Explicit Arakelov geometry

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Chapter 3

Hyperelliptic Riemann surfaces I

The purpose of this and the next chapter is to make the analytic theory from Chapter 2 explicit in the case of a hyperelliptic Riemann surface $X$. We will prove two theorems (Theorem 3.1.2 and Theorem 3.1.3) expressing the Arakelov-Green function $G$ of $X$, evaluated at pairs of Weierstrass points, in terms of the invariant $T(X)$ and a second natural invariant of $X$, which is introduced in Section 3.2 below. As corollaries, we find simple closed formulas for the invariant $T(X)$ and Faltings’ delta-invariant $\delta(X)$ of $X$. The main part of the present chapter is devoted to a proof of Theorem 3.1.2. We finish with a section dealing with some more special results in the case $g = 2$. The proof of Theorem 3.1.3 will be given in the next chapter. Although our two theorems look very similar, the techniques used in the proofs are very different. The proof of Theorem 3.1.2 uses only complex function theory, but for the proof of Theorem 3.1.3 we need to take a broader perspective and consider hyperelliptic curves over arbitrary base schemes. A special role is then played by hyperelliptic curves which are defined over a discrete valuation ring with residue characteristic equal to 2.

3.1 Results

Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$. In Section 3.2 we introduce a non-zero invariant $\|\varphi_g\|(X)$, the Petersson norm of the modular discriminant associated to $X$. As we will see, this is a very natural invariant to consider for hyperelliptic Riemann surfaces. Unfortunately, it is not so clear how to extend its definition to the general Riemann surface of genus $g$.

Definition 3.1.1. We denote by $G'$ the modified Arakelov-Green function

$$G'(P,Q) := S(X)^{-1/3} \cdot G(P,Q)$$

on $X \times X$.

In the present chapter we prove the following theorem dealing with $G'$ and $T(X)$. Recall that the Weierstrass points of $X$ are just the branch points of a hyperelliptic map $X \to \mathbb{P}^1$.

Theorem 3.1.2. Let $W$ be a Weierstrass point of $X$. Let $n = \binom{2g}{g+1}$. Consider the product $\prod_{W' \neq W} G'(W,W')$ running over all Weierstrass points $W'$ different from $W$, ignoring their weights. Then $\prod_{W' \neq W} G'(W,W')$ is independent of the choice of $W$ and the formula

$$\prod_{W' \neq W} G'(W,W')^{(g-1)/2} = 2^{(g-1)^2} \pi^{2g+2} \cdot T(X)^{\frac{g-1}{3}} \cdot \|\varphi_g\|(X)^{\frac{1}{2g}}$$

holds.
The following theorem will be derived in the next chapter.

**Theorem 3.1.3.** Let \( m = (2g+2) \). Then we have

\[
\prod_{(W,W')} G'(W, W')^{n(g-1)} = \pi^{-2g(g+2)m} \cdot T(X)^{-g(g+2)m} \cdot \|\varphi_g\|(X)^{-\frac{3}{2}(g+1)},
\]

the product running over all ordered pairs of distinct Weierstrass points of \( X \).

Combining the above two theorems yields a simple closed formula for the invariant \( T(X) \) in terms of \( \|\varphi_g\|(X) \). This formula should be compared with the formula in Theorem 2.3.3 above.

**Theorem 3.1.4.** Let \( \Delta_g(X) \) be the modified discriminant \( \|\Delta_g\|(X) = 2^{-(4g+4)h} \cdot \|\varphi_g\|(X) \). Then the formula

\[
T(X) = (2\pi)^{-2g} \cdot \|\Delta_g\|(X)^{-\frac{3}{2}g+1}
\]

holds.

From the viewpoint of arithmetic geometry, the modified invariant \( \|\Delta_g\| \) is definitely the right one to consider. As we will see below, it has an integral structure which causes it to behave well in all characteristics. In this sense it is the right generalisation of the discriminant \( \Delta \) for elliptic curves. The visual presence of the factors \( 2\pi \) and \( \|\Delta_g\| \) in the above formula suggests the existence of a certain “motivic” interpretation of the invariant \( T \). However, at present we do not know such an interpretation.

With Theorem 2.1.3 we obtain the following corollary.

**Corollary 3.1.5.** For Faltings’ delta-invariant \( \delta(X) \) of \( X \), the formula

\[
\exp(\delta(X)/4) = (2\pi)^{-2g} \cdot S(X)^{-(g-1)/g^3} \cdot \|\Delta_g\|(X)^{-\frac{3}{2}g+1}
\]

holds.

We remark that in the case \( g = 2 \), an explicit formula for the delta-invariant has been given already by Bost [Bo]. We will turn to the relation between his and our formula in Section 3.8.

The idea of the proof of Theorem 3.1.2 is quite straightforward: we start with the formula for \( T(X) \) in Proposition 2.2.7 and the formula for \( G \) in Theorem 2.1.2 and observe what happens if we let \( P \) approach the Weierstrass point \( W \) on \( X \). Thus, we have to perform a local study around \( W \) of the function \( \prod_{W'} |\vartheta|(gP - W') \) and of the functions \( |F_z|(P) \) and \( W_z(\omega)(P) \) for a suitable local coordinate \( z \). In Section 3.3 we find a suitable local coordinate on an embedding of \( X \) into its jacobian. In Section 3.6 we collect the local information that we need in order to complete the proof in Section 3.7. Some preliminary work on this local information is carried out in the Sections 3.4 and 3.5. These two sections form the technical heart of the present chapter.

### 3.2 Modular discriminant

In this section we introduce the modular discriminant \( \varphi_g \) and its Petersson norm \( \|\varphi_g\| \). The modular discriminant generalises the usual discriminant function \( \Delta \) for elliptic curves.

Let \( g \geq 2 \) be an integer and let \( \mathcal{H}_g \) be the Siegel upper half-space of symmetric complex \( g \times g \)-matrices with positive definite imaginary part. For \( z \in \mathbb{C}^g \) (viewed as a column vector), a matrix \( \tau \in \mathcal{H}_g \) and \( n, n' \in \frac{1}{2}\mathbb{Z}^g \) we have the theta function with characteristic \( \eta = \left[ \frac{n}{\eta'} \right] \) given by

\[
\vartheta[\eta](z; \tau) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i^2(n + n')\tau(n + n') + 2\pi i^2(n + n')(z + \eta')).
\]
We agree that we always choose the entries of $\eta'$ and $\eta''$ to be in the set $\{0, 1/2\}$. For an analytic theta characteristic $\eta$, the corresponding theta function $\vartheta[\eta](z; \tau)$ is either odd or even as a function of $z$. We call the analytic theta characteristic $\eta$ odd if the corresponding theta function $\vartheta[\eta](z; \tau)$ is odd, and even if the corresponding theta function $\vartheta[\eta](z; \tau)$ is even.

For any subset $S$ of $\{1, 2, \ldots, 2g + 1\}$ we define a theta characteristic $\eta_S$ as in [Mu2], Chapter IIIa: let

\[
\eta_{2k-1} = \left[ \begin{array}{c} 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \\ 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \end{array} \right], \quad 1 \leq k \leq g + 1,
\]

\[
\eta_{2k} = \left[ \begin{array}{c} 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \\ 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \end{array} \right], \quad 1 \leq k \leq g,
\]

where the non-zero entry in the top row occurs in the $k$-th position. Then we put $\eta_S := \sum_{k \in S} \eta_k$ where the sum is taken modulo 1.

**Definition 3.2.1.** (Cf. [Lo], Section 3.) Let $T$ be the collection of subsets of $\{1, 2, \ldots, 2g + 1\}$ of cardinality $g + 1$. Write $U = \{1, 3, \ldots, 2g + 1\}$ and let $\Delta$ denote the symmetric difference. The modular discriminant $\varphi_\Delta$ is defined to be the function

\[ \varphi_\Delta(\tau) := \prod_{T \in T} \vartheta[\eta_{T \Delta}](0; \tau)^8 \]

on $\mathcal{H}_\Delta$. The function $\varphi_\Delta$ is a modular form on $\Gamma_\Delta(2) = \{ \gamma \in \text{Sp}(2g, \mathbb{Z}) | \gamma \equiv I_{2g} \text{ mod } 2 \}$ of weight $4r$ where $\tau = (2g + 1)^{1/2}$. Consider an equation $y^2 = f(x)$ where $f \in \mathbb{C}[X]$ is a monic and separable polynomial of degree $2g + 1$. Write $f(x) = \prod_{k=1}^{2g+1} (x - a_k)$ and denote by $D = \prod_{k < l} (a_k - a_l)^2$ the discriminant of $f$. Let $X$ be the hyperelliptic Riemann surface of genus $g$ defined by $y^2 = f(x)$. Then $X$ carries a basis of holomorphic differentials $\mu_k = x^{k-1} dx/2y$ where $k = 1, \ldots, g$. Further, in [Mu2], Chapter IIIa, §5 it is shown how, given an ordering of the roots of $f$, one can construct a canonical symplectic basis of the homology of $X$. Throughout this chapter, we will always work with such a canonical basis of homology, i.e., a certain ordering of the roots of a hyperelliptic equation will always be taken for granted.

Let $(\mu | \mu')$ be the period matrix of the differentials $\mu_k$ with respect to a chosen canonical basis of homology, and let $\tau = \mu^{-1} \mu'$.

**Proposition 3.2.2.** We have the formula

\[ D^n = \pi^{4rg} (\det \mu)^{-4r} \varphi_\Delta(\tau) \]

relating the discriminant $D$ of the polynomial $f$ to the value $\varphi_\Delta(\tau)$ of the modular discriminant.

**Proof.** We follow the proof of [Lo], Proposition 3.2. Let $S$ be a subset of $\{1, 2, \ldots, 2g + 1\}$ with $|S \Delta U| = g + 1$. Then Thomae's formula (cf. [Mu2], Chapter IIIa, §8) holds:

**Theorem 3.2.3.** (Thomae's formula) We have

\[ \vartheta[\eta_S](0; \tau)^8 = (\det \mu)^4 \pi^{-4g} \prod_{k < l} (a_k - a_l)^2 \prod_{k < l} (a_k - a_l)^2. \]

If $T \in T$ then obviously $T \Delta U$ is a set $S$ with $|S \Delta U| = g + 1$, and conversely, every such set $S$ can be obtained in this way by taking a $T \in T$. Taking the product over all $T \in T$ we obtain by Thomae's formula

\[ \varphi_\Delta(\tau) = (\det \mu)^{4r} \pi^{-4gr} \prod_{T \in T} \left( \prod_{k < l} (a_k - a_l)^2 \prod_{k < l} (a_k - a_l)^2 \right). \]
The number of times a term \((a_k - a_l)^2\) appears on the right hand side is easily seen to be \(n\), hence 
\[
\varphi_g(\tau) = (\det^4 \pi^{-4\rho}) \prod_{k < l} (a_k - a_l)^{2n} \quad \text{which is what we wanted.}
\]

**Definition 3.2.4.** Let \(X\) be a hyperelliptic Riemann surface of genus \(g \geq 2\) and let \(\tau\) be a period matrix for \(X\) formed on a canonical symplectic basis, given by an ordering of the roots of an equation \(y^2 = f(x)\) for \(X\). Then we write \(\|\varphi_g\|(|\tau|)\) for the Petersson norm \((\det^4 \pi^{-4\rho}) |\varphi_g(\tau)|\) of \(\varphi_g(\tau)\). This does not depend on the choice of \(\tau\) and hence it defines an invariant \(\|\varphi_g\|(X)\) of \(X\).

It follows from Proposition 3.2.2 that the invariant \(\|\varphi_g\|(X)\) is non-zero.

### 3.3 Local coordinate

For our local computations on our hyperelliptic Riemann surface we need a convenient local coordinate. We find one by embedding the Riemann surface into its jacobian and by taking one of the euclidean coordinates.

Let \(X\) be a hyperelliptic Riemann surface of genus \(g \geq 2\), let \(y^2 = f(x)\) with \(f\) monic of degree \(2g + 1\) be an equation for \(X\), let \(\mu_k\) be the differential given by \(\mu_k = x^{k-1}dx/2y\) for \(k = 1, \ldots, g\), and let \((\mu|\mu')\) be their period matrix formed on a canonical basis of homology. Let \(L\) be the lattice in \(\mathbb{C}^g\) generated by the columns of \((\mu|\mu')\). We have an embedding \(\iota : X \hookrightarrow \mathbb{C}^g/L\) given by integration \(P \mapsto \int_{\infty}^P (\mu_1, \ldots, \mu_g)\). We want to express the coordinates \(z_1, \ldots, z_g\), restricted to \(\iota(X)\), in terms of a local coordinate about \(0 = \iota(\infty)\). This is established by the following lemma. In general, we denote by \(O(w_1, \ldots, w_s; d)\) a Laurent series in the variables \(w_1, \ldots, w_s\) all of whose terms have total degree at least \(d\).

**Lemma 3.3.1.** The coordinate \(z_g\) is a local coordinate about 0 on \(\iota(X)\), and we have

\[
z_k = \frac{1}{2(g-k) + 1} z_2^{(g-k)+1} + O(z_g; 2(g-k) + 2)
\]

on \(\iota(X)\) for \(k = 1, \ldots, g\).

**Proof.** We can choose a local coordinate \(t\) about \(\infty\) on \(X\) such that \(x = t^{-2}\) and \(y = -t^{-(2g+1)} + O(t; -2g)\). For \(P \in X\) in a small enough neighbourhood of \(\infty\) on \(X\) and for a suitable integration path on \(X\) we then have

\[
z_k(P) = \int_{\infty}^P \frac{x^{k-1}dx}{2y} = \int_0^{t(P)} \frac{t^{2(k-1)} \cdot (-2t^{-3}dt)}{-2t^{2(g+1)} + O(t; -2g)}
\]

\[
= \int_0^{t(P)} \left(t^{2(g-k)} + O(t; 2(g-k) + 1)\right) dt
\]

\[
= \frac{1}{2(g-k) + 1} t(P)^{2(g-k)+1} + O(t(P); 2(g-k) + 2).
\]

By taking \(k = g\) we find \(z_g = t + O(t; 2)\) and for \(k = 1, \ldots, g - 1\) then

\[
z_k = \frac{1}{2(g-k) + 1} z_2^{(g-k)+1} + O(z_g; 2(g-k) + 2),
\]

which is what we wanted. \(\square\)

### 3.4 Schur polynomials

In this section we assemble some facts on Schur polynomials. We will need these facts at various places in the next sections. Fix a positive integer \(g\). Consider the ring of symmetric polynomials
with integer coefficients in the variables \(x_1, \ldots, x_g\). Let \(e_r\) be the elementary symmetric functions given by the generating function \(E(t) = \sum_{r \geq 0} e_r t^r = \prod_{k=1}^d (1 + x_k t)\).

**Definition 3.4.1.** Let \(d\) be a positive integer and let \(\pi = \{\pi_1, \ldots, \pi_h\}\) with \(\pi_1 \geq \ldots \geq \pi_h\) be a partition of \(d\). The Schur polynomial associated to \(\pi\) is the polynomial

\[
S_\pi := \det(e_{k+i})_{1 \leq k, l \leq h},
\]

where \(h\) is the length of the partition \(\pi\), and where \(\pi'\) is the conjugate partition of \(\pi\) given by \(\pi'_k = \#\{l : \pi_l \geq k\}\), i.e., the partition obtained by switching the associated Young diagram around its diagonal. The polynomial \(S_\pi\) is symmetric and has total degree \(d\). We denote by \(S_g\) the Schur polynomial in \(g\) variables associated to the partition \(\pi = \{g, g-1, \ldots, 2, 1\}\). Thus, the formula

\[
S_g = \det(e_{g-2k+l+1})_{1 \leq k, l \leq g}
\]

holds, and the polynomial \(S_g\) has total degree \(g(g + 1)/2\).

Let \(p_r\) be the elementary Newton functions (power sums) given by the generating function \(P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{k \geq 1} x_k/(1 - x_k t)\). The following proposition is then a special case of Theorem 4.1 of [BEL2].

**Proposition 3.4.2.** The Schur polynomial \(S_g\) can be expressed as a polynomial in the \(g\) functions \(p_1, p_3, \ldots, p_{2g-1}\) only. This polynomial is unique.

**Definition 3.4.3.** We define \(s_g\) to be the unique polynomial in \(g\) variables given by the above proposition.

The next proposition is a special case of Theorem 6.2 of [BEL2].

**Proposition 3.4.4.** Let \(s(x_1, \ldots, x_g) \in \mathbb{C}[x_1, \ldots, x_g]\) be a polynomial in \(g\) variables such that for any set of \(g\) complex numbers \(w_1, \ldots, w_g\), the polynomial \(s(z_1 - w, z_2 - w^3, \ldots, z_g - w^{2g-1})\) in \(w\) either has exactly \(g\) roots \(w_1, \ldots, w_g\), or vanishes identically, if we give \(z\) the value \(z = (p_1(w_1, \ldots, w_g), p_3(w_1, \ldots, w_g), \ldots, p_{2g-1}(w_1, \ldots, w_g))\). Then \(s\) is equal to the polynomial \(s_g\) up to a constant factor.

**Definition 3.4.5.** We define \(\sigma_g\) to be the polynomial in \(g\) variables given by the equation

\[
\sigma_g(z_1, \ldots, z_g) = s_g(z_2, 3z_g-1, \ldots, (2g-1)z_1).
\]

The following proposition is then the result of a simple calculation.

**Proposition 3.4.6.** Up to a sign, the homogeneous part of least total degree of \(\sigma_g\) is equal to the Hankel determinant

\[
H(z) = \det \begin{pmatrix}
z_1 & z_2 & \cdots & z_{(g+1)/2} 
z_2 & z_3 & \cdots & z_{(g+3)/2} 
\vdots & \vdots & \ddots & \vdots 
z_{(g+1)/2} & z_{(g+3)/2} & \cdots & z_g
\end{pmatrix}
\]

if \(g\) is odd, or

\[
H(z) = \det \begin{pmatrix}
z_1 & z_2 & \cdots & z_{g/2} 
z_2 & z_3 & \cdots & z_{(g+2)/2} 
\vdots & \vdots & \ddots & \vdots 
z_{g/2} & z_{(g+2)/2} & \cdots & z_{g-1}
\end{pmatrix}
\]

if \(g\) is even.
We conclude with some more general facts. These can all be found for example in Appendix A to [Fu].

**Proposition 3.4.7.** Let \( \pi = \{\pi_1, \ldots, \pi_h\} \) with \( \pi_1 \geq \ldots \geq \pi_h \) be a partition. Then the formula

\[
S_{\pi}(1, \ldots, 1) = \prod_{k<l} \frac{\pi_k - \pi_l + l - k}{l - k}
\]

holds. In particular, \( S_2(1, \ldots, 1) = 2^{g(g-1)/2} \).

**Definition 3.4.8.** Let \( i = (i_1, \ldots, i_d) \) be a \( d \)-tuple of non-negative integers. The \( i \)-th generalised Newton function \( p^i \) is defined to be the polynomial

\[
p^i := p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_d^{i_d},
\]

where the \( p_r \) are the elementary Newton functions.

**Proposition 3.4.9.** The set of generalised Newton functions \( p^i \), where \( i \) runs through the \( d \)-tuples \( i = (i_1, \ldots, i_d) \) of non-negative integers with \( \sum i_r = d \), forms a basis of the \( \mathbb{Q} \)-vector space of symmetric polynomials of total degree \( d \) with rational coefficients.

**Proposition 3.4.10.** For a partition \( \pi \) of \( d \) and a \( d \)-tuple \( i = (i_1, \ldots, i_d) \), denote by \( \omega^i(\pi) \) the coefficient of the monomial \( x_1^{i_1} \cdot \ldots \cdot x_d^{i_d} \) in \( p^i \). Then the polynomial \( S_{\pi} \) can be expanded on the basis \( \{p^i\} \) of generalised Newton functions of total degree \( d \) as

\[
S_{\pi} = \sum \omega^i(\pi) \cdot p^i.
\]

3.5 Sigma function

We consider again hyperelliptic Riemann surfaces of genus \( g \geq 2 \), defined by equations \( y^2 = f(x) \) with \( f \) monic and separable of degree \( 2g + 1 \). We write \( f(x) = x^{2g+1} + \lambda_1 x^{2g} + \ldots + \lambda_{2g} + \lambda_{2g+1} \) and denote by \( \lambda \) the vector of coefficients \( (\lambda_1, \ldots, \lambda_{2g+1}) \). In this section we study the sigma function \( \sigma(z; \lambda) \) with argument \( z \in \mathbb{C}^g \) and parameter \( \lambda \). This is a modified theta function, studied extensively in the nineteenth century. Klein observed that the sigma function serves very well to study the function theory of hyperelliptic Riemann surfaces. For us it will be a convenient technical tool for obtaining the local expansions that we need. We will give the definition of the sigma function, as well as its power series expansion in \( z, \lambda \). For more details we refer to the *Enzyklopädie der mathematischen Wissenschaften*, Band II, Teil 2, Kapitel 7.XII. A modern reference is [BEL1], where one also finds applications of the sigma function in the theory of the Korteweg-de Vries differential equation.

As before, let \( \mu \) be the holomorphic differential given by \( \mu_k = x^k dx/2y \) for \( k = 1, \ldots, g \), and let \( (\mu_1') \) be their period matrix formed on a canonical basis of homology. Let \( L \) be the lattice in \( \mathbb{C}^g \) generated by the columns of \((\mu_1')\). By the theorem of Abel-Jacobi we have a bijective map \( \text{Pic}_{g-1}(X) \rightarrow \mathbb{C}^g/L \) given by \( \sum m_k P_k \mapsto \sum m_k f^\infty_k(\mu_1, \ldots, \mu_g) \). Denote by \( \Theta \) the image of the theta divisor of classes of effective divisors of degree \( g - 1 \), and let \( q : \mathbb{C}^g \rightarrow \mathbb{C}^g/L \) be the projection map. Let \( \tau = \mu_1\mu' \). By Theorem 1.4.2, there exists a unique theta-characteristic \( \delta \) such that \( \vartheta[\delta](z; \tau) \) vanishes to order one precisely along \( q^{-1}(\Theta) \). The characteristic \( \delta \) is odd if \( g \equiv 1 \) or \( 2 \mod 4 \), and even if \( g \equiv 0 \) or \( 3 \mod 4 \).

**Definition 3.5.1.** Let \( \nu \) be the matrix of \( A \)-periods of the differentials of the second kind \( \nu_k = -\frac{1}{4\pi} \sum l \lambda_{l+k+1} x^k dx \) for \( k = 1, \ldots, g \). These differentials have a second order pole at \( \infty \) and no other poles. The sigma function is then the function

\[
\sigma(z; \lambda) := \exp(-\frac{1}{2} \nu \mu^{-1} z) \cdot \vartheta[\delta](\mu^{-1} z; \tau).
\]
Using some of the facts on Schur polynomials from the previous section, we can give the power series expansion of \( \sigma(z; \lambda) \). The result is probably well-known to specialists, but we couldn't find an explicit reference in the literature. For the special case \( g = 2 \), a somewhat stronger version of the formula from the proposition below has been obtained by Grant, see [Gr], Theorem 2.11.

**Proposition 3.5.2.** The power series expansion of \( \sigma(z; \lambda) \) about \( z = 0 \) is of the form

\[
\sigma(z; \lambda) = \gamma \cdot \sigma_g(z) + O(\lambda),
\]

where \( \sigma_g \) is the polynomial given by Definition 3.4.5 and where the symbol \( O(\lambda) \) denotes a power series in \( z, \lambda \) in which each term contains \( \lambda_k \) raised to a positive integral power. The constant \( \gamma \) satisfies the formula

\[
\gamma = \pi A(r-n)(\det \mu)^{-4(r-n)} \varphi_2(\tau).
\]

If we assign the variable \( z_k \) a weight \( 2(g-k) + 1 \), and the variable \( \lambda_k \) a weight \(-2k\), then the power series expansion in \( z, \lambda \) of \( \sigma(z; \lambda) \) is homogeneous of weight \( g(g+1)/2 \).

**Proof.** First of all, the homogeneity of the power series expansion in \( z, \lambda \) with respect to the assigned weights follows from an explicit formula for \( \sigma(z; \lambda) \) given in [BE3]. This homogeneity is also mentioned there, cf. the concluding remarks after Corollary 1. Write \( \sigma(z; \lambda) = \sigma_g(z) + O(\lambda) \) where \( O(\lambda) \) denotes a power series in \( z, \lambda \) in which each term contains \( \lambda_k \) raised to a positive integral power. Because of the homogeneity, the series \( \sigma_0(z) \) is necessarily a polynomial in the variables \( z_1, \ldots, z_g \). By the Riemann vanishing theorem, there is a dense open subset \( U \subset \mathbb{C}^{2g+1} \) such that for any \( \lambda \in U \), the function \( \sigma(z; \lambda) \) satisfies the following property: for any set of \( g \) points \( P_1, \ldots, P_g \) on the hyperelliptic Riemann surface \( X = X_1 \) corresponding to \( \lambda \), the function \( \sigma(z - \int_0^1 (\mu_1, \ldots, \mu_g); \lambda) \) in \( P \) on \( X \) either has exactly \( g \) roots \( P_1, \ldots, P_g \), or vanishes identically, when we give the argument \( z \) the value \( z = \sum_k \int_0^1 (\mu_1, \ldots, \mu_g) \). In the limit \( \lambda \to 0 \) we find then, as in the proof of Lemma 3.3.1, that for any set of \( g \) complex numbers \( w_1, \ldots, w_g \) the polynomial

\[
\sigma_0 \left( \frac{1}{2g-1} (z_g - w^{2g-1}), \frac{1}{2g-3} (z_{g-1} - w^{2g-3}), \ldots, \frac{1}{3} (z_2 - w^3), z_1 - w \right)
\]

in \( w \) either has exactly \( g \) roots \( w_1, \ldots, w_g \), or vanishes identically, for the value

\[
z = (p_1(w_1, \ldots, w_g), p_3(w_1, \ldots, w_g), \ldots, p_{2g-1}(w_1, \ldots, w_g)).
\]

By Proposition 3.4.4, the polynomial \( \sigma_0 \) must be equal to the polynomial \( \sigma_g \) up to a constant factor \( \gamma \). As to this constant \( \gamma \), we find in [Ba], Section IX a calculation of a constant \( \gamma' \) such that \( \sigma(z; \lambda) = \gamma' \cdot H(z) + O(\lambda; [(g+3)/2]) \), where \( H(z) \) is the Hankel determinant from Proposition 3.4.6 and where now we consider the power series expansion only with respect to the variables \( z_1, \ldots, z_g \) and with respect to their usual weight \( \deg(z_k) = 1 \). By Proposition 3.4.6, this \( \gamma' \) is equal to our constant \( \gamma \), up to a sign. We just quote the result of Baker's computation:

\[
\gamma' = \vartheta(0; \tau)^4 \cdot \prod_{k \notin \mathbb{C}} (a_k - a_i)^2 / (\ell_1 \ell_3 \cdots \ell_{2g+1}),
\]

where \( \ell_r \) := \( -i \cdot \prod_{k \in U} (a_k - a_r) / \prod_{k \notin U} (a_k - a_r) \).

By Thomae's formula Theorem 3.2.3 we have

\[
\vartheta(0; \tau)^8 = (\det \mu)^4 \pi^{-4g} \prod_{k \notin \mathbb{C}} (a_k - a_i)^2 \prod_{k \notin \mathbb{C}} (a_k - a_i)^2.
\]

Combining, we obtain \( \gamma^8 = D \cdot \pi^{-4g} \cdot (\det \mu)^4 \). The formula for \( \gamma \) that we gave then follows from Proposition 3.2.2.

\[ \square \]
Example 3.5.3. By way of illustration, we have computed $\sigma_g$ for small $g$:

<table>
<thead>
<tr>
<th>$g$</th>
<th>$\sigma_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z_1$</td>
</tr>
<tr>
<td>2</td>
<td>$-z_1 + \frac{1}{3}z_2^2$</td>
</tr>
<tr>
<td>3</td>
<td>$z_1z_3 - z_2^2 - \frac{1}{8}z_2z_3^2 + \frac{1}{15}z_2^2z_3 + \frac{1}{108}z_2^3z_4 + \frac{1}{4725}z_4^4$</td>
</tr>
<tr>
<td>4</td>
<td>$z_1z_3 - z_2^2 - z_3^2z_4 + z_2z_3^2 - \frac{1}{3}z_1z_3^2 - \frac{15}{16}z_2z_3^2 - \frac{1}{108}z_2^3z_4 + \frac{1}{4725}z_4^4$</td>
</tr>
</tbody>
</table>

Remark 3.5.4. As can be seen from Proposition 3.4.6, the homogeneous part of least total degree (with respect to the usual weight $\deg(z_k) = 1$) of $\sigma_g(z)$ has degree $\left\lceil (g + 1)/2 \right\rceil$. Hence, by a fundamental theorem of Riemann, the theta-characteristic $\delta$ gives rise to a linear system of dimension $\left\lceil (g - 1)/2 \right\rceil$ on $X$.

3.6 Leading coefficients

In this section we calculate the leading coefficients of the power series expansions in $z_g$ of the holomorphic functions $\theta[\delta](gz; \tau)|_{\mu}(X)$ and $W_{z_g}(\mu)$, the Wronskian in $z_g$ of the basis $\{\mu_1, \ldots, \mu_g\}$.

Proposition 3.6.1. The leading coefficient of the power series expansion of $\sigma(gz; \lambda)|_{\mu}(X)$, and hence of $\theta[\delta](gz; \tau)|_{\mu}(X)$, is equal to $\gamma \cdot 2^{g(g-1)/2}$, where $\gamma$ is the constant from Proposition 3.5.2.

Proof. By Lemma 3.3.1 and Proposition 3.5.2, the power series expansion of $\sigma(gz; \lambda)|_{\mu}(X)$ has the form

$$\sigma(gz; \lambda)|_{\mu}(X) = \gamma \cdot \sigma_g \left( \frac{g}{2g-1} z_g^{2g-1}, \frac{g}{2g-3} z_g^{2g-3}, \ldots, \frac{g}{3g-3} z_g^{2g-3}, z_g z_g \right) + O(z_g; g(g+1)/2 + 1).$$

Hence we need to calculate $\sigma_g \left( \frac{g}{2g-1}, \frac{g}{2g-3}, \ldots, \frac{g}{3g-3}, g \right)$. By Definition 3.4.5 this is $s_g(g, g, \ldots, g)$. But by Proposition 3.4.2 and Definition 3.4.3 we have $s_g(g, g, \ldots, g) = S_g(1, 1, \ldots, 1)$, and by Proposition 3.4.7 we have $S_g(1, 1, \ldots, 1) = 2^g(g-1)/2$. The proposition follows.

Proposition 3.6.2. The leading coefficient of the power series expansion of the Wronskian $W_{z_g}(\mu)$ is equal to $2^{g(g-1)/2}$.

Proof. Expanding the Wronskian yields

$$W_{z_g}(\mu) = \det \left( \frac{1}{(k-1)!} \frac{d^k z_1}{dz_1^k} \right)_{1 \leq k, l \leq g} = \det \begin{pmatrix} z_2^{2g-2} & z_3^{2g-4} & \cdots & z_g^2 & 1 \\ (2g-2)z_2^{2g-3} & (2g-4)z_3^{2g-5} & \cdots & 2z_g^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (2g-2)z_2^g & (2g-4)z_3^g & \cdots & z_g^{2g-2} & 0 \\ \end{pmatrix} + O(z_g; g(g-1)/2 + 1).$$

Let $A$ be the matrix of binomial coefficients $A := \binom{2g-2k}{g-1}$. From the expansion of the Wronskian it follows that, up to a sign, the required leading coefficient is equal to $\det A$. We will compute this number. First of all note that

$$\det A = \frac{(2g-2)!}{(g-1)!(g-2)! \cdots 2!} \det \left( \frac{1}{(g-2k+l)!} \right)_{1 \leq k, l \leq g-1}.$$
where we define $1/n! := 0$ for $n < 0$. Now let $d = g(g - 1)/2$ and consider the ring of symmetric polynomials with integer coefficients in $g - 1$ variables. It is well known that for the elementary symmetric functions $e_r$ we have an expansion

$$e_r = \frac{1}{r!} \det \left( \begin{array}{cccc}
p_1 & 1 & 0 & \cdots & 0 \\
p_2 & p_1 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{r-1} & p_{r-2} & p_{r-3} & \cdots & r - 1 \\
p_r & p_{r-1} & p_{r-2} & \cdots & p_1
\end{array} \right),$$

with $p_r$ the elementary Newton functions. From Definition 3.4.1 and this expansion it follows that $\det(1/(g - 2k + l)!)$ is the coefficient of $p_l^k$ in the expansion of $S_{g-1}$ with respect to the basis of generalised Newton functions. By Proposition 3.4.10, this coefficient is equal to $\omega_{g-1}(d)/d!$, where $\omega_{g-1}(d)$ is the coefficient of $x_1^{g-1}x_2^{g-2} \cdots x_{g-1}^2x_g$ in $p_l^k$. Writing this out, it immediately follows that $\det(1/(g - 2k + l)! = 1/(g - 1)!(g - 2)! \cdots 1!$. Combining one finds $\det A = 2^{g(g-1)/2}$.

### 3.7 Proof of Theorem 3.1.2

Now we are ready to prove Theorem 3.1.2. Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$, and let $W$ be one of its Weierstrass points.

**Proof of Theorem 3.1.2.** Fix a hyperelliptic equation $y^2 = f(x)$ for $X$ with $f$ monic and separable of degree $2g + 1$ that puts $W$ at infinity. Choose a canonical basis of the homology of $X$, and form the period matrix $(\mu|\mu')$ of the differentials $x_k dx/2y$ for $k = 1, \ldots, g$ on this basis. Let $L$ be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu|\mu')$, and embed $X$ into $\mathbb{C}^g/L$ with base point $W$ as in Section 3.3. We have the standard euclidean coordinates $z_1, \ldots, z_g$ on $\mathbb{C}^g/L$ and according to Lemma 3.3.1 we have that $z_g$ is a local coordinate about $W$ on $X$. The weight $w$ of $W$ is given by $w = g(g - 1)/2$, cf. Remark 2.2.9. Consider then the following quantities:

$$A(W') = \lim_{Q \to W} \frac{\|\mathbf{\omega}(Q-W')\|}{|z_g|^g} \quad \text{for Weierstrass points } W' \neq W;$$

$$A(W) = \lim_{Q \to W} \frac{\|\mathbf{\omega}(Q-W)\|}{|z_g|^{w+g}} = \lim_{Q \to W} \frac{\|F_{z_g}(Q)\|}{|z_g|^w};$$

$$B(W) = \lim_{Q \to W} \frac{|W_{z_g}(\omega)(Q)|}{|z_g|^w},$$

where $W_{z_g}(\omega)$ is the Wronskian in $z_g$ of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega_X^1)$. We have by Theorem 2.1.2

$$G'(W, W')^g = \frac{A(W')}{\prod_{W''} A(W'')^{w/2}} \quad \text{for Weierstrass points } W' \neq W,$$

hence

$$\prod_{W' \neq W} G'(W, W')^g = \frac{1}{A(W)} \cdot \left( \prod_{W'} A(W') \right)^{\frac{g(g-1)}{2}}.$$

Further we have by Proposition 2.2.7, letting $P$ approach $W$,

$$T(X) = A(W)^{-(g+1)} \cdot \left( \prod_{W'} A(W') \right)^{\frac{w(g-1)}{2}} \cdot B(W)^2.$$
Eliminating the factor $\prod_{W'} A(W')$ yields

$$\prod_{W' \neq W} G'(W, W')(g^{-1})^2 = A(W)^4 \cdot B(W)^{\frac{2g+1}{g} \cdot T(X)^{\frac{2g+1}{g}}}.$$ 

Now we use the results obtained in Section 3.6. Let $\tau = \mu^{-1} \mu'$. A simple calculation gives that $A(W)$ is $(\det \Im \tau)^{1/4}$ times the absolute value of the leading coefficient of the power series expansion of $\varphi[(\mu^{-1} z; \tau)](X)$ in $z_0$. Hence by Propositions 3.5.2 and 3.6.1 we have

$$A(W) = 2^{g(s-1)/2} \pi^{g(s-1)/2} \cdot (\det \Im \tau)^{1/4} \cdot |\det \mu|^{-\frac{s-1}{2g}} \cdot |\varphi_2(\tau)|^{1/g}.$$ 

Further we have by Proposition 1.4.1 that $|\mu_1 \wedge \ldots \wedge \mu_g|^2 = (\det \Im \tau) \cdot |\det \mu|^2$. This gives that $|W_{\omega}(\omega)| = |W_{\omega}(\mu)| \cdot (\det \Im \tau)^{-1/2} \cdot |\det \mu|^{-1}$. From Proposition 3.6.2 we derive then

$$B(W) = 2^{g(s-1)/2} \cdot (\det \Im \tau)^{-1/2} \cdot |\det \mu|^{-1}.$$ 

Plugging in our results for $A(W)$ and $B(W)$ finally gives the theorem. $\square$

**Remark 3.7.1.** The fact that the product from Theorem 3.1.2 is independent of the choice of the Weierstrass point $W$ follows a fortiori from the computations in the above proof. It would be interesting to have an a priori reason for this independence.

### 3.8 The case $g = 2$

We can say a little bit more if we specialise to the case of a Riemann surface $X$ of genus $g = 2$. Note that such a Riemann surface is always hyperelliptic, and that it has 6 Weierstrass points, each of weight 1. The Arakelov theory of Riemann surfaces of genus 2 has been studied in quite some detail before, see especially the papers [Bo] and [BMM]. It will be convenient to work with the function

$$\varphi_2(\tau) := \prod_{\eta \text{ even}} \varphi(\eta)(0; \tau)^2$$

on $\mathcal{H}_2$. This is a modular form on the full symplectic group $\text{Sp}(4, \mathbb{Z})$ of weight 10. It relates to our $\varphi_2$ by the formula $\varphi_2 = (\varphi_2)^4$. If $\tau \in \mathcal{H}_2$ is associated to a Riemann surface $X$ of genus 2 then we write $||\varphi_2||^2(X) = (\det \Im \tau)^3 |\varphi_2(\tau)|$. This definition is independent of the choice of $\tau$. Also we will work with the modified $||\Delta_2||^2(X) = 2^{-12} ||\varphi_2||^2(X)$. We remark that this $||\Delta_2||$ is the $||\Delta_2||$ from the papers [Bo] and [BMM]. Our aim in this section is to prove the following two theorems. Recall the function $||J||$ from Definition 1.4.11 above.

**Theorem 3.8.1.** Let $W, W'$ be two Weierstrass points of $X$. Then the formula

$$G'(W, W')^2 = 2^{1/4} \pi^{-2} \cdot ||\varphi_2||^2(X)^{-3/16} \cdot ||J||^2(W, W')$$

holds.

Define the invariant $||H||^2(X)$ by

$$\log ||H||^2(X) = \frac{1}{2} \int_{\text{Pic}^0(X)} \log ||\varphi|| \cdot \nu^2.$$ 

This invariant has been introduced by Bost in [Bo].
Theorem 3.8.2. The formula
\[ S(X) = \|\Delta_2\|^{-1/4}(X) \cdot \|H\|^{-4}(X) \]
holds.

From Theorems 3.8.1 and 3.8.2 we obtain the following corollary.

Corollary 3.8.3. (i) Let \( W, W' \) be two Weierstrass points of \( X \). Then the formula
\[ G(W, W')^2 = (2\pi)^{-2} \cdot \|\Delta_2\|^{-1/4}(X) \cdot \|H\|^{-1}(X) \cdot \|J\|(W, W') \]
holds.

(ii) For Faltings' delta-invariant \( \delta(X) \) of \( X \), the formula
\[ \delta(X) = -16 \log(2\pi) - \log \|\Delta_2\|^{-1}(X) - 4 \log \|H\|^{-1}(X) \]
holds.

Proof. The first statement is a consequence of Theorems 3.8.1 and 3.8.2 and the definition of \( G' \). The second statement follows from Theorem 3.1.4, Theorem 3.8.2 and Theorem 2.1.3 relating the delta-invariant to the invariants \( S(X) \) and \( T(X) \).

The above corollary is also obtained by Bost in [Bo], Proposition 4, and is more or less proved in the appendix to [BMM]. Our approach is slightly different; in particular we think that in our approach the appearance of the function \( \|J\| \) is more natural (see especially the proof of Lemma 3.8.6 below).

It would be interesting to have explicit formulas for \( S(X) \) in higher genera. For this we probably need a generalisation of Lemma 3.8.8 below, but this seems difficult. A possible approach is suggested in [JK1].

We will frequently make use of Rosenhain's identity, cf. also Theorem 4.5.1 below.

Theorem 3.8.4. (Rosenhain [Ro]) Let \( W, W' \) be two Weierstrass points of \( X \). Then the formula
\[ \|J\|(W, W') = \pi^2 \cdot \prod_{W'' \neq W, W'} \|\theta\|(W - W' + W'') \]
holds.

Corollary 3.8.5. (i) Let \( W, W' \) be two Weierstrass points of \( X \). Then the formula
\[ G(W, W')^2 = 2^{-2} \cdot \|\Delta_2\|^{-1/4}(X) \cdot \|H\|^{-1}(X) \cdot \prod_{W'' \neq W, W'} \|\theta\|(W - W' + W'') \]
holds.

(ii) Suppose \( y^2 = f(x) \) is a hyperelliptic equation for \( X \) with \( f \) monic of degree 5. Choose an ordering of its roots, and consider the canonical symplectic basis of homology that corresponds to this ordering. Let \( (\mu, \mu') \) be the period matrix of the differentials \( dx/2y, xdx/2y \) on this basis, and let \( \tau = \mu^{-1} \mu' \). Let \( W = (\alpha_1, 0) \) and let \( W' = (\alpha_2, 0) \). Then the formula
\[ G(W, W')^2 = \frac{(2\pi) \cdot |\alpha_1 - \alpha_2| \cdot \|H\|^{-1}(X)}{|f'(\alpha_1)f'(\alpha_2)|^{1/4}(\det \text{Im} \tau)^{1/4} \det \mu^{1/2}} \]
holds.

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Proof. We obtain (i) from Corollary 3.8.3 by Rosenhain's formula. The formula in (ii) follows then from the first by an application of Thomae's formula, Theorem 3.2.3. 

For the proof of Theorem 3.8.1 we need the following lemma. It is a specialisation to the case $g = 2$ of some of the results from Section 3.6. Choose a hyperelliptic equation $y^2 = f(x)$ for $X$ with $f$ monic and separable of degree 5. Choose an ordering of its roots, and consider the canonical symplectic basis of homology that corresponds to this ordering. Let $(\mu, \mu')$ be the period matrix of the differentials $dx/2y$, $xdx/2y$ on this basis, and let $\tau = \mu^{-1} \mu'$. Let $L$ be the lattice in $\mathbb{C}^2$ generated by the columns of $(\mu, \mu')$, and make an embedding $\iota: X \hookrightarrow \mathbb{C}^2/L$ as in Section 3.5, taking the point at infinity as a base point. Let $z = (z_1, z_2)$ be the standard euclidean coordinates on $\mathbb{C}^2/L$. Let $\delta$ be the odd analytic theta characteristic such that $\vartheta[\delta](\mu^{-1} z; \tau)$ vanishes identically on $\iota(X)$. Let $\gamma$ be the constant from Proposition 3.5.2.

**Lemma 3.8.6.** We have

$$\vartheta[\delta](2\mu^{-1} z; \tau)|_{\iota(X)} = 2\gamma z_2^3 + O(z_2; 5).$$

Further, for odd $\delta'$ different from $\delta$ let $J(\delta, \delta')(\tau)$ be the Jacobian

$$J(\delta, \delta')(\tau) = (\partial(\vartheta[\delta], \vartheta[\delta'])/\partial(z_1, z_2))(0; \tau).$$

Then the expansion

$$\vartheta[\delta'](2\mu^{-1} z; \tau)|_{\iota(X)} = -2\gamma^{-1} J(\delta, \delta')(\tau) \cdot (\det \mu)^{-1} z_2 + O(z_2; 3)$$

holds.

**Proof.** The first expansion follows directly from Propositions 3.5.2 and 3.6.1. As to the second, observe that

$$\vartheta[\delta'](2\mu^{-1} z; \tau) = 2 \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_1}|_{z=0} \cdot z_1 + 2 \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} \cdot z_2 + O(z_1, z_2; 3)$$

locally about 0. When restricted to $\iota(X)$, we know by Lemma 3.3.1 that $z_2$ becomes a local coordinate about 0 and that $z_1 = \frac{1}{2} z_2^3 + O(z_2; 4)$ locally about 0. Thus when expanded with respect to the coordinate $z_2$ we get

$$\vartheta[\delta'](2\mu^{-1} z; \tau)|_{\iota(X)} = 2 \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} \cdot z_2 + O(z_2; 3)$$

about 0. It remains to compute the constant $\frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0}$. From Proposition 3.5.2 and the table accompanying this proposition we get that $\frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} = 0$, and that $\frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} = -\gamma$. This gives

$$-\gamma \cdot \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} = \det \begin{pmatrix} \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_1}|_{z=0} & \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_1}|_{z=0} \\ \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} & \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} \end{pmatrix}.$$ 

But on the other hand we have

$$\det \begin{pmatrix} \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_1}|_{z=0} & \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_1}|_{z=0} \\ \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} & \frac{\partial \vartheta[\delta'](\mu^{-1} z; \tau)}{\partial z_2}|_{z=0} \end{pmatrix} = (\det \mu)^{-1} \cdot J(\delta, \delta')(\tau).$$

Together this gives the required constant. □

**Proof of Theorem 3.8.1.** As in the proof of Theorem 3.1.2, we fix a hyperelliptic equation $y^2 = f(x)$
for $X$ with $f$ monic of degree 5 that puts $W$ at infinity. We choose a canonical basis of the homology of $X$, and form the period matrix $(\mu_1 \mu')$ of the differentials $dx/2y, xdx/2y$ on this basis. Let $\tau = \mu^{-1} \mu'$ and let $\kappa$ be the Riemann vector from Theorem 1.4.2 corresponding to infinity. The Abel-Jacobi map $\tau \cdot u: \text{Pic}_1(X) \to \mathbb{C}^2/\mathbb{Z}^2 + \tau \mathbb{Z}^2$ from Theorem 1.4.2 induces an identification of the set of Weierstrass points of $X$ with the set of odd analytic theta characteristics in dimension 2, a Weierstrass point $P$ corresponding to the characteristic $\eta = [\eta']$ such that $(\tau \cdot u)(P) = [\eta' + \tau \cdot \eta']$. In particular, the Weierstrass point $W$ corresponds to the characteristic $\delta$. Let $\delta'$ be the analytic theta characteristic corresponding to $W'$, and for a general Weierstrass point $W''$, denote by $\delta''$ the corresponding analytic theta characteristic. From the definition of $G'$ and Theorem 2.1.2 it follows that

$$G'(W, W')^2 = \lim_{P \to W} \frac{\|\varphi\|(2P - W')}{\prod_{W'' \neq W} \|\varphi\|(2P - W'')}^{1/8}.$$ 

We compute the right hand side with Lemma 3.8.6; we find that it is equal to

$$\frac{2|\gamma|^{-1}(\det \text{Im} \tau)^{1/4} \cdot |J(\delta, \delta')(\tau)| \cdot |\det \mu|^{-1}}{\left(2|\gamma| \prod_{|\mu'| \neq \delta} (2|\gamma|^{-1}(\det \text{Im} \tau)^{1/4} \cdot |J(\delta, \delta')(\tau)| \cdot |\det \mu|^{-1})^{1/8},}
$$

where $\gamma$ is the constant from Proposition 3.5.2. Using the formula for $\gamma$ from Proposition 3.5.2 we can rewrite this as

$$2^{1/4} \pi^{-3/4} \cdot \|\varphi\|_2^2 (X)^{-1/16} \left( \prod_{W'' \neq W} \|J\|(W, W'') \right)^{-1/8} \cdot \|\varphi\|_2((W, W')).$$

Rosenhain's formula Theorem 3.8.4 gives that $\prod_{W'' \neq W} \|J\|(W, W'') = \pi^{10} \|\varphi\|_2^2((X))$. Plugging this in finally gives the theorem.

We next proceed to the proof of Theorem 3.8.2. We will make use of the fact, special to the case $g = 2$, that the theta divisor in the jacobian of $X$ can be identified with $X$ itself. We need two lemmas.

**Lemma 3.8.7.** Let $W, W', W''$ be distinct Weierstrass points on $X$. Then

$$\lim_{P \to W} \frac{\|\varphi\|(P - W + W')}{\|\varphi\|(2\sigma(P) - W'')} = \|J\|(W, W'),$$

where $\sigma$ is the hyperelliptic involution of $X$.

**Proof.** This follows from the second expansion in Lemma 3.8.6. \qed

The next lemma is Proposition 14 in [BMM]. The proof is by no means trivial, and seems difficult to generalise to higher genera.

**Lemma 3.8.8.** Let $W, W'$ be two distinct Weierstrass points of $X$. Then the equality

$$\sum_{W'' \neq W, W'} \log \|\varphi\|(W - W' + W'') - \int_{\Phi + W'} \log \|\varphi\| \cdot \nu = 2 \log 2 + 2 \log \|H\|(X)$$

holds, the sum running over the Weierstrass points different from $W$ and $W'$. 

**Proof of Theorem 3.8.2.** Let $R, R'$ be two points on $X$ and let $W'$ be a Weierstrass point of $X$. We apply Green's formula Lemma 1.1.6 to the functions $f_1(P) = \|\varphi\|(R - R' + P) = (\|\varphi\| \cdot \Phi_{R-R'})(P)$ and $f_2(P) = \|\varphi\|(2P - W') = (\|\varphi\| \cdot \Phi_{W'})(P)$. Here for a divisor $D$ on $X$ we use the notation
\( \varphi_D \) introduced in Proposition 1.4.5. The divisor of \( f_1 \) on \( X \) is \( R' + \sigma(R) \), and the divisor of \( f_2 \) on \( X \) is \( W + 2W'' \), where \( W \) is the divisor of Weierstrass points on \( X \). By Proposition 1.4.5 we have \( \frac{1}{2\pi i} \partial \overline{\partial} \log f_1^2 = 2\mu \) and \( \frac{1}{2\pi i} \partial \overline{\partial} \log f_2^2 = 8\mu \) outside the zeroes of \( f_1 \) and \( f_2 \), respectively. Green's formula gives

\[
-8 \int_X \log \|\vartheta\|(R - R' + P) \cdot \mu(P) + 2 \int_X \log \|\vartheta\||(2P - W'') \cdot \mu(P) \\
= \log \|\vartheta\|(2R' - W'') + \log \|\vartheta\|(2\sigma(R) - W'') - \sum_{W \in W} \log \|\vartheta\|(R - R' + W) \\
- 2 \log \|\vartheta\||(R - R' + W''),
\]

in other words,

\[
4 \int_{\Theta + R - R'} \log \|\vartheta\| \cdot \nu + 2 \log S(X) \\
= - \sum_{W'' \neq W, W'} \log \|\vartheta\|(2R' - W'') - \sum_{W'' \neq W, W'} \log \|\vartheta\|(2\sigma(R) - W'') \\
+ 4 \sum_{W \in W} \log \|\vartheta\|(R - R' + W) + 2 \sum_{W'' \neq W, W'} \log \|\vartheta\|(R - R' + W''),
\]

where \( \nu \) is the canonical translation invariant \((1,1)\)-form on \( \text{Pic}_1(X) \) introduced in Section 1.4. We have used that \( \Theta \) can be identified with \( X \) and that \( \nu \) restricts to \( 2\mu \) on \( \Theta \). Now fix two distinct Weierstrass points \( W, W' \). Summing the above equation over the 4 Weierstrass points \( W'' \neq W, W' \) we obtain

\[
16 \int_{\Theta + W - W'} \log \|\vartheta\| \cdot \nu + 8 \log S(X) \\
= - \sum_{W'' \neq W, W'} \log \|\vartheta\|(2R' - W'') - \sum_{W'' \neq W, W'} \log \|\vartheta\|(2\sigma(R) - W'') \\
+ 4 \sum_{W \in W} \log \|\vartheta\|(R - R' + W) + 2 \sum_{W'' \neq W, W'} \log \|\vartheta\|(R - R' + W'').
\]

Now let \( R \to W \) and \( R' \to W' \). We obtain from Lemma 3.8.7 and Theorem 3.8.4

\[
16 \int_{\Theta + W - W'} \log \|\vartheta\| \cdot \nu + 8 \log S(X) \\
= \sum_{W'' \neq W, W'} \log \left( \frac{2\|\vartheta\|(W', W'')^2}{4\|\vartheta\|(W, W')(W', W'')} \right) + 6 \sum_{W'' \neq W, W'} \log \|\vartheta\|(W - W' + W'') \\
= 16 \sum_{W'' \neq W, W'} \log \|\vartheta\|(W - W' + W'') - 32 \log 2 - 2 \log \|\Delta_2\|(X).
\]

The theorem then follows by plugging in the result mentioned in Lemma 3.8.8. \( \square \)