(n-(x_1+\cdots+x_{n-1}))} \lambda^n e^{-\lambda(x_1+\cdots+x_n)} dx_n \cdots dx_1.
\]

where

\[
 \Gamma = \{ 0 \leq x_1 \leq 1, \ldots, 0 \leq x_{n-1} \leq n-1-(x_1+\cdots+x_{n-2}), x_n \geq n-(x_1+\cdots+x_{n-1}) \}.
\]

Integration w.r.t. \( x_n \), and using \( P(\tau = n) = p_n = \lambda^{n-1}e^{-\lambda n n^{-1}/n!} \), gives

\[
 L_n(s) = \frac{\lambda^n e^{-\lambda n^{n-1}}}{p_n} \int_0^1 \cdots \int_0^1 e^{-s(n-(x_1+\cdots+x_{n-1}))} dx_{n-1} \cdots dx_1
\]

\[
 = \frac{n!}{n^{n-1}} \int_0^1 \cdots \int_0^1 e^{-s(y_{n-1}+1)} dy_{n-1} \cdots dy_1 
\]

(4.12)

\[
 = \frac{n!}{n^{n-1}} e^{-s r_n(0)}, \quad (4.13)
\]

where the second equality follows from substituting \( y_k = k \cdot (x_1 + \cdots + x_k) \), and \( r_n(0) \) is as in Lemma 4.6. Note that \( \lambda \) has disappeared!

Using Lemma 4.6 with \( y = 0 \) and \( q_n(0) = n^{-1-n!} \) yields

\[
 r_n(0) = s^{-1} \sum_{k=0}^{n-2} \frac{(n-1-k)^{n-2-k}}{(n-1-k)!} (se^s)^{-k} + (se^s)^{-n+1} 
\]

(4.14)

\[
 = s^{-1} \sum_{m=1}^{n-1} \frac{m^{m-1}}{m!} (se^s)^{m-n+1} + (se^s)^{-n+1}
\]

\[
 = (se^s)^{-n+1} \left( 1 - \sum_{m=1}^{n-1} \frac{m^{m-1}}{m!} (-s)^{m-1} e^{sm} \right).
\]

By (4.13) we conclude

\[
 L_n(s) = \frac{n!}{n^{n-1}} (-s)^{-n+1} e^{-sn} \left( 1 - \sum_{m=1}^{n-1} \frac{m^{m-1}}{m!} (-s)^{m-1} e^{sm} \right)
\]

\[
 = \frac{1}{p_n(-s)} \left( 1 - \sum_{m=1}^{n-1} p_m(-s) \right).
\]
where \( p_m(x) = \frac{m^{m-1}}{m!} x^{m-1} e^{-mx} \).

### Example 4.7
The transform \( \mathcal{L}_n(s) \) satisfies

1. \( \mathcal{L}_1(s) = \frac{1}{p_1(-s)} = e^{-s} \).
2. \( \mathcal{L}_2(s) = \frac{1 - p_1(-s)}{p_2(-s)} = \frac{1 - e^s}{-se^{2s}} = s^{-1}e^{-s} - s^{-1}e^{-2s} \).
3. \( \mathcal{L}_3(s) = \frac{1 - p_1(-s) - p_2(-s)}{p_3(-s)} = \frac{1 - e^s + se^{2s}}{3e^{3s}} = \frac{2}{3} \left( \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^3} \right) \).

From Proposition 4.4 we may derive a relation that will be very useful later on in this chapter. Let \( n \in \mathbb{N} \). Then by Proposition 4.4,

\[
p_{n+1}(-s)\mathcal{L}_{n+1}(s) = 1 - \sum_{m=1}^{n} p_m(-s) = p_n(-s)\mathcal{L}_n(s) - p_n(-s).
\]

Hence

\[
1 - \mathcal{L}_n(s) = -\frac{p_{n+1}(-s)}{p_n(-s)}\mathcal{L}_{n+1}(s) = -\frac{(n+1)^n(-s)^n+1 e^{(n+1)s}}{(n+1)!} p_{n+1}(-s)\mathcal{L}_{n+1}(s)
\]

\[
= se^{s} \left( \frac{n+1}{n} \right)^{n-1} \mathcal{L}_{n+1}(s). \tag{4.15}
\]

The section is concluded with the following remark. There exists a probabilistic interpretation of the transform \( \mathcal{L}_n \) which is widely used in queueing theory, see Runnenburg [55]. The value of the transform \( \mathcal{L}_n(s) \) is the probability that \( Z \), an exponentially distributed random variable with parameter \( s \), independent of \( D \), is larger than \( D \), given that \( \tau = n \). Indeed.

\[
P(Z > D \mid \tau = n) = \int_0^\infty e^{-x} d\mathcal{L}_{[\tau=n]}(x) = E[e^{-sD} \mid \tau = n] = \mathcal{L}_n(s). \tag{4.16}
\]

### 4.5 The M/G/1 queue with a different clock

In this section we find a (necessary and sufficient) condition, in terms of the transform \( \mathcal{L}_n \), under which \( P(M_1 = n) > P(M_2 = n) \) holds. For this we need the following two observations. Consider the M/G/1 FB queue \( Q_2 \) with service times 0+ and 1+ with probability \( c/(1+c) \) and \( 1/(1+c) \) respectively. Denote in the sequel the M/D/1 queue with service times equal to \( a, a > 0 \). by M/D(a)/1.
Proposition 4.8 Let $M(a), a > 0$, be the maximum queue length in a busy period in the $M/D(a)/1$ queue with arrival intensity $\lambda/a$. Then

$$M(a) \overset{d}{=} M(1).$$

Proof By shrinking the time axis by a factor $a$, we obtain a one to one correspondence between (the sample paths of) an $M/D(a)/1$ queue with arrival intensity $\lambda/a$ and an $M/D(1)/1$ queue with arrival intensity $\lambda$: the former queue just has a clock that ticks a factor $a$ slower than the clock of the latter. The proof is finished by observing that the maximum queue length in a busy period is independent of the speed of the clock.

The following calculation may serve as an illustration of the concept used in the proof of Proposition 4.8. The probability that in the $M/D(a)/1$ queue the first interarrival time $A_1$ is larger than the service time $a$ of the first customer, is equal to $e^{-a(\lambda/a)} = e^{-\lambda}$. This probability is the same as the probability that an $\text{exp}(\lambda)$ distributed interarrival time is larger than 1.

By the theory of Poisson processes, the arrival process of the queue $Q_2$ may be seen as consisting of two independent arrival processes: one for the $(1+c)$-customers and one for the null-customer. These processes have intensity $\lambda/(1+c)$ and $\lambda c/(1+c)$ respectively, since $P(B = 1 + c) = 1/(1+c)$. We may therefore consider $Q_2$ as a queue with (constant) service times equal to $1+c$ and arrival intensity $\lambda/(1+c)$ in which every now and then a null-customer arrives. The null-customers arrive according to an independent Poisson process with intensity $\lambda c/(1+c)$. By the nature of the FB discipline, such null-customers leave again immediately. Let $Q_3$ be the queue $Q_2$ with a clock that ticks $1+c$ times faster. Then $Q_3$ is a queue with service times equal to 1 and arrival intensity $\lambda$, in which null-customers arrive according to an independent Poisson process with intensity $c\lambda$. Applying Proposition 4.8 yields the following analogy.

Proposition 4.9 The distribution of the maximum queue length in a busy period in the queues $Q_2$ and $Q_3$ is the same.

This property is exploited in the next proposition. Recall that $M_2$ is the maximum queue length in $Q_2$.

Proposition 4.10 Let $n \geq 2$. Then

$$P(M_2 = n) = \frac{p_n}{c+1} \mathcal{L}_n(c\lambda) + \frac{p_{n-1}}{c+1} (1 - \mathcal{L}_{n-1}(c\lambda)).$$
Proof Let $M_3$ be the maximum queue length in the first busy period of the queue $Q_3$ defined above. Below we calculate the distribution of $M_3$. By Proposition 4.9, this distribution is the same as the distribution of $M_2$. Let $B_1$ be the first service time in the queue $Q_3$. First note that $P(M_3 = n. B_1 = 0+) = 0$ for $n \geq 2$, since the busy period ends immediately if the first customer is a null-customer. Hence for $n \geq 2$ we only need to calculate $P(M_3 = n. B_1 = 1)$.

Let $Z$ be an $\exp(c\lambda)$ distributed random variable, independent of the busy period length $\tau$. Let $\tau - Z$ be the last arrival time of a null-customer in the busy period and recall that $D$ is the length of the period in which the maximum queue length is attained. If $M_3 = n$, then the maximum length may be formed in the following two ways: 1) the maximum number of large customers, i.e. customers with service time 1, is $n$ and no null-customers arrive in the period this maximum is attained, or 2) the maximum number of large customers is $n - 1$ and a null-customer arrives during the period this maximum is attained. Hence

$$\{M_3 = n\} = \{\tau = n. \tau - Z < \tau - D\} \cup \{\tau = n - 1. \tau - Z \geq \tau - D\}.$$ 

Hence

$$P(M_3 = n \mid B_1 = 1) = P(\tau = n, Z > D) + P(\tau = n - 1, Z \leq D)$$

$$= P(Z > D \mid \tau = n)P(\tau = n) + P(Z \leq D \mid \tau = n - 1)P(\tau = n - 1)$$

$$= \mathcal{L}_n(c\lambda)p_n + (1 - \mathcal{L}_{n-1}(c\lambda))p_{n-1}$$

by equation (4.16). The proposition now follows from $P(B_1 = 1) = 1/(1 + c)$. \qed

Lemma 4.11 Fix an $n \in \{2, 3, 4, \ldots\}$. The inequality $P(M_1 = n) \geq P(M_2 = n)$ holds if and only if

$$\mathcal{L}_n(c\lambda) \leq \frac{c + 1}{1 + ce^{(c+1)\lambda}}.$$ (4.17)

Proof Proposition 4.10 gives us

$$P(M_1 = n) - P(M_2 = n) = p_n - \frac{p_n}{c + 1}\mathcal{L}_n(c\lambda) - \frac{p_{n-1}}{c + 1}(1 - \mathcal{L}_{n-1}(c\lambda)).$$

$$= \frac{p_{n-1}}{c + 1}\left(\frac{p_n}{p_{n-1}}(c + 1 - \mathcal{L}_n(c\lambda)) - (1 - \mathcal{L}_{n-1}(c\lambda))\right)$$

$$= \frac{p_{n-1}}{c + 1}\left(\lambda e^{-\lambda} \left(\frac{n}{n - 1}\right)^{n-2} (c + 1 - \mathcal{L}_n(c\lambda)) - (1 - \mathcal{L}_{n-1}(c\lambda))\right).$$ (4.18)

since by Proposition 4.3

$$\frac{p_n}{p_{n-1}} = \frac{\lambda^{n-1}e^{-\lambda n} n^{n-1}}{\lambda^{n-2}e^{-\lambda(n-1)} (n-1)!} = \frac{\lambda n^{n-2}}{e^\lambda (n-1)^{n-2}}.$$
From (4.15) it then follows that
\[ 1 - \mathcal{L}_{n-1}(c\lambda) = c\lambda e^{c\lambda} \left( \frac{n}{n-1} \right)^{n-2} \mathcal{L}_n(c\lambda). \]

Substituting this in (4.18) gives
\[
\frac{p_{n-1}}{c+1} \left( \lambda e^{-\lambda} \left( \frac{n}{n-1} \right)^{n-2} \left( c + 1 - \mathcal{L}_n(c\lambda) \right) - c\lambda e^{c\lambda} \left( \frac{n}{n-1} \right)^{n-2} \mathcal{L}_n(c\lambda) \right)
\]
\[ = \frac{\lambda p_{n-1}}{(c+1)e^\lambda} \left( \frac{n}{n-1} \right)^{n-2} \left( c + 1 - \mathcal{L}_n(c\lambda) - ce^{(c+1)\lambda} \mathcal{L}_n(c\lambda) \right). \]

The statement follows. \qed

### 4.6 Technicalities

In this section we prove by patient calculation that the condition in Lemma 4.11 holds for \( n \geq 3 \).

**Lemma 4.12** For all \( x \geq 0 \)

\[ m(x) := 3x^2(x+1)e^{3x} - 2(xe^{2x} - e^x + 1)(1 + xe^{x+1}) \geq 0. \]

**Proof** We write out

\[ m(x) = 3x^3e^{3x} + 3x^2e^{3x} - 2xe^{2x} - 2x^2e^{3x+1} + 2e^x + 2xe^{2x+1} - 2 - 2xe^{x+1}. \]

(4.19)

To prove that \( m(x) \) is positive we expand \( m(x) \) in a power series and show that all the coefficients are positive. First we see that for \( x \downarrow 0 \),

\[ m(x) = O(x^3) + x^2(-2 + 2 + 2e - 2e) + x(-2 + 2 + 2e - 2e) + 2 - 2 \]
\[ = O(x^3). \]

Hence expression (4.19) may be rewritten as

\[ m(x) = \sum_{k=3}^\infty x^k \left[ \frac{3^{k-2}}{(k-3)!} + \frac{3^{k-1} - 2e3^{k-2}}{(k-2)!} + \frac{e2^k - 2k - 2e}{(k-1)!} + \frac{2}{k!} \right] \]
\[ = \sum_{k=3}^\infty \frac{x^k}{(k-1)!} b_k, \]

where

\[ b_k = 3^{k-2}(k-2)(k-1) + 3^{k-1}(k-1) + 2^k - 2e3^{k-2}(k-1) + e2^k - 2e + \frac{2}{k}. \]
We show that $b_k \geq 0$ for $k \geq 3$. First we compute $b_3 = 26/3 + 2e$ and $b_4 = 207/2 - 20e$. Then for $k \geq 5$ we have

$$b_k = 3^{k-2}(k-1)(k-2 + 3 - 2e) + 2^k(e - 1) - 2e + 2/k \geq 2^k - 2e > 0.$$ 

We conclude that the power series expansion of $m(x)$ only allows positive coefficients, and hence $m(x)$ is positive. \hfill \square

**Lemma 4.13** For $0 \leq \lambda \leq 1$

$$\mathcal{L}_3(\lambda c) \leq \frac{1 + c}{1 + ce^{(c+1)/c}}.$$ 

**Proof** Write $\lambda c = a$. With use of Example 4.7 we then have to show that

$$\mathcal{L}_3(\lambda c) = \mathcal{L}_3(a) = \frac{2}{3} \left( \frac{1}{a} e^{-a} - \frac{1}{a^2} e^{-2a} + \frac{1}{a^3} e^{-3a} \right) \leq \frac{1 + c}{1 + ce^{(1+1)/c}}$$

holds for $0 \leq a \leq c$. This is equivalent with

$$2(ac^{2a} - e^a + 1) \leq \frac{(3ac^2e^{3a})(1 + c)}{1 + ce^{(1+1)/c}} \quad \iff \quad 3a^2c^{3a}(1 + c) - 2(ac^{2a} - e^a + 1)(1 + ce^{(1+1)/c}) \geq 0. \quad (4.20)$$

Denote the LHS of (4.20) by $h(a, c)$. We first show that for $0 \leq a \leq c$ it holds that

$$\frac{d^2 h(a, c)}{dc^2} \geq 0 \quad \text{and} \quad \frac{dh(a, c)}{dc} \big|_{a,c=0} = 0.$$ 

This implies that $\frac{dh(a, c)}{dc} \geq 0$. Then we show that $h(a, a) \geq 0$. This is enough to show that (4.20) holds and to prove the theorem. We compute

$$\frac{d}{dc} h(a, c) = 3a^2c^{3a} - 2(ac^{2a} - e^a + 1)e^{a(1+1/c)} \left( 1 + c \cdot \frac{-a}{c^2} \right)$$

$$= 3a^2c^{3a} + 2(ac^{2a} - e^a + 1)e^{a(1+1/c)} \left( \frac{a}{c} - 1 \right)$$

and

$$\frac{d^2}{dc^2} h(a, c) = -2(ac^{2a} - e^a + 1)e^{a/c} \left( -a \cdot \frac{1}{c^2} + (\frac{a}{c} - 1) \cdot \frac{-a}{c^2} \right)$$

$$= 2(ac^{2a} - e^a + 1)e^{a/c} \frac{a^2}{c^3}.$$ 

Now $v(a) = (ac^{2a} - e^a + 1) \geq 0$ since $v(0) = 0$ and $v'(a) = e^{2a} + 2ae^a - e^a \geq 0$. Hence $\frac{d^2 h(a, c)}{dc} \geq 0$.

To show that $\frac{dh(a, c)}{dc} \big|_{a,c=0} = 0$, we let $a$ and $c$ go to zero under the condition that $a \leq c$. The fact that $\lim_{a \to 0} ac^{2a} - e^a + 1 = 0$ then yields $\frac{dh(a, c)}{dc} \big|_{a,c=0} = 0$.

Finally, the inequality $h(a, a) \geq 0$ is given by Lemma 4.12. Hence (4.20) holds and the proof is finished. \hfill \square
Lemma 4.14 For all \( n \geq 3 \),
\[
\mathcal{L}_n(s) \leq \mathcal{L}_3(s).
\]  
(4.21)

Proof We prove that for all \( n \geq 3 \),
\[
P(D > x | \tau = 3) \leq P(D > x | \tau = n).
\]
The inequality (4.21) then follows from (2.4).

Let \( F_n \) be the conditional distribution function of \( D \) given that \( \tau = n \), i.e.
\[
F_n(x) = P(D < x | \tau = n).
\]

Denote the density of \( F_n \) by \( f_n \). Let \( 1 \leq s \leq 2 \). If \( D = s \) and \( \tau = n \), then \( A_{n-1} = s - (A_1 + \cdots + A_{n-2}) \). To calculate \( f_n \), we only integrate over the allowed values for \( A_1 \) up to \( A_{n-2} \). Using the same transformation as in (4.12), \( f_n \) satisfies
\[
f_n(s) = \frac{1}{p_n} \lambda^{n-1} e^{-\lambda n} \int_0^1 \int_0^{1+y_1} \cdots \int_0^{1+y_{n-3}} dy_{n-2} \cdots dy_1
\]
\[
= \frac{1}{p_n} \lambda^{n-1} e^{-\lambda n} q_{n-1}(0) = \frac{n!}{n^{n-1}} \frac{(n-1)^{n-2}}{(n-1)!} = \left( \frac{n-1}{n} \right)^{n-2}.
\]
Hence \( f_n(s) \) is constant for \( 1 \leq s \leq 2 \). For \( n \geq 4 \) we have
\[
f_n(s) = \left( \frac{n-1}{n} \right)^{n-2} = \left( \frac{n-1}{n} \right)^n \left( \frac{n}{n-1} \right)^2 \leq \frac{116}{e^9} \leq \frac{2}{3} = f_3(s),
\]  
(4.22)

since \((1 - 1/n)^n\) increases to \(1/e\) and \(18e > 48\).

For \( 2 \leq s \leq 3 \) we have
\[
f_n(s) = \frac{1}{p_n} \lambda^{n-1} e^{-\lambda} \int_0^{3-s+y_{n-3}} \int_0^{1+y_1} \cdots \int_0 dy_{n-2} \cdots dy_1
\]
\[
= \frac{1}{p_n} \lambda^{n-1} e^{-\lambda n} (q_{n-1}(0) - (s-2)q_{n-2}(0)).
\]  
(4.23)

Hence \( f_n(s) \) is a linear polynomial for \( 2 \leq s \leq 3 \). Moreover, by Lemma 4.5 we have \( q_2(0) - q_1(0) = 0 \). Hence (4.23) implies that \( f_3(3) = 0 \).

We assume that \( F_n(s) > F_3(s) \) for some \( 2 \leq s < 3 \) and derive a contradiction. By (4.22), the assumption implies that \( f_n(t) > f_3(t) \) for some \( 2 \leq t \leq s \). But since \( f_k(x) \) is a linear polynomial for \( 2 \leq x \leq 3 \) for every \( k \) and \( f_3(3) = 0 \) by the remark made above, it must hold that \( f_n(x) > f_3(x) \) for \( t \leq x < 3 \). This implies \( F_n(3) > F_3(3) = 1 \), which is the desired contradiction. Hence \( F_n(s) \leq F_3(s) \) for \( 2 \leq s \leq 3 \).

We conclude that \( F_3(s) \geq F_n(s) \) for all \( n \geq 3 \) and the lemma is proved. \( \square \)
4.7 Comparing the queue lengths

In this section we enjoy the fruits of our labour by proving the main theorem of this chapter. After that we derive some other results from the lemmas and propositions above.

**Proof of Theorem 4.2** We first show that

\[ P(M_1 = 1) \leq P(M_2 = 1), \quad \lambda > 0. \tag{4.24} \]

Let \( B_1 \) be the service time of the first customer in \( Q_2 \). Let \( A_1 \) and \( A'_1 \) be the \( \exp(\lambda) \) distributed first interarrival times for \( Q_1 \) respectively \( Q_2 \). Since \( A'_1 \) and \( B_1 \) are independent, we have

\[
P(M_2 = 1) - P(M_1 = 1) = P(A'_1 > B_1) - P(A_1 > 1)
= P(B_1 = 0+) + P(A'_1 > 1 + c)P(B_1 = 1 + c) - e^{-\lambda}
= \frac{e}{1 + c} + e^{-(1+c)\lambda}(1 + c)^{-1} - e^{-\lambda} =: g(\lambda).
\]

Then \( g(0) = 0 \) and \( g'(\lambda) = e^{-\lambda} - e^{-(1+c)\lambda} \geq 0 \), giving \( g(\lambda) > 0 \) for all \( \lambda > 0 \), and (4.24) is proved.

To prove the stochastic domination in (4.2) and complete the proof, it is enough to show that \( P(M_1 = n) \geq P(M_2 = n) \) for all \( n \geq 3 \). By Proposition 4.11, \( P(M_1 = n) \geq P(M_2 = n) \) holds if and only if

\[
\mathcal{L}_n(c\lambda) \leq \frac{e + 1}{1 + ce(c+1)\lambda}.
\tag{4.25}
\]

This condition holds for \( n \geq 3 \) by Lemmas 4.13 and 4.14 and the proof is finished. \( \square \)

Using (4.15), the joint transform of \( \tau \) and \( D \) can be found as follows. Let \( \tau \) and \( D \) be as in (4.4) and (4.7), and let \( \phi(x) = E x^\tau \) be the generating function of \( \tau \). Let as before \( p_n(x) = x^{n-1}e^{-nx} \frac{n-1}{n!} \) and \( P(\tau = n) = p_n = p_n(\lambda) \). By definition (4.8)

\[
\phi(x) = E x^\tau e^{-sD} = \sum_{n=1}^{\infty} x^n P(\tau = n) - \sum_{n=1}^{\infty} x^n P(\tau = n)E[e^{-sD} | \tau = n]
= \sum_{n=1}^{\infty} x^n P(\tau = n)(1 - \mathcal{L}_n(s)).
\]
Using (4.15) and Example 4.7, we find

\[
\phi(x) - E x^r e^{−sD} = \sum_{n=1}^{\infty} \frac{x^n n^{n-1}}{n!} \lambda^{n-1} e^{-\lambda n} se^s \left( \frac{n+1}{n} \right)^{n-1} \mathcal{L}_{n+1}(s) \\
= \frac{se^s e^\lambda}{\lambda x} \sum_{n=1}^{\infty} \frac{x^{n+1} (n+1)^n}{(n+1)!} \lambda^n e^{-\lambda(n+1)} \mathcal{L}_{n+1}(s) \\
= \frac{se^s e^\lambda}{\lambda x} \left( E x^r e^{−sD} - xe^{-\lambda e^{-s}} \right).
\]

Hence

\[
E x^r e^{−sD} = \frac{\phi(x) + s/\lambda}{1 + se^s e^\lambda/(\lambda x)} = \frac{\lambda x \phi(x) + sx}{\lambda x + se^s e^\lambda}.
\] 

(4.26)

From (4.26) we may compute the covariance of \(\tau\) and \(D\). We compute

\[
E\tau = \phi'(1) = 1/(1 - \rho) = 1/(1 - \lambda),
\]

\[
E\tau D = -\frac{\partial}{\partial x \partial s} E x^r e^{−sD} \bigg|_{x=1, s=0} = \frac{e^\lambda (\phi'(1) - 1)}{\lambda} = \frac{e^\lambda}{(1 - \lambda)}
\]

and

\[
ED = -\frac{\partial}{\partial s} \frac{s + \lambda}{se^s + \lambda} \bigg|_{s=0} = \frac{e^\lambda - 1}{\lambda}.
\]

Hence

\[
Cov(\tau, D) = \frac{e^\lambda}{(1 - \lambda)} - \frac{1}{1 - \lambda} \left( \frac{e^\lambda - 1}{\lambda} \right) = \frac{1 - (1 - \lambda)e^\lambda}{\lambda(1 - \lambda)}.
\] 

(4.27)

The chapter is concluded with the following result. The result is striking since the calculated expectation does not depend on \(\lambda\).

**Theorem 4.15** Let \(A_1, A_2, \ldots\) be independent exponentially distributed random variables with parameter \(\lambda > 0\). Consider the random walk \(S_n\) with \(S_0 = 0\) and increments \(X_n = A_n - 1\) for \(n = 1, 2, \ldots\). Then, independently of \(\lambda\),

\[
E [S_{n-1} | S_1 < 0, \ldots, S_n < 0, S_n > 0] = 1 - \left( \frac{n+1}{n} \right)^{n-1} \rightarrow 1 - e.
\]

**Proof** By definition of \(Y_n\), \(\tau\) and \(D\) in (4.3), (4.4) and (4.7), we find

\[
E [S_{n-1} | S_1 < 0, \ldots, S_{n-1} < 0, S_n > 0] = E [-Y_{\tau-1} | \tau = n]
\]

\[
= 1 - E [D | \tau = n].
\] 

(4.28)
Using the transform $\mathcal{L}_n(s)$ defined in (4.8), and with the help of (4.15), we have

\begin{equation}
E[D \mid \tau = n] = -\frac{d}{ds} \mathcal{L}_n(s) \bigg|_{s=0} = \frac{d}{ds} (1 - \mathcal{L}_n(s)) \bigg|_{s=0} \tag{4.29}
\end{equation}

\begin{align*}
&= \frac{d}{ds} \left( se^{s} \left( 1 + \frac{1}{n} \right)^{n-1} \mathcal{L}_{n+1}(s) \right) \bigg|_{s=0} \\
&= \left[ (1 + s)e^{s} \left( 1 + \frac{1}{n} \right)^{n-1} \mathcal{L}_{n+1}(s) + se^{s} \frac{d}{ds} \mathcal{L}_{n+1}(s) \right]_{s=0} \\
&= \left( 1 + \frac{1}{n} \right)^{n-1} - \epsilon. \tag{4.30}
\end{align*}

since $\frac{d}{ds} \mathcal{L}_{n+1}(s) \big|_{s=0} = ED_{n+1} \leq n+1$. Combining (4.28) and (4.30) completes the proof.  \qed