Explicit Arakelov geometry

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Introduction

(i) Arakelov geometry is a technique for studying diophantine problems from a geometrical point of view. In short, given a diophantine problem, one considers an arithmetic scheme associated with that problem, and adds in the complex points of that scheme by way of "compactification". Next, one endows all arithmetic bundles on the scheme with an additional structure over the complex numbers, meaning one endows them with certain hermitian metrics. It is well-known from traditional topology or geometry that compactifying a space often introduces a convenient structure to it, which makes a study of it easier generally. The same holds in our case: by introducing an additional Arakelov structure to a given arithmetic situation one ends up with a convenient set-up to formulate, study and even prove diophantine properties of the original situation. For instance one could think of questions dealing with the size of the solutions to a given diophantine problem. Fermat’s method of descent can perhaps be viewed as a prototype of Arakelov geometry on arithmetic schemes.

(ii) Probably the best way to start an introduction to Arakelov geometry is to consider the simplest type of arithmetic scheme possible, namely the spectrum of a ring of integers in a number field, for instance Spec(ℤ). In the nineteenth century, some authors, like Kummer, Kronecker, Dedekind and Weber, drew attention to the remarkable analogy that one has between the properties of rings of integers in a number field, on the one hand, and the properties of coordinate rings of affine non-singular curves on the other. In particular, they started the parallel development of a theory of “places” or “prime divisors” on both sides of the analogy. Most important, morally speaking, was however that the success of this theory allowed mathematicians to see that number theory on the one hand, and geometry on the other, are unified by a bigger picture. This way of thinking continued to be stressed in the twentieth century, most notably by Weil, and it is fair to say that the later development of the concept of a scheme by Grothendieck is directly related to these early ideas.

The idea of “compactifying” the spectrum of a ring of integers can be motivated as follows. We start at the geometric side. Let ℂ be an affine non-singular curve over an algebraically closed field. The first thing we do is to “compactify” it: by making an appropriate embedding of ℂ into projective space and taking the Zariski closure, one gets a complete non-singular curve ̄ℂ. This curve is essentially unique. Now we consider divisors on ̄ℂ: a divisor is a finite formal integral linear combination \( D = \sum_{P} n_{P}P \) of points on ̄ℂ. The divisors form in a natural way a group \( \text{Div}(\overline{\mathbb{C}}) \). We obtain a natural group homomorphism \( \text{Div}(\overline{\mathbb{C}}) \to \mathbb{Z} \) by taking the degree \( \deg D = \sum_{P} n_{P} \). In order to obtain an interesting theory from this, one associates to any non-zero rational function \( f \) on ̄ℂ a divisor \( (f) = \sum_{P} v_{P}(f)P \), where \( v_{P}(f) \) denotes the multiplicity of \( f \) at \( P \). By factoring out the divisors of rational functions one obtains the so-called Picard group \( \text{Pic}(\overline{\mathbb{C}}) \) of ̄ℂ. Now a fundamental result is that the degree of the divisor of a rational function is 0, and hence the degree factors through a homomorphism \( \text{Pic}(\overline{\mathbb{C}}) \to \mathbb{Z} \). It turns out that the kernel \( \text{Pic}^{0}(\overline{\mathbb{C}}) \) of this homomorphism can be given a natural structure of projective algebraic variety. This variety is a fundamental invariant attached to ̄ℂ and is studied extensively in algebraic geometry. The
fundamental property that the degree of a divisor of a rational function is 0 is not true in general when we consider only affine curves. This makes the step of compactifying $C$ so important.

Turning next to the arithmetic side, given the success of compactifying a curve at the geometric side, one wants to define analogues of divisor, degree and compactification, in such a way that the degree of a divisor of a rational function is 0. This leads us to an arithmetic analogue of the degree 0 part of the Picard group. The compactification step is as follows: let $B = \text{Spec}(O_K)$ be the spectrum of the ring of integers $O_K$ in a number field $K$. We formally add to $B$ the set of embeddings $\sigma : K \rightarrow \mathbb{C}$ of $K$ into $\mathbb{C}$. By algebraic number theory this set is finite of cardinality $[K : \mathbb{Q}]$. Now we consider Arakelov divisors on this enlarged $B$: an Arakelov divisor on $B$ is a finite formal linear combination $D = \sum_{P} n_P P + \sum_{\sigma} \alpha_{\sigma} \cdot \sigma$, with the first sum running over the non-zero prime ideals of $O_K$, with $n_P \in \mathbb{Z}$, and with the second sum running over the complex embeddings of $K$, with $\alpha_{\sigma} \in \mathbb{R}$. Note that the non-zero prime ideals of $O_K$ correspond to the closed points of $B$. The set of Arakelov divisors forms in a natural way a group $\text{Div}(B)$. On it we have an Arakelov degree $\deg D = \sum_{P} n_P \log \#(O_K/P) + \sum_{\sigma} \alpha_{\sigma}$ which takes values in $\mathbb{R}$. The Arakelov divisor associated to a non-zero rational function $f \in K$ is given as $(f) = \sum_{P} v_P(f) \log \#(O_K/P) + \sum_{\sigma} v_{\sigma}(f) \sigma$ with $v_P(f)$ the multiplicity of $f$ at $P$, i.e., the multiplicity of $P$ in the prime ideal decomposition of $f$, and with $v_{\sigma}(f) = -\log |f|_{\sigma}$. The crucial idea is now that the product formula accounts for the fact that $\deg(f) = 0$ for any non-zero $f \in K$. So indeed, by factoring out the divisors of rational functions, we obtain a Picard group $\hat{\text{Pic}}(B)$ with a degree $\hat{\text{Pic}}(B) \rightarrow \mathbb{R}$. To illustrate the use of these constructions, we refer to Tate's thesis: there Tate showed that the degree 0 part $\hat{\text{Pic}}^0(B)$, the analogue of the $\text{Pic}^0(C)$ from geometry, can be seen as a natural starting point to prove finiteness theorems in algebraic number theory, such as Dirichlet's unit theorem, or the finiteness of the class group. In fact, Tate uses a slight variant of our $\hat{\text{Pic}}^0(B)$, but we shall ignore this fact.

(iii) Shafarevich asked for an extension of the above idea to varieties defined over a number field. In particular he asked for this extension in the context of the Mordell conjecture. Let $C$ be a curve over a field $k$. The statement that the set $C(k)$ of rational points of $C$ is finite, is called the Mordell conjecture for $C/k$. Now for curves over a function field in characteristic 0, the Mordell conjecture (under certain trivial conditions on $C$) was proven to be true in the 1960s by Manin and Grauert. However, the Mordell conjecture for curves over a number field was by then still unknown, and the technique of proof could not be straightforwardly generalised. A different approach to the Mordell conjecture for function fields was given by Parshin and Arakelov. The main feature of their approach is that it leads to an effective version of the conjecture: they define a function $h$, called a height function, on the set of rational points, with the property that for all $A$, the set of $P$ with $h(P) \leq A$ is finite, and can in principle be explicitly enumerated. Now what they prove is that the height of a rational point can be bounded $a \text{ priori}$. Hence, it is possible in principle to construct an exhaustive list of the rational points of a given curve.

In order to prove this result, the essential step is to associate to the curve $C/k$ a model $p : \mathcal{X} \rightarrow B$ with $\mathcal{X}$ a complete algebraic surface, and with $B$ a non-singular projective curve with function field $k$, such that the generic fiber of $\mathcal{X}$ is isomorphic to $C$. The rational points of $C/k$ correspond then to the sections $P : B \rightarrow \mathcal{X}$ of $p$. The essential tool, then, is classical intersection theory on $\mathcal{X}$. It turns out that certain inequalities between the canonical classes of this surface can be derived, and these inequalities make it possible to bound the height of a section.

The obvious question, in the light of the Mordell conjecture for number fields, is whether this set-up can be carried over to the case of curves defined over a number field. As was said before, Shafarevich asked for such an analogue, but eventually it was Arakelov who, building on ideas of Shafarevich and Parshin, came up with a promising solution. His results are written down in the important paper *An intersection theory for divisors on an arithmetic surface*, published in 1974.

Let us describe the idea of that paper. Let $C/K$ be a curve over a number field $K$. To it there
is associated a scheme \( p : X \to B = \text{Spec}(O_K) \), called an arithmetic surface, which is a fibration in curves over \( B \), just as in the classical context of function fields mentioned above. The generic fiber of \( p : X \to B \) is isomorphic to \( C \), and for almost all non-zero primes \( P \) of \( O_K \), the fiber at the corresponding closed point is equal to the reduction of \( C \) modulo \( P \). Again, the set of rational points of \( C/K \) corresponds to the set of sections \( P : B \to X \). In order to attack the Mordell conjecture for \( C \), one wants to have an intersection theory for divisors on \( X \). The first idea, as always, is to compactify the scheme \( X \). We do this by formally adding in, for each complex embedding \( \sigma \) of \( K \), the complex points of \( C \), base changed along \( \sigma \) to \( C \). These complex points come with the natural structure of a Riemann surface, and yield the so-called “fibers at infinity” \( F_\sigma \) of \( X \). Now, an Arakelov divisor on \( X \) is a sum \( D = D_{\text{fn}} + D_{\text{inf}} \) with \( D_{\text{fn}} \) a traditional Weil divisor on \( X \), and with \( D_{\text{inf}} = \sum \alpha_\sigma F_\sigma \) an “infinite” contribution with \( \alpha_\sigma \in \mathbb{R} \). The set of such divisors forms in a natural way a group \( \text{Div}(X) \). The main result of Arakelov is that one has a natural symmetric and bilinear intersection pairing on this group, and that this pairing factors through the Arakelov divisors of rational functions of \( X \). The crucial case to consider is the intersection of two distinct sections \( P, Q \) of \( p : X \to B \), viewed as divisors on \( X \). We have a finite contribution \( (P, Q)_{\text{fn}} \) which is given using the traditional intersection numbers on \( X \), but we also have an “infinite” contribution \( (P, Q)_{\text{inf}} \), which is defined to be a sum \(- \sum_\sigma \log G(P_\sigma, Q_\sigma)\) over the complex embeddings \( \sigma \). Here \( G \) is a kind of “distance” function on \( X_\sigma \), the Riemann surface corresponding to \( \sigma \). Arakelov defines \( G \) by writing down the axioms that it is supposed to satisfy, and by observing that these axioms allow a unique solution. The function \( G \), called the Arakelov-Green function, is a very important invariant attached to each (compact and connected) Riemann surface. One of the properties of Arakelov’s intersection theory is that an adjunction formula holds true, as in the classical function field case.

Given Arakelov’s intersection theory on arithmetic surfaces, the set-up appears to be present to try to attack the Mordell conjecture. Unfortunately, no proof exists yet which translates the original ideas of Parshin and Arakelov into the number field setting. The major problem is that as yet there seem to exist no good arithmetic analogues of the classical canonical class inequalities. However, we do have an ineffective proof of the Mordell conjecture for number fields, due to Faltings. He was inspired by Szpiro to work on this conjecture using Arakelov theory, but ultimately he found a proof which runs, strictly speaking, along different lines. Nevertheless, Faltings obtained many interesting results in Arakelov intersection theory, and he wrote down these results in his 1984 landmark paper *Calculus on arithmetic surfaces*. Here Faltings shows that, besides the adjunction formula, also other theorems from classical intersection theory on algebraic surfaces have a true analogue for arithmetic surfaces, such as the Riemann-Roch theorem, the Hodge index theorem, and the Noether formula. The formulation of the Noether formula requires the introduction of a new fundamental invariant \( \delta \) of Riemann surfaces, and in his paper Faltings asks for a further study of the properties of this invariant.

(jv) As we said above, the major difficulty in translating the classical techniques for effective Mordell into the number field setting is the lack of good canonical class inequalities. For example, one would like to formulate and prove a convenient analogue of the classical Bogomolov-Miyaoka-Yau-inequality for algebraic surfaces, and attempts to do this have been made by for example Parshin and Moret-Bailly in the 1980s. It was shown by Bost, Mestre and Moret-Bailly, however, that a certain naïve analogue of the classical inequality is false. But parallel to this it also became clear that besides effective Mordell, also other major diophantine conjectures, such as Szpiro’s conjecture and the abc-conjecture, would follow if one had good canonical class inequalities for arithmetic surfaces. No doubt it is very worthwhile to look further and better for such inequalities.

Unfortunately, during the last decades not much progress seems to have been made on this problem. The difficulties generally arise because of the difficult complex differential geometry that
one encounters while dealing with the contributions at infinity. Also, we have no good idea how the canonical classes of an arithmetic surface can be calculated, and neither do we have any good idea how to relate them to other, perhaps easier, invariants. Many authors therefore continue to stress the importance of finding ways to calculate canonical classes of arithmetic surfaces, and of making up an inventory of the possible values that may occur. It is clear that a better understanding of the invariants associated to "infinity" is much needed.

Several authors have done Arakelov intersection theory from this point of view. A first important step was taken by Bost, Mestre and Moret-Bailly, who studied the explicit and calculational aspects of the first non-trivial case, namely of curves of genus 2 (the Arakelov theory of elliptic curves is well-understood, see for instance Faltings' paper). After that, several other isolated examples have been considered: for example Ullmo et al. studied the Arakelov theory of the modular curves \( X_0(N) \), and Guàrdia in his thesis covered a certain class of plane quartic curves admitting many automorphisms.

In the present thesis we wish to contribute to the problem of doing explicit Arakelov geometry by trying to find a description of the main numerical invariants of arithmetic surfaces that makes it possible to calculate them efficiently. We give explicit formulas for the Arakelov-Green function as well as for the Faltings delta-invariant, where it should be remarked that these invariants are defined only in a very implicit way. We show how we can make things even more explicit in the case of elliptic and hyperelliptic curves. Finally, we indicate how efficient calculations are to be done, and in fact we include some explicit numerical examples.

(v) We now turn to a more specialised description of the main results of this thesis. For an explanation of the notation we refer to the main text.

Chapter 1 is an introduction to Arakelov theory. We introduce the main characters, such as the Arakelov-Green function, the delta-invariant, the Faltings height and the relative dualising sheaf, and we prove some fundamental properties about them. The results described in this chapter are certainly not new, although our proofs sometimes differ from the standard ones.

In Chapter 2 we state and prove our explicit formulas for the Arakelov-Green function and Faltings' delta-invariant. Let \( X \) be a compact and connected Riemann surface of genus \( g > 0 \), and let \( G \) be the Arakelov-Green function of \( X \). Let \( \mu \) be the fundamental \((1,1)\)-form of \( X \) and let \( ||\theta|| \) be the normalised theta function on \( \text{Pic}_g(X) \). Let \( S(X) \) be the invariant defined by

\[
\log S(X) := -\int_X \log ||\theta|| \cdot (gP - Q) \cdot \mu(P),
\]

with \( Q \) an arbitrary point on \( X \). It can be checked that the integral is well-defined and does not depend on the choice of \( Q \). Let \( W \) be the classical divisor of Weierstrass points on \( X \). We have then the following explicit formula for the Arakelov-Green function.

**Theorem.** For \( P, Q \) points on \( X \), with \( P \) not a Weierstrass point, we have

\[
G(P, Q)^g = S(X)^{1/g^2} \cdot \frac{||\theta|| \cdot (gP - Q)}{\prod_{W \in W} ||\theta|| \cdot (gP - W)^{1/g^2}}.
\]

Here the product runs over the Weierstrass points of \( X \), counted with their weights. The formula is valid also for Weierstrass points \( P \), provided that we take the leading coefficients of a power series expansion about \( P \) in both numerator and denominator.

As to Faltings' delta-invariant \( \delta(X) \) of \( X \), we prove the following result. Let \( \Phi : X \times X \to \text{Pic}_{g-1}(X) \) be the map sending \((P, Q)\) to the class of \((gP - Q)\). For a fixed \( Q \in X \), let \( i_Q : X \to X \times X \) be the map sending \( P \) to \((P, Q)\), and put \( \phi_Q = \Phi \cdot i_Q \).
Theorem. Define the line bundle $L_X$ by

$$L_X := \left( \bigotimes_{w \in W} \phi'_w (O(\Theta)) \right)^{\otimes (g-1)/g^3} \otimes \phi^*(O(\Theta)) \bigwedge \Omega_{X} \otimes O_X \otimes \left( \Omega_{X}^{\otimes (g+1)/g^3} \right)^{\otimes (g+1)/g^3} \otimes \left( \bigwedge \Omega_{X}^{(g+1)/g^2} \otimes \bigwedge \Omega_{X}^{(g+1)/g^2} \bigotimes O_X \right)^{\otimes 2}.$$ 

Then the line bundle $L_X$ is canonically trivial. If $T(X)$ is the norm of the canonical trivialising section of $L_X$, the formula

$$\exp(\delta(X)/4) = S(X)^{(g-1)/g^3}. T(X)$$

holds.

We have the following explicit formula for $T(X)$. For $P$ on $X$, not a Weierstrass point, and $z$ a local coordinate about $P$, we put

$$\|F_z\|(P) := \lim_{Q \to P} \|gP - Q\| |z(P) - z(Q)|^g.$$ 

Further we let $W_z(\omega)(P)$ be the Wronskian at $P$ in $z$ of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of the differentials $H^0(X, \Omega_X^1)$ with respect to the hermitian inner product $(\omega, \chi) \mapsto \frac{1}{2} \int_X \omega \wedge \chi$.

Theorem. The invariant $T(X)$ satisfies the formula

$$T(X) = \|F_z\|^{(g+1)/g^3} \prod_{w \in W} \|\theta\|(gP - W)^{(g-1)/g^3}. |W_z(\omega)(P)|^2,$$

where again the product runs over the Weierstrass points of $X$, counted with their weights, and where $P$ can be any point of $X$ that is not a Weierstrass point.

It follows that the invariant $T(X)$ can be given in purely classical terms.

Chapters 3 and 4 are devoted to the proof of the following result, specialising to hyperelliptic Riemann surfaces.

Theorem. Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$, and let $\|\Delta_g\|(X)$ be its modified modular discriminant. Then for the invariant $T(X)$ of $X$, the formula

$$T(X) = (2\pi)^{-2g} \cdot \|\Delta_g\|^{-\frac{3g-3}{2}}$$

holds.

The proof of this theorem follows by combining two results relating the Arakelov-Green function to the invariants $T(X)$ and $\|\Delta_g\|(X)$. Although these results look quite similar, the proofs that we give of these results use very different techniques. For the first result, which we prove in Chapter 3, we only use function theory on hyperelliptic Riemann surfaces. For the second result, which we prove in Chapter 4, we broaden our perspective and consider hyperelliptic curves over an arbitrary base scheme. The result follows then from a consideration of a certain isomorphism of line bundles over the moduli stack of hyperelliptic curves. Special care is needed to deal with its specialisation to characteristic 2, where the locus of Weierstrass points behaves in an atypical way.

In Chapter 5 we focus on the Arakelov theory of elliptic curves. Mainly because the fundamental $(1,1)$-form $\mu$ behaves well under isogenies, a fruitful theory emerges in this case. We give a reasonably self-contained and fairly elementary exposition of this theory. We recover some well-known results, due to Faltings, Szpiro and Autissier, but with alternative proofs. In particular, we base our discussion on a complex projection formula for isogenies, which seems new. The main new results that we derive from this formula are as follows.

Theorem. Let $X$ and $X'$ be Riemann surfaces of genus 1. Let $\|\eta\|(X)$ and $\|\eta\|(X')$ be the values
of the normalised eta-function associated to \(X\) and \(X'\), respectively. Suppose we have an isogeny \(f : X \to X'\). Then we have

\[
\prod_{P \in \text{Ker} f, P \neq 0} G_X(0,P) = \frac{\sqrt{N} \cdot \|\eta\|(X')^2}{\|\eta\|(X)^2},
\]

where \(N\) is the degree of \(f\).

The above theorem answers a question posed by Szpiro.

**Theorem.** Let \(E\) and \(E'\) be elliptic curves over a number field \(K\), related by an isogeny \(f : E \to E'\). Let \(p : \mathcal{E} \to B\) and \(p' : \mathcal{E}' \to B\) be arithmetic surfaces over the ring of integers of \(K\) with generic fibers isomorphic to \(E\) and \(E'\), respectively. Suppose that the isogeny \(f\) extends to a \(B\)-morphism \(\mathcal{E} \to \mathcal{E}'\); for example, this is guaranteed if \(\mathcal{E}'\) is a minimal arithmetic surface. Let \(D\) be an Arakelov divisor on \(\mathcal{E}\) and let \(D'\) be an Arakelov divisor on \(\mathcal{E}'\). Then the equality of intersection products \((f^*D', D) = (D', f_*D)\) holds.

In the final Chapter 6 we explain how our explicit formulas can be used to effectively calculate examples of canonical classes. It turns out that the major difficulty is always the calculation of the invariant \(S(X)\).

**Theorem.** Consider the hyperelliptic curve \(X\) of genus 3 and defined over \(\mathbb{Q}\), with hyperelliptic equation

\[
y^2 = x(x-1)(4x^5 + 24x^4 + 16x^3 - 23x^2 - 21x - 4).
\]

Then \(X\) has semi-stable reduction over \(\mathbb{Q}\) with bad reduction only at the primes \(p = 37, p = 701\) and \(p = 14717\). For the corresponding Riemann surface (also denoted by \(X\)) we have

\[
\log T(X) = -4.44361200473681284...
\]
\[
\log S(X) = 17.57...
\]
\[
\delta(X) = -33.40...
\]

and for the curve \(X/\mathbb{Q}\) we have

\[
h_F(X) = -1.280295247656532068...
\]
\[
e(X) = 20.32...
\]

for the Faltings height and the self-intersection of the relative dualising sheaf, respectively.

The main results of this thesis are also described in the following papers.


**References for the introduction**


