Appendix

A.1 The queue length in unstable queues

In this section we discuss the asymptotic queue length in unstable systems for several service disciplines. Assume $\rho > 1$. For the $M/G/1$ FB queue Balkema and Verwijmeren [6] showed that

$$\frac{Q(t)}{t} \to \lambda(1 - F(c^*)) =: \gamma_{FB} \quad a.s.$$  

where the critical service time $c^*$ is the unique solution of

$$\lambda \int_0^c (1 - F(x))dx = 1. \quad (A.1)$$

This solution exists since $\lambda E(B \wedge c)$ is continuous in $c$ and increases from 0 to $\rho > 1$.

In the PS queue the server capacity allotted to a single customer converges to zero since asymptotically the number of customers in the queue grows linearly. However, this does not imply that the fraction of the customers that leave the queue is negligible for $t \to \infty$. By a nice argument Jean-Marie and Robert [28] showed that $Q(t)/t$ converges a.s. to $\gamma_{PS}$, the unique (strictly) positive solution of $\theta = \lambda(1 - E^{-\theta B})$.

From this we find by partial integration that the asymptotic growth rate $\gamma_{PS}$ is also the unique solution of the equation

$$\lambda \int_0^\infty e^{-\theta x}(1 - F(x))dx = 1. \quad (A.2)$$

It would be interesting to find conditions on the service-time distribution $F$ under which the growth rates for FB and PS can be ordered. Perhaps this can be done by combining (A.1) and (A.2). It would be interesting as well to compare $\gamma_{FB}$ and $\gamma_{PS}$ with $\gamma_{FIFO}$ given below. For some service-time distributions we have the following numerical results.
Suppose the service times have a Pareto distribution, i.e. the tail of $F$ is given by $1 - F(x) = (cx + 1)^{-a}, x \geq 0, a > 1, c > 0$. Numerical calculations show that for these service times $\gamma_{FB} = \lambda(1 - F(c^*)) < \gamma_{PS}$, and hence FB has a smaller asymptotic growth rate than PS. This shows again that the FB discipline is very efficient for heavy-tailed distributions. For exponential distributions the queue lengths under FB and PS are equal. For uniformly distributed service times, numerical calculations showed that $\gamma_{FB} > \gamma_{FIFO}$.

Finally, we describe the growth rate of the queue length under the FIFO discipline. For every discipline the asymptotic rate is given by $\lim_{t \to \infty} (A(t) - D(t))/t$, where $A(t)$ and $D(t)$ denote the number of arrivals and departures up to time $t$. Define $N(t) = \sup\{n : B_1 + \cdots + B_n \leq t\}$. By the renewal theorem $N(t)/t \to 1/(EB)$ a.s. as $t \to \infty$. In the unstable queue with probability one after a finite time a busy period starts that will never end. Hence for the FIFO discipline,

$$\frac{D(t)}{t} \to \frac{1}{EB} \quad \text{a.s.}$$

Since $A(t)/t \to \lambda$ a.s., we conclude that the queue length $Q(t)$ grows asymptotically linearly with rate

$$\gamma_{FIFO} := \lim_{t \to \infty} \frac{A(t) - D(t)}{t} = \lambda - \frac{1}{EB}.$$

Note that $\lambda - 1/EB > 0$ if and only if $\rho = \lambda EB > 1$.

If $EB = \infty$, then $Q(t)/t \to \lambda$ a.s. and the number of customers that manage to leave the queue is negligible. In the extreme case that $P(B = \infty) > 0$, a.s. only a finite number of customers ever leaves the system. The remaining customers are jammed behind a customer with an infinite service time. The other disciplines described in this section do not suffer noticeably from the presence of customers with an infinite service demand.