Explicit Arakelov geometry

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Chapter 2

Analytic invariants

The purpose of this chapter is to give explicit formulas for the Arakelov-Green function and the delta-invariant, introduced in Chapter 1. In order to do this, we introduce two new invariants $S$ and $T$ of Riemann surfaces. These invariants are reasonably explicit and can be efficiently calculated. In Section 2.1 we state our results. After giving the proofs in Section 2.2, we specialise to the case of elliptic curves in Section 2.3. In particular we obtain Faltings’ formula for the delta-invariant for elliptic curves, given in [Fa2]. The asymptotic behavior of the invariants $S$ and $T$ is considered in Section 2.4. In Section 2.5 we give some applications of our formulas in intersection theory. Among other things we prove a lower bound for the self-intersection of the relative dualising sheaf. Finally we comment upon the use of Arakelov geometry in a recent bound for the complexity of an algorithm, due to Edixhoven, for computing certain Galois representations.

2.1 Results

Let $X$ be a compact and connected Riemann surface of genus $g > 0$. Our first result deals with the Arakelov-Green function $G$ of $X$. Let $P$ be a generic point on $X$. By the remarks after the proof of Lemma 1.4.8, there is a constant $c = c(P)$ depending only on $P$ such that for all $Q \in X$ we have $G(P, Q)^g = c(P) \cdot \| \vartheta \| (gP - Q)$. This has been observed by some authors before, see for instance the remarks in [Jo], p. 229. Our contribution is that we make the dependence on $P$ of the constant $c(P)$ clear. Our result involves the divisor $W$ of Weierstrass points on $X$. This is a divisor of degree $g^2 - g$ on $X$, given as the divisor of a Wronskian differential formed out of a basis of the holomorphic differentials $H^0(X, \Omega^1_X)$. For each point $P \in X$, the multiplicity of $P$ in $W$ is given by a weight $w(P)$, which can also be calculated by means of the classical gap sequence at $P$ (see Remark 2.2.9).

Definition 2.1.1. We define the invariant $S(X)$ of $X$ by means of the formula

$$\log S(X) := - \int_X \log \| \vartheta \| (gP - Q) \cdot \mu(P),$$

where $Q$ can be any point in $X$.

We will see later (Proposition 2.2.3) that the integrand has logarithmic singularities only at the Weierstrass points of $X$, which are integrable. Hence the integral is well-defined. That the definition does not depend on the choice of $Q$ follows from the translation-invariance of the form $\nu$ on $\text{Pic}_{g-1}(X)$.

The invariant $S(X)$ appears as a normalisation constant in the formula that we propose for the Arakelov-Green function.
Theorem 2.1.2. Let \( P, Q \in X \) with \( P \) not a Weierstrass point. Then the formula
\[
G(P, Q)^g = S(X)^{1/g^2} \cdot \frac{\|\theta\| (gP - Q)}{\prod_{W \in W} \|\theta\| (gP - W)^{1/g^2}}
\]
holds. Here the Weierstrass points are counted with their weights.

For \( P \) a Weierstrass point, and \( Q \neq P \), both numerator and denominator in the formula of Theorem 2.1.2 vanish with order \( w(P) \), the weight of \( P \). The formula remains true also in this case, provided that we take the leading coefficients of the appropriate power series expansions about \( P \) in both numerator and denominator. Note that apart from the normalisation term involving \( S(X) \), the Arakelov-Green function can be expressed in terms of certain values of the \( \|\theta\| \)-function. These values are very easy to calculate numerically. The (real) 2-dimensional integral involved in computing \( S(X) \) is harder to carry out in general, but it is still not difficult.

Other ways of expressing the Arakelov-Green function in terms of quantities associated to \( X \) and \( \mu \) have been given, for instance one might use the eigenvalues and eigenfunctions of the Laplacian (see [Fa2], Section 3), or one might use abelian differentials of the second and third kind (see [La], Chapter II). There is also a closed formula due to Bost [Bo]
\[
\log G(P, Q) = \frac{1}{g^2} \int_{\Phi + P - Q} \log \|\theta\| \cdot \nu^{g-1} + A(X),
\]
expressing the Arakelov-Green function in terms of an integral over the translated theta divisor. Here \( \nu \) is the canonical translation-invariant \((1,1)\)-form on \( \text{Pic}_{g-1}(X) \), and the quantity \( A(X) \) is a certain normalisation constant, perhaps comparable to our \( S(X) \).

One of our motives for finding a new explicit formula was the need to have a formula that makes the efficient calculation of the Arakelov-Green function possible. The other approaches that we mentioned are perhaps less suitable for this objective. For instance, the formula given by Bost involves a (real) \( 2g - 2 \)-dimensional integral over a region which seems not easy to parametrise. Also, for each new pair of points \( (P, Q) \) one has to calculate such an integral again, whereas in our approach one only has to calculate a certain integral once.

Our second result deals with Faltings' delta-invariant \( \delta(X) \). Let \( \Phi : X \times X \to \text{Pic}_{g-1}(X) \) be the map sending \( (P, Q) \) to the class of \( (gP - Q) \). For a fixed \( Q \in X \), let \( i_Q : X \to X \times X \) be the map sending \( P \) to \( (P, Q) \), and put \( \phi_{-Q} = \Phi \cdot i_Q \). This coincides with the definition of \( \phi_D \) in Proposition 1.4.5 for divisors\( D \), where we take \( D = -Q \). Define the line bundle \( L_X \) by
\[
L_X := \left( \bigotimes_{W \in W} \phi^*_W (O(\Theta)) \right)^{(g-1)/g^2} \otimes O_X \otimes O_X^{\Omega^g X} \otimes (\Omega^g X)^{(g+1)/g^2} \otimes O_X 
\]
We have then the following theorem.

Theorem 2.1.3. The line bundle \( L_X \) is canonically trivial. Let \( T(X) \) be the norm of the canonical trivialising section of \( L_X \). Then the formula
\[
\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)
\]
holds.

Despite appearances to the contrary, the invariant \( T(X) \) admits a very concrete description, see Propositions 2.2.7 and 2.2.8. In fact, the computation of \( T(X) \) involves only elementary operations on special values of the functions \( \|\theta\| \) and Guàrdia's \( \|J\| \). Thus, we have now a very simple closed
formula for the delta-invariant, reducing its calculation to the calculation of the invariants $S(X)$ and $T(X)$, the former involving a (real) 2-dimensional integral, and the latter being elementary to calculate. We shall demonstrate the practical significance of our formulas for calculating Arakelov invariants in Chapter 6.

It seems an important problem to relate the invariants $S(X)$ and $T(X)$ to more classical invariants. In Chapters 3 and 4 we prove a result that does this for $T(X)$ with $X$ a hyperelliptic Riemann surface. This is already quite involved.

Next, it seems worthwhile to study whether our invariants $S(X)$ and $T(X)$ give rise to proper, strongly $(g-2)$-pseudoconvex functions on $\mathcal{M}_g(\mathbb{C})$. This notion arises in the context of Morse theory on manifolds. The importance of finding such functions is stressed by Hain and Looijenga (private communication); indeed, if such functions would be seen to exist, numerous interesting results (both known and still conjectural) on the geometry of $\mathcal{M}_g(\mathbb{C})$ would be implied. Perhaps the explicit nature of our invariants opens a way to constructing such functions.

Our inspiration to study Weierstrass points in order to obtain results in Arakelov theory stems from the papers [Ar1], [Bu] and [Jo]. Especially the latter paper has been useful. For example, our formula for the delta-invariant in Theorem 2.1.3 is closely related to the formula from Theorem 2.6 of that paper. Our improvement on that formula is perhaps that we give an explicit splitting of the delta-invariant in a new invariant $S(X)$ involving an integral, and a new invariant $T(X)$ which is purely "classical". These invariants seem to be of interest in their own right.

### 2.2 Proofs

In this section we prove Theorems 2.1.2 and 2.1.3. The major idea will be to give Arakelov-theoretic versions of classical results on Weierstrass points.

First we recall the Wronskian differential that defines the divisor of Weierstrass points on $X$. An alternative approach is sketched in Remark 2.2.9 below. Let $\{\psi_1, \ldots, \psi_g\}$ be a basis of $H^0(X, \Omega^1_X)$. Let $P$ be a point on $X$ and let $z$ be a local coordinate about $P$. Write $\psi_k = f_k \cdot dz$ for $k = 1, \ldots, g$. The Wronskian determinant about $P$ is then the holomorphic function

$$W_z(\psi) := \det \left( \frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \leq k, l \leq g}.$$

Let $\tilde{\psi}$ be the $g(g+1)/2$-fold holomorphic differential

$$\tilde{\psi} := W_z(\psi) \cdot (dz)^{g(g+1)/2}.$$

Then $\tilde{\psi}$ is independent of the choice of the local coordinate $z$, and extends to a non-zero global section of $\Omega^g(\mathbb{P}^{g+1})$. A change of basis changes the Wronskian differential by a non-zero scalar factor, so that the divisor of a Wronskian differential $\tilde{\psi}$ on $X$ is unique: we denote this divisor by $\mathcal{W}$, the divisor of Weierstrass points.

The Wronskian differential leads to a canonical sheaf morphism

$$\left( \wedge^g H^0(X, \Omega^1_X) \otimes \mathcal{O}_X \right) \longrightarrow \Omega^{g(g+1)/2}_X$$

given by

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\frac{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi}}.$$

This gives a canonical section in $\Omega^{g(g+1)/2}_X \otimes \mathcal{O}_X \left( \wedge^g H^0(X, \Omega^1_X) \otimes \mathcal{O}_X \right)^{\vee}$ whose divisor is $\mathcal{W}$. 

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Proposition 2.2.1. The canonical isomorphism
\[ \Omega^{g(g+1)/2}_X \otimes_{\mathcal{O}_X} \left( \wedge^g H^0(X, \Omega_X^1) \otimes_{\mathcal{O}_X} \mathcal{O}_X \right)^\vee \overset{\sim}{\longrightarrow} \mathcal{O}_X(W) \]
has a constant norm on \( X \).

Proof. This follows since both sides have the same curvature form, and the divisors of the canonical sections are equal. \( \square \)

Definition 2.2.2. We shall denote by \( R(X) \) the norm of the isomorphism from Proposition 2.2.1. In more concrete terms we have \( \prod_{W \in W} G(P, W) = R(X) \cdot ||\omega||_{\mathcal{A}_R}(P) \) for any \( P \in X \), where \( \{\omega_1, \ldots, \omega_g\} \) is an orthonormal basis of \( H^0(X, \Omega_X^1) \), and where the norm of the Wronskian differential \( \omega \) is taken in the line bundle \( \Omega^{g(g+1)/2}_X \) with its canonical metric induced from the canonical metric on \( \Omega_X^1 \).

Taking logarithms and integrating against \( \mu(P) \) gives, by property (iii) of the Arakelov-Green function, the formula \( \log R(X) = -\int_X \log ||\omega||_{\mathcal{A}_R}(P) \cdot \mu(P) \).

Recall from Section 2.1 the map \( \Phi : X \times X \to \text{Pic}_{g-1}(X) \) sending \( (P, Q) \) to the class of \( (gP - Q) \). A classical result on the divisor of Weierstrass points is that the equality of divisors
\[ \Phi^*(\Theta) = W \times X + g \cdot \Delta_X \]
holds on \( X \times X \), see for example [Fay], p. 31. Denote by \( p_1 : X \times X \to X \) the projection on the first factor. Using Proposition 2.2.1, the above equality of divisors yields a canonical isomorphism of line bundles
\[ \Phi^*(O(\Theta)) \overset{\sim}{\longrightarrow} p_1^* \left( \Omega^{g(g+1)/2}_X \otimes \left( \wedge^g H^0(X, \Omega_X^1) \otimes_{\mathcal{O}_X} \mathcal{O}_X \right)^\vee \right) \otimes \mathcal{O}_{X \times X}(\Delta_X)^{\otimes g} \]
on \( X \times X \). We will prove this isomorphism in the next proposition, and show that its norm is constant on \( X \times X \). After Corollary 2.2.5 to this proposition, the proofs of Theorems 2.1.2 and 2.1.3 are just a few lines.

Proposition 2.2.3. On \( X \times X \), there exists a canonical isomorphism of line bundles
\[ \Phi^*(O(\Theta)) \overset{\sim}{\longrightarrow} p_1^* \left( \Omega^{g(g+1)/2}_X \otimes \left( \wedge^g H^0(X, \Omega_X^1) \otimes_{\mathcal{O}_X} \mathcal{O}_X \right)^\vee \right) \otimes \mathcal{O}_{X \times X}(\Delta_X)^{\otimes g} \]

The norm of this isomorphism is everywhere equal to \( \exp(\delta(X)/8) \).

Proof. We are done if we can prove that
\[ \exp(\delta(X)/8) \cdot ||\omega||((gP - Q)) = ||\omega||_{\mathcal{A}_R}(P) \cdot G(P, Q)^g \]
for all \( P, Q \in X \), where \( \{\omega_1, \ldots, \omega_g\} \) is an orthonormal basis of \( H^0(X, \Omega_X^1) \). But this follows from the formula in Theorem 1.4.9, by a computation which is performed in [Jo], p. 233. Let \( P \) be a point on \( X \), and choose a local coordinate \( z \) about \( P \). By definition of the canonical metric on \( \Omega_X^1 \) we have then that \( \lim_{P \to P} ||z(P) - z(Q)||/G(P, Q) = ||dz||_{\mathcal{A}_R}(P) \). Letting \( P_1, \ldots, P_g \) approach \( P \) in Theorem 1.4.9 we get
\[
\lim_{P \to P} \frac{||\det \omega_k(P)||_{\mathcal{A}_R}}{G(P_k, P)} = \lim_{P \to P} \left\{ \frac{||\det \omega_k(P)||_{\mathcal{A}_R}}{G(P_k, P)} \cdot \frac{||z(P_k) - z(P)||}{G(P_k, P)} \right\} = \left\{ \frac{||\det \omega_k(P)||_{\mathcal{A}_R}}{G(P_k, P)} \cdot ||dz||_{\mathcal{A}_R}^{g(g+1)/2}(P) \right\} = ||\omega||_{\mathcal{A}_R}(P).
\]
The required formula is therefore just a limiting case of Theorem 1.4.9 where all $P_k$ approach $P$. □

**Corollary 2.2.4.** The formula $S(X) = R(X) \cdot \exp(\delta(X)/8)$ holds.

**Proof.** This follows easily by taking logarithms in the formula

$$\exp(\delta(X)/8) \cdot \|\vartheta\|(gP - Q) = \|\tilde{\omega}\|_{\Lambda^*} (P) \cdot G(P, Q)^g$$

and integrating against $\mu(P)$. Here we use again property (iii) of the Arakelov-Green function and the formula $\log R(X) = -\int_X \log \|\tilde{\omega}\|_{\Lambda^*} (P) \cdot \mu(P)$, which was noted above. □

**Corollary 2.2.5.** (i) Let $Q \in X$. Then we have a canonical isomorphism

$$\phi^*_Q(O(\Theta)) \sim O_X(W + g \cdot Q)$$

of constant norm $S(X)$ on $X$.

(ii) We have a canonical isomorphism

$$\Phi^*(O(\Theta)) \otimes_{O_X} \Omega^g \sim O_X(W)$$

of constant norm $S(X)$ on $X$.

**Proof.** We obtain the isomorphism in (i) by restricting the isomorphism from Proposition 2.2.3 to a slice $X \times \{Q\}$, and using Proposition 2.2.1. Its norm is then equal to $R(X) \cdot \exp(\delta(X)/8)$, which is $S(X)$ by Corollary 2.2.4. For the isomorphism in (ii) we restrict the isomorphism from Proposition 2.2.3 to the diagonal, and apply the canonical adjunction isomorphism $O_{XX}(-\Delta X) \sim \Omega^g_X$. Again we get norm $R(X) \cdot \exp(\delta(X)/8)$, since the adjunction isomorphism is an isometry. □

Note that Corollary 2.2.5 gives an alternative interpretation to the invariant $S(X)$.

**Proof of Theorem 2.1.2.** By taking norms of canonical sections on left and right in the isomorphism from Corollary 2.2.5 (i) we obtain

$$G(P, Q)^g \cdot \prod_{W \in W} G(P, W) = S(X) \cdot \|\vartheta\|(gP - Q)$$

for any $P, Q \in X$. Now take the (weighted) product over $Q \in W$. This gives

$$\prod_{W \in W} G(P, W)^g = S(X)^{g^2} \cdot \prod_{W \in W} \|\vartheta\|(gP - W).$$

Plugging this in in the first formula gives

$$G(P, Q)^g \cdot S(X)^{g^2} \cdot \prod_{W \in W} \|\vartheta\|(gP - W)^{1/s^3} = S(X) \cdot \|\vartheta\|(gP - Q),$$

from which the theorem follows. □

**Proof of Theorem 2.1.3.** From Corollary 2.2.5 (i) we obtain, again by taking the (weighted) product over $Q \in W$, a canonical isomorphism

$$\left( \bigotimes_{W \in W} \phi^*_W O(\Theta) \right) \sim O_X(g^3 \cdot W)$$
of norm $S(X)^{g^2-g}$. It follows that we have a canonical isomorphism

$$\left( \bigotimes_{w \in W} \phi^*_w O(\Theta) \right)^{\otimes(g-1)/g^3} \simeq O_X((g-1) \cdot W)$$

of norm $S(X)^{(g-1)(g^2-g)/g^3}$. From Corollary 2.2.5 (ii) we obtain a canonical isomorphism

$$((\Phi^*(O(\Theta))|_{\Delta_X}) \otimes_{O_X} \Omega^g_X)^{\otimes-(g+1)} \simeq O_X(-(g+1)W)$$

of norm $S(X)^{-g+1}$. Finally from Proposition 2.2.1 and Corollary 2.2.4 we have a canonical isomorphism

$$\left( \Omega^g_X(\otimes_{W \in W} \phi^*_w O(\Theta)) \right)^{\otimes 2} \simeq O_X(2W)$$

of norm $S(X)^2 \exp(-\delta(X)/4)$. It follows that indeed the line bundle $L_X$ is canonically trivial, and that its canonical trivialising section has norm

$$S(X)^{-(g-1)(g^2-g)/g^3} \cdot S(X)^{g+1} \cdot S(X)^{-2} \cdot \exp(\delta(X)/4) = S(X)^{(g-1)/g^3} \cdot \exp(\delta(X)/4).$$

By definition this is $T(X)$, so the theorem follows.

Theorem 2.1.2 leads to an alternative formula for $S(X)$.

Proposition 2.2.6. Let $P$ be a point on $X$, not a Weierstrass point. Then the formula

$$\log S(X) = -g^2 \cdot \int_X \log \|\Theta\|(gP-Q) \cdot \mu(Q) + \frac{1}{g} \sum_{W \in W} \log \|\Theta\|(gP-W)$$

holds. Here the sum is over the Weierstrass points of $X$, counted with their weights.

Proof. Take logarithms in Theorem 2.1.2 and integrate against $\mu(Q)$.

It remains for us to give an explicit formula for the invariant $T(X)$. Let $P \in X$ not a Weierstrass point and let $z$ be a local coordinate about $P$. Define $F_z(P)$ as

$$\|F_z(P)\| := \lim_{Q \to P} \frac{\|\Theta\|(gP-Q)}{|z(P) - z(Q)|^g}.$$

Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega^1_X)$.

Proposition 2.2.7. The formula

$$T(X) = \|F_z\|^{-(g+1)} \cdot \prod_{W \in W} \|\Theta\||(gP-W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2$$

holds.

Proof. Let $F$ be the canonical section of $(\Phi^*(O(\Theta))|_{\Delta_X}) \otimes \Omega^g_X$ given by the canonical isomorphism in Corollary 2.2.5 (ii). For its norm we have $\|F\| = \|F_z\| \cdot \|dz\|^g_{Ar}$ in the local coordinate $z$. The canonical section of $\bigotimes_{W \in W} \phi^*_w O(\Theta)$ has norm $\prod_{W \in W} \|\Theta\||(gP-W)$ at $P$. Finally, the canonical section of $\Omega^g_X(\otimes_{W \in W} \phi^*_w O(\Theta)) \otimes_{O_X} (\wedge^g H^0(X, \Omega^1_X))^{\otimes 2}$ has norm $\|\omega\|_{Ar} = |W_z(\omega)| \cdot \|dz\|_{Ar}^{(g+1)/2}$.

We next give a formula for $T(X)$ in which only first order partial derivatives of the theta function occur.
Proposition 2.2.8. Let $P_1, \ldots, P_g, Q$ be generic points on $X$. Then the formula

\[
T(X) = \left( \prod_{k=1}^g \|\vartheta(S_{P_k} - Q)\|^{1/g} \right) \cdot \left( \prod_{k \neq l} \|\vartheta((gP_k - P_l)/\vartheta((P_1, \ldots, P_g)^2) \right) \cdot \prod_{w \in W} \prod_{k=1}^g \|\vartheta((gP_k - W)^{(g-1)/g})^{1/9} \cdot \prod_{w \in W} \prod_{k=1}^g \|\vartheta((gP_k - W)^{(g-1)/g})^{1/9}
\]

holds.

Proof. The formula follows from Theorem 1.4.12, using Theorem 2.1.2 to eliminate the occurring values of the Arakelov-Green function $G$, and using Theorem 2.1.3 to eliminate the factor $\exp(\delta(X)/8)$. The factors involving $S(X)$ that are introduced in this way cancel out. \qed

Remark 2.2.9. An alternative way to obtain the divisor of Weierstras points $W$ on $X$ is to use gap-sequences. Let $P \in X$ be a point.

Definition 2.2.10. The gap-sequence $\Gamma(P)$ at $P$ is the set

\[
\Gamma(P) = \{ a \geq 1 \mid \text{there is no meromorphic function } f \text{ with } (f)_\infty = a \cdot P \}
\]

= \{ a \geq 1 \mid \text{there exists a holomorphic 1-form with a zero of exact order } a - 1 \text{ at } P \}.

Here $(f)_\infty$ denotes the polar part of a meromorphic function $f$. The equality implied by the definition follows from the Riemann-Roch theorem.

The following facts are then not difficult to see:

(i) $\mathbb{N} \setminus \Gamma(P)$ is a semi-group;

(ii) $1 \in \Gamma(P)$;

(iii) For $a \in \Gamma(P)$ we have $a \leq 2g - 1$;

(iv) The set $\Gamma(P)$ has cardinality $g$.

Let $\Gamma(P) = \{ a_1, \ldots, a_g \}$ with $a_1 < \ldots < a_g$. We then define the weight of $P$ to be the deviation of the gap-sequence from the sequence $\{1, \ldots, g\}$:

Definition 2.2.11. The weight $w(P)$ of $P$ is the number $w(P) = \sum_{k=1}^g (a_k - k)$.

It follows that always $w(P) \leq g(g-1)/2$.

Definition 2.2.12. We call $P$ a Weierstrass point if $\Gamma(P)$ differs from $\{1, \ldots, g\}$. Equivalently, we call $P$ a Weierstrass point if $w(P) > 0$ or if $h^0(gP) > 1$.

In [Gun], pp. 123–125 we find a proof of the following proposition.

Proposition 2.2.13. Let $\psi = W_\varphi(\psi) \cdot (dz)^{g(g+1)/2}$ be a Wronskian differential in $H^0(X, \Omega_X^g(g+1)/2)$. Then we have an equality of divisors $\text{div } \psi = \sum_{P \in X} w(P) \cdot P$.

As an example, consider a hyperelliptic Riemann surface $X$ of genus $g \geq 2$. A hyperelliptic map $X \rightarrow \mathbb{P}^1$ has $2g + 2$ ramification points, and for each ramification point $P$, the gap-sequence $\Gamma(P)$ at $P$ equals $\Gamma(P) = \{1, 3, \ldots, 2g - 1\}$. Hence, each $P$ has weight $(g-1)/2$, and the ramification points are exactly the Weierstrass points of $X$. 

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2.3 Elliptic curves

In this section we make the invariants $S(X)$ and $T(X)$ explicit for a Riemann surface $X$ of genus 1. We can write $X = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$ where $\tau$ is an element in the complex upper half plane. It is determined up to a transformation with an element of $\text{SL}(2, \mathbb{Z})$. Since

$$\frac{i}{2} \int_X dz \wedge d\bar{z} = \text{Im} \tau,$$

the holomorphic differential $dz/\sqrt{\text{Im} \tau}$ is an orthonormal basis of $H^0(X, \Omega^1_X)$.

As usual we write $q = \exp(2\pi i \tau)$ and then we have the eta-function $\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k)$ and the modular discriminant $\Delta(\tau) = \eta(\tau)^{24} = q \prod_{k=1}^{\infty} (1 - q^k)^2$. The latter is a modular form on $\text{SL}(2, \mathbb{Z})$ of weight 12. We put $\|\eta\|(X) = (\text{Im} \tau)^{1/4} \|\eta(\tau)\|$ and $\|\Delta\|(X) = \|\eta\|(X)^{24} = (\text{Im} \tau)^6 \|\Delta(\tau)\|$. These definitions do not depend on the choice of $\tau$.

**Theorem 2.3.1.** The formula

$$S(X) = \frac{1}{\|\eta\|(X)}$$

holds.

As an immediate consequence we find a formula for the Arakelov-Green function, given already in [Fa2], Section 7.

**Corollary 2.3.2.** The formula

$$G(P, Q) = \frac{\|\theta\|(P - Q)}{\|\eta\|(X)}$$

holds.

**Proof.** Apply the previous result to the formula in Theorem 2.1.2. \(\square\)

**Proof of Theorem 2.3.1.** We follow an analogous computation in [La], Chapter II, §5. The fundamental (1,1)-form $\mu$ is given by $\mu = \frac{i}{2}(dz \wedge d\bar{z})/\text{Im} \tau$. We will perform our integrals over the fundamental domain $A$ for $X$ given by $z = \alpha \tau + \beta$ with $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$ and $\beta \in [0, 1]$. Write $y = \text{Im} z$. We find

$$\int_A -\pi y^2 \cdot (\text{Im} \tau)^{-1} \cdot \mu = \int_{\alpha=-\frac{1}{2}}^{\frac{1}{2}} \int_{\beta=0}^{1} -\pi \alpha^2 \cdot \text{Im} \tau \cdot d\alpha d\beta = -\frac{\pi}{12} \cdot \text{Im} \tau$$

and we shall prove

$$\int_A \log |\theta(z; \tau)| \cdot \mu(z) = \log \prod_{k=1}^{\infty} (1 - \exp(2\pi i k \tau)).$$

Together this gives

$$\log S(X) = -\int_X \log \|\theta\| \cdot \mu = -\log \|\eta\|(X)$$

as required. Let us prove the integral formula. We will make use of the product expansion (cf. [Mu2], p. 68)

$$\theta(z; \tau) = \prod_{k=1}^{\infty} (1 - \exp(2\pi i k \tau)) \cdot \prod_{k=0}^{\infty} \{(1 + \exp(\pi i (2k + 1) \tau - 2\pi i z)) (1 + \exp(\pi i (2k + 1) \tau + 2\pi i z))\}.$$
Fix an index $k \geq 0$. In order to compute
\[\int_A \log |(1 + \exp(\pi i(2k + 1)\tau - 2\pi iz))| \cdot \mu(z)\]
we observe the following: for $\alpha < 1/2$ we have
\[|\exp(\pi i(2k + 1)\tau - 2\pi i(\alpha + \beta))| < 1,
\]
and next for $w \in \mathbb{C}$ with $|w| < 1$ we have
\[-\log(1 - w) = \sum_{m=1}^{\infty} \frac{w^m}{m},\]
where the convergence is uniform on compact subsets. This gives
\[\int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \log |(1 + \exp(\pi i(2k + 1)\tau - 2\pi i(\alpha + \beta)))| d\alpha d\beta = \]
\[\int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \Re \left\{ \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \cdot \exp(\pi i(2k + 1)m\tau - 2m\pi i(\alpha + \beta)) \right\} d\alpha d\beta = \]
\[= \Re \sum_{m=1}^{\infty} \int_{\alpha=-1/2}^{1/2} \int_{\beta=0}^{1} \frac{(-1)^{m+1}}{m} \cdot \exp(\pi i(2k + 1)m\tau - 2m\pi i(\alpha + \beta)) d\alpha d\beta = 0,
\]
where the latter equality holds since for any $m$,
\[\int_{\beta=0}^{1} \exp(\pi i(2k + 1)m\tau - 2m\pi i(\alpha + \beta)) d\beta = 0
\]
as one sees directly. In a similar vein one proves that
\[\int_{\beta=0}^{1} \log |(1 + \exp(\pi i(2k + 1)\tau + 2\pi iz))| \cdot \mu(z) = 0
\]
for any fixed $k \geq 0$. Together this gives the required integral formula.

In Chapter 5, where we study the Arakelov theory of elliptic curves more closely, we give an alternative proof of Corollary 2.3.2. This proof relies on special properties of the Arakelov-Green function of $X$, which we discover later on.

Next we turn to the invariant $T(X)$.

**Theorem 2.3.3.** The formula
\[T(X) = (2\pi)^{-2} \cdot \|\Delta\|(X)^{-1/4}\]
holds.

Using Theorem 2.1.3 we find the following corollary, which is also in Section 7 of [Fa2].

**Corollary 2.3.4.** (Faltings [Fa2]) For Faltings' delta-invariant $\delta(X)$ of $X$, the formula
\[\delta(X) = -\log \|\Delta\|(X) - 8 \log(2\pi)\]

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holds.

Proof of Theorem 2.3.3. We make use of the explicit formula for $T(X)$ in Proposition 2.2.7. Take the euclidean coordinate $z$ as a local coordinate on $X = C/\mathbb{Z} + \tau \mathbb{Z}$, and choose $\omega = dz/\sqrt{\text{Im} \tau}$ as an orthonormal basis of $H^0(X, \Omega^1_X)$. Choose an arbitrary point $P \in X$. The Riemann vector is given by $\kappa = \frac{1 + \tau}{2}$. An explicit computation yields

$$T(X) = \|F_z\|^2 \cdot |W_z(\omega)(P)|^2 = (\text{Im} \tau)^{-3/2} \cdot \exp(\pi \cdot \text{Im} \tau / 2) \cdot \left| \frac{\partial^2}{\partial z^2} \left( \frac{1 + \tau}{2^2}; \tau \right) \right|^{-2}.$$  

The proposition follows by the formula

$$\left( \exp(\pi i \tau / 4) \cdot \frac{\partial}{\partial z} \left( \frac{1 + \tau}{2}; \tau \right) \right)^8 = (2\pi)^8 \cdot \Delta(\tau)$$

which is a consequence of Jacobi's derivative formula (cf. [Mu2], Chapter I, §13). \hfill \Box

We could circumvent the computation in the above proof and apply Proposition 2.2.8 directly. However, the idea of using the explicit formula from Proposition 2.2.7 will be applied again in the next chapter, where we compute $T(X)$ for hyperelliptic Riemann surfaces (see especially the proof of Theorem 3.1.2 in Section 3.7). In fact the above proof is a special case of the arguments developed in Chapter 3.

We will give a proof of Jacobi's derivative formula in Section 4.6, using Arakelov theory.

## 2.4 Asymptotics

In [Fa2], Faltings asked for the behavior of the delta-invariant in a family of Riemann surfaces degenerating to a surface with a single node. An answer to this problem has been formulated by, among others, Jorgenson [Jo] and Wentworth [We]. Given our splitting of the delta-invariant in the invariants $S(X)$ and $T(X)$ (Theorem 2.1.3), it seems natural to ask for the asymptotic behavior of these new invariants. One expects that the question is more subtle than for the delta-invariant as a whole, and indeed this turns out to be the case. In fact, the asymptotic behavior of these invariants depends on the structure of the limit divisor of Weierstrass points. In the present section we give an asymptotic formula only in a generic case (Theorem 2.4.2). We start however by recalling the result of Jorgenson and Wentworth.

**Theorem 2.4.1.** (Jorgenson [Jo], Wentworth [We]) Let $X_t$ be a holomorphic family of compact and connected Riemann surfaces of genus $g > 0$ over the punctured disc as in [Fay], Chapter 3, degenerating as $t \to 0$ to a surface with a single node. If the degenerate surface is the union of two compact and connected Riemann surfaces of positive genera $g_1, g_2$ meeting at a single point, the formula

$$\delta(X_t) = -\frac{4g_1 g_2}{g} \log |t| + O(1)$$

holds. If the degenerate surface remains connected upon removing the node, the formula

$$\delta(X_t) = -\frac{4g-1}{3g} \log |t| - 6 \log(-\log |t|) + O(1)$$

holds.

In particular, the asymptotic behavior of the delta-invariant is the same no matter what specific degenerate surface we choose (of the type mentioned in the theorem). This also accounts for the fact (cf. [Jo], Theorem 6.2) that the delta-invariant is, up to a log log-term associated to the locus
of degenerate surfaces with a non-separating node, a Weil function on \( \overline{\mathcal{M}}_g(\mathbb{C}) \). As we will see in a minute, this is not true for the invariants \( \log S(X) \) and \( \log T(X) \). However, we have the following result for a "generic" degenerate surface of separate type.

**Theorem 2.4.2.** Suppose that the degenerate surface is the union of two Riemann surfaces of positive genera \( g_1, g_2 \) with two points identified, and suppose furthermore that neither of these two points was a Weierstrass point on each of the two separate Riemann surfaces. Then the formulas

\[
\log S(X_t) = -\frac{g_1 g_2}{g} \log |t| + O(1)
\]

and

\[
\log T(X_t) = -\frac{g_1 g_2 (g^2 + g - 1)}{g^3} \log |t| + O(1)
\]

hold.

**Proof.** We review from [Fay], Chapter 3 the description of the holomorphic family \( X_t \) in the separating case. We fix two compact and connected Riemann surfaces \( X_1 \) and \( X_2 \) of positive genera \( g_1, g_2 \), respectively. Further we fix coordinate neighbourhoods \( U_k \) about \( P_k \) and local coordinates \( z_k : U_k \to D \), where \( D \) is the unit disk. We let \( W'_t = \{ z_k \in X_k \mid x_k \in X_k \setminus U_k \ or \ |z_k(x_k)| > |t| \} \) for \( t \in D \) and \( C_t = \{(X,Y) \in D \times D \mid XY = t\} \). The family \( X_t \) of genus \( g = g_1 + g_2 \) is then built from these data by putting \( X_t = W'_t \cup \cup_{C_t} \cup W'_t \) with the following identifications: \( z_1 \in W'_1 \cap U_1 \) is identified with \( (z_1(x_1), t/z_1(x_1)) \in C_t \) and \( z_2 \in W'_2 \cap U_2 \) is identified with \( (z_2(x_2), t/z_2(x_2)) \in C_t \). For \( t = 0 \), we obtain a singular surface \( X_0 \) which is just \( X_1 \cup X_2 \) with the points \( P_1, P_2 \) identified.

From Section 3 of [Jo] we deduce the formulas

\[
\log \|\vartheta\|(gP - Q) = \begin{cases} 
g_2 \log |t|, & P, Q \in X_1 \setminus \{P_1\} 
g_1 \log |t|, & P, Q \in X_2 \setminus \{P_2\} 
0, & \text{otherwise} \end{cases} + O(1),
\]

\[
\log \|F_x\|(P) = \begin{cases} 
g_2 \log |t|, & P \in X_1 \setminus \{P_1\} 
g_1 \log |t|, & P \in X_2 \setminus \{P_2\} \end{cases} + O(1),
\]

\[
\log |W_x(\omega)(P)| = \begin{cases} 
\frac{1}{2} g_2 (g_2 + 1) \log |t|, & P \in X_1 \setminus \{P_1\} 
\frac{1}{2} g_1 (g_1 + 1) \log |t|, & P \in X_2 \setminus \{P_2\} \end{cases} + O(1),
\]

\[
g \int_X \log \|\vartheta\|(gP - Q) \cdot \mu(Q) = g_1 g_2 \log |t| + O(1).
\]

By Theorem 3.1 in [EH], under the condition stated in the theorem the limit Weierstrass divisor \( \mathcal{W}_0 \) on \( X_0 \), i.e., the intersection of the closure of the Weierstrass divisor on the generic fiber with \( X_0 \), is equal to the union of a part \( \mathcal{W}_1 \) consisting of the ramification points outside \( P_1 \) of the linear system \( |K_{X_1}((g_2 + 1)P_1)| \) on \( X_1 \), and a part \( \mathcal{W}_2 \) consisting of the ramification points outside \( P_2 \) of the linear system \( |K_{X_2}((g_1 + 1)P_2)| \) on \( X_2 \). Here \( K_{X_1} \) and \( K_{X_2} \) denote canonical divisors on \( X_1 \) and \( X_2 \), respectively. In particular, by the Plücker formulas we have \( \deg(\mathcal{W}_1) = g_1 (g_2 - 1) \) and \( \deg(\mathcal{W}_2) = g_2 (g^2 - 1) \). Using the first formula above we obtain from this that

\[
\sum_{W \in \mathcal{W}} \log \|\vartheta\|(gP - W) = (g^2 - 1) g_1 g_2 \log |t| + O(1).
\]

We obtain the limit formula for \( \log S(X_t) \) by applying Corollary 2.2.6, and the limit formula for \( \log T(X_t) \) by applying Proposition 2.2.7. \( \square \)

If, contrary to the conditions in the theorem, one of the identified points is a Weierstrass point, the limit Weierstrass divisor is in general different from the divisor described in the above proof. It seems interesting to investigate the asymptotic behavior of the invariants \( T(X) \) and \( S(X) \) in various cases that can occur. For example, using Theorem 3.1.4 below and a result of Cornalba.
and Harris [CH] it is easy to compute the asymptotic behavior of \( T(X) \) in a holomorphic family of hyperelliptic Riemann surfaces degenerating to the union of two hyperelliptic Riemann surfaces meeting in a single point. In this case, the two identified points must be Weierstrass points since they are fixed by the hyperelliptic involution. We are outside the scope of Theorem 2.4.2, and indeed we find a different asymptotic behavior.

It also seems interesting to study the degeneration of the Weierstrass points further in the case that the degenerate surface has a non-separating node. This problem was posed already by Eisenbud and Harris in [EH].

### 2.5 Applications

In this section we use Proposition 2.2.1 to give a formula for the relative dualising sheaf on a semi-stable arithmetic surface (Proposition 2.5.2). As consequences we derive, among other things, a lower bound for the self-intersection of the relative dualising sheaf (Proposition 2.5.4) and a formula for the self-intersection of a point (Proposition 2.5.8).

Let \( p : X \to B \) be a semi-stable arithmetic surface over the spectrum \( B \) of the ring of integers in a number field \( K \). We assume that the generic fiber \( X_K \) is a geometrically connected, smooth proper curve of genus \( g > 0 \). Denote by \( W \) the Zariski closure in \( X \) of the divisor of Weierstrass points on \( X_K \), and denote by \( \omega_{X/B} \) the relative dualising sheaf of \( p \). We will first deduce some properties of \( W \) on \( X \).

**Lemma 2.5.1.** There exists an effective vertical divisor \( V \) on \( X \) such that we have a canonical isomorphism

\[
\omega_{X/B}^{\otimes g(g+1)/2} \otimes_{O_X} (p^*(\det p_*\omega_{X/B}))^\vee \cong O_X(V + W)
\]

of line bundles on \( X \).

**Proof.** We have on \( X \) a canonical sheaf morphism \( p^*(\det p_*\omega_{X/B}) \to \omega_{X/B}^{\otimes g(g+1)/2} \) given locally by

\[
\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \psi
\]

for a basis \( \{\psi_1, \ldots, \psi_g\} \) of differentials on the generic fiber of \( X \). Multiplying by \( (p^*(\det p_*\omega_{X/B}))^\vee \)

we obtain a morphism

\[
O_X \to \omega_{X/B}^{\otimes g(g+1)/2} \otimes_{O_X} (p^*(\det p_*\omega_{X/B}))^\vee.
\]

The image of 1 is a section whose divisor is an effective divisor \( V + W \) where \( V \) is vertical. This gives the required isomorphism. \( \square \)

We will now turn to the Arakelov intersection theory on \( X \). For a complex embedding \( \sigma : K \into \mathbb{C} \) we denote by \( F_\sigma \) the “fiber at infinity” associated to \( \sigma \). The corresponding compact and connected Riemann surface is denoted by \( X_\sigma \). The next proposition is an analogue of Lemma 3.3 in [Ar1].

**Proposition 2.5.2.** Let \( V \) be the effective vertical divisor from Lemma 2.5.1. Then we have

\[
\frac{1}{2} g(g+1)\omega_{X/B} = V + W + \sum_{\sigma : K \into \mathbb{C}} \log R(X_\sigma) \cdot F_\sigma + p^*(\det p_*\omega_{X/B})
\]

as Arakelov divisors on \( X \). Here the sum runs over the embeddings of \( K \) in \( \mathbb{C} \).

**Proof.** Consider the canonical isomorphism from Lemma 2.5.1. The restriction of this isomorphism to \( X_\sigma \) is the isomorphism of Proposition 2.2.1. In particular it has norm \( R(X_\sigma) \). The proposition follows. \( \square \)
Remark 2.5.3. Lemma 2.5.1 shows how the canonical isomorphism from Proposition 2.2.1 over the “fibers at infinity” of \( X \) extends over \( X \) itself. The “difference” between the left and right hand side is measured by the divisor \( V \), which can therefore be seen as a finite analogue of the numbers \( \log R(X_\sigma) \) associated with infinity.

We shall deduce three consequences from Proposition 2.5.2. We assume for the moment that \( g \geq 2 \). We define \( R_b \) for a closed point \( b \in B \) by the equation \((2g - 2) \cdot \log R_b = (V_b, \omega_{X/B})\), where the intersection is taken in the sense of Arakelov. The assumption that \( p : X \to B \) is semi-stable implies that the quantity \( \log R_b \) is always non-negative.

**Proposition 2.5.4.** Assume that \( g \geq 2 \). Then the lower bound
\[
(\omega_{X/B}, \omega_{X/B}) \geq \frac{8(g - 1)}{(2g - 1)(g + 1)} \left( \sum_b \log R_b + \sum_{\sigma : K \to \mathbb{C}} \log R(X_\sigma) + \deg \det p_* \omega_{X/B} \right)
\]
holds. Here the first sum runs over the closed points \( b \in B \), and the second sum runs over the embeddings of \( K \) in \( \mathbb{C} \).

**Proof.** Intersecting the equality from Proposition 2.5.2 with \( \omega_{X/B} \) we obtain
\[
\frac{1}{2} g(g + 1)(\omega_{X/B}, \omega_{X/B}) = (W, \omega_{X/B}) + (2g - 2) \left( \sum_b \log R_b + \sum_{\sigma : K \to \mathbb{C}} \log R(X_\sigma) + \deg \det p_* \omega_{X/B} \right).
\]
Now since the generic degree of \( W \) is \( g^3 - g \) we obtain by Proposition 1.5.2 the lower bound
\[
(W, \omega_{X/B}) \geq \frac{g^3 - g}{2g(2g - 2)} (\omega_{X/B}, \omega_{X/B}).
\]
Using this in the first equality gives the result. \( \square \)

We remark that lower bounds of a similar type have been given by Burnol, cf. [Bu], Section 3.3. He defines, for a compact and connected Riemann surface \( X \) of genus \( g \geq 2 \), the constants
\[
A_k(X) := - \int_X \log \| \partial \| (k \Omega_X - (2k - 1)(g - 1)P) \cdot \mu(P)
\]
for \( k \geq 2 \). The integrands have only a finite number of logarithmic singularities, and hence the integrals are well-defined. Burnol arrives then, for a semi-stable arithmetic surface \( p : X \to B \) of genus \( g \geq 2 \), at the lower bound
\[
\left( \frac{k^2 - k}{2} + \frac{6g - 5}{48(g - 1)} \right) (\omega_{X/B}, \omega_{X/B}) \geq \frac{1}{2g - 2} \sum_{\sigma} \left( A_k(X_\sigma) - \frac{1}{24} \delta(X_\sigma) - \frac{g \log(2\pi)}{3} \right)
\]
\[
+ \frac{1}{12(2g - 2)} \sum_b \delta_b \log \#k(b)
\]
for any \( k \geq 2 \). He remarks with respect to this lower bound that it only becomes non-trivial (i.e. better than the classical lower bound from Proposition 1.5.2) if for all complex embeddings \( \sigma \) we would have \( A_k(X_\sigma) \geq \frac{1}{24} \delta(X_\sigma) + g \log(2\pi)/3 \). In order to get an idea of how often this may occur, one might start by making a study of the asymptotic behavior of the analytic invariants \( A_k \). This was not carried out in [Bu]. However, with respect to the analytic terms in our lower bound for \( (\omega_{X/B}, \omega_{X/B}) \) we have by Theorem 2.4.2 and Corollary 2.2.4 the following result.
Proposition 2.5.5. Let $X_t$ be a holomorphic family of compact and connected Riemann surfaces of genus $g \geq 2$ over the punctured disk, degenerating to the union of two Riemann surfaces of positive genera $g_1, g_2$ with two points identified. Suppose that neither of these two points was a Weierstrass point on each of the two separate Riemann surfaces. Then the formula

$$\log R(X_t) = -\frac{g_1 g_2}{2g} \log |t| + O(1)$$

holds.

In particular, the value $\log R(X_t)$ goes to plus infinity under the conditions described in the theorem. It would be interesting to have a more precise, quantitative version of Proposition 2.5.5.

Our second result deals with an upper bound for $\sum \log S(X_\sigma)$ for a semi-stable arithmetic surface $p : X \to B$ of genus $g \geq 2$. Edixhoven has recently found an application of Arakelov theory in a study of the complexity of a certain algorithm that computes Galois representations associated to modular forms. In order to obtain a bound for this complexity, it turned out to be necessary to know how to bound the Arakelov-Green function $\sum \log G(P_\sigma, Q_\sigma)$ from above for a semi-stable arithmetic surface $p : X \to B$. This bound should depend on as few parameters as possible, and should be polynomial in the parameters that measure the length of the input of the algorithm. The present author has tried to attack this problem by looking at the explicit formula in Theorem 2.1.2. He expected that the classical part involving the values of the theta function would not be too difficult to bound from above, and that instead the normalisation constant $S(X)$ could be difficult. Indeed, Edixhoven informed him that Zagier had had these experiences on a similar problem. Things turned out to be otherwise: we can prove a bound for $\sum \log S(X_\sigma)$ that meets Edixhoven's demands, but as yet we cannot deal with the classical term. Fortunately, at Edixhoven's request, other authors have searched for bounds on the Arakelov-Green function; we now have satisfactory answers due to Merkl (private communication) and Jorgenson-Kramer [JK2], [JK3], [JK4].

Proposition 2.5.6. Let $p : X \to B$ be a semi-stable arithmetic surface of genus $g \geq 2$. Then the upper bound

$$\sum \log S(X_\sigma) \leq \frac{1}{2} \deg \det p_* \omega_{X/B} + \frac{g^2}{4(g-1)}(\omega_{X/B} \cdot \omega_{X/B}) + \frac{g}{2}[K : Q] \log(2\pi)$$

holds.

Proof. In the Noether formula Corollary 1.6.3 we eliminate the terms involving $\delta$ by using the formula $\delta(X)/8 = \log S(X) - \log R(X)$, which is Corollary 2.2.4. We eliminate then the term involving $\log R$ by using the formula from Proposition 2.5.4.

In Section 2.6 we describe, by way of appendix, Edixhoven's algorithm.

The final result of this section deals with the self-intersection of a point $P$. This self-intersection gives, upon dividing by the degree of the field of definition of $P$, the height of $P$ with respect to the relative dualising sheaf. A major problem in diophantine geometry is to obtain certain bounds for this height. We want to contribute to this problem by giving an explicit expression for the self-intersection of a point. Perhaps it turns out to be possible to give bounds of the required shape for each of the summands in the expression.

We can assume that $g \geq 1$ again. We first state a lemma.

Lemma 2.5.7. Let $P$ be a section of $p$, not a Weierstrass point on the generic fiber. Then we have a canonical isomorphism

$$P^*(O_X(V + W))^{\otimes 2} \cong (\det R_p O_X(gP))^{\otimes 2}$$
of line bundles on $B$.

**Proof.** Applying Riemann-Roch to the line bundle $O_X(gP)$ we obtain a canonical isomorphism

$$(\det R_{\ast}O_X(gP))^\otimes 2 \cong (O_X(gP), O_X(gP) \otimes \omega_X^{-1}) \otimes (\det p_{\ast}\omega_{X/B})^\otimes 2$$

of line bundles on $B$. The line bundle at the right hand side is, by the adjunction formula, canonically isomorphic to the line bundle $(P, P)^{\otimes g(g+1)} \otimes (\det p_{\ast}\omega_{X/B})^\otimes 2$. On the other hand, pulling back the isomorphism from Lemma 2.5.1 along $P$ and using once more the adjunction formula gives a canonical isomorphism

$$(P, P)^{\otimes (g+1)/2} \cong (V + W, P) \otimes \det p_{\ast}\omega_{X/B}.$$ 

The lemma follows by a combination of these observations. \hfill $\square$

**Proposition 2.5.8.** Let $P$ be a section of $p$, not a Weierstrass point on the generic fiber. Then

$$-\frac{1}{2}g(g+1)(P, P)$$

is given by the expression

$$-\sum_{\sigma:K\hookrightarrow \mathbb{C}} \log G(P, \omega_\sigma) + \log \#R^1 p_{\ast}O_X(gP) + \sum_{\sigma:K\hookrightarrow \mathbb{C}} \log R(X_\sigma) + \deg \det p_{\ast}\omega_{X/B},$$

where $\sigma$ runs through the complex embeddings of $K$.

**Proof.** Intersecting the equality from Proposition 2.5.2 with $P$, and using the adjunction formula $(\omega, P) = -(P, P)$, we obtain the equality

$$-\frac{1}{2}g(g+1)(P, P) = (V + W, P) + \sum_{\sigma:K\hookrightarrow \mathbb{C}} \log R(X_\sigma) + \deg \det p_{\ast}\omega_{X/B}.$$ 

It remains therefore to see that $(V + W, P)_{\text{ln}} = \log \#R^1 p_{\ast}O_X(gP)$. For this we consider the isomorphism in Lemma 2.5.7. Note that $p_{\ast}O_X(gP)$ is canonically trivialised by the function 1. This gives a canonical section at the right hand side with norm the square of $\#R^1 p_{\ast}O_X(gP)$. Under the isomorphism, it is identified with the canonical section on the left-hand side, which has norm the square of $\exp((V + W, P)_{\text{ln}})$. The required equality follows. \hfill $\square$

We see that minus the self-intersection of a point $P$ is large if $P$ is close to a Weierstrass point, either in the $p$-adic or in the complex topology.

### 2.6 Edixhoven’s algorithm

To conclude this chapter we describe, in a few words, the essentials of Edixhoven’s algorithm to compute Galois representations efficiently. We thank Edixhoven for explaining to us these ideas.

Consider for example the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation on the motive $M_\Delta$ associated to the discriminant modular form $\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n$. For a prime number $p$, the integer $\tau(p)$ is the trace of the Frobenius at $p$ acting on $M_\Delta$. Our goal is an algorithm that, given a prime number $p$, computes the integer $\tau(p)$, and we want that algorithm to run in time polynomial in $\log p$. Earlier algorithms to compute $\tau(p)$ are exponential in $\log p$.

By a famous argument due to Schoof, since the integers $\tau(p)$ can be bounded as $|\tau(p)| \leq 2p^{1/2}$, it suffices to give an algorithm that, given a prime number $p$ and a prime number $\ell$, computes the trace of Frobenius at $p$ modulo $\ell$ in time polynomial in $\ell$. Now it can be shown that the mod $\ell$ étale realisation of $M_\Delta$ is the dual of a certain 2-dimensional $\mathbb{F}_\ell$-vector space $V_\ell$ contained in the $\ell$-torsion $J_1(\ell)(\overline{\mathbb{Q}})[\ell]$ of the jacobian $J_1(\ell)$ of the modular curve $X_1(\ell)$. In fact, this $V_\ell$ is the intersection of 39
the kernels of the endomorphisms \( T_q - \tau(q) \), with \( q \) running over the primes up to about \( \ell^2/24 \), acting on \( \tilde{J}_1(\ell)(\overline{\mathbb{Q}})[\ell] \). Here \( T_q \) is the \( q \)-th Hecke operator. We are basically through if, given a prime \( \ell \), we can compute, in a time polynomial in \( \ell \), the minimum polynomial of a generator of the field of definition of a non-zero point in \( V_\ell \).

Using explicit estimates in Arakelov intersection theory it can be shown that such an algorithm exists. In fact, the actual algorithm is probabilistic with an expected running time polynomial in \( \ell \), but we shall ignore this aspect here. Let us describe the main idea, which is surprisingly simple. Consider a prime \( \ell \) and let \( x \) be a non-zero point in \( V_\ell \). We want to compute the minimum polynomial of a generator of the field of definition of \( x \). First of all, it can be shown that we can explicitly construct an effective divisor \( D \) of degree \( g \) on \( X_1(\ell) \), supported on the cusps, such that \( x \) is equal to the class of \( D' - D \) for a unique effective divisor \( D' = P_1 + \cdots + P_g \) on \( X_1(\ell) \). Here \( g \) is the genus of \( X_1(\ell) \), which is a polynomial function of \( \ell \). Since the field of definition of \( D \) is small, we are reduced to finding the minimum polynomial of a generator of the field of definition \( K \) of \( D' \). The essential idea is to do this by numerical methods. Using \( p \)-adic methods in the sense of Couveignes, or using numerical integration over the complex numbers, it is possible to write down an approximation \( \tilde{D}' \) of \( D' \). Having found this approximation, one obtains also an approximation of a generator \( \alpha \) of the field of definition of \( D' \). This is seen by the following lemma: there is an explicit finite sequence of morphisms \( j_1, \ldots, j_N : X_1(\ell) \to \mathbb{P}^1 \), defined over \( \overline{\mathbb{Q}} \), such that at least one \( j \) has the property that \( j(P_1) + \cdots + j(P_g) \) generates \( K \) (in fact, for this we need to work on \( X_1(5\ell) \), but we shall ignore this fact). It is virtually no extra effort to compute approximations to all Galois conjugates of \( \alpha \), and hence we find approximations of the rational numbers that form the coefficients of the minimum polynomial of \( \alpha \). If we could prove that the height of these coefficients is bounded by a polynomial in \( \ell \), we would have the required algorithm: indeed, the polynomial bound on the height implies that it is sufficient to carry out all the approximations in our earlier steps with an accuracy that is polynomial in \( \ell \), and hence they can be made to require a running time that is polynomial in \( \ell \). Now a bound on the height of the coefficients of the required shape follows from the following general proposition. The proof uses only arithmetic intersection theory as explained in Chapter 1.

**Proposition 2.6.1.** Let \( X \) be a proper connected non-singular curve of genus \( g \geq 1 \) over \( \overline{\mathbb{Q}} \), and let \( D \) be an effective divisor of degree \( g \) on \( X \). For any torsion line bundle \( L \) on \( X \) that satisfies \( h^0(L(D)) = 1 \) we have the following. Let \( K \) be a number field such that both \( L \) and \( D \) are defined over \( K \), such that \( X \) has semi-stable reduction over \( K \), and such that \( X \) has a rational point \( P \) over \( K \). Let \( D' \) be the unique effective divisor on \( X \) such that \( L \) is isomorphic to \( O_X(D' - D) \). Extend \( D, D' \) and \( P \) to horizontal divisors on the semi-stable model \( p : X \to B \) of \( X \) over \( K \). Then for the Arakelov intersection \( (D' - D, P) \) the upper bound

\[
(D' - D, P) \leq -\frac{1}{2} (D, D - \omega_{X/B}) + 2g^2 \sum_b \nu_b \log \# k(b) + \frac{1}{2} \deg \det p_* \omega_{X/B} \log(2\pi)
\]

\[
+ \sum_{\nu} \log \| \vartheta \|_{L^2, \nu} + \frac{g}{2} [K : \mathbb{Q}] \log(2\pi)
\]

holds. Here \( \nu_b \) is the number of irreducible components in the fiber at \( b \) if the fiber is reducible, and \( \nu_b = 0 \) otherwise.

It is not difficult to show that a polynomial bound for \( (D' - D, P) \) in our set-up implies a polynomial bound for the height of the coefficients of the minimum polynomial of \( \alpha \). Hence we are done if we could see that the terms on the right hand side in the above lemma are bounded by a polynomial in \( \ell \). This is again not difficult, except for the first term \(-\frac{1}{2} (D, D - \omega_{X/B})\), which requires that we bound the Arakelov-Green function by a polynomial in \( \ell \). We have commented upon this particular problem in the previous section.