Explicit Arakelov geometry

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Chapter 1

Review of Arakelov geometry

In this chapter we review the fundamental notions of Arakelov geometry, as developed in Arakelov’s paper [Ar2] and Faltings’ paper [Fa2]. These papers will serve as the basic references throughout the whole chapter.

In Section 1.1 we discuss the complex differential geometric notions that are needed to provide the “contributions at infinity” in Arakelov intersection theory. In Section 1.2 we turn then to this intersection theory itself, and discuss its formal properties. In Section 1.3 we recall the defining properties of the determinant of cohomology and the Deligne bracket, and show how they are metrised over the complex numbers. These metrisations allow us to give an arithmetic version of the Riemann-Roch theorem. In Section 1.4 we introduce Faltings’ delta-invariant, and give two fundamental formulas in which this invariant occurs. In Section 1.5 we recall the definition and basic properties of semi-stable curves and show how they are used to define Arakelov invariants for curves over number fields. Finally in Section 1.6 we discuss the arithmetic significance of the delta-invariant by stating and sketching a proof of the arithmetic Noether formula, due to Faltings and Moret-Bailly.

1.1 Analytic part

Let $X$ be a compact and connected Riemann surface of genus $g > 0$, and let $\Omega_X^1$ be its holomorphic cotangent bundle. On the space of holomorphic differential forms $H^0(X, \Omega_X^1)$ we have a natural hermitian inner product given by

$$(\omega, \eta) = \frac{i}{2} \int_X \omega \wedge \bar{\eta}.$$ 

Here we use the notation $i = \sqrt{-1}$. We use this inner product\(^1\) to form an orthonormal basis $\\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$. Then we define a canonical $(1,1)$-form $\mu$ on $X$ by setting

$$\mu := \frac{i}{2g} \sum_{k=1}^g \omega_k \wedge \bar{\omega}_k.$$ 

Clearly the form $\mu$ does not depend on the choice of orthonormal basis, and we have $\int_X \mu = 1$.

**Definition 1.1.1.** The canonical Arakelov-Green function $G$ is the unique function $X \times X \to \mathbb{R}_{\geq 0}$ such that the following properties hold:

1. $G(P, Q)^2$ is $C^\infty$ on $X \times X$ and $G(P, Q)$ vanishes only at the diagonal $\Delta_X$. For a fixed $P \in X$, an open neighbourhood $U$ of $P$ and a local coordinate $z$ on $U$ we can write $\log G(P, Q) =$

\(^1\)We warn the reader that some authors use the normalisation $\frac{1}{2g}$ instead of $\frac{1}{2}$. 

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defined as follows. Consider the line bundle of admissible line bundles to arithmetic surfaces, and define an intersection product for admissible line bundles.

Lemma 1.1.6. (Green's formula) Let $\phi, \psi$ be functions on $X$ such that for any $P \in X$, any small enough open neighbourhood $U$ of $P$ and any local coordinate $z$ on $U$ we can write $\log |z(P)| + \log f(Q)$ for $P \neq Q$ in $U$, with $f$ a $C^\infty$-function;

(ii) for all $P \in X$ we have $\partial_\theta \partial_\theta \log G(P, Q)^2 = 2\pi i \mu(Q)$ for $Q \neq P$;

(iii) for all $P \in X$ we have $\int_X \log G(P, Q) \mu(Q) = 0$.

Of course, the existence and uniqueness of such a function require proof. Such a proof is given in [Ar2]. However, that proof relies on methods from the theory of partial differential equations, and is ineffective in the sense that it does not give a way to construct $G$. One of the results in this thesis is an explicit formula for $G$ which is well-suited for concrete calculations (see Theorem 2.1.2).

The defining properties of $G$ imply the symmetry relation $G(P, Q) = G(Q, P)$ for all $P, Q \in X$. This follows by an easy application of Green's formula, which we state at the end of this section. The symmetry of $G$ will be crucial for obtaining the symmetry of the Arakelov intersection product that we shall define in Section 1.2.

We now describe how the Arakelov-Green function gives rise to certain canonical metrics on the line bundles $O_X(D)$, where $D$ is a divisor on $X$. It suffices to consider the case of a point $P \in X$, for the general case follows from this by taking tensor products. Let $s$ be the canonical generating section of the line bundle $O_X(P)$. We then define a smooth hermitian metric $\|\cdot\|$ on $O_X(P)$ by putting $\|s\|_{O_X(P)} = G(P, Q)$ for any $Q \in X$. By property (ii) of the Arakelov-Green function, the curvature form (cf. [GH], p. 148) of $O_X(P)$ is equal to $\mu$, and in general, the curvature form of $O_X(D)$ is deg($D$) · $\mu$, with deg($D$) the degree of $D$.

**Definition 1.1.2.** A line bundle $L$ with a smooth hermitian metric $\|\cdot\|$ is called admissible if its curvature form is a multiple of $\mu$. We also call the metric $\|\cdot\|$ itself admissible in this case.

We will frequently make use of the following observation.

**Proposition 1.1.3.** Let $\|\cdot\|$ and $\|\cdot\|$' be admissible metrics on a line bundle $L$. Then the quotient $\|\cdot\|/\|\cdot\|$' is a constant function on $X$.

**Proof.** The logarithm of the quotient is a smooth harmonic function on $X$, hence it is constant. □

It follows that any admissible line bundle $L$ is, up to a constant scaling factor, isomorphic to the admissible line bundle $O_X(D)$ for a certain divisor $D$. In Section 1.2 we will generalise the notion of admissible line bundle to arithmetic surfaces, and define an intersection product for admissible line bundles.

An important example of an admissible line bundle is the holomorphic cotangent bundle $\Omega^1_X$. We define a metric on it as follows. Consider the line bundle $O_{X \times X}(\Delta_X)$ on $X \times X$. By the adjunction formula, we have a canonical residue isomorphism $O_{X \times X}(\Delta_X) \cong \Omega^1_X$. We obtain a smooth hermitian metric $\|\cdot\|$ on $O_{X \times X}(\Delta_X)$ by putting $\|s\|_{(P, Q)} = G(P, Q)$, where $s$ is the canonical generating section.

**Definition 1.1.4.** We define the metric $\|\cdot\|_{Ar}$ on $\Omega^1_X$ by requiring that the residue isomorphism be an isometry.

**Theorem 1.1.5.** (Arakelov [Ar2]) The metric $\|\cdot\|_{Ar}$ is admissible.

It remains to state Green's formula. We will use this formula once more in Section 3.8. It can be proved in a straightforward way using Stokes' formula.

**Lemma 1.1.6.** (Green's formula) Let $\phi, \psi$ be functions on $X$ such that for any $P \in X$, any small enough open neighbourhood $U$ of $P$ and any local coordinate $z$ on $U$ we can write $\log |z(P)| + f(Q)$ and $\log |z(Q)| + g(Q)$ for all $P \neq Q$ in $U$ with $v_P(\phi), v_P(\psi)$ integers and $f, g$ two $C^\infty$-functions on $U$. Then the formula

$$\int_X \left( \log \phi \cdot \partial_\theta \log \psi - \log \psi \cdot \partial_\theta \log \phi \right) = \sum_{P \in X} \left( v_P(\phi) \log \psi(P) - v_P(\psi) \log \phi(P) \right)$$
1.2 Intersection theory

In this section we describe the intersection theory on an arithmetic surface in the original style of Arakelov [Ar2]. For the general facts that we use on arithmetic surfaces we refer to [Li].

**Definition 1.2.1.** An arithmetic surface is a proper flat morphism $p : \mathcal{X} \to B$ of schemes with $\mathcal{X}$ regular and with $B$ the spectrum of the ring of integers in a number field $K$, such that the generic fiber $\mathcal{X}_K$ is a geometrically connected curve. If $\mathcal{X}_K$ has genus $g$, we also say that $\mathcal{X}$ is of genus $g$.

The arithmetic genus is constant in the fibers of an arithmetic surface, and all geometric fibers except finitely many are non-singular. Further we have $p_*O_\mathcal{X} = O_B$ for an arithmetic surface $p : \mathcal{X} \to B$, and hence, by the Zariski connectedness theorem, all fibers of $p$ are connected.

**Definition 1.2.2.** An arithmetic surface $p : \mathcal{X} \to B$ of positive genus is called minimal if every proper birational $B$-morphism $\mathcal{X} \to \mathcal{X}'$ with $p' : \mathcal{X}' \to B$ an arithmetic surface, is an isomorphism.

For any geometrically connected, non-singular proper curve $C$ of positive genus defined over a number field $K$ there exists a minimal arithmetic surface $p : \mathcal{X} \to B$ together with an isomorphism $\mathcal{X}_p \cong C$. This minimal arithmetic surface is unique up to isomorphism.

We now proceed to discuss the Arakelov divisors on an arithmetic surface $p : \mathcal{X} \to B$.

**Definition 1.2.3.** (Cf. [Ar2]) An Arakelov divisor on $\mathcal{X}$ is a finite formal integral linear combination of irreducible closed subschemes on $\mathcal{X}$ (i.e., a Weil divisor), plus a contribution $\sum \alpha_\sigma : F_\sigma$ running over the embeddings $\sigma : K \hookrightarrow \mathbb{C}$ of $K$ into the complex numbers. Here the $\alpha_\sigma \in \mathbb{R}$, and the $F_\sigma$ are formal symbols, called the "fibers at infinity", corresponding to the Riemann surfaces $X_\sigma = (\mathcal{X} \otimes_{\mathcal{O}_B} \mathbb{C})/(\mathbb{C})$. We have a natural group structure on the set of such divisors, denoted by $\text{Div}(\mathcal{X})$.

Given an Arakelov divisor $D$, we write $D = D_{\text{fin}} + D_{\text{inf}}$ with $D_{\text{fin}}$ its finite part, i.e., the underlying Weil divisor, and with $D_{\text{inf}} = \sum \alpha_\sigma : F_\sigma$ its infinite part. To a non-zero rational function $f$ on $\mathcal{X}$ we associate an Arakelov divisor $(f) = (f)_{\text{fin}} + (f)_{\text{inf}}$ with $(f)_{\text{fin}}$ the usual divisor of $f$ on $\mathcal{X}$, and $(f)_{\text{inf}} = \sum \alpha_\sigma v_\sigma(f) \cdot F_\sigma$ with $v_\sigma(f) = -\int_{X_\sigma} \log |f|_\sigma \cdot \mu_\sigma$. Here $\mu_\sigma$ is the fundamental $(1,1)$-form on $X_\sigma$ given in Section 1.1. The infinite contribution $v_\sigma(f) \cdot F_\sigma$ is supposed to be an analogue of the contribution to $(f)$ in the fiber above a closed point $b \in B$, which is given by $\sum_C v_C(f) \cdot C$ where $C$ runs through the irreducible components of the fiber above $b$, and where $v_C$ denotes the normalised discrete valuation on the function field of $\mathcal{X}$ defined by $C$. The "fiber at infinity" $F_\sigma$ should be seen as "infinitely degenerate", with each point $P$ of $X_\sigma$ corresponding to an irreducible component, such that the valuation $v_P$ of $f$ along this component is given by $v_P(f) = - \log |f|_\sigma(P)$.

**Definition 1.2.4.** We say that two Arakelov divisors $D_1, D_2$ are linearly equivalent if their difference is of the form $(f)$ for some non-zero rational function $f$. We denote by $\text{Cl}(\mathcal{X})$ the group of Arakelov divisors on $\mathcal{X}$ modulo linear equivalence.

Next we discuss the intersection theory of Arakelov divisors, and show that this intersection theory respects linear equivalence. A vertical divisor on $\mathcal{X}$ is a divisor which consists only of irreducible components of the fibers of $p$. A horizontal divisor on $\mathcal{X}$ is a divisor which is flat over $B$. For typical cases $D_1, D_2$ of Arakelov divisors, the intersection product $(D_1, D_2)$ is then defined as follows: (i) if $D_1$ is a vertical divisor, and $D_2$ is a Weil divisor, without any components in common with $D_1$, then the intersection $(D_1, D_2)$ is defined as $(D_1, D_2) = \sum_b (D_1, D_2)_b \log \#k(b)$ where $b$ runs through the closed points of $B$ and where $(D_1, D_2)_b$ denotes the usual intersection multiplicity (cf. [Li], Section 9.1) of $D_1, D_2$ above $b$. (ii) if $D_1$ is a horizontal divisor, and $D_2$ is a
"fiber at infinity" $F_\tau$, then $(D_1, D_2) = \deg(D_1)$ with $\deg(D_1)$ the generic degree of $D_1$. (iii) if $D_1$ and $D_2$ are distinct sections of $p$, then $(D_1, D_2)$ is defined as $(D_1, D_2) = (D_1, D_2)_{\text{fin}} + (D_1, D_2)_{\text{inf}}$ with $(D_1, D_2)_{\text{fin}} = \sum (D_1, D_2)_{\alpha} \log \# k(b)$ as in (i) and with $(D_1, D_2)_{\text{inf}} = - \sum \log G_\alpha(D_1^\alpha, D_2^\alpha)$ with $G_\alpha$ the Arakelov-Green function (cf. Section 1.1) on $X_\alpha$. Note that $- \log G(P, Q)$ becomes a kind of intersection multiplicity "at infinity". The intersection numbers defined in this way extend by linearity to a pairing on $\text{Div}(\mathcal{X})$.

**Theorem 1.2.5.** (Arakelov [Ar2]) There exists a natural bilinear symmetric intersection pairing $\text{Div}(\mathcal{X}) \times \text{Div}(\mathcal{X}) \to \mathbb{R}$. This pairing factors through linear equivalence, giving an intersection pairing $\hat{\text{Pic}}(\mathcal{X}) \times \hat{\text{Pic}}(\mathcal{X}) \to \mathbb{R}$.

Morally speaking, by "compactifying" the arithmetic surface by adding in the "fibers at infinity", and by "compactifying" the horizontal divisors on the arithmetic surface by allowing also for their complex points, we have created a framework that allows us to define a natural intersection theory respecting linear equivalence. This makes for a formal analogy with the classical intersection theory that we have on smooth proper surfaces defined over an algebraically closed field.

Let us sketch a proof of the second statement of Theorem 1.2.5 by showing that for a section $D$ of $p$, and a non-zero rational function $f$ on $\mathcal{X}$, we have $(D, (f)) = 0$. First let us determine, in general, the Arakelov-Green function $G(\text{div}(f), P)$ for a non-zero meromorphic function $f$ on a compact and connected Riemann surface $X$ of positive genus. We note that $\partial_p \partial_p \log G(\text{div}(f), P)^2 = 0$ outside $\text{div}(f)$, since the degree of $\text{div}(f)$ is 0. But we also have $\partial \bar{\partial} \log |f|^2 = 0$ outside $\text{div}(f)$, since $f$ is holomorphic outside $\text{div}(f)$. This implies that $G(\text{div}(f), P) = e^\alpha \cdot |f|(P)$ for some constant $\alpha$, and after taking logarithms and integrating against $\mu$ we find, by property (iii) of Definition 1.1.1, that $\alpha = - \int_X \log |f| \cdot \mu = u(f)$. We compute then

$$(D, (f)) = (D, (f))_{\text{fin}} + \sum_\alpha v_\alpha(f) \cdot F_\alpha$$

$$= (D, (f))_{\text{fin}} + (D, (f))_{\text{inf}} + \sum_\alpha v_\alpha(f)$$

$$= \sum_b v_b(f|_D) \log \# k(b) - \sum_\alpha \log \left( e^{v_\alpha(f)} \cdot |f|_\alpha(D^\alpha) \right) + \sum_\alpha v_\alpha(f)$$

$$= \sum_b v_b(f|_D) \log \# k(b) - \sum_\alpha \log |f|_\alpha(D^\alpha),$$

which is zero by the product formula for $K$.

Finally, we connect the notion of Arakelov divisor with the notion of admissible line bundle.

**Definition 1.2.6.** An admissible line bundle $L$ on $\mathcal{X}$ is the datum of a line bundle $L$ on $\mathcal{X}$, together with smooth hermitian metrics on the restrictions of $L$ to the $X_\alpha$, such that these restrictions are all admissible in the sense of Section 1.1. The group of isomorphism classes of admissible line bundles on $\mathcal{X}$ is denoted by $\hat{\text{Pic}}(\mathcal{X})$.

To each Arakelov divisor $D = D_{\text{fin}} + D_{\text{inf}}$ with $D_{\text{inf}} = \sum_\alpha \alpha_\alpha \cdot F_\alpha$ we can associate an admissible line bundle $O_{\mathcal{X}}(D)$, as follows. For the underlying line bundle, we take $O_{\mathcal{X}}(D_{\text{fin}})$. For the metric on $O_{\mathcal{X}}(D_{\text{fin}})|_{X_\alpha}$ we take the canonical metric on $O_{\mathcal{X}}(D_{\text{fin}})|_{X_\alpha}$ as in Section 1.1, multiplied by $e^{-\alpha_\alpha}$. Clearly, for two Arakelov divisors $D_1$ and $D_2$ which are linearly equivalent, the corresponding admissible line bundles $O_{\mathcal{X}}(D_1)$ and $O_{\mathcal{X}}(D_2)$ are isomorphic. The proof of the following theorem is then a rather formal exercise.

**Theorem 1.2.7.** (Arakelov [Ar2]) There exists a canonical isomorphism of groups $\hat{\text{Pic}}(\mathcal{X}) \to \hat{\text{Pic}}(\mathcal{X})$.

Theorem 1.2.7, together with Theorem 1.2.5, allows us to speak of the intersection product of two admissible line bundles, and we often do this.
1.3 Determinant of cohomology

The determinant of cohomology for an arithmetic surface $p : \mathcal{X} \to B$ is a gadget on the base $B$ which allows us to formulate an arithmetic Riemann-Roch theorem for $p$ (Theorem 1.3.8). In the present section we will describe the determinant of cohomology in full generality. Our Riemann-Roch theorem will be a formal analogue of the Riemann-Roch that one obtains by taking the determinant of cohomology on a proper morphism $p : \mathcal{X} \to B$ with $\mathcal{X}$ a smooth proper surface and $B$ a smooth proper curve, both defined over an algebraically closed field. With the help of arithmetic Riemann-Roch, we will be able to formulate and prove an arithmetic analogue of the Noether formula (see Section 1.6). References for this section are [De2] and [Mo1].

The determinant of cohomology is determined by a set of uniquely defining properties.

**Definition 1.3.1.** (Cf. [Mo1], §1) Let $p : \mathcal{X} \to B$ be a proper morphism of Noetherian schemes. To each coherent $O_X$-module $F$ on $\mathcal{X}$, flat over $O_B$, we associate a line bundle $\det Rp_* F$ on $B$, called the determinant of cohomology of $F$, satisfying the following properties:

(i) The association $F \mapsto \det Rp_* F$ is functorial for isomorphisms $F \overset{\sim}{\to} F'$ of coherent $O_X$-modules.

(ii) The construction of $\det Rp_* F$ commutes with base change, i.e., each cartesian diagram

$$
\begin{array}{ccc}
\mathcal{X}' & \overset{u'}{\to} & \mathcal{X} \\
\downarrow p' & & \downarrow p \\
B' & \overset{u}{\to} & B
\end{array}
$$

gives rise to a canonical isomorphism $u^*(\det Rp_* F) \overset{\sim}{\to} \det Rp'_*(u'^* F)$.

(iii) Each exact sequence

$$
0 \to F' \to F \to F'' \to 0
$$

of flat coherent $O_X$-modules gives rise to an isomorphism

$$
\det Rp_* F \overset{\sim}{\to} \det Rp_* F' \otimes \det Rp_* F''
$$

compatible with base change and with isomorphisms of exact sequences.

(iv) Let $E = (0 \to E^0 \to E^1 \to \cdots \to E^n \to 0)$ be a finite complex of $O_B$-modules which are locally free of finite rank, and suppose there is given a quasi-isomorphism $E \to Rp_* F$. Then one has a canonical isomorphism

$$
\det Rp_* F \overset{\sim}{\to} \bigotimes_{k=0}^n (\det E^k)^{\otimes (-1)^k},
$$

compatible with base change. Here $\det E$ denotes the maximal exterior power of a locally free $O_B$-module $E$ of finite rank.

(v) In particular, when the $O_B$-modules $R^k p_* F$ are locally free, one has a canonical isomorphism

$$
\det Rp_* F \overset{\sim}{\to} \bigotimes_{k=0}^n (\det R^k p_* F)^{\otimes (-1)^k},
$$

compatible with base change.
(vi) Let \( \chi_{X/B}(F) \) be the locally constant function \( x \mapsto \chi(F_x) \) on \( B \). Let \( u \in \Gamma(B, O_B^\times) \) be multiplication by \( u \) in \( F \). By (i), this gives an automorphism of \( \det R_p_* F \); this automorphism is multiplication by \( u^{x \times \eta}(F) \).

(vii) If \( M \) is a line bundle on \( B \) then one has a canonical isomorphism

\[
\det R_p_* (F \otimes p^* M) \sim \tau \det R_p_* F \otimes M^{\otimes x \times \eta}(F)
\]

of line bundles on \( B \).

In the case \( B = \text{Spec}(\mathbb{C}) \), we will often use the shorthand notation \( \lambda(F) \) for the determinant of cohomology of \( F \). Explicitly, we have \( \lambda(F) = \otimes_{k=0}^n (\det H^k(X, F))^\otimes (-1)^k \), where \( n \) is the dimension of \( X \).

An important canonical coherent sheaf in the situation where \( p : X \rightarrow B \) is proper, flat and locally a complete intersection, is the relative dualising sheaf \( \omega_{X/B} \), cf. [Li], Section 6.4. In fact, the sheaf \( \omega_{X/B} \) is invertible, and satisfies the following important duality relation (Serre duality): let \( F \) be any coherent sheaf on \( X \), flat over \( O_B \). Then we have a canonical isomorphism

\[
\det R_p_* F \sim \det R_p_*(\Omega_{X/B}^1 \otimes F^\vee)
\]

of line bundles on \( B \). The relative dualising sheaf behaves well with respect to base change: let \( u : B' \rightarrow B \) be a morphism, let \( X' = X \times_B B' \) and let \( u' : X' \rightarrow X \) be the projection onto the first factor. Then we have a canonical isomorphism \( u^*\omega_{X/B} \sim u^*\omega_{X'/B'} \). As a consequence, by property (ii) in Definition 1.3.1 we have a canonical isomorphism \( u^*(\det \omega_{X/B}) \sim \det \omega_{X'/B'} \) on \( B' \). Here \( p' : X' \rightarrow B' \) is the projection on the second factor. If \( p : X \rightarrow B \) is a smooth curve, the relative dualising sheaf \( \omega_{X/B} \) can be identified with the sheaf \( \Omega_{X/B}^1 \) of relative differentials. A convenient description is also possible if the fibers of \( p \) are nodal curves, see [DM], §1.

For our Riemann-Roch theorem we need a metric on the determinant of cohomology \( \det R_p_* L \), where \( L \) is an admissible line bundle on an arithmetic surface \( p : X \rightarrow B \). So, let us restrict for the moment to the case that \( B = \text{Spec}(\mathbb{C}) \), and consider the determinant of cohomology \( \lambda(L) \), where \( L \) is an admissible line bundle on a compact and connected Riemann surface \( X \) of positive genus \( g \). The following theorem gives a satisfactory answer to our question.

**Theorem 1.3.2.** (Faltings [Fa2]) For every admissible line bundle \( L \) there exists a unique metric on \( \lambda(L) \) such that the following axioms hold:

(i) any isomorphism \( L_1 \sim L_2 \) of admissible line bundles induces an isometry \( \lambda(L_1) \sim \lambda(L_2) \);

(ii) if we scale the metric on \( L \) by a factor \( \alpha \), the metric on \( \lambda(L) \) is scaled by a factor \( \alpha^\chi(L), \) where \( \chi(L) = \deg L - g + 1 \);

(iii) for any admissible line bundle \( L \) and any point \( P \), the exact sequence

\[
0 \rightarrow L \rightarrow L(P) \rightarrow P_* P^* L(P) \rightarrow 0
\]

induces an isometry

\[
\lambda(L(P)) \sim \lambda(L) \otimes P^* L(P);
\]

here \( L(P) \) carries the metric coming from the canonical isomorphism \( L(P) \sim L \otimes_{O_X} O_X(P) \);

(iv) for \( L = \Omega_X^1 \), the metric on \( \lambda(L) = \wedge^g H^0(X, \Omega_X^1) \) is defined by the hermitian inner product \( (\omega, \eta) \mapsto \frac{1}{2} \int_X \omega \wedge \overline{\eta} \) on \( H^0(X, \Omega_X^1) \).
We will refer to the metric in the theorem as the Faltings metric on the determinant of cohomology. For the proof of Theorem 1.3.2 we shall use the so-called Deligne bracket. Since we will make essential use of this tool later on, we define it here in detail.

**Definition 1.3.3.** (Cf. [De2]) Let \( p : \mathcal{X} \to B \) be a proper, flat curve which is locally a complete intersection. Let \( L, M \) be two line bundles on \( \mathcal{X} \). Then \( \langle L, M \rangle \) is to be the \( \mathcal{O}_B \)-module which is generated, locally for the étale topology on \( B \), by the symbols \( \langle l, m \rangle \) for local sections \( l, m \) of \( L, M \), with relations

\[
\langle l, fm \rangle = f(\text{div}(l)) \cdot \langle l, m \rangle, \quad \langle fl, m \rangle = f(\text{div}(m)) \cdot \langle l, m \rangle.
\]

Here \( f(\text{div}(l)) \) should be interpreted as a norm: for an effective relative Cartier divisor \( D \) on \( \mathcal{X} \) we set \( f(D) = \mathcal{N}_{D/B}(f) \) and then for \( \text{div}(l) = D_1 - D_2 \) with \( D_1, D_2 \) effective we set \( f(\text{div}(l)) = f(D_1) \cdot f(D_2)^{-1} \). One checks that this is independent of the choices of \( D_1, D_2 \). Furthermore, it can be shown that the \( \mathcal{O}_B \)-module \( \langle L, M \rangle \) is actually a line bundle on \( B \).

We have the following properties for the Deligne bracket.

(i) For given line bundles \( L_1, L_2, M_1, M_2, L, M \) on \( \mathcal{X} \) we have canonical isomorphisms

\[
(L_1 \otimes L_2, M) \sim (L_1, M) \otimes (L_2, M), \quad (L, M_1 \otimes M_2) \sim (L, M_1) \otimes (L, M_2),
\]

and \( (L, M) \sim (M, L) \);

(ii) The formation of the Deligne bracket commutes with base change, i.e., each cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{u'} & \mathcal{X} \\
\downarrow & & \downarrow \\
B' & \xrightarrow{u} & B
\end{array}
\]

gives rise to a canonical isomorphism \( u^*(L, M) \sim (u^*L, u^*M) \);

(iii) For \( P : B \to \mathcal{X} \) a section of \( p \) we have a canonical isomorphism \( P^*L \sim (\mathcal{O}_X(P), L) \);

(iv) If the \( B \)-morphism \( q : \mathcal{X}' \to \mathcal{X} \) is the blowing-up of a singular point on \( \mathcal{X} \), then we have a canonical isomorphism \( \langle q^*L, q^*M \rangle \sim (L, M) \);

(v) For the relative dualising sheaf \( \omega_{\mathcal{X}/B} \) of \( p \) and any section \( P : B \to \mathcal{X} \) of \( p \) we have a canonical adjunction isomorphism \( (P, P)^{\mathcal{O}_B^{-1}} \sim (P, \omega_{\mathcal{X}/B}) \).

The relation with the determinant of cohomology is given by the following formula: let \( L, M \) be line bundles on \( \mathcal{X} \), then we have a canonical isomorphism

\[
(L, M) \sim \text{det} \, Rp_* (L \otimes M) \otimes (\text{det} \, Rp_* L)^{\mathcal{O}_B^{-1}} \otimes (\text{det} \, Rp_* M)^{\mathcal{O}_B^{-1}} \otimes \text{det} \, p_* \omega_{\mathcal{X}/B}.
\]

This formula gives us new information on the determinant of cohomology, namely, it follows from the formula that we have a canonical isomorphism

\[
(* \quad (\text{det} \, Rp_* L)^{\mathcal{O}_B^{\text{grade}}} \sim (L, L \otimes \omega_{\mathcal{X}/B}^{-1}) \otimes (\text{det} \, p_* \omega_{\mathcal{X}/B})^{\mathcal{O}_B^{\text{grade}}}
\]

of line bundles on \( B \). This isomorphism can be interpreted as Riemann-Roch for the morphism \( p : \mathcal{X} \to B \). We will use Riemann-Roch to put metrics on the \( \lambda(L) \). First of all we show how the Deligne bracket can be metrised in a natural way.

**Definition 1.3.4.** (Cf. [De2]) Let \( L, M \) be admissible line bundles on a Riemann surface \( X \). Then for local sections \( l, m \) of \( L \) and \( M \) we put

\[
\log \| (l, m) \| = \langle \log \| m \| \rangle [\text{div}(l)].
\]
It can be checked that this gives a well-defined metric on \( \langle L, M \rangle \), and in fact the isomorphisms from (i), (iii) and (v) above are isometries for this metric.

**Proof of Theorem 1.3.2.** We will construct a metric on \( \lambda(L) \) such that axioms (i)-(iv) are satisfied. First of all we use property (iv) from Theorem 1.3.2 to put a metric on \( \lambda(\omega) \). Next we use Definition 1.3.4 above to put a metric on the brackets \( \langle L, L \otimes \omega^{-1} \rangle \). Then by Riemann-Roch \((*)\) we obtain a metric on \( \lambda(L) \). From this construction, the axioms (i) and (ii) are clear; it only remains to see that property (iii) is satisfied. But this we can see by the following argument due to Mazur: we have isometries

\[
\lambda(L)^{\otimes 2} \sim \langle L, L \otimes \omega^{-1} \rangle \otimes \lambda(\omega)^{\otimes 2} \quad \text{and} \quad \lambda(L(P))^\otimes \sim \langle L(P), L(P) \otimes \omega^{-1} \rangle \otimes \lambda(\omega)^{\otimes 2}.
\]

Combining, we obtain an isometry

\[
\lambda(L(P))^{\otimes 2} \otimes \lambda(L)^{\otimes -2} \sim \langle L(P), L(P) \otimes \omega^{-1} \rangle \langle L, L \otimes \omega^{-1} \rangle^{\otimes -1}.
\]

By expanding the brackets, we see that the latter is isometric to \( P^*L(P) \otimes P^*(L \otimes \omega^{-1}) \). By the adjunction formula, this is isometric with \((P^*L(P))^{\otimes 2}\). Hence property (iii) also holds, and Theorem 1.3.2 is proven. \(\square\)

Note that the Riemann-Roch isomorphism \((*)\), which is by now an isometry given the various metrisations, gives us that the canonical Serre duality isomorphism \( \lambda(L) \rightarrow \lambda(\Omega_X^1 \otimes L^{-1}) \) is an isometry.

To conclude this section, we explain what all this means for admissible line bundles on arithmetic surfaces. Using the metrisation of the determinant of cohomology, one obtains, for any arithmetic surface \( p: X \rightarrow B = \text{Spec}(R) \) and any admissible line bundle \( L \) on \( X \), the determinant of cohomology \( \text{det} R_p L \) as a metrised line bundle (or metrised projective \( R \)-module) on \( B \).

**Definition 1.3.5.** For a metrised projective \( R \)-module \( M \) we define a degree as follows: choose a non-zero element \( s \) of \( M \), then

\[
\widetilde{\deg} M = \log \#(M/R \cdot s) - \sum_{\sigma} \log \|s\|_\sigma.
\]

One can check using the product formula that this definition is independent of the choice of \( s \).

It follows directly from Definitions 1.3.4 and 1.3.5 that for two admissible line bundles \( L, M \) on \( X \) we have \( \widetilde{\deg} \langle L, M \rangle = \langle L, M \rangle \), the intersection product from Section 1.2.

We are now ready to reap the fruits of our work. Let \( \omega_{X/B} \) be the admissible line bundle on \( X \) whose underlying line bundle is the relative dualising sheaf of \( p \), and where the metrics at infinity are the canonical ones as in Section 1.1.

**Proposition 1.3.6.** (Adjunction formula, Arakelov [Ar2]) For any section \( P: B \rightarrow X \) we have an equality \(-\langle P, P \rangle = \langle P, \omega_{X/B} \rangle \).

**Proof.** This follows immediately from property (iii) of the Deligne bracket and the definition of the admissible metric on \( \Omega^1_X \) for a compact and connected Riemann surface \( X \), given in Section 1.1. \(\square\)

**Proposition 1.3.7.** Let \( q: B' \rightarrow B \) be a finite morphism with \( B' \) the spectrum of the ring of integers in a finite extension \( F \) of the quotient field \( K \) of \( R \). Let \( \chi' \rightarrow X \times_B B' \) be the minimal desingularisation of \( X \times_B B' \), and let \( r: \chi' \rightarrow X \) be the induced morphism. Then we have, for any two admissible line bundles \( L, M \) on \( X \), an equality \( \langle r^*L, r^*M \rangle = [F: K]\langle L, M \rangle \).

**Proof.** This follows from properties (ii) and (iv) of the Deligne bracket. \(\square\)
Proposition 1.3.8. \((\text{Riemann-Roch theorem, Faltings [Fa2]})\) \(L\) be an admissible line bundle on \(X\). Then the formula
\[
\deg \det R^p_* L = \frac{1}{2} (\langle L, L \otimes \omega_{X/B} \rangle) + \deg \det p_* \omega_{X/B}
\]
holds.

Proof. This follows directly from the fact that Riemann-Roch (*) is an isometry. \(\Box\)

### 1.4 Faltings’ delta-invariant

The definition of the Faltings metric on the determinant of cohomology (see Theorem 1.3.2) is rather implicit, since it is given as the unique metric satisfying a certain set of axioms. In this section we want to make the Faltings metric more explicit. It turns out that there is a close relationship with theta functions, which we briefly review first. The connection is provided by Faltings’ delta-invariant, which is defined in Theorem 1.4.6. We end this section by giving two fundamental formulas in which the delta-invariant occurs.

Let again \(X\) be a compact and connected Riemann surface of genus \(g > 0\). Let \(\text{Pic}_{g-1}(X)\) be the degree \(g - 1\) part in the Picard variety of isomorphism classes of line bundles on \(X\). Choose a symplectic basis for the homology \(H_1(X, \mathbb{Z})\) of \(X\) and choose a basis \(\{\omega_1, \ldots, \omega_g\}\) of the holomorphic differentials \(H^0(X, \Omega_X^1)\). Let \(\Omega = (\Omega_1|\Omega_2)\) be the period matrix given by these data. By Riemann’s first bilinear relations, the matrix \(\Omega_1\) is invertible and the matrix \(\tau = \Omega_1^{-1}\Omega_2\) lies in \(\mathcal{H}_g\), the Siegel upper half-space of complex symmetric \(g \times g\)-matrices with positive definite imaginary part.

**Lemma 1.4.1.** (Riemann’s second bilinear relations) The matrix identity
\[
\frac{i}{2} \int_X \omega_k \wedge \overline{\omega_l} \big|_{1 \leq k, l \leq g} = \frac{i}{2} (\overline{\Omega_2^t} \Omega_1 - \overline{\Omega_1^t} \Omega_2) = \overline{\Omega}_1 (\text{Im} \tau)^t \Omega_1
\]
holds.

Proof. For the first equality, see for instance [GH], pp. 231–232. The second follows from the first by the fact that \(\tau\) is symmetric. \(\Box\)

Choose a point \(P_0 \in X\), and let \(\{\eta_1, \ldots, \eta_g\} = \{\omega_1, \ldots, \omega_g\} \cdot \Omega_1^{-1}\). Then by a classical theorem of Abel and Jacobi, the map
\[
\text{Div}_{g-1}(X) \longrightarrow \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g, \quad \sum n_k P_k \mapsto \sum n_k \int_{P_0}^{P_k} (\eta_1, \ldots, \eta_g)
\]
descends to well-defined bijective map
\[
u : \text{Pic}_{g-1}(X) \sim \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g.
\]

Let \(\vartheta(z; \tau)\) be Riemann’s theta function given by
\[
\vartheta(z; \tau) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i \tau n^2 + 2\pi i n z).
\]

Due to its transformation properties under translation of \(z\) by an element of the lattice \(\mathbb{Z}^g + \tau \mathbb{Z}^g\), the function \(\vartheta\) can be viewed as a global section of a line bundle on \(\mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g\). We denote by \(\Theta_0\) the divisor of this section. Let \(\Theta \subset \text{Pic}_{g-1}(X)\) be the divisor given by the classes of line bundles admitting a global section. Riemann has shown that there is a close relationship between these two “theta-divisors”.

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Theorem 1.4.2. (Riemann) There is an element $\kappa = \kappa(P_0)$ in $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$ such that the following holds. Let $t_\kappa$ denote translation by $\kappa$ in $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$. Then the equality of divisors $(t_\kappa \cdot u)^*\Theta_0 = \Theta$ holds. In particular we have a canonical isomorphism of line bundles $(t_\kappa \cdot u)^*\mathcal{O}(\Theta_0) \sim \mathcal{O}(\Theta)$ on $\text{Pic}_{g-1}(X)$. Furthermore, for a divisor $D$ of degree $g-1$ on $X$ we have $(t_\kappa \cdot u)(K - D) = -(t_\kappa \cdot u)(D)$, where $K$ is a canonical divisor on $X$. In particular, the map $t_\kappa \cdot u$ identifies the set of classes of semi-canonical divisors (i.e., divisors $D$ with $2D$ linearly equivalent to $K$) with the set of $2$-division points on $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$.

We want to put a metric on the line bundle $\mathcal{O}(\Theta)$. By Riemann's theorem, it suffices to put a metric on the line bundle $\mathcal{O}(\Theta_0)$ on $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$. Let $s$ be the canonical section of $\mathcal{O}(\Theta_0)$, and let $\nu$ be the canonical translation-invariant $(1,1)$-form on $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$ given by

$$\nu := \frac{i}{2} \sum_{1 \leq k, l \leq g} (\text{Im} r)^{-1}_k d\bar{z}_k \wedge dz_l.$$ 

The $2g$-form $\frac{i}{2} \nu^g$ gives the Haar measure on $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$. We let $\| \cdot \|_0$ be the metric on $\mathcal{O}(\Theta_0)$ uniquely defined by the following properties:

(i) the curvature form of $\| \cdot \|_0$ is equal to $\nu$;

(ii) $\frac{1}{g} \int_{\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g} |s|^2_0 \nu^g = 2^{-9/2}$.

Definition 1.4.3. We denote by $\| \cdot \|_e$ the metric on $\mathcal{O}(\Theta)$ induced by $\| \cdot \|_0$ via Riemann's theorem, and we write $\| \cdot \|_e$ as a shorthand for $\|(t_\kappa \cdot u)^* s\|_e$, or, by abuse of notation, for $\| s \|_0$.

Note that $\| \vartheta \|_e(K - D) = \| \vartheta \|_e(D)$ for any divisor $D$ of degree $g - 1$, and that $\| \vartheta \|_e(D)$ vanishes if and only if $D$ is linearly equivalent to an effective divisor.

By checking the properties (i) and (ii) one finds the following explicit formula for $\| \vartheta \|_e$.

Proposition 1.4.4. Let $z \in \mathbb{C}^g$ and $\tau \in \mathcal{H}_g$, the Siegel upper half-space of degree $g$. Then the formula

$$\| \vartheta \|_e(z; \tau) = (\text{det Im} \tau)^{1/4} \exp(-\pi^2 y \cdot (\text{Im} \tau)^{-1} \cdot y) \cdot \| \vartheta \|_e(z; \tau)$$

holds. Here $y = \text{Im} z$.

It is not difficult to check using Lemma 1.4.1 that if we embed $X$ into $\mathbb{C}^g/\mathbb{Z}^g + r\mathbb{Z}^g$ by integration $j : P \mapsto \int_{P_0} (\eta_1, \ldots, \eta_g)$, we have $j^* \nu = g \cdot \mu$. One can view this as an alternative definition of the form $\mu$.

Proposition 1.4.5. Let $D$ be a divisor on $X$, and consider the map $\phi_D : X \to \text{Pic}_{g-1}(X)$ given by $P \mapsto [D - \chi(D) \cdot P]$, where $\chi(D) = \text{deg} D - g + 1$. Then the line bundle $\phi_D^* (\mathcal{O}(\Theta))$ on $X$ is admissible and has degree $g \cdot \chi(D)^2$.

Proof. A computation using the formula in Proposition 1.4.4 shows that outside $\phi_D^* (\mathcal{O}(\Theta))$ we have $\partial P \partial \bar{P} \text{log} \| \vartheta \|_e (D - \chi(D) \cdot P)^2 = 2\pi i \chi(D)^2 \cdot j^* \nu = 2\pi i g \chi(D)^2 \cdot \mu$. Thus, the curvature form of $\phi_D^* (\mathcal{O}(\Theta))$ is a multiple of $\mu$, and the degree of $\phi_D^* (\mathcal{O}(\Theta))$ is $g \cdot \chi(D)^2$. \qed

The following theorem introduces Faltings' delta-invariant, connecting Faltings' metric on the determinant of cohomology with the metric on $\mathcal{O}(\Theta)$ defined in Definition 1.4.3. It follows from axiom (ii) in Theorem 1.3.2 that for an admissible line bundle $L$ of degree $g - 1$, the metric on $\lambda(L)$ is in fact independent of the metric on $L$.

Theorem 1.4.6. (Faltings [Fa2]) There is a constant $\delta = \delta(X)$ such that the following holds. Let $L$ be an admissible line bundle of degree $g - 1$. Then there is a canonical isomorphism $\lambda(L) \sim \mathcal{O}(-\Theta)[L]$, and the norm of this isomorphism is equal to $\exp(\delta/8)$. 

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For the proof we need the following lemma. For the general definition of the scheme $\text{Pic}_{g-1}(\mathcal{X}/B)$, its theta divisor $\Theta$, and for the existence of the universal bundle, we refer the reader to [Mo1], Section 2.

**Lemma 1.4.7.** Let $B$ be a noetherian scheme and let $p: \mathcal{X} \to B$ be a smooth proper curve admitting a section. There is, up to a unique isomorphism, a unique universal line bundle $U$ on the product $\mathcal{X} \times \text{Pic}_{g-1}(\mathcal{X}/B)$. Let $g: \mathcal{X} \times \text{Pic}_{g-1}(\mathcal{X}/B) \to \text{Pic}_{g-1}(\mathcal{X}/B)$ be the projection onto the second factor. Then there is a canonical isomorphism $\det Rg_*U \to O(-\Theta)$ of line bundles on $\text{Pic}_{g-1}(\mathcal{X}/B)$, compatible with base change.

**Proof.** This is in [Mo1], Section 2.4.

**Sketch of the proof of Theorem 1.4.6.** Let $r$ be a non-negative integer, let $E$ be a divisor of degree $r + g - 1$ on $X$, and consider the map $\varphi_E: X^r \to \text{Pic}_{g-1}(X)$ given by $(P_1, \ldots, P_r) \mapsto O_X(E - (P_1 + \cdots + P_r))$. Let $U$ be the universal line bundle on $X \times \text{Pic}_{g-1}(X)$, and consider the pullback diagram

$$
\begin{array}{ccc}
X^r \times X & \xrightarrow{\varphi_E} & \text{Pic}_{g-1}(X) \times X \\
p \downarrow & & q \downarrow \\
X^r & \xrightarrow{\varphi_E} & \text{Pic}_{g-1}(X)
\end{array}
$$

with $p, q$ the projections on the first factor and with $\varphi_E = (\varphi_E, \text{id}_X)$. By Lemma 1.4.7 and Definition 1.3.1 we have a canonical isomorphism $\det R\varphi_{E*}U \cong \varphi_{E*}(O(-\Theta))$ of line bundles on $X^r$. It clearly suffices for our purposes to prove that the norm of this isomorphism is constant. But this follows from a calculation as performed in [Fa2], p. 397, showing that the curvature forms of the line bundles at both sides of the isomorphism are equal.

In order to perform the calculation referred to at the end of the above proof, Faltings makes use of the following lemma. We, in turn, will use this lemma to derive an explicit formula from Theorem 1.4.6.

**Lemma 1.4.8.** Let $L$ be an admissible line bundle on $X$ and let $P_1, \ldots, P_r$ be $r$ points on $X$. Then we have a canonical isomorphism

$$
\lambda(L \otimes O_X(P_1 + \cdots + P_r)^\vee) \cong \lambda(L) \otimes \bigotimes_{k=1}^{r} P_k^* L^\vee \otimes \bigotimes_{k<l} P_k^* O_X(P_l),
$$

and this isomorphism is an isometry.

**Proof.** This follows just by iteration of axiom (iii) from Theorem 1.3.2. 

A fundamental theorem of Riemann states that if $D = P_1 + \cdots + P_g$ is an effective divisor of degree $g$ such that $\phi_D(X)$ is not contained in $\Theta$, we have an equality of divisors $\phi_D^n(\Theta) = D$ on $X$. By Propositions 1.1.3 and 1.4.5, the canonical isomorphism $\phi_D^n(O(\Theta)) \cong O_X(P_1 + \cdots + P_g)$ has constant norm on $X$. In other words, there is a constant $c = c(P_1, \ldots, P_g)$ depending only on $P_1, \ldots, P_g$ such that $\|\phi_D^n(\Theta) - O_X(P_1 + \cdots + P_g)\| = c \cdot \prod_{j=1}^{g} G(P_j, Q)$ for all $Q \in X$. We will now compute this constant. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis of the differentials $H^0(X, \Omega_X^1)$ and let $P_1, \ldots, P_g$ be $g$ points on $X$. Let $z_1, \ldots, z_g$ be local coordinates about $P_1, \ldots, P_g$ and write $\omega_k = f_{k\ell} \cdot dz_\ell$ locally at $P_\ell$. Then we write $\|\det \omega_k(P_\ell)\|_A = |\det(f_{k\ell}(0))| \cdot \prod_{k=1}^{g} \|dz_k\|_A$. This definition does not depend on the choices of the local coordinates $z_1, \ldots, z_g$.

**Theorem 1.4.9.** (Faltings [Fa2]) Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega_X^1)$ provided with the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \overline{\eta}$. Let $P_1, \ldots, P_g, Q$ be generic points on $X$. 

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Then the formula
\[ \|\vartheta\|(P_1 + \cdots + P_g - Q) = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_i)\|\frown}{\prod_{k=1}^g G(P_k, P_1)} \cdot \prod_{k=1}^g G(P_k, Q) \]
holds.

Proof. We apply Lemma 1.4.8 to the admissible line bundle \( L = \Omega_X^1(Q) = \Omega_X^1 \otimes_X O_X(Q) \) and the points \( P_1, \ldots, P_g \). We obtain the required formula by computing the norm of a canonical section on the left and the right hand side of the isomorphism in Lemma 1.4.8. By Serre duality we have \( \lambda(\Omega_X^1(Q) \otimes O_X(P_1 + \cdots + P_g)) \cong \lambda(O_X(P_1 + \cdots + P_g - Q)). \) For generic points \( P_1, \ldots, P_g, Q \), the line bundle \( O_X(P_1 + \cdots + P_g - Q) \) has no global sections. In this case, the determinant \( \lambda(O_X(P_1 + \cdots + P_g - Q)) \) is canonically isomorphic to \( \mathbb{C} \) and hence has a canonical section \( 1 \). By Theorem 1.4.6, it has norm \( \exp(-\delta(X)/8) \cdot \|\vartheta\|(P_1 + \cdots + P_g - Q)^{-1} \). Now let's look at the right hand side of the isomorphism in Lemma 1.4.8. We have a canonical isomorphism
\[ \lambda(\Omega_X^1(Q)) \cong \bigotimes_{k=1}^g P_k^* \Omega_X^1(Q) \]
given by taking the determinant of the evaluation map
\[ H^0(X, \Omega_X^1(Q)) = H^0(X, \Omega_X^1) \cong \bigoplus_{k=1}^g P_k^* \Omega_X^1(Q). \]
The norm of this isomorphism is \( \|\det \omega_k(P_i)\|\frown \cdot \prod_{k=1}^g G(P_k, Q) \), and hence we have a canonical element in \( \lambda(\Omega_X^1(Q))^\vee \otimes \bigotimes_{k=1}^g P_k^* \Omega_X^1(Q) \) of that same norm. We end up with a canonical element in
\[ \lambda(\Omega_X^1(Q)) \otimes \left( \bigotimes_{k=1}^g P_k^* \Omega_X^1(Q) \right)^\vee \otimes \bigotimes_{k<l} P_k^* O_X(P_k) \]
of norm
\[ \|\det \omega_k(P_i)\|^{-1} \cdot \prod_{k=1}^g G(P_k, Q)^{-1} \cdot \prod_{k<l} G(P_k, P_l). \]
The theorem follows by equating the two norms. \( \square \)

An important counterpart to Faltings' formula has been proved byGuàrdia [Gu1]. We will make essential use of this formula in Section 4.5 where, as an appendix to our work involved in determining a certain auxiliary Arakelov invariant for hyperelliptic Riemann surfaces, we prove a relation between products of certain Jacobian Nullwerte and products of certain Theta-nullwerte. The new ingredient in Guàrdia's formula is a function \( ||J|| \) on Sym^2 X, which we shall introduce first.

Recall that we have fixed for our Riemann surface \( X \) a symplectic basis of its homology and a basis \( \{\omega_1, \ldots, \omega_g\} \) of \( H^0(X, \Omega_X^1) \), giving rise to a period matrix \( \Omega = (\Omega_1, \Omega_2) \). We have put \( \tau = \Omega_1^{-1} \Omega_2 \) and \( \{\eta_1, \ldots, \eta_g\} = \{\omega_1, \ldots, \omega_g\} \cdot \tau^{-1}. \)

Lemma 1.4.10. Consider \( \wedge^g H^0(X, \Omega_X^1) \) with its metric derived from the hermitian inner product
\[ (\omega, \eta) \mapsto \frac{1}{2} \int_X \omega \wedge \overline{\eta} \] on \( H^0(X, \Omega_X^1) \). Then the formula \( ||\omega_1 \wedge \ldots \wedge \omega_g||^2 = (\det \operatorname{Im} \tau) \cdot ||\Omega_i||^2 \) holds.

Proof. Note that \( ||\omega_1 \wedge \ldots \wedge \omega_g||^2 = \det ((\omega_k, \omega_l))_{k,l} \). The formula follows then from Lemma 1.4.1. \( \square \)
Definition 1.4.11. For \(w_1, \ldots, w_g \in \mathbb{C}^g\) we put
\[
J(w_1, \ldots, w_g) := \det \left( \frac{\partial}{\partial z_k}(w_l) \right),
\]
\[
\|J\|(w_1, \ldots, w_g) := (\det \text{Im} \tau)^{\frac{1}{2g^2}} \cdot \exp(-\pi \sum_{k=1}^g y_k \cdot (\text{Im} \tau)^{-1} \cdot y_k) \cdot |J(w_1, \ldots, w_g)|,
\]
where \(y_k = \text{Im} w_k\) for \(k = 1, \ldots, g\). The latter definition depends only on the classes of the vectors \(w_k\) in \(\mathbb{C}^g/\mathbb{Z}^g + \tau \mathbb{Z}^g\). Next, fix \(g\) points \(P_1, \ldots, P_g\) on \(X\) and choose \(g\) vectors \(w_1, \ldots, w_g\) in \(\mathbb{C}^g\) by requiring that for each \(k = 1, \ldots, g\), the divisor \(\sum_{k=1}^g P_k\) corresponds by Riemann's theorem 1.4.2 to the class \([w_k]\) in \(\mathbb{C}^g/\mathbb{Z}^g + \tau \mathbb{Z}^g\). We then define \(\|J\|(P_1, \ldots, P_g) := \|J\|(w_1, \ldots, w_g)\). One may check that this definition does not depend on the choice of the matrix \(\tau\).

We have \(\|J\|(P_1, \ldots, P_g) = 0\) if and only if the points \(P_1, \ldots, P_g\) are linearly dependent on the image of \(X\) under the canonical map \(X \to \mathbb{P}(H^0(X, \Omega_X^1))^g\).

The following theorem is Corollary 2.6 in [Gu1].

**Theorem 1.4.12.** (Guàrdia [Gu1]) Let \(P_1, \ldots, P_g, Q\) be generic points on \(X\). Then the formula
\[
\|\theta\|(P_1 + \cdots + P_g - Q)^{g-1} = \exp(\delta(X)/8) \cdot \|J\|(P_1, \ldots, P_g) \cdot \prod_{k=1}^g G(P_k, Q)^{g-1} / \prod_{k \neq l} G(P_k, P_l)
\]
holds.

**Proof.** If \(P\) is a point on \(X\) and \(t\) is a local coordinate about \(P\), then by definition of the Arakelov metric on \(\Omega_X^1\) we have
\[
\lim_{Q \to P} \|\theta\|(D + P - Q)/|t(P) - t(Q)| = (\det \text{Im} \tau)^{1/4} \cdot \exp(-\pi \cdot (\text{Im} \tau)^{-1} \cdot y) \cdot \sum_{k=1}^g \frac{\partial \theta}{\partial z_k}(w) \cdot \eta_k(P),
\]
by the formula in Proposition 1.4.4. Here \(y = \text{Im} w\) and \(w \in \mathbb{C}^g\) lifts a class that corresponds to \(D\) in \(\text{Pic}_{g-1}(X)\). Let us assume that \(\{\omega_1, \ldots, \omega_g\}\) was an orthonormal basis of \(H^0(X, \Omega_X^1)\). We are going to apply the above to the equation
\[
\|\theta\|(P_1 + \cdots + P_g - Q) = \exp(\delta(X)/8) \cdot \frac{\|\det \omega_k(P_i)\|_{\text{Ar}}}{\prod_{k \neq l} G(P_k, P_l)} \cdot \prod_{k=1}^g G(P_k, Q)
\]
which is Faltings' fundamental formula from Theorem 1.4.9. Let \(t_1, \ldots, t_g\) be local coordinates about the points \(P_1, \ldots, P_g\), and let \(w_k\) for each \(k = 1, \ldots, g\) correspond to the divisor \(\sum_{i \neq k} P_i\). Dividing through \(|t_k(P_k) - t_k(Q)|\) and taking the limit \(Q \to P_k\) we obtain
\[
(\det \text{Im} \tau)^{1/4} \cdot \exp(-\pi \cdot y_k \cdot (\text{Im} \tau)^{-1} \cdot y_k) \cdot \sum_{i=1}^g \frac{\partial \bar{\theta}}{\partial z_l}(w_k) \cdot \eta_l(P_k)
\]

\[
= \exp(\delta(X)/8) \frac{\|\det \omega_k(P_i)\|_{\text{Ar}}}{\prod_{k \neq l} G(P_k, P_l)} \cdot \prod_{i \neq k} G(P_k, P_l) \cdot \frac{1}{\|dt_k\|_{\text{Ar}}(P_k)}.
\]
Multiplying over $k = 1, \ldots, g$ we obtain

$$(\det \text{Im} r)^{g/4} \cdot \exp(-\pi \sum_{k=1}^{g} y_k \cdot (\text{Im} r)^{-1} \cdot y_k) \cdot \prod_{k=1}^{g} \prod_{l=1}^{g} \frac{\partial \eta_l}{\partial z_k} (w_k) \cdot \eta_l(P_k)$$

$$= \exp(-g\delta(X)/8) \cdot \left( \frac{\| \det \omega_k(P_l) \|_{\Lambda^2}}{\prod_{k<l} G(P_k, P_l)} \right)^g \cdot \prod_{k<l} G(P_k, P_l)^2 \cdot \prod_{k=1}^{g} \frac{1}{\| dt_k \|_{\Lambda^2}(P_k)}.$$

Riemann's singularity theorem (see [GH], pp. 341–342) says that for any effective divisor $D$ on $X$, the projectivised tangent space $\mathbb{P}T_{g,D}$ at the class of $D$ in $0 \subset \text{Pic}_{g-1}(X)$ contains the image of the divisor $D$ on $X$ under the canonical map $X \to \mathbb{P}T_{\text{Pic}_{g-1}(X), D} \cong \mathbb{P}(H^0(X, \Omega_X^1)^\vee)$. For us this means that $\sum_{m=1}^{g} \frac{\partial \eta_l}{\partial z_m} (w_k) \cdot \eta_m(P_l) = 0$ whenever $k \neq l$. As a consequence, we can write

$$\prod_{k=1}^{g} \prod_{l=1}^{g} \frac{\partial \eta_l}{\partial z_k} (w_k) \cdot \eta_l(P_k) = J(w_1, \ldots, w_g) \cdot \det \eta_k (P_k).$$

Plugging this in we obtain

$$(\det \text{Im} r)^{-1/2} \cdot \| J \| (P_1, \ldots, P_g) \cdot \| \det \eta_k (P_k) \|$$

$$= \exp(-g\delta(X)/8) \cdot \left( \frac{\| \det \omega_k(P_l) \|_{\Lambda^2}}{\prod_{k<l} G(P_k, P_l)} \right)^g \cdot \prod_{k<l} G(P_k, P_l)^2 \cdot \| \det \omega_k(P_l) \|_{\Lambda^2}.$$

It follows from Lemma 1.4.10 that $\| \det \eta_k (P_k) \| = (\det \text{Im} r)^{1/2} \| \det \omega_k (P_k) \|$. Hence we arrive at

$$\| J \| (P_1, \ldots, P_g) = \exp(-g\delta(X)/8) \cdot \left( \frac{\| \det \omega_k(P_l) \|_{\Lambda^2}}{\prod_{k<l} G(P_k, P_l)} \right)^{g-1}.$$

The required formula is obtained by eliminating the factor $\| \det \omega_k(P_l) \|_{\Lambda^2}$ using Faltings' fundamental formula again. \qed

It is not so clear either from Theorem 1.4.6 or from the formulas of Faltings and Guàrdia derived above, how one can compute the delta-invariant for a given Riemann surface $X$. In fact, in the introduction to his paper [Fa2], Faltings says that he cannot give an explicit formula for it, except in the case of elliptic curves. However, as will become apparent in Section 1.6, the delta-invariant plays a very fundamental role in the function theory of the moduli space of curves, and therefore it deserves to be studied further. In Chapter 2 we will answer Faltings' question by giving a simple closed formula for the delta-invariant which holds in arbitrary genus.

### 1.5 Semi-stability

In this section and the next we formulate results that hold only in general for semi-stable arithmetic surfaces. We start by recalling the definition of a semi-stable curve.

**Definition 1.5.1.** Let $B$ be a locally Noetherian scheme. A proper flat curve $p : X \to B$ is called semi-stable if all geometric fibers of $p$ are reduced, connected and have only ordinary double points as singularities, the arithmetic genus of the fibers is positive, and each non-singular rational component of a geometric fiber meets the other components in at least 2 points. For a semi-stable curve $p : X \to B$ and a closed point $b \in B$ we denote by $\delta_b$ the number of singular points in the fiber at $b$. If $p : X \to B$ is a semi-stable arithmetic surface, we denote by $\Delta_{X/B}$ the divisor $\sum_b \delta_b \cdot b$ on $B$, where $b$ runs through the closed points of $B$. 

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We will need the following result in Section 2.5. The proof uses the celebrated Hodge index theorem for arithmetic surfaces (cf. [Fa2], §5). This is well-documented and we will not discuss this further.

**Proposition 1.5.2.** (Faltings [Fa2]) Let $p : X \to B$ be a semi-stable arithmetic surface of genus $g > 0$, and let $D$ be an effective Arakelov divisor on $X$. Then

(i) $(\omega_{X/B}, \omega_{X/B}) \geq 0$.

(ii) $4g(g - 1) \cdot (\omega_{X/B}, D) \geq (\omega_{X/B}, \omega_{X/B}) \cdot \deg D$.

**Proof.** This is Theorem 5 in [Fa2].

We next consider the properties of semi-stable arithmetic surfaces with respect to base change.

**Proposition 1.5.3.** Let $q : B' \to B$ be a finite morphism with $B'$ the spectrum of the ring of integers in a finite extension $L$ of the quotient field $K$ of $R$. Let $X' = X \times_B B'$ be the minimal desingularisation of $X \times_B B'$, and let $r : X' \to X$ be the induced morphism.

(i) The arithmetic surface $X' \to B'$ is again semi-stable.

(ii) We have an equality of divisors $\Delta_{X'/B'} = q^* \Delta_{X/B}$ on $B'$.

(iii) There exists a canonical isomorphism $r^* \omega_{X/B} \cong \omega_{X'/B'}$ on $X'$.

(iv) There exists a canonical isomorphism $r^* \det p_\ast \omega_{X'/B'} \cong q^* \det p_\ast \omega_{X/B}$ on $B'$.

**Proof.** As to (i) and (ii), these follow from the fact (cf. [La], Theorem V.5.1) that a double point in the fiber of $X \times_B B'$ at a closed point $b'$ is resolved by a chain of $e - 1$ irreducible components isomorphic to $\mathbb{P}^1$ and having geometric self-intersection $-2$. Here $e$ is the ramification index of $q : B' \to B$ at $b'$. Statement (iii) is in [La], Proposition V.5.5. Finally (iv) follows from (iii) and the defining properties of the determinant of cohomology.

**Proposition 1.5.3** makes it possible to define invariants of curves defined over a number field.

**Theorem 1.5.4.** (Stable reduction theorem, Grothendieck, Deligne-Mumford et al. [DM]) For any geometrically connected, non-singular proper curve $C$ of positive genus over a number field $K$ there exists a finite extension $L$ of $K$ and a semi-stable arithmetic surface $p : X \to B$ over the ring of integers of $L$ such that the generic fiber of $p$ is isomorphic to $X \otimes_K L$.

We note that a semi-stable arithmetic surface is a minimal model of its generic fiber.

**Proposition 1.5.5.** Let $C/K$ be a curve of positive genus, and let $L$ be a finite extension of $K$ over which $C$ acquires semi-stable reduction. Let $p : X \to B$ be a semi-stable arithmetic surface over the ring of integers of $L$. Then the quantities $\deg \det p_\ast \omega_{X/B}/[L : Q]$ and $(\omega_{X/B}, \omega_{X/B})/|L : Q|$ do not depend on the choice of $L$, hence they define invariants of $C$.

**Proof.** This follows from Propositions 1.5.3, 1.3.7 and 1.3.8.

**Definition 1.5.6.** We denote by $h_F(C)$ the quantity $\deg \det p_\ast \omega_{X/B}/[L : Q]$ from the above proposition. It is often referred to as the Faltings height of $C$. We denote by $e(C)$ the quantity $(\omega_{X/B}, \omega_{X/B})/[L : Q]$ from the above proposition.

In [Fa1] it is proved that for a fixed number field $K$, the set of isomorphism classes of $C/K$ of fixed positive genus and of bounded Faltings height, is finite.
1.6 Noether's formula

In this section we demonstrate the importance of the delta-invariant by showing that it can be seen as the norm of the so-called Mumford-isomorphism on the moduli space of curves (Theorem 1.6.1). This fundamental isomorphism was first obtained in [Mu1] by an application of the Grothendieck-Riemann-Roch theorem. In [Fa2] and [Mo2] we find an explicit construction of this isomorphism. We briefly discuss this construction in the proof of Theorem 1.6.1, leaving it to the reader to check the details in the aforementioned papers. As a consequence of the calculation of the norm of the Mumford-isomorphism we obtain the celebrated Noether formula in Arakelov theory (Corollary 1.6.3). Throughout this section we will freely use the language of stacks as in [Fa2] and [Mo2].

Let $g > 0$ be an integer. Let $M_g$ be the moduli stack of smooth curves of genus $g$, and let $p : U_g \to M_g$ be the universal curve. For line bundles on $U_g$ we have as in Section 1.3 the notion of Deligne bracket and determinant of cohomology on $M_g$. In particular, if $\omega$ is the relative dualising sheaf of $p : U_g \to M_g$, then we have the line bundles $\det p_* \omega$ and $(\omega, \omega)$ on $M_g$.

**Theorem 1.6.1.** (Mumford [Mu1], Faltings [Fa2], Moret-Bailly [Mo2]) There exists an isomorphism

$$\mu : (\det p_* \omega)^{\otimes 2} \cong (\omega, \omega)$$

of line bundles on $M_g$. This isomorphism is unique up to a sign. Its norm on $M_g(\mathbb{C})$ is equal to $(2\pi)^{-4g} \exp(\delta)$.

**Sketch of the proof.** In order to prove existence it suffices, roughly speaking, to construct for each smooth proper curve $p : \mathcal{X} \to B$ of genus $g$ an isomorphism $((\det p_* \omega_{\mathcal{X}/B})^{\otimes 2} \cong (\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B})$ which is compatible with base change. We will sketch such a construction only for $p : \mathcal{X} \to B$ which come equipped with a theta-characteristic $L$, i.e. a line bundle with an isomorphism $L^{\otimes 2} \cong \omega_{\mathcal{X}/B}$.

The general case requires a more subtle argument. Using $L$, we make $J := \text{Pic}_{g-1}(\mathcal{X}/B)$ into an abelian scheme over $B$. For this we refer to [Mo1], Section 2. Let $\epsilon : B \to J$ be its zero-section, and let $\Omega^1_{J/B}$ be the sheaf of relative 1-forms. In the case $B = \text{Spec}(\mathbb{C})$, the global sections $H^0(J, \Omega^1_J)$ come equipped with a hermitian inner product $(\alpha, \beta) \mapsto (i/2)^g (-1)^{(g-1)/2} \int_J \alpha \wedge \overline{\beta}$. The next four steps give then the required isomorphism. (i) Let $\Theta$ be the theta divisor of $\text{Pic}_{g-1}(\mathcal{X}/B)$, see once more [Mo1], Section 2. Then there is a canonical isomorphism $\epsilon^*(\Omega^1_{J/B}) \cong \epsilon^* (O(\Theta))^{\otimes 2}$, compatible with base change. This is Moret-Bailly's *formule clé*, see [Mo2] and [Mo3]. (ii) Let $j : \mathcal{X} \to J$ be the usual embedding, unique up to translation, which exists locally for the étale topology on $B$. Then there is a canonical isomorphism $\epsilon^*(\Omega^1_{J/B}) \cong \det p_* \omega_{\mathcal{X}/B}$, compatible with base change. (iii) There is a canonical isomorphism $\det R^*_p \omega \cong \epsilon^* (O(-\Theta))$, compatible with base change. This follows directly from Lemma 1.4.7. (iv) There are canonical isomorphisms

$$\det R^*_p (\omega^{\otimes 2}) \cong (\omega, \omega) \otimes \det p_* \omega \quad \text{and} \quad (\det R^*_p \omega)^{\otimes 8} \cong (\omega, \omega)^{\otimes -1} \otimes (\det p_* \omega)^{\otimes 8},$$

compatible with base change. These isomorphisms follow from the Riemann-Roch theorem for $p : \mathcal{X} \to B$, discussed in Section 1.3. The uniqueness up to sign of the isomorphism $\mu$ follows from the fact (see [Mo2], Lemma 2.2.3) that $H^0(M_g, \mathbb{G}_m) = \{ \pm 1, -1 \}$. The statement on the norm of $\mu$ follows from the fact that the isomorphism in (i) has norm $(2\pi)^{-4g}$ (this is the main result of [Mo3]), the isomorphism in (iii) has norm $\exp(\delta/8)$ by definition of the delta-invariant, and the other isomorphisms are isometries.

As was shown in [DM], for any $g \geq 1$ we have a moduli stack $\overline{M}_g$ classifying stable curves of genus $g$. It contains the moduli stack $M_g$ of smooth proper curves of genus $g$ as an open substack. It is customary to denote by $\Delta$ the closed subset $\overline{M}_g - M_g$, provided with its reduced structure; this
is a normal crossings divisor in \( \overline{M}_g \) (see [DM]). The divisor \( \Delta \) is the union of different components

\[
\Delta = \Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_r, \quad r = \lfloor g/2 \rfloor,
\]

where \( \Delta_0 \) denotes the closure of the locus corresponding to irreducible curves with a single node, and where \( \Delta_k \) for \( k > 0 \) denotes the closure of the locus corresponding to reducible curves with components of genus \( k \) and genus \( g - k \). Mumford [Mu1] has shown that the isomorphism \( \mu \) extends over \( \overline{M}_g \).

**Theorem 1.6.2.** (Mumford [Mu1]) There exists an isomorphism

\[
\mu : (\det p_*(\omega))^\otimes 12 \rightarrow (\omega \otimes O_{\overline{M}_g}(\Delta))
\]

of line bundles on \( \overline{M}_g \). This isomorphism is unique up to sign.

By considering the Mumford-isomorphism on the base of a semi-stable arithmetic surface \( p : X \rightarrow B \) and taking degrees on left and right we obtain the arithmetic Noether formula.

**Corollary 1.6.3.** (Noether's formula, Faltings [Fa2], Moret-Bailly [Mo2]) Let \( p : X \rightarrow B \) be a semi-stable arithmetic surface of genus \( g > 0 \), with \( B \) the spectrum of the ring of integers in a number field \( K \). Then the formula

\[
12 \deg \det p_*(\omega_{X/B}) = (\omega_{X/B}, \omega_{X/B}) + \sum_b \delta_b \log \#k(b) + \sum_{\sigma : K \rightarrow \mathbb{C}} \delta(X_{\sigma}) - 4g[K : \mathbb{Q}] \log(2\pi)
\]

holds. Here \( b \) runs through the closed points in \( B \), and \( \sigma \) runs through the complex embeddings of \( K \).

A detailed investigation as in [Jo] and [We] shows that when viewed as a function on the moduli space \( M_g(\mathbb{C}) \), the delta-invariant acquires logarithmic singularities along the components of the boundary divisor \( \Delta \). We will come back to this in Section 2.4. As was remarked by Faltings in his introduction to [Fa2], the delta-invariant can be viewed as the minus logarithm of a "distance" to \( \Delta \). This interpretation is supported by the Noether formula.