Financial Time and Volatility
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Chapter 1

Introduction

Financial markets pass through hectic and calm periods. In hectic markets the fluctuations in prices are large. In calm markets price fluctuations tend to be moderate. The variable that quantifies the size of fluctuations is called volatility in finance. It is related to the standard deviation of relative changes in the daily prices of the assets. Usually the volatility is denoted by the Greek letter $\sigma$.

Volatility plays a crucial rôle in financial markets. It is the essential variable in the pricing of options. It is also important for determining the risk of portfolios or investments. Higher volatility incurs, in general, greater risks. This thesis centers around the volatility of stock markets.

It is generally believed that prices in the future in financial markets are uncertain. The best estimate for tomorrow’s price is today’s price. This is an implication of the Efficient Market Hypothesis (EMH), see Fama (1965) and Samuelson (1965). The EMH formally states that all the available information is incorporated in today’s prices. The obvious mathematical concept to model such a price process is a martingale.

In practice stock prices increase, on average, over time. Over the past century the average annual increase was approximately 10% for U.S. stocks including dividends. The risk free return on a savings account was (of course) less. See for example Malkiel (1996) and his references. Expected returns are higher for more volatile stocks, see French, Schwert and Stambauch (1987). Investors are rewarded for taking risk. So over periods of a year or longer instead of a martingale, the general model for a stock price process is a semi-martingale: the sum of a martingale and a trend process, which is unknown but assumed to be fairly smooth, continuous and locally of finite variation.

The idea of using stochastic processes for modelling stock prices goes back to the pioneering work of the French mathematician Bachelier (1900). He used a random walk, with Gaussian increments, for modelling stock prices.
In the limit a random walk becomes a Brownian motion. Osborne (1959) introduced the geometric Brownian motion as a more realistic model for stock prices. In the setting of a geometric Brownian motion Black and Scholes (1973) were able to derive a pricing formula for European options.

In the Black and Scholes framework volatility is assumed to be constant over time. As several financial institutions have learned to their cost this is not the case. This has led to the introduction of models, both continuous and discrete-time, in which volatility is itself random. See Taylor (1994) for an overview.

Figure 1.1 depicts the time series of the daily log-returns of the S&P 500 over the period 1988-2001, and on the right hand side the histogram of the values attained by this time series. From these two figures we observe two well known characteristics of financial time series. First, periods of high volatility (large fluctuations) tend to cluster, and secondly the distribution of the return series has fat tails in comparison with the normal distribution. These characteristics are common to financial processes. They have been observed by many authors, see for example Mandelbrot (1963). A closer look at figure 1.1 also reveals that the empirical distribution is skewed. Stock prices tend to move downwards faster then they move upwards. This phenomenon is known as asymmetric volatility, see Black (1976) and Wu (2001).

![Figure 1.1](image)

**Figure 1.1.** Left: the time series of the daily returns of the S&P 500. Right: the histogram of the daily returns of the S&P 500. The dotted line represents the normal density with the same mean and standard deviation. The right tail is enlarged in the corner of the right figure.

One approach to model a non-constant volatility is by introducing a clock which measures financial time. The intuitive idea is attractive. The clock runs fast if trading is brisk and it runs slowly if trading is light. The idea of using such a clock for analyzing financial processes was introduced by Clark
(1973). He applied the clock to cotton futures price data. As a measure of financial time he proposed to use the number of contracts traded.

In the past fifteen years high frequency data for financial markets have become available. In this thesis high frequency data are used for running the financial clock. The financial time is the partial sum of squared log returns. Obviously this clock runs fast in hectic periods, where price fluctuations are large, and slowly in calm periods, where price fluctuations are moderate.

In chapter 2 we analyze high frequency data of the Dutch AEX stock index. Our data set contains the values of the AEX at 15 second intervals over a period of four years. Figure 2.9 on page 28 shows the graph of the partial sums of squared log returns. The vertical axis in this graph gives the value of financial time for the AEX index.

If instead of 15 second returns one uses 30 second returns, or one or two minute returns of the AEX a different graph is obtained. But the difference is at most a few percent. This is surprising. For a bounded smooth function over a fixed interval the sum of squared increments decreases as the number of subintervals increases. For small subintervals the squared increments are much smaller than the increments themselves. In the limit, the sum of squares over any fixed bounded interval vanishes as the length of the subintervals tends to zero.

The sample path of the AEX is so wriggly that the sum of squared increments does not decrease to zero when taking smaller time-intervals. This sample path behaviour is typical for continuous semi-martingales. In this thesis it is assumed that the financial process has continuous sample paths. The use of partial sums of squares to determine financial time is then justified by the Time-Change for Martingales Theorem. This famous theorem states that every continuous martingale is a standard Brownian motion on a new time scale. The time-change which transforms the process to this new time scale is the quadratic variation, which is the limit of the partial sum of squares. For standard Brownian motion the quadratic variation over any interval is equal to the length of the interval. If the correct limiting procedure is applied, then the sum of squared increments of a continuous martingale converges to the quadratic variation.

From a theoretical point of view, the quadratic variation may be computed if one knows the entire sample path. We, however, have only a large, but finite, number of observations. Therefore, only an estimate of the time-change is obtained. Since there is no information on the underlying stochastic model for the quadratic variation, the size of the error in the estimate is unclear. For the AEX we have more than a thousand observations per day. We therefore may expect that the difference between the computed quadratic variation and the real quadratic variation will be relatively small over periods
of a day or more.

The quadratic variation measured by our clock is related to volatility. The squared volatility is the derivative of the graph of the time-change. If volatility is constant, then the slope of the time-change equals the squared volatility. This is the volatility in the Black and Scholes formula. This relation between the time-change and volatility holds in general. Chapter 2.2 gives the mathematical details.

There are various interpretations of volatility in financial markets. Distinction has to be made between historical volatility and implied volatility. Historical volatility is usually associated to the standard deviation of the daily log returns over a certain period. Implied volatility is obtained by inverting the Black and Scholes formula and inserting the market price of the option. It provides information on how much the markets expect the underlying process to fluctuate. This is the volatility traders use.

Traditionally theorists regard volatility as a latent variable. This causes problems in estimating parameters of the so-called stochastic volatility models. In most cases volatility disappears by integration or, in other cases, imperfect proxies for volatility are used.

The discrete-time ARCH-type models, introduced by Nobel prize laureate Engle (1982), handle latent volatility by assuming that volatility is stochastic but predictable one time-step ahead. In this specific approach it is possible to obtain estimates for the parameters by maximizing the likelihood function. The ARCH-type models are able to capture characteristics such as the volatility clustering of financial time series and also the persistence of volatility. Today's literature on ARCH is vast. For an overview on applications of ARCH models to financial data see Bollerslev, Chou and Kroner (1992) and its references.

The literature on high frequency data is growing at an increasing rate. Anderson, Bollerslev, Diebold and Labys (2000) analyze high frequency data of the US$-DM exchange rate. The volatility they derive is known as realized volatility and is obtained from summing 48 squared half hour exchange rate returns. It is shown that the hypothesis that the distribution of daily returns standardized by realized volatility is Gaussian can not be rejected. In their approach the daily returns are constructed from a sequence of i.i.d. standard Gaussian random variables together with a sequence of daily volatilities. The daily return is the product of a Gaussian variable and the daily volatility. It is explicitly assumed that the daily volatility and the Gaussian variable are independent. This implies that the distribution of the daily returns is symmetric. Bollerslev and Zhou (2002) use high frequency data in order to estimate parameters of certain stochastic volatility models. They use the realized volatility series to obtain parameter estimates by means of the
method of moments.

The approach of Ané and Geman (2000) is along the lines of Clark. Their clock measures the number of transactions in certain assets. Intuitively speaking, using the number of transactions for running the financial clock is reasonable. In practice, high activity goes along with many transactions and vice versa. In contrast to Clark, Ané and Geman do not assume that the time-change has independent identically distributed increments. They assume independence between the time-change and the Brownian motion. It is shown for the technology stocks IBM and Cisco that the hypothesis of normality of the log returns, scaled by the square root of the increase of financial time, can not be rejected.

In this thesis high frequency data are used to make volatility visible. For the AEX we have 15 second interval observations and for the S&P 500 two minute interval observations. We find no signs of discontinuities in the sample paths of the AEX and the S&P 500. Under the assumption that the financial process is continuous, it is possible to test the hypothesis that the price process is a semi-martingale. To test the hypothesis of a semi-martingale the Time-Change for Martingales Theorem is used to squeeze and stretch the physical time axis in order to obtain the sample path of a Brownian motion in financial time.

Given a Brownian sample path \( \varphi \) on the interval \([0, t]\) it is straightforward to test whether it is a realization of a Brownian motion by using the well known property of Brownian motion that increments are independent normally distributed with variance equal to the length of the interval: choose a partition \( t_0, \ldots, t_N \) with \( 0 = t_0 < \cdots < t_N = t \) and test whether the sequence \((\varphi(t_i) - \varphi(t_{i-1}))/\sqrt{t_i - t_{i-1}}, i = 1, \ldots, N\), is statistically distinguishable from the realization of a sequence of i.i.d. standard normal random variables. There exist various statistical procedures for testing normality and independence.

If the price process is a Geometric Brownian motion then the quadratic variation of the log price process will be linear. The graph of our estimate \( \hat{q} \) of the quadratic variation of the log price process of the AEX is given in figure 2.9 (left) on page 28 and for the S&P 500 in figure 3.3 on page 46. One could describe these functions as piecewise linear but not as linear. We consider the following question. If one runs the log price process in financial time does one obtain a standard Brownian motion?

In chapter 2 the Time-Change for Martingales Theorem is applied to the log price process of the AEX index. It is shown that the hypothesis of i.i.d. standard normality of the centered intra-day returns, scaled by the square root of the increase of financial time, can not be rejected. This result indicates that a continuous semi-martingale gives a good description of the
AEX index.

In chapter 3 it is shown that the hypothesis that the increments of the log price process of the S&P 500 corrected for a linear drift, over fixed financial time intervals, scaled by the square root of the length of the financial time interval, are i.i.d. standard normal can not be rejected. This indicates that the model of a continuous semi-martingale holds for the S&P 500. However, for the S&P 500 centered intra-day returns, scaled by the square root of the increase in financial time, the hypothesis of normality is rejected. This is in line with the results in Thomakos and Wang (2003). The dependence between the time-change process and the underlying Brownian motion seems to be more apparent in S&P 500 than in the AEX. This can possible be explained by the distinct ways the data sets are collected. A consequence of this dependence is that it destroys the normality in the S&P 500 for intervals of random length in financial time, such as daily log returns. The dependence also explains the asymmetric distribution of the daily returns of the S& P 500 which is evident from the negative third moment.

A natural question one may ask is whether fixed financial time increments of the log AEX, corrected for drift, standardized by the length of the financial time interval, are also i.i.d. standard normal. Indeed, normality can not be rejected. However, there is significant autocorrelation in periods of high volatility for fixed but small financial time intervals. This can be understood by realizing that if volatility is high, fixed financial time intervals may only capture physical time periods of half an hour or even less. Under these circumstances non-synchronous trading effects may become 'visible' introducing positive autocorrelation. This effect only occurs for cash indices and not for future data, explaining why it is absent for the S&P500 data.

Using the time-change of the AEX we are able to observe the volatility process. In chapter 2 it is shown that the volatility process is highly variable. Daily changes in the volatility are large. Even if we calculate volatility over time-intervals as short as 20 minutes, the magnitude of the relative changes does not decrease as the time interval becomes shorter, see figure 2.11. We are not able to observe the volatility on much smaller time-intervals, but the unstable behaviour of intraday volatility suggests that it is unlikely that the volatility process is continuous.

Both the graphs of the time-change of the AEX and the S&P 500 exhibit long periods in which the graph is linear. At certain points, breakpoints, the slope changes abruptly. In chapter 4 it is shown that a piecewise linear function gives a good fit to the graph of the time-change of the S& P 500. A natural question is whether it is possible to link news items to the breakpoints. To study this question we constructed confidence intervals around the breakpoints and looked through the headlines of the Wall Street Journal
and the New York Times dated within these intervals. We found that 16 out of the 26 breakpoints can be linked more or less convincingly to news shocks.

In chapter 5 a more theoretically oriented topic is addressed. It discusses the asymptotic behaviour of the variation of the hedging strategy of a European call option in a binomial tree model. Binomial trees are widely used in practice for their simplicity and diversity. Cox, Ross and Rubinstein (1979) used the binomial tree to derive option prices. The binomial tree that is used in chapter 5 is derived from a simple trinomial tree. It is shown that the price process, the option price, and the hedging strategy converge to the continuous analogs. However, the quadratic variation of the hedging strategy may diverge.

Organization

The remainder of the thesis consists of four self contained chapters.

Chapter 2 focuses on the Dutch AEX stock exchange. This chapter is based on Peters and de Vilder (2002b) and has been submitted for publication. It formulates the hypothesis that the daily returns, open-close, centered by their mean, and scaled by the square root of the increment of quadratic variation for that day, are i.i.d. standard normal. This hypothesis can not be rejected. Subsequently we calculate over 800 days the average volatility per two minute period. We then obtain the specific daily volatility pattern in figure 2.7 on page 25. The spike at 14:30 and the jump at 15:30 are linked to events in the U.S. We also discuss the variability of volatility over periods of less than a day. In the section thereafter the observed volatility is used to estimate the parameters of two popular ARCH-type models. Our findings are not in agreement with the parameter estimates found by using maximum likelihood applied to daily log returns. There is no strong evidence of long term memory due to coefficients close to one. We also find evidence of dependence between the daily volatility and the sign and size of the daily standardized return.

Chapter 3 analyses two minute observations of the S&P 500 future stock index. It has been submitted for publication. See Peters and de Vilder (2002a). We formulate the hypothesis that the S&P 500 constitutes, in financial time, a Brownian motion. This hypothesis can not be rejected. In the second part of this chapter we formulate the hypothesis that the centered daily returns, standardized by the square root of the increase of financial time over that day, are i.i.d. standard normal. This hypothesis has to be rejected. We conclude that the time-change and the Brownian motion are dependent. We estimate a regression model. The log of the volatility over fixed financial time intervals is negatively related with the sign and positively with the size...
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of the increments of the Brownian motion over the time intervals.

Chapter 4 is based on Peters (2003b). In this chapter a continuous piece-wise linear function is fitted to the graph of the time-change of the S&P 500. First we show how Fisher's optimization principle can be used to find the location of breakpoints for a given number of breakpoints. By introducing a probabilistic model we are able to distinguish between spurious and non-spurious breakpoints. A pruning procedure returns the configuration of breakpoints that determines our approximation to the time-change of the S&P 500. We give two methods for determining confidence intervals for the breakpoints. Subsequently we look for news items within the confidence intervals in order to investigate the relation between news and breakpoints. We also compare the breakpoints found in the AEX index with the breakpoints of the S&P 500 and find a number of close coincidences.

In the final chapter 5 the asymptotic properties of the quadratic and higher order variations of the hedging strategy of a European call option are discussed. This chapter is has been published. See Peters (2003a). We present conditions for convergence of the quadratic variation of the hedging strategy. Finally, it is shown that the quadratic variation and higher order variations of the hedging strategy may diverge by choosing particular configurations of the binomial tree which is used to model the underlying asset.