Explicit Arakelov geometry

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Chapter 6

Numerical examples

As was explained in the Introduction, it is important to know how to calculate Arakelov invariants explicitly. Our Theorems 2.1.2 and 2.1.3 provide a solution to this problem. We illustrate this in the present chapter by computing examples of Arakelov invariants of hyperelliptic curves of small genus. In Section 6.1 we say some words on implementation. In Section 6.2 we focus on curves of genus 2. The computational aspects of this case are well-documented in [BMM]. Our approach in Section 6.2 will be different, but we do not pretend to be able to attain significantly better results. In Section 6.3 we consider a hyperelliptic curve of genus 3. In particular we find an explicit result for its delta-invariant. As far as we know, no explicit values of Arakelov invariants in genus 3 have been obtained so far, and it seems that the method and results in Section 6.3 are new.

6.1 Implementation

The difficulties in computing Arakelov invariants are usually caused by the analytic contributions at infinity. In this section we explain what we need to compute exactly, and how one can do this, given the results in this thesis.

Let \( X \) be a compact and connected Riemann surface of genus \( g > 0 \). First of all we need a period matrix \( (\Omega_1 | \Omega_2) \) for \( X \). It is well-known that if \( X \) has many automorphisms, it is possible to compute such a period matrix purely theoretically. For example, there is a beautiful theory dealing with periods of elliptic curves with complex multiplication, as we saw in the previous chapter. An exact period matrix for the genus 2 Riemann surface associated to the equation \( y^2 + y = x^5 \), which visibly admits at least 10 automorphisms, was given in [BMM].

Next, when exact computations turn out not to be possible, one can often resort to a long tradition going back at least to Gauss which is concerned with finding algorithms to give rapidly converging series of approximations to periods. These algorithms can be very efficient for special types of curves. In general, however, there is no other method than to approximate the occurring line integrals directly. If one does this, one has various numerical integration methods at one's disposal, and nowadays many of these have been implemented in computer algebra packages such as Maple or Mathematica. These allow one to approximate periods very efficiently.

Once one has a period matrix, one has the associated matrix \( \tau = \Omega_1^{-1} \Omega_2 \) in the Siegel upper half space of degree \( g \) and if the period matrix was on the basis \( \{\omega_1, \ldots, \omega_g\} \) of \( H^0(X, \Omega_X^1) \), we also find the length of \( \omega_1 \wedge \cdots \wedge \omega_g \) with respect to the Faltings metric on \( \wedge^g H^0(X, \Omega_X^1) \), by the formula

\[
\|\omega_1 \wedge \cdots \wedge \omega_g\|^2 = (\det \text{Im} \tau) \cdot |\det \Omega_1|^2
\]

from Proposition 1.4.10. These results allow one to calculate the analytic contributions to the
Faltings height of a curve.

Next we want to calculate the delta-invariant and certain values of the Arakelov-Green function. These we need in order to be able to calculate Arakelov intersection numbers, such as the self-intersection of the relative dualising sheaf, or the height of a rational point. A suitable formula for the self-intersection of the relative dualising sheaf follows for instance from the proof of Proposition 2.5.4.

As is clear from Theorems 2.1.2, 2.1.3 and 2.2.8, we need to be able to calculate certain values of the function \( \|\vartheta\| \) on \( \text{Pic}_{g-1}(X) \) and of the function \( \|J\| \) on \( \text{Sym}^2(X) \), but also we need to calculate the integral

\[
\log S(X) = - \int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P)
\]

over the Riemann surface \( X \).

The first problem is not difficult by the explicit formulas for \( \|\vartheta\| \) and \( \|J\| \) given in Chapter 1. We work with the usual identification

\[
\text{Pic}_{g-1}(X) \cong \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g , \quad \sum m_k P_k \mapsto \sum m_k \int_{P_0}^{P_k} \left( \eta_1, \ldots, \eta_g \right) + \kappa(P_0),
\]

with \( \kappa(P_0) \) the Riemann vector for a base point \( P_0 \). Here the basis \( \{\eta_1, \ldots, \eta_g\} = \{\omega_1, \ldots, \omega_g\} \cdot \Omega_1^{-1} \), and the Riemann vector \( \kappa(P_0) = (\kappa(P_0)_1, \ldots, \kappa(P_0)_g) \) can be made explicit by the classical formula

\[
\kappa(P_0)_k = \frac{1 + \tau_{kk}}{2} - \sum_{i \neq k} \int_{A_i} \eta_i(x) \int_{P_0}^{x} \eta_k
\]

for \( k = 1, \ldots, g \), see [Fay], p. 43. The \( A_1, \ldots, A_g \) are the \( A \)-chains in homology leading to the part \( \Omega_1 \) of the period matrix. Using the explicit formulas in Chapter 1 it is not difficult to carry out an a priori investigation which shows how many terms in the defining series for \( \vartheta \) and \( \frac{\partial \vartheta}{\partial z} \) we have to compute in order to approximate a value of \( \|\vartheta\| \) or \( \|J\| \) with a prescribed accuracy.

The second problem, to calculate the integral, is more difficult. First of all, one needs to make the form \( \mu \) explicit. This can be done using our basis \( \{\omega_1, \ldots, \omega_g\} \) of holomorphic differentials: it follows from the definition and the Riemann's bilinear relations that if one puts \( h = (\Omega_1(1m^\tau)\Omega_1)^{-1} \) then the form \( \mu \) can be written as \( \mu = \frac{1}{2g} \sum_{k=1}^g \omega_k \wedge \bar{\omega}_k \). Using a local coordinate and writing out the differentials \( \omega_1, \ldots, \omega_g \) in this local coordinate one next tries to convert the integral into an integral over a domain in \( C \), using the standard euclidean coordinates. The main problem is, however, that the integrand has singularities at the Weierstrass points of \( X \). This means that any numerical approximation has to take special care of these points. If the weights of the Weierstrass points are not too large, one can perhaps safely resort to the defining equation of \( \log S(X) \). Otherwise, one probably does better by using the formula in Proposition 2.2.6, which involves a similar integral, but this integral has only a singularity at the chosen point \( P \), and the order of vanishing of \( \|\vartheta\|(gP - Q) \) at \( Q = P \) is equal to \( g \). However, one has to note that the error produced in calculating the integral will be multiplied by \( g^2 \) if one wants to obtain \( \log S(X) \) in this way. In the computer algebra package Mathematica, it is possible to specify the points in an integration domain at which the evaluation of an integral needs special care, for instance because of the presence of logarithmic singularities in the integrand. There are special packages available particular suited for integrands with logarithmic singularities, also in 2 dimensions.

Let's make the above more explicit in the case of hyperelliptic Riemann surfaces, which seems the easiest case from the computational point of view. Our numerical examples in Sections 6.2 and 6.3 below deal with this case. Suppose that we deal with a hyperelliptic Riemann surface \( X \) of genus \( g \geq 2 \) given by an equation \( y^2 = f(x) \) with \( f(x) \in \mathbb{C}[x] \) separable of degree \( 2g + 1 \). Fix an
ordering of the roots of \( f \). Recall that in [Mu2], Chapter IIIa, §5 a traditional and canonical way is given to build a symplectic basis \( \{A_1, \ldots, A_g, B_1, \ldots, B_g\} \) for the homology of \( X \). We take this basis as a starting point, and with Mathematica we compute the periods of, say, the differentials \( \omega_1 = dx/y, \ldots, \omega_g = x^{g-1} dx/y \). This involves making appropriate branch cuts in \( C \), and then taking line integrals over paths that become the loops \( A_1, \ldots, B_g \) on the 2-sheeted cover \( X \) of \( \mathbb{P}^1 \), reversing the orientation each time one crosses a branch cut. The line integrals involved in the Abel-Jacobi map are carried out in a similar way. We only still need the Riemann vector, but this is done in [Mu2], Chapter IIIa, §5: if we take oo as a base-point on \( X \), then \( k \) is given by \( k = k_1 + \tau \cdot k_2 \mod \mathbb{Z}^2 + \tau \mathbb{Z}^2 \) with \( k_1 = (\frac{2}{5}, \frac{3}{2}, \ldots, 1, \frac{1}{2}) \) and \( k_2 = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \). We will turn to specific details concerning the computation of \( \log S(X) \) in the sections below.

6.2 Example with \( g = 2 \)

In broad lines, the computational aspects of Arakelov theory for genus 2 curves have been discussed already in [BMM]. For concrete calculations, however, the authors specialise to the case of semi-stable arithmetic surfaces whose singular fibers are irreducible curves with a single double point, cf. Section 3 of [BMM]. We want to give formulas for the Arakelov invariants of an arbitrary semi-stable arithmetic surface of genus 2. Although not worked out in detail in [BMM], it is certainly well-known among experts how to do this.

For a Riemann surface \( X \) of genus 2, we denote by \( \|\Delta_2\|(|X|) \) the invariant of \( X \) defined in Section 3.8. This is the \( \|\Delta_2\|(|X|) \) occurring in [BMM].

Proposition 6.2.1. Let \( p : X \to B = \text{Spec}(R) \) be a semi-stable arithmetic surface of genus 2 with good reduction at all primes dividing 2. Then the formulas

\[
10 \deg \det p_* \omega_{X/B} = \sum_b \varepsilon_b \delta_b \log \# k(b) - \sum_{\sigma} \log((2\pi)^{20} \|\Delta_2\|(|X_{\sigma}|))
\]

and

\[
(\omega_{X/B} \cdot \omega_{X/B}) = \sum_b (\frac{6}{5} \varepsilon_b - 1) \delta_b \log \# k(b) + \frac{1}{20} \sum_{\sigma} \log \|\Delta_2\|(|X_{\sigma}|) + 4 \sum_{\sigma} \log S(X_{\sigma})
\]

hold, where \( b \) runs through the closed points of \( B \) and where \( \varepsilon_b = 2 \) if the stable geometric fiber at \( b \) is the union of two curves of genus 1 meeting at a single point, and \( \varepsilon_b = 1 \) otherwise.

Proof. We can assume that the generic fiber of \( X \) is given by an equation \( y^2 = f(x) \), with \( f(x) \) a separable polynomial of degree 6 defined over the quotient field of \( R \). Let \( D \) be the discriminant of \( f \). In [Ue], Proposition 2.1 it is shown that the element \( \Lambda_{X/B} = D \cdot (dx/y \wedge dx/y)^{\otimes 10} \) defines a rational section of \((\det p_* \omega_{X/B})^{\otimes 10}\) independent of the choice of equation \( y^2 = f(x) \). By an argument as in Lemma 4.3.1 to deal with the infinite contributions we obtain

\[
10 \deg \det p_* \omega_{X/B} = \sum_b d_b \log \# k(b) - \sum_{\sigma} \log((2\pi)^{20} \|\Delta_2\|(|X_{\sigma}|)),
\]

where \( d_b = \text{ord}_{b}(\Lambda_{X/B}) \). According to the Table in §5 of [Ue] one has \( d_b = \varepsilon_b \delta_b \) with \( \varepsilon_b \) as in the proposition. This gives the first formula. The second formula follows from Noether's formula, where we eliminate the factor \( \sum_{\sigma} \delta(X_{\sigma}) \) by using Corollary 3.1.5.

Let us turn to a concrete example. We take smooth projective curves \( X_t \) given by the equation

\[
y^2 + (x^2 + 1)y = x^5 + t,
\]

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with $t \in \mathbb{Z}$. One can check that the $X_t$ are curves of genus 2 defined over $\mathbb{Q}$ and having good reduction at 2. Moreover, if $t \not\equiv 3 \pmod{7}$ then $X_t$ has semi-stable reduction over $\mathbb{Q}$. Contrary to the family of curves considered in Section 3 of [BMM], various types of reduction will occur.

Let us specialise for example to $t = 7$. We find (in the standard Kodaira notation employed in [Ue]) reduction $I_{1-0}$ (an irreducible curve with 2 double points) at 3, reduction $I_{2-0-0}$ (a union of a smooth curve of genus 1 and a $\mathbb{P}^1$ of self-intersection -2) at 5, and reduction $I_{1-0-0}$ (an irreducible curve with a single double point) at 29 and 339617. Thus, $\delta_3 = \delta_5 = 2$, $\delta_{29} = \delta_{339617} = 1$ and all $\varepsilon$'s are 1.

Let us proceed by computing the Arakelov invariants of $X = X_7$. We take an equation $y^2 = f(x)$ for $X$ with $f$ monic and separable of degree 5. We compute the period matrix $(\Omega_1|\Omega_2)$ on the differentials $dx/y$ and $xdx/y$ as described in Section 6.1. We obtain

$$\|A_2\|(X) = 2.067079790957566 \cdots 10^{-5}$$

With Proposition 6.2.1 we find

$$h_F(X) = -0.44517827222228057 \cdots$$

Using Theorem 3.1.4 we compute

$$\log T(X) = -3.9806368335392663 \cdots$$

In order to calculate $\log S(X)$ we make use of the formula

$$\log S(X) = -4 \int_X \log \|\varpi\|(2P - Q) \cdot \mu(Q) + \frac{1}{2} \sum_{W \in \mathcal{W}} \log \|\varpi\|(2P - W)$$

derived from Corollary 2.2.6. We do this since the integrand in the defining equation of $\log S(X)$ diverges at infinity. Write $x = u + iv$ with $u, v$ real. We want to express $\mu$ in terms of the coordinates $u, v$. This is done by the following lemma.

**Lemma 6.2.2.** Let $h$ be the matrix given by $h = (\Omega_1(\operatorname{Im}r)^t \Omega_1)^{-1}$. Then we can write

$$\mu = (h_{11} + 2h_{12}u + h_{22}(u^2 + v^2)) \frac{dudv}{2|f|}$$

in the coordinates $u, v$.

**Proof.** Let $\omega_k = x^{k-1} dx/y$ for $k = 1, 2$. As we have noted above, the form $\mu$ is given by $\mu = \frac{1}{4} \sum k_{k=1}^2 h_{kl} \cdot \omega_k \wedge \bar{\omega}_l$. Expanding this expression gives the result, where we note that the matrix $h$ is real symmetric, since our defining equation for $X$ is defined over the real numbers. \qed

We can now carry out the integral, choosing an arbitrary point $P$ and taking care of the singularity of the integrand at this point $P$. We find the approximation

$$\log S(X) = 0.77 \cdots$$

leading to

$$\delta(X) = -16.69 \cdots$$

by Theorem 2.1.3 and finally to

$$e(X) = 4.53 \cdots$$

by Proposition 6.2.1.
We have checked the computation by also calculating the invariant $\log \|H\|(X)$ and using the formulas in Section 3.8. It turns out that calculating the invariant $\log \|H\|$ is done much faster by Mathematica. Hence, it seems that for the computations on the analytic side in genus 2 it is better to stick to the approach in [BMM].

In [BMM] the curve $Y/Q$ given by $y^2 + y = x^5$ is discussed. The results imply that

$$\|\Delta_2\|(Y) = 2.07046497\ldots \cdot 10^{-5}$$

and

$$\delta(Y) = -16.68\ldots$$

The reader will notice that these values are rather close to the values for $\|\Delta_2\|(X)$ and $\delta(X)$ found above. This is no coincidence: a calculation shows that the family $X_t$ over $\mathbb{P}^1(\mathbb{C})$ has potentially good reduction at infinity, with smooth fiber isomorphic to $Y$.

Using Proposition 6.2.1 and the fact that $Y$ has potentially everywhere good reduction, one finds (as in [BMM])

$$h_F(Y) = -2.597239125\ldots, \quad e(Y) = 0.2152\ldots$$

On the other hand, for $t \in \mathbb{Z}$ one finds that $h_F(X_t)$ and $e(X_t)$ tend to infinity as $|t|$ tends to infinity. This illustrates the complicated behaviour of the functions $h_F$ and $e$ on the moduli space of curves.

Finally, we remark that a PARI program for computing the reduction and the potential stable reduction of curves of genus 2 defined over $\mathbb{Q}$ is available at the homepage of Qing Liu.

### 6.3 Example with $g = 3$

In this section we turn again to the methods developed in Section 3 of [BMM]. We generalise some of the results there to hyperelliptic curves of higher genera, and conclude with a numerical example in genus 3.

First of all, we prove a result on the self-intersection of the relative dualising sheaf. Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface whose generic fiber is a hyperelliptic curve of genus $g \geq 2$. According to [DM], Theorem 1.11, the hyperelliptic involution on the generic fiber extends uniquely to an involution $\sigma \in \text{Aut}_B(\mathcal{X})$.

**Proposition 6.3.1.** Assume that $p : \mathcal{X} \to B$ has two $\sigma$-invariant sections $P, Q : B \to \mathcal{X}$. Assume furthermore that the fibers of $p$ are irreducible. Then the formula

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = -4g(g - 1) \cdot (P, Q)$$

holds.

**Proof.** We follow the argument in [BMM], Section 1.3. Let $U$ be the largest open subset of $B$ over which $p$ is smooth. According to Lemma 4.2.1, the line bundle $\omega_{\mathcal{X}/B} \otimes O_{\mathcal{X}}(-(2g - 2)P) \otimes p^*(P, P)^{\otimes (2g-1)}$ has a nowhere vanishing section $s$ when restricted to $\mathcal{X}_U$. Thus, this $s$ can be seen as a rational section of that same bundle on $\mathcal{X}$. Let $V_F$ its divisor. Its support is disjoint with $\mathcal{X}_U$, and we have a canonical isomorphism

$$\omega_{\mathcal{X}/B} \simeq O_{\mathcal{X}}((2g - 2)P) \otimes p^*(P, P)^{\otimes (2g-1)} \otimes O_{\mathcal{X}}(V_F).$$

Pulling back along $P$ we find a canonical isomorphism $\langle \omega_{\mathcal{X}/B}, P \rangle \simeq (P, P)^{\otimes -1} \otimes (P, O_{\mathcal{X}}(V_F))$, extending the canonical adjunction isomorphism $\langle \omega_{\mathcal{X}/B}, P \rangle_U \simeq (P, P)^{\otimes -1}$ over $U$. But we know the adjunction isomorphism extends over $B$, so we must have that $\langle P, O_{\mathcal{X}}(V_F) \rangle$ is trivial. Since $\mathcal{X}$
is normal and since by assumption all fibers of $p$ are irreducible, we find that $V_p = 0$. The formula follows then by a calculation as in the proof of Corollary 4.2.2 above.

Let $K$ be a number field, and let $A$ be its ring of integers. Let $F \in A[x]$ be monic of degree $2g + 1$ with $F(0)$ and $F(1)$ a unit in $A$. Put $R(x) = x(x - 1) + 4F(x)$. Suppose that the following conditions hold for $R$: (i) the discriminant $D$ of $R$ is non-zero; (ii) for every prime $p$ of $A$ of residue characteristic $\neq 2$ we have $v_p(D) = 0$ or $1$; (iii) if $\text{char}(p) \neq 2$ and $v_p(D) = 1$, then $R \pmod{p}$ has a unique multiple root, and its multiplicity is $2$. As in [BMM], Section 3 one may then prove the following statement.

**Proposition 6.3.2.** The equation

$$C_F : y^2 = x(x - 1)R(x)$$

defines a hyperelliptic curve of genus $g$ over $K$. It extends to a semi-stable arithmetic surface $p : X \to B = \text{Spec}(A)$. We have that $X$ has bad reduction at $p$ if and only if $\text{char}(p) \neq 2$ and $v_p(D) = 1$. In this case, the bad fiber is an irreducible curve with a single double point. The differentials $dx/y, \ldots, x^{g-1}dx/y$ form a basis of the $O_B$-module $p_*\omega_{X/B}$. The points $W_0, W_1$ on $C_F$ given by $x = 0$ and $x = 1$ extend to disjoint $\sigma$-invariant sections of $p$.

As for the Arakelov invariants of $C_F$, we find from this the following result.

**Proposition 6.3.3.** At a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $\Omega_{\sigma} = (\Omega_{1\sigma}, \Omega_{2\sigma})$ be a period matrix for the Riemann surface corresponding to $C_F \otimes_{\sigma, K} \mathbb{C}$, formed on the basis $dx/y, \ldots, x^{g-1}dx/y$. Further, let $\tau_{\sigma} = \Omega_{1\sigma}^{-1}\Omega_{2\sigma}$. Then

$$\deg \det p_*\omega_{X/B} = -\frac{1}{2} \sum_{\sigma} \log (|\det \Omega_{1\sigma}|^2 (\det \text{Im}\tau_{\sigma})),$$

where the sum runs over the complex embeddings of $K$. Further, the formula

$$(\omega_{X/B}, \omega_{X/B}) = 4g(g - 1) \sum_{\sigma} \log G_{\sigma}(W_0, W_1)$$

holds.

**Proof.** The first statement follows by Lemma 1.4.1 and Proposition 6.3.2. The second follows from Proposition 6.3.1 and Proposition 6.3.2. □

For our numerical example, we choose the polynomial $F(x) = x^5 + 6x^4 + 4x^3 - 6x^2 - 5x - 1$ defined over $\mathbb{Q}$. Then the corresponding $R(x) = x(x - 1) + 4F(x)$ satisfies the conditions described above. The corresponding hyperelliptic curve (which we will call $X$ from now on) of genus 3 has bad reduction at the primes $p = 37, p = 701$ and $p = 14717$. An equation is given by

$$X : y^2 = x(x - 1)(4x^5 + 24x^4 + 16x^3 - 23x^2 - 21x - 4).$$

We choose an ordering of the Weierstrass points of $X$. We construct from this a canonical symplectic basis of the homology of (the Riemann surface corresponding to) $X$. Using Mathematica, we compute the periods of the differentials $dx/y, xdx/y, x^2dx/y$. This leads, by Proposition 6.3.3, to the numerical approximation

$$h_F(X) = -1.280295247656532068...$$

With Theorem 3.1.4 we find the following numerical approximation to $\log T(X)$:

$$\log T(X) = -4.44361200473681284...$$

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It remains then to calculate the invariant $\log S(X)$. Again we compute it by using Corollary 2.2.6. Write $x = u + iv$ with $u, v$ real. The analogue of Lemma 6.2.2 is as follows, with basically the same proof.

**Lemma 6.3.4.** Let $h$ be the $3 \times 3$-matrix given by $h = (\Omega_1 (\text{Im} \tau)^4 \Omega_1)^{-1}$. Then we can write

$$
\mu = (h_{11} + 2h_{12}u + 2h_{13}(u^2 - v^2) + h_{22}(u^2 + v^2) + 2h_{23}u(u^2 + v^2) + h_{33}(u^2 + v^2)^2) \cdot \frac{dudv}{3|f|}
$$

in the coordinates $u, v$.

Using this, and taking care of the singularities of the integrand, we find the approximation

$$
\log S(X) = 17.57...
$$

In order to check this result, we have taken several choices for $P$. Also, to exclude a possible systematic error, we have checked that $\mu$ integrates to 1 over $X$.

By Theorem 2.1.3 we have

$$
\delta(X) = -33.40...
$$

and using Theorem 2.1.2 we can approximate, by taking $Q = W_1$ and letting $P$ approach $W_0$,

$$
G(W_0, W_1) = 2.33...
$$

By Proposition 6.3.3 we finally find

$$
e(X) = 20.32...
$$

The running times of the computations were negligible, except for the computation of the integral involved in $\log S(X)$, which took about 7 hours on the author's laptop.