Measurement of the W boson mass and width with the L3 detector
Baldew, S.V.

Citation for published version (APA):
Baldew, S. V. (2004). Measurement of the W boson mass and width with the L3 detector
4.1 Kinematic fit

The energy $E$ and the angles $\theta$ and $\phi$ of the jets and leptons are measured in the detector with finite resolution. In general, these quantities will not satisfy energy-momentum conservation. In the four-jet case, four-momentum conservation using the measured jets is not necessarily satisfied. In the semi-leptonic channel, $q\bar{q}\nu$ and $qq\mu\nu$, the outgoing neutrino escapes undetected. The momentum of the neutrino follows from the three-momentum conservation, but energy conservation is not automatically satisfied.

To resolve these problems and to increase the resolution, a kinematic fit is used. As a further advantage, additional constraints, such as equal mass of the decaying $W$'s, can be imposed.

The kinematic fit is an iterative procedure, which exploits the Least-Squares Principle and the Lagrange method (following the procedure of reference [59]).

Let:

- $\eta$ be the vector of $N$ observables containing the energy $E$ and the angles $\theta$ and $\phi$ of the detected particles.
- $\gamma$ be the measurements of these variables. This will be the initial guess in the iteration procedure.
- $V(\gamma)$ be the covariance matrix, which contains the estimated errors on the measured variables and their correlation. This matrix needs not be diagonal, but in our case we simplify the problem and assume this matrix to be initially diagonal, which implies that the measurements are uncorrelated.
- $\xi$ be the additional set of $J$ unmeasured variables. For the semi-leptonic channel in $WW$ production these are the energy $E$ and the angles $\theta$ and $\phi$ of the undetected neutrino.
- $f(\eta, \xi)$ be $K$ constraint equations, relating the measured and unmeasured variables.

In the case under study the constraint equations come from the energy and momentum conservation,

$$f_i = \sum_{i=1}^{4} E_i = E_{CM},$$

(4.1)
4.1. Kinematic fit

\[ f_2 = \sum_{i=1}^{4} \beta_i E_i \sin(\theta_i) \cos(\phi_i) = 0, \quad (4.2) \]

\[ f_3 = \sum_{i=1}^{4} \beta_i E_i \sin(\theta_i) \sin(\phi_i) = 0, \quad (4.3) \]

\[ f_4 = \sum_{i=1}^{4} \beta_i E_i \cos(\theta_i) = 0, \quad (4.4) \]

where

\[ \beta_i = \frac{\sqrt{E_i^2 - m_i^2}}{E_i}. \quad (4.5) \]

If also the equal mass condition of the two W’s in the event is imposed, an additional constraint has to be added,

\[ f_5 = (E_1 + E_2)^2 - (\beta_1 E_1 \sin(\theta_1) \cos(\phi_1) + \beta_2 E_2 \sin(\theta_2) \cos(\phi_2))^2 - (\beta_1 E_1 \sin(\theta_1) \sin(\phi_1) + \beta_2 E_2 \sin(\theta_2) \sin(\phi_2))^2 - (\beta_1 E_1 \cos(\theta_1) + \beta_2 E_2 \cos(\theta_2))^2 - (E_3 + E_4)^2 + (\beta_3 E_3 \sin(\theta_3) \cos(\phi_3) + \beta_4 E_4 \sin(\theta_4) \cos(\phi_4))^2 + (\beta_3 E_3 \sin(\theta_3) \sin(\phi_3) + \beta_4 E_4 \sin(\theta_4) \sin(\phi_4))^2 + (\beta_3 E_3 \cos(\theta_3) + \beta_4 E_4 \cos(\theta_4))^2 = 0. \quad (4.6) \]

The best estimate of the variables is, from the Least-Squares Principle, determined by minimising the \( \chi^2 \), respecting the constraints

\[ \chi^2(\eta) = (y - \eta)^T V^{-1}(y - \eta) = \text{minimum} \]

\[ f(\eta, \xi) = 0 \]

Because the constraint equations are non-linear we adopt the method of the Lagrangian multipliers and use an iterative procedure solving

\[ \chi^2(\eta, \xi, \lambda) = (y - \eta)^T V^{-1}(y - \eta) + 2\lambda f(\eta, \xi) = \text{minimum}, \quad (4.8) \]

where \( \lambda \) are \( k \) Lagrangian multipliers.

If we define

\[ (F_\eta)_{ki} = \frac{\partial f_k}{\partial \eta_i}, \quad (4.9) \]

\[ (F_\xi)_{kj} = \frac{\partial f_k}{\partial \xi_j}, \quad (4.10) \]
equating the derivatives of $\chi^2$ with respect to all unknowns to zero leads to

$$V^{-1}(\eta - y) + F_\eta^T \Delta = 0, \quad (4.11)$$

$$F_\xi^T \Delta = 0, \quad (4.12)$$

$$f(\eta, \xi) = 0. \quad (4.13)$$

Assuming that the $\nu$-th iteration is performed, the constraint equations can be expanded in a Taylor series in the point $(\eta^{\nu}, \xi^{\nu})$. Neglecting the second and higher orders terms in (4.13) results in

$$\xi^{\nu+1} = \xi^{\nu} - (F_\xi^T S^{-1} F_\eta) F_\xi^T S^{-1} \xi^{\nu}.$$

Together with equations (4.11) and (4.12), for the $(\nu + 1)$-th iteration and introducing

$$E^{\nu} \equiv F_\eta^T (\eta^{\nu+1} - \eta^{\nu}), \quad (4.15)$$

$$S^{\nu} \equiv F_\eta^T V(F_\eta^T)^{\nu}, \quad (4.16)$$

we find,

$$\xi^{\nu+1} = \xi^{\nu} - (F_\xi^T S^{-1} F_\eta) F_\xi^T S^{-1} \xi^{\nu}, \quad (4.17)$$

$$\Delta^{\nu+1} = S^{-1} E^{\nu}. \quad (4.18)$$

$$\eta^{\nu+1} = y - V F_\eta^T \Delta^{\nu+1}. \quad (4.19)$$

The $\chi^2$ in the $(\nu + 1)$-th step is

$$(\chi^2)^{\nu+1} = (\Delta^{\nu+1})^T S \Delta^{\nu+1} + 2(\Delta^{\nu+1})^T E^{\nu+1}. \quad (4.20)$$

The iteration stops when one of the following conditions is satisfied:

- the constraint equations are balanced to better than a required precision.
- the derivatives $\partial \chi^2/\partial \eta$, equation (4.11), and $\partial \chi^2/\partial \xi$, equation (4.12), are sufficiently close to 0.
- the $\chi^2$ change per iteration step is small.

According to the number of constraints $K$ and the number of unmeasured variables $J$, the fit is called a $nC$-fit where $n = K - J$. In case of the fully hadronic channel there are four constraints from the three-momentum and energy conservation. This fit is a $4C$-fit. Imposing equal mass constraint on the two $W$'s changes this fit into a $5C$-fit. Similar to this, in the semi-leptonic case, without equal mass constraint, a $1C$-fit is used ($K = 4, J = 3$). Due to the undetected neutrino the momentum conservation can be fulfilled by assigning the missing three-momentum to the neutrino. However, energy conservation is not automatically satisfied. Only energy conservation remains as a constraint. In the $2C$-fit case an additional equal mass constraint is imposed for the semi-leptonic events.
4.2 Error propagation

After the iterative procedure of the kinematic fit the error on the improved variables, \( E, \theta \) and \( \phi \), can be estimated with the law of propagation of errors. This estimation will be used in the calculation of the error on the mass.

The general principle of the law of propagation will be explained in section 4.2.1. It will be illustrated with the simple application of the mass error estimation. In section 4.2.2 the error calculation on the kinematic fit variables will be given.

4.2.1 Error on the mass

From the measured energy \( E \) and the angles \( \theta \) and \( \phi \) of the jets and leptons of an event, or in case of a neutrino taking into consideration the constraint equations in determining its \( E, \theta \) and \( \phi \), one can calculate the mass of the hypothetical W,

\[
M_W = \left[ (E_1 + E_2)^2 - (p_1 \sin(\theta_1) \cos(\phi_1) + p_2 \sin(\theta_2) \cos(\phi_2))^2 - (p_1 \sin(\theta_1) \sin(\phi_1) + p_2 \sin(\theta_2) \sin(\phi_2))^2 - (p_1 \cos(\theta_1) + p_2 \cos(\theta_2))^2 \right]^{\frac{1}{2}}.
\]

with \( i = \sqrt{E_i^2 - m_i^2} \) the momentum of the lepton or jet and \( m_i \) the mass. The estimated error on this calculated mass \( M_W \) comes from the propagated errors of the variables, \( E, \theta \) and \( \phi \).

The error is propagated by means of the law of propagation of errors, which takes the general form,

\[
V(z) = AV(y)A^T,
\]

where \( z \) depends on the random variables \( x \),

\[
z_k(x) = z_k(y_1, y_2, \ldots, y_n), \quad k = 1, 2, \ldots, m.
\]

The matrix \( V(x) \) is the covariance matrix and the matrix \( A \) is defined as,

\[
\frac{\partial z_k}{\partial y_i} = A_{ki}.
\]

To go from the general equation (4.22) to the special case, where we want to calculate the error on the mass, the random variable \( z_k \) is substituted by the mass \( M_W \) from equation (4.21). The random variables \( y \) is \( (E_1, \theta_1, \phi_1, E_2, \theta_2, \phi_2) \) and the matrix \( A \) is the derivatives of \( M_W \) with respect to \( y = (E_1, \theta_1, \phi_1, E_2, \theta_2, \phi_2) \).

Using the improved \( E, \theta \) and \( \phi \) values found by the kinematic fit not only improves the resolution of the mass, but also the error on the mass,
4.2.2 Errors on the kinematic fit variables

To improve the resolution of a measurement and to satisfy certain physics constraints, the kinematic fit is used as discussed in section 4.1.

The errors on the variables, \( E, \theta \) and \( \phi \) are found by applying equation (4.22). If \( \eta = \eta^{\nu+1} \) and \( \xi = \xi^{\nu+1} \) then equation (4.17) - (4.19) and equation (4.15) yields [59], if all the mentioned equations are expressed in terms of \( y \),

\[
\begin{align*}
\hat{\eta} &= g(y) = y - V F_\eta^T S^{-1}[I_K - F_\xi(F_\xi^T S^{-1} F_\xi)^{-1} F_\xi^T S^{-1}][f + F_\eta(y - \eta)], \\
\hat{\xi} &= h(y) = \xi - (F_\xi^T S^{-1} F_\xi)^{-1} F_\xi^T S^{-1}[f + F_\eta(y - \eta)],
\end{align*}
\] (4.25)-(4.26)

where \( I_K \) is the identity matrix.

The total covariance matrix for \( \eta \) and \( \xi \) from equation (4.22), is given by

\[
\begin{pmatrix}
V(\hat{\eta}) & \text{cov}(\hat{\eta}, \hat{\xi}) \\
\text{cov}(\hat{\eta}, \hat{\xi}) & V(\hat{\xi})
\end{pmatrix} =
\begin{pmatrix}
\left( \frac{da}{dy} \right) V(y) \left( \frac{da}{dy} \right)^T & \left( \frac{da}{dy} \right) V(y) \left( \frac{dh}{dy} \right)^T \\
\left( \frac{da}{dy} \right) V(y) \left( \frac{dh}{dy} \right)^T & \left( \frac{dh}{dy} \right) V(y) \left( \frac{dh}{dy} \right)^T
\end{pmatrix}.
\] (4.27)

When we define the abbreviations,

\[
\begin{align*}
G &= F_\eta^T S^{-1} F_\eta, \\
H &= F_\xi^T S^{-1} F_\xi, \\
U^{-1} &= F_\xi^T S^{-1} F_\xi.
\end{align*}
\] (4.28)-(4.30)

the explicit form of the covariance matrix is,

\[
\begin{align*}
V(\hat{\eta}) &= V(y)[I_N - (G - HUH^T)V(y)], \\
V(\hat{\xi}) &= U, \\
\text{cov}(\hat{\eta}, \hat{\xi}) &= -V(y)HU.
\end{align*}
\] (4.31)-(4.33)

The fit procedure reduces the variance on the measurement and correlates the quantities even if the measurements are independent.

4.3 Example of an event fit

As an example of the kinematic fit I will show one \( q\bar{q}e\nu \) event at a centre-of-mass energy of 189 GeV. Figure 4.1 shows an event in which the electron and the two jets can be clearly distinguished.

Table 4.1 gives the measured values of the energy, polar angle and azimuthal angle for the jets and electron in the selected candidate event which is shown in the event display in figure 4.1. Since the neutrino cannot be detected, it is reconstructed from the momentum conservation.

The estimated errors on these measured values, which are determined from the detector resolution as discussed in section 4.4, are shown in table 4.1. Due to the fact that the neutrino is
4.4 Resolution

The kinematic fit requires the covariance matrix $V(y)$ as input parameter. This matrix contains the estimated errors on the measured variables and their correlations. To simplify the calculations it is assumed that the measurements are uncorrelated, i.e. the non-diagonal elements are zero.

The resolutions used in the covariance matrix for electrons, muons and jets are determined from Monte Carlo sample studies and verified with real data using $Z \rightarrow f\bar{f}$.

Uncertainties on the energy and angle measurements depend on which sub-detectors are hit by the particles. Also the energy of the particle is a factor in the precision of the measurement, due to the fact that the detector is optimised for certain energy ranges. To account for these facts
Table 4.1: The $E$, $\theta$ and $\phi$ of the two jets, the electron and the neutrino in a certain $qq\nu$ event before the kinematic fit. The parameter values of the electron neutrino are are not measured but derived from the momentum conservation. The associated errors of the measurement on $E$, $\theta$ and $\phi$ of the jets and leptons in before the kinematic fit are also given. There is no error assigned to the neutrino variables since these variables are derived from the momentum conservation.

<table>
<thead>
<tr>
<th></th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$e$</th>
<th>$\nu_\text{e}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^{\text{before fit}}$ [GeV]</td>
<td>57.46 ± 6.47</td>
<td>27.08 ± 4.56</td>
<td>48.56 ± 0.73</td>
<td>49.21</td>
</tr>
<tr>
<td>$\theta^{\text{before fit}}$</td>
<td>1.837 ± 0.033</td>
<td>2.080 ± 0.058</td>
<td>1.3188 ± 0.0100</td>
<td>1.255</td>
</tr>
<tr>
<td>$\phi^{\text{before fit}}$</td>
<td>4.640 ± 0.030</td>
<td>2.078 ± 0.053</td>
<td>0.0307 ± 0.0100</td>
<td>2.339</td>
</tr>
</tbody>
</table>

Table 4.2: The $E$, $\theta$ and $\phi$ of the jets and electron in the candidate $qq\nu$ event and the errors on the $E$, $\theta$ and $\phi$ of jets and leptons after the kinematic fit. The errors on the variables of the neutrino are calculated by means of the law of error propagation.

<table>
<thead>
<tr>
<th></th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$e$</th>
<th>$\nu_\text{e}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^{\text{2Cfit}}$ [GeV]</td>
<td>60.08 ± 1.62</td>
<td>34.25 ± 1.64</td>
<td>48.39 ± 0.71</td>
<td>45.94 ± 0.71</td>
</tr>
<tr>
<td>$\theta^{\text{2Cfit}}$</td>
<td>1.833 ± 0.033</td>
<td>2.061 ± 0.056</td>
<td>1.3182 ± 0.0099</td>
<td>1.155 ± 0.063</td>
</tr>
<tr>
<td>$\phi^{\text{2Cfit}}$</td>
<td>4.632 ± 0.029</td>
<td>2.082 ± 0.053</td>
<td>0.0315 ± 0.0099</td>
<td>2.309 ± 0.069</td>
</tr>
</tbody>
</table>
Figure 4.2: The difference of the reconstructed mass before the kinematic fit and the generated mass shows a much broader distribution than the distribution in which the 2C fit mass is compared with the generated mass. The 1C fit also improves the resolution of the mass but less than the 2C fit. The kinematic fit clearly enhances the resolution of the mass.
the energy, $\theta$ and $\phi$ resolution parametrisations are a function of $E$ and $\theta$. The azimuthal angle is neglected in the parametrisation, because the detector is symmetric in the $\phi$ direction.

### 4.4.1 Muon resolution

The muon energy resolution depends on the number of muon chambers that are hit and the region of the detector, i.e. barrel or forward region, as well as on the momentum of the muon itself. Muons are divided in different classes according to the number of segments reconstructed in the barrel and forward chambers. If a muon is reconstructed in all three chambers of the barrel it is denoted as '3p'. If only two barrel segments and one forward segment are reconstructed it is classified as a '2p1f'. In a similar way other types of muons are classified. Figure 4.3 [41] shows the difference between two different types of muons. The spectrum on the left-hand side is the

![Figure 4.3: The difference between two types of muons is shown: the resolution of the so-called 3p muons on the left-hand side is compared with the 2p muons on the right-hand side. The number of segments reconstructed in the barrel changes the resolution. The momentum resolution for the 3p class muons is 3.4% while for the 2p1f class it is 27% for data at the Z peak.](image)

3p muon category. The broader spectrum on the right-hand side are the events with only two segments in the barrel, 2p0f, or shortly 2p. The difference in resolution for these different types of muons can be clearly observed from these examples. For each different muon type the energy or momentum resolution is dependent on the momentum itself. The momentum dependence for a 3p class muon is shown in figure 4.4 [41]. A muon with a momentum of about 45 GeV in the 3p class has a relative resolution of 3.7% while the other muon categories are around 30% and the MIP resolution is about 40%

The polar and azimuthal angle resolutions depend on whether there is a matching TEC track or not. If there is a matching TEC track the angular measurements are much better. For $\theta$ the resolution ranges from 0.15° to 0.23°, while if there is no matching track it rises to 0.46°. If no
4.4. Resolution

4.4.1 Kinematic fit and error propagation

Figure 4.4: Momentum dependence of the resolution of 3p class muons. The parametrisation is determined from Monte Carlo simulation. The square is the resolution of the Z peak data for a muon momentum of 45 GeV.

The track is matched φ has an accuracy of 0.17° which improves to 0.01° with a track match.

With a Z decaying into a muon pair one can test the description of the angular resolutions. The two muons are produced practically back-to-back at the Z peak. One can extract the distributions \((θ_1 + θ_2 - 180°)\) and \((φ_1 - φ_2)\) for the two produced muons as is shown in figure 4.5 [41]. The θ resolution for a muon at 45 GeV is 0.26°, whereas the φ resolution is about 0.021° for all types of measured muons conjointly. The MIP resolution is slightly worse at around 0.34° and 0.026° for θ and φ. A more detailed description of the Z calibration test for the muon resolution can be found in [41].

4.4.2 Electron resolution

In a similar way one may assume that the electron resolutions also depend on the direction of the electron in the detector. Moreover, if the electron has passed the EGAP it is expected to have a worse measurement. However, in the data as provided for the electron channel in this thesis the electron resolutions are fixed throughout the detector. The energy resolution has been determined to be 1.5% of the measured energy for the electron. The error on the measured polar and azimuthal angle are set to be 0.56°. In section 4.5 it will be shown that the resolution assumptions for the electron are reasonable.
4.5 Testing the kinematic fit

To test whether the estimated values of the parameters $E$, $\theta$ and $\phi$ are reliable or not, the $\chi^2$ from equation (4.20) provides a measure of the goodness-of-fit. The $\chi^2$ is converted into a chi-square probability $P_{\chi^2}$.

The probability $P_{\chi^2}$ is uniformly distributed between 0 and 1, under the assumption that the measurements are normally distributed. Any deviation of the $P_{\chi^2}$ from a uniform distribution indicates that either the measurement or model is unsatisfactory.
4.5. Testing the kinematic fit

Kinematic fit and error propagation

Figure 4.6: The energy resolution for a jet is a function of energy $E$ and polar angle $\theta$.

Figure 4.7: The polar angle resolution of a jet.
4.5. Testing the kinematic fit

Figure 4.8: The azimuthal angle resolution of a jet.
4.5. Testing the kinematic fit

Kinematic fit and error propagation

Data 91 GeV
M.C. qq(Y)

Figure 4.9: Left: Monte Carlo and data jet energy spectrum normalised to the beam energy at $\sqrt{s} = 91$ GeV. The relative energy jet energy resolution is 14.6% for 45 GeV jets.

Data 91 GeV
M.C. qq(Y)

Figure 4.10: Left: The polar angle resolution for jets measured at the Z peak. Right: The azimuthal resolution for 45 GeV jets.
The probability distributions of the electron, muon and four-jet channels in figure 4.11 have a flat distribution. However, they exhibit a peak at low probability. Background events which in general do not satisfy the hypothesis of a signal event have a low probability and are revealed as contamination in this peak. But also events with bad reconstructed jets for which the kinematic fit could not properly converge can be found here. Also the measured jet momentum distribution has non-Gaussian tails. These tails can distort the chi-square probability distribution.

Aside from the peak at low probability the $P_{\chi^2}$ distribution shows a uniform distribution. To study the fit more attentively, one can measure the deviations between the observed and fitted values with respect to the uncertainty on observed and fitted quantities. The pull of a kinematic fit for the $i^{th}$ observation of a variable $v$ is defined as:

$$pull(v) = \frac{v_i^{meas} - v_i^{fit}}{\sqrt{(\sigma_i^{meas})^2 - (\sigma_i^{fit})^2}}.$$  \hspace{1cm} (4.34)

The parameters $v_i^{meas}$ and $v_i^{fit}$ are the measured and fitted values of $v$, while $\sigma_i^{meas}$ is the resolution of the measured variable and $\sigma_i^{fit}$ the error on the fitted value.

The pull of each variable should be a normal distribution with mean zero and standard deviation one if the measurements are normally distributed. A relative shift from zero indicates a bias in the observation. Similarly, a narrower or broader pull distribution indicates that the errors on the measurements are probably taken too large or too small respectively. Thus by examining the pull distribution it is possible to check whether the input resolutions used in the kinematic fit are reasonable.

The pull distribution for the energy, polar and azimuthal angle for the electron, muon and hadronic jet are shown in figures 4.12, 4.13 and 4.14 respectively. If the resolution functions are non-Gaussian the estimated error on the fit value $\sigma_i^{fit}$ may be larger than the input resolution $\sigma_i^{meas}$. The events are put in the last bin of the plots if this happens. A good agreement between data and Monte Carlo can be seen in the examples, indicating no major error in the modelling and resolutions.
4.5. Testing the kinematic fit

Kinematic fit and error propagation

Figure 4.11: The chi-square probability for electron, muon and four-jet channel are shown. The practically flat distributions indicate that the measurements are approximately normally distributed and that the kinematic fit model is well described.
4.5. Testing the kinematic fit

Figure 4.12: The energy, $\theta$ and $\phi$ pull distributions for the electron. The last bin represents the events for which the estimated error after the fit is larger than the estimated error of the input variable. The pull is determined from a Gaussian fit through the data points.
4.5. Testing the kinematic fit

Kinematic fit and error propagation

Figure 4.13: The energy, $\theta$ and $\phi$ pull distribution for a muon. The last bin represents the events for which the estimated error after the fit is larger than the estimated error of the input variable. The pull is determined from a Gaussian fit through the data points.
4.5. Testing the kinematic fit

Figure 4.14: The energy, $\theta$ and $\phi$ pull distribution for a jet. The last bin represents the events for which the estimated error after the fit is larger than the estimated error of the input variable. The pull is determined from a Gaussian fit through the data points.
4.5. Testing the kinematic fit

Kinematic fit and error propagation