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Peters, R.T.

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Chapter 5

Nonconvergence in the Variation of the Hedging Strategy of a European Call Option

5.1 Introduction

The binomial tree, introduced by Cox, Ross and Rubinstein (1979), is a widely used discrete stochastic model to describe movements of financial assets. By exploiting this structure Cox, Ross and Rubinstein were able to deduce an option pricing model, which converges to the celebrated Black, Scholes (1973) and Merton (1973) option pricing model when the step size tends to zero. Binomial tree models yield good discrete time approximations to financial markets. These models are widely used, both in practice and theory, because of their versatility and the ease of numerical simulation. It is well known that for binomial trees the market is complete: one can always find a hedging strategy and hence it is possible to price options.

He (1990) and Amin and Khanna (1994) have investigated under which conditions the option price in discrete models converges to the option price in the corresponding continuous limit model. A natural question then is to characterize the rate of convergence of the discrete time option price processes. This interesting question is addressed in Heston and Zhou (2000) who show that the rate of convergence depends on the smoothness of the option payoff function. In Hubalek and Schachermayer (1998) it is shown that the convergence of option prices is related to the contiguity of the sequence of physical measures with respect to the sequence of risk-neutral measures.
Less attention has been given to the asymptotic behaviour of the hedging strategy under binomial trees. The hedging strategy \( H \) in a complete market is determined by the option and by the price process of the underlying stock. At any moment \( t \) the variable \( H_t \) denotes the amount of stock which is needed to hedge the given option. In Duffie and Protter (1992) weak convergence of the financial gain process is investigated. Recently, Jacod, Méléard and Protter (2000) obtained general results on convergence of stochastic integral representations of martingales. These results yield conditions for convergence of the discrete hedging strategy to the continuous hedging strategy of contingent claims. The present chapter deals with the asymptotic behaviour of the variation of the hedging strategy. For simplicity we focus on convergence to the Bachelier model. We shall construct examples where the hedging strategy of a European call option converges, but where the variation of the hedging strategy may diverge. This result is somewhat counter intuitive. The limit process of the stock (Brownian motion) as well as the limit hedging strategy are of finite quadratic variation. The quadratic variation of the stock price process converges to the quadratic variation of its limit process. For the hedging strategy this needs not hold.

Greater variation would seem to increase transaction costs. Note however that the linear variation of the asymptotic hedge process is infinite over any time interval of positive length since the hedge process is a continuous semi-martingale and the quadratic variation does not vanish. Under the assumption of nonvanishing transaction costs hedging according to the theory will almost surely entail infinite transaction costs over any time interval of positive length. So in practice traders hedge on an epsilon grid rather than continuously. Our result on higher order variations has no influence on this epsilon grid which is determined by weak convergence of the hedge process. See for details on hedging with transaction costs Soner, Shreve and Cvitanić (1995) and their references.

The phenomenon that convergence of a sequence of stochastic processes does not imply convergence of the quadratic variation is well-known. Jacod (1980) gives general conditions for convergence of the quadratic variation of a sequence of processes to the quadratic variation of the limit process. The situation in the present chapter is different. We are concerned with conditions on the underlying sequence of stock price processes from which the sequence of hedging strategies is derived.

The chapter is organized as follows. In section 5.2 we formally introduce the model. Section 5.3.2 contains information on the continuous hedging strategy. In section 5.3.3 we will derive the hedging strategy in discrete time and analyze the variation of the hedging strategy and the asymptotic behaviour of the variation. In theorem 5.2 we present the main result. It is
shown how the parameters in the binomial tree model may be chosen so as to blow up the variation of the hedging strategy at any desired rate. We end with some concluding remarks.

5.2 The Model

In this section we formally introduce the model. We first present the general setting and then introduce the symmetric trinomial tree model.

5.2.1 General setting

In the binomial tree model the basic time interval \([0, T]\) is subdivided by a finite number of time points \(0 = t_0 < t_1 < \cdots < t_N = T\). The price of a risky asset increases from \(s_n\) to \(s_n + a_{n+1}\) or decreases to \(s_n + b_{n+1}\) over the corresponding time interval \([t_n, t_{n+1}]\). So \(a_n > 0 > b_n\). For simplicity we assume the following: the price of the risk free asset is constant (interest rate \(r = 0\)), the price process \(S_n\) of the risky asset is a martingale, and the limit price process is a Brownian motion (Bachelier model) rather than a geometric Brownian motion.

Let \(\mathcal{D}\) denote the space of \(\mathbb{R}\)-valued càdlàg functions on the interval \([0, T]\). We use the Skorohod topology on \(\mathcal{D}\) throughout this chapter. See Billingsley (1968). The binomial tree yields a sequence of right-continuous piecewise constant processes on this interval by setting \(S_N(t) = S_n\) on \([t_n, t_{n+1})\) for \(n = 0, \ldots, N - 1\) and \(S_N(T) = S_{N-1}\) by continuity. When we speak of convergence of binomial trees we mean weak convergence of these processes in \(\mathcal{D}\) for \(N \to \infty\).

In principle \(a_{n+1}\) and \(b_{n+1}\) may depend on \(N\), on \(n\) and on the value of \(S_n\), or even on the whole price process \(S_0, \ldots, S_n\) up to time \(t_n\). In this chapter we assume that the time points are equidistant, \(t_n = nT/N\), and that \(a_n\) and \(b_n\) depend only on \(n\) and \(N\). In addition we assume

\[
\begin{align*}
  a_n &\to 0, \quad b_n \to 0, \quad |a_nb_n| = \sigma^2 T/N, \quad n = 0, \ldots, N, \quad N \to \infty, \quad (5.1)
\end{align*}
\]

where \(\sigma^2\) is fixed. By the Martingale Central Limit Theorem it then follows that the price process \(S_n\) of the underlying asset converges in distribution to a Brownian motion on \([0, T]\), with variance function \(\sigma^2 t\) when \(N \to \infty\). It is well known that a Brownian motion has finite quadratic variation a.s. and for any given sequence of partitions with width tending to 0 the discrete approximation of the quadratic variation converges in probability and in \(L^2\) to the quadratic variation of Brownian motion, see for example Revuz and Yor (1994) page 27. Theorem 1.4 of Jacod (1980) ensures that under condition
(5.1) the quadratic variation of the discrete price process converges to the quadratic variation of the continuous limit process.

Introduce independent random variables \( X_n = X_{N,n} \), \( n = 1, \ldots, N \), with expectation zero and taking values \( a_n > 0 \) and \( b_n < 0 \). Then \( S_n \) may be expressed as: \( S_n = S_0 + \sum_{k=1}^{n} X_k \), \( n = 0, \ldots, N \), with \( S_0 = s_0 \) a.s., where \( s_0 \) is the deterministic price of the asset at time \( t = 0 \). We define the filtration \((\mathcal{F}_n)_{n=0,\ldots,N}\) by setting \( \mathcal{F}_n = \sigma(S_k, k = 0, \ldots, n) \).

If the price process of an asset is modelled by such a binomial tree, it is well known that it is possible to replicate, hence hedge, claims: the market is complete. That is, there exists a unique equivalent risk neutral probability measure \( Q \) such that the (discounted) price process w.r.t. \( Q \) is a martingale. By assumption, the increments are independent centered random variables and hence \( Q = P \) is the martingale probability measure of the variables \( S_n \). As shown in Harrison and Kreps (1979), the price process \( F_t(S_t) \) of any contingent claim \( F(S_T) \) under this measure also is a martingale and hence the price of the claim \( F \) at time \( t \) is given by the conditional expectation:

\[
F_t(S_t) = E(F(S_T)|\mathcal{F}_t), \quad t \in \{t_0, \ldots, t_N\}.
\]

**Limit Price of the Call**

Consider a European call option \( C(T,K) \), with exercise price \( K \) and expiration date \( T \). The price \( C_n(s_n) \) at time \( t_n \) for \( N \) fixed depends on the realization \( s_n \) of the underlying stock \( S_n \):

\[
C_n(s_n) = E\left((S_N - K)_+|S_n = s_n\right).
\]

In the next Proposition we will show that the price of a European call option converges if condition (5.1) holds.

**Proposition 5.1** Let \( C(T,K) \) be a European call option with price \( C_n(s_n) \). Let \( N \to \infty \). If we choose \( n \) so that \( nT/N \to t \) and \( s_n \to s \) then \( C_n(s_n) \) converges to

\[
C_t(s) = \sigma \sqrt{T-t} \left( \frac{(K-s)}{\sigma \sqrt{T-t}} \right) + (s - K) \Phi \left( \frac{s - K}{\sigma \sqrt{T-t}} \right)
\]

with

\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(z)dz, \quad x \in \mathbb{R}.
\]
The right hand side in equation (5.3) is exactly the pricing formula of a call option in the Bachelier model.

Proof. In discrete time the price of the call option is given by

\[ E\left((S_N - K)_+ \mid S_n = s_n\right) = E\left((s_n - K + \sum_{m=n+1}^{N} X_m)_+\right). \]

Let \( U \sim N(0, 1) \) and \( s_n \rightarrow s \). Since \( X_n \) vanishes uniformly for \( N \rightarrow \infty \), the moment generating function (mgf) of \( Z_n := \sum_{m=n+1}^{N} X_m \) converges to the mgf of \( \sigma \sqrt{(T-t)U} \) for \( N \rightarrow \infty \) and \( n/N \rightarrow t/T \). Hence \( E\psi(Z_n) \rightarrow E\psi(\sigma \sqrt{(T-t)U}) \) for any continuous function \( \psi \) of polynomial growth in particular for \( \psi(x) = (x - K)_+ \). Thus

\[ C_t(s) = E\left((s - K + \sigma \sqrt{(T-t)U})_+\right) \]
\[ = \int_{K-s/\sigma \sqrt{T-t}}^{\infty} \sigma \sqrt{(T-t)u} \varphi(u)du + (s - K) \int_{K-s/\sigma \sqrt{T-t}}^{\infty} \varphi(u)du \]
\[ = \sigma \sqrt{(T-t)} \varphi\left(\frac{(K-s)}{\sigma \sqrt{T-t}}\right) + (s - K) \Phi\left(\frac{s - K}{\sigma \sqrt{(T-t)}}\right). \]

The expression on the right hand side of (5.3) differs slightly from the Black, Scholes (1973) and Merton (1973) pricing formula. This is due to the fact that the binomial tree model converges to the Bachelier model instead of the Black and Scholes model. The simpler model has been chosen to keep the calculations below tractable.

### 5.2.2 The odd-even binomial model

Our initial interest is a discrete model in which an asset price increases, decreases or will remain constant with certain probabilities. We consider a symmetric trinomial tree, such that the change in the stock price, \( \Delta S \), takes the values \(-d_N, 0, d_N\) with probability \( p_0, 1 - 2p_0, p_0\) respectively, with \( p_0 \in (0, 1/4]\). A trinomial tree model is not complete and hence we are not able to hedge options. However, it is possible to embed our trinomial tree into a binomial tree model. This can be done by splitting up one step of the trinomial tree into two steps of a binomial tree and expressing \( \Delta S \) as the sum of two independent centered random variables \( X \) and \( X' \) where \( X \) takes values \( a > 0 \) and \( b < 0 \) and \( X' \) is distributed like \(-X\). (If \( a \geq -b \), then this embedding is unique.) Since \( EX = EX' = 0 \) it follows that \( P(X = a) = P(X' = -a) = -b/d_N \) and \( P(X = b) = P(X' = -b) = a/d_N. \)
The values $a$ and $b$ are determined by: $p_0 = -abd_N^{-2}$ and $d_N = a - b$. Under the restriction $a \geq -b$ this yields

$$a = \frac{1}{2} d_N \left(1 + \sqrt{1 - 4p_0}\right), \quad b = -\frac{1}{2} d_N \left(1 - \sqrt{1 - 4p_0}\right),$$

provided $p_0 \leq 1/4$. We assume that condition (5.1) holds, $|ab| = \sigma^2 T/N$, so $p_0 = \sigma^2 T/(Nd_N^2)$ and

$$a = \frac{1}{2} d_N \left(1 + \sqrt{1 - \frac{4\sigma^2 T}{Nd_N^2}}\right), \quad b = -\frac{1}{2} d_N \left(1 - \sqrt{1 - \frac{4\sigma^2 T}{Nd_N^2}}\right).$$

The parameter $d_N$ is the only parameter in this model and via $p_0 \leq 1/4$ it is restricted by

$$d_N^2 \geq \frac{4\sigma^2 T}{N}. \quad (5.4)$$

In this construction we have to assume that $N$ is even. As we have shown above, the increments of the binomial tree, $X_n$, have the following property, $X_{2n}$ is distributed like $-X_{2n-1}$, and $X_{2n-1}$ has values $a > 0$ and $b < 0$ with $a \geq -b$ and $d_N = a - b$. With the notation $p_j = P(X_j = a)$, $q_j = P(X_j = b) = 1 - p_j$, the law of $X_{2n-1}$ is given by

$$\tilde{p}_N = P(X_{2n-1} = a) = -\frac{b}{d_N} = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4\sigma^2 T}{Nd_N^2}}\right),$$

$$\tilde{q}_N = P(X_{2n-1} = b) = \frac{a}{d_N} = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4\sigma^2 T}{Nd_N^2}}\right). \quad (5.5)$$

Note $p_{2n} = \tilde{q}_N$ and $q_{2n} = \tilde{p}_N$.

In this setting we are able to calculate the hedge ratio and the changes in the hedge ratio explicitly. Therefore this model is well suited to analyze the limiting behaviour of the hedging strategy.

We shall refer to this model as the odd-even binomial model. The odd-even model in the paper of Hubalek and Schachermayer (1998) differs slightly from our model since their odd-even binomial model converges to the Black and Scholes model rather than the Bachelier model.

If we fix $a$ then $b = -a^2 T/N \to 0$ and $d_N = a - b \to a$ for $N \to \infty$, and the price process converges to a symmetric Poisson process. We note that in this limiting model it is not possible to hedge call options. This is due to the fact that there exist infinitely many equivalent martingale measures. Moreover, this limit model admits only the trivial sub- and superreplicating strategies for call options, see for example El Karoui and Quenez (1995) for details.
5.3 Hedging

In this section we investigate the asymptotic behaviour of the variation of the hedging strategy in the odd-even binomial model. First we will show that under condition (5.1) the discrete hedging strategy converges to a limiting strategy as the number $N$ of steps in the subdivision increases. We will give some general information on the limiting hedging strategy. Thereafter we study the variation of the hedging strategy.

5.3.1 Convergence of the hedging strategy

In the binomial tree model, it follows from the Martingale Representation Theorem that:

\[
(S_N - K)_+ = C_0^N(s_0) + \sum_{n=1}^{N} H_{n-1} \Delta S_n, \quad \Delta S_n = S_n - S_{n-1},
\]

where the process $H_{n-1}$ is $\mathcal{F}_{n-1}$ measurable and $C_0^N(s_0)$ is given by equation (5.2). In continuous time

\[
(S_T - K)_+ = C_0(s_0) + \int_0^T H_t dS_t.
\]

where $H_t$ is $\mathcal{F}_t$ measurable. The processes $H_n, H_t$ denote the hedge ratio in respectively discrete and continuous time. If we embed the sequence of $H_n$ in a piecewise constant càdlàg process, then Theorem 4.3 of Jacod, Méléard and Protter (2000) implies that the process $H_n$ converges in distribution to the process $H_t$. The conditions of this Theorem are satisfied under condition (5.1).

5.3.2 Continuous Hedging

We have shown that in the configuration of section 5.2.1 the price of a call option is given by equation (5.3). In the continuous setting the hedge ratio $H_t$ is equal to

\[
H_t = \frac{\partial C_t}{\partial S_t} = \Phi\left(\frac{S_t - K}{\sigma \sqrt{T - t}}\right),
\]

see for example Merton (1992), page 323. The right hand side of equation (5.8) is obtained by differentiating the right hand side of equation (5.3) with respect to $s$. Making use of the latter expression and Itô's formula one can easily derive the stochastic differential equation for the hedging strategy:

\[
dH_t = \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2} \left(\frac{B_t - K/\sigma}{\sqrt{T-t}}\right)^2\right) dB_t,
\]
with $B_t$ a standard Brownian motion. This equation implies that the hedging strategy is a continuous martingale with respect to the natural filtration of $B_t$. Hence the quadratic variation of the continuous hedging strategy exists and is given by

$$
\int_0^{\theta T} (dH_t)^2 = \int_0^{\theta T} \frac{1}{2\pi(T-t)} \exp \left( - \left( \frac{B_t - K/\sigma}{\sqrt{T-t}} \right)^2 \right) dt, \quad \theta \in (0, 1).
$$

(5.10)

The integral on the right is finite a.s.

### 5.3.3 Discrete Hedging

**The Hedge Ratio**

We consider the self-financing replicating strategy $(H_n^0, H_n)$ for the call option $C(T, K)$ with price process $C_n = C_n(S_n)$

$$
\begin{align*}
H_{n-1}^0 + H_{n-1} S_n &= C_n \\
H_n^0 + H_n S_n &= C_n.
\end{align*}
$$

(5.11)

It will be useful to introduce the random variables

$$
Z_n := \sum_{i=n}^N X_i.
$$

(5.12)

In the next Lemma we derive an expression for the hedge ratio $H_n$ by exploiting the self-financing property of the strategy.

**Lemma 5.1** For a binomial tree model as in section 5.2.1 the hedge ratio for a European call option $C(T, K)$ at time $t = t_n$ conditional on $S_n = s_n$ is given by

$$
H_n = \frac{\int_{K-s_n-a_{n+1}^+}^{K-s_n-b_{n+1}^-} P(Z_{n+2} > z)dz}{a_{n+1} - b_{n+1}}
$$

(5.13)

with $Z_{n+2}$ as in (5.12).

**Proof.** From equation (5.11) it follows that $H_n^0 + H_n(s_n + a_{n+1}) = C_{n+1}(s_n + a_{n+1})$ and $H_n^0 + H_n(s_n + b_{n+1}) = C_{n+1}(s_n + b_{n+1})$ holds. Subtraction of these two equations leads to

$$
H_n = \frac{C_{n+1}(s_n + a_{n+1}) - C_{n+1}(s_n + b_{n+1})}{a_{n+1} - b_{n+1}},
$$

(5.14)

see also Merton (1992), page 341.
The price of a call option at time \( t = t_{n+1} \) is given by equation (5.2) and may be written as

\[
C_{n+1}(s_{n+1}) = E(S_N - K)_+|S_{n+1} = s_{n+1}) = E(s_{n+1} + Z_{n+2} - K)_+
\]

\[
= \int_0^\infty P(s_{n+1} + Z_{n+2} - K > z) dz
\]

\[
= \int_{K-s_{n+1}}^\infty P(Z_{n+2} > z) dz.
\]  \hfill (5.15)

The combination of equations (5.14) and (5.15) gives equation (5.13).

**Increments of the Hedging Strategy**

Lemma 5.1 yields a simple formula for the increments in the hedging strategy in the odd-even binomial model introduced in section 5.2.2.

**Proposition 5.2** Consider the odd-even binomial model in section 5.2.2. Then \( a_n = -b_{n+1} \) and \( b_n = -a_{n+1} \) for all \( n \). Choose the parameter \( d_N \) in the trinomial tree so that \((K - s_0)/d_N\) is an integer. Then the increment of the hedging strategy, \( H_{n+1} - H_n \), of a European call option \( C(T, K) \), conditional on \( S_n = s_n \), can be expressed as

\[
H_{n+1} - H_n = (p_{n+1}1_{A_{n+1}} - q_{n+1}1_{B_{n+1}}) P(Z_{n+3} = K - s_n)
\]  \hfill (5.16)

with

\[ A_n = \{X_n = a_n\}, \quad B_n = \{X_n = b_n\}, \]

and \( p_n \) and \( q_n \) given above equation (5.5).

**Proof.** Note that \( a_{n+1} + b_{n+2} = 0 \) and \( a_{n+1} + a_{n+2} = -b_{n+1} - b_{n+2} = d_N \). Furthermore, conditionally on \( S_n = s_n \), \( S_{n+1} \) assumes the value \( s_n + a_{n+1} \) on \( A_{n+1} \) and \( S_{n+1} \) assumes the value \( s_n + b_{n+1} \) on \( B_{n+1} \). Now, conditionally on \( S_n = s_n \), Lemma 5.1 gives,

\[
H_{n+1} = 1_{A_{n+1}} \frac{1}{d_N} \int_{K-s_n}^{K-s_{n+d_N}} P(Z_{n+3} > z) dz +
\]

\[
1_{B_{n+1}} \frac{1}{d_N} \int_{K-s_n}^{K-s_{n-d_N}} P(Z_{n+3} > z) dz.
\]  \hfill (5.17)
Using Lemma 5.1 again, we see that $H_n$ may be written as

$$H_n = P(A_{n+2}) \frac{1}{d_N} \int_{K-s_n-a_{n+1}}^{K-s_n-b_{n+1}} P(Z_{n+3} + a_{n+2} > z)dz +$$

$$P(B_{n+2}) \frac{1}{d_N} \int_{K-s_n-a_{n+1}}^{K-s_n-b_{n+1}} P(Z_{n+3} + b_{n+2} > z)dz$$

$$= p_{n+2} \frac{1}{d_N} \int_{K-s_n-d_N}^{K-s_n} P(Z_{n+3} > z)dz +$$

$$q_{n+2} \frac{1}{d_N} \int_{K-s_n}^{K-s_n+d_N} P(Z_{n+3} > z)dz.$$

(5.18)

Subtracting (5.18) from (5.17) and then rearranging terms we obtain

$$H_{n+1} - H_n = (p_{n+2} 1_{A_{n+1}} - p_{n+2} 1_{B_{n+1}}) \times$$

$$\frac{1}{d_N} \left( \int_{K-s_n-d_N}^{K-s_n} P(Z_{n+3} > z)dz - \int_{K-s_n}^{K-s_n+d_N} P(Z_{n+3} > z)dz \right).$$

(5.19)

The difference between the two integrals in (5.19) is represented by the shaded area in figure 5.1. Here we use that $(K - s_0)$ is a multiple of $d_N$.

![Figure 5.1. $P(Z_{n+3} > z)$](image)

From the figure it follows that the difference of the two integrals equals

$$\int_{K-s_n-d_N}^{K-s_n} P(Z_{n+3} > z)dz - \int_{K-s_n}^{K-s_n+d_N} P(Z_{n+3} > z)dz = d_N P(Z_{n+3} = K-s_n).$$

(5.20)
5.3 Hedging

Combining equations (5.19) and (5.20) with \( p_{n+2} = q_{n+1} \), \( q_{n+2} = p_{n+1} \) we get (5.16).

We may use the previous Proposition to calculate the expected change in the \( r \)-th power of the increments of the hedging strategy conditional on \( S_n = s_n \).

**Corollary 5.1** Suppose that \((K - s_0)/d_N \) is an integer. Then in the odd-even binomial model of section 5.2.2, for any \( r \in \mathbb{R} \), the expected change in the hedge ratio of a European call option \( C(T, K) \) conditioned on \( S_n = s_n \) satisfies

\[
E(|H_{n+1} - H_n|^r | S_n = s_n) = (\tilde{p}_N^{r+1} + \tilde{q}_N^{r+1}) (P(Z_{n+3} = K - s_n))^r, \tag{5.21}
\]

where \( \tilde{p}_N \) and \( \tilde{q}_N \) are given in (5.5) and \( Z_n \) in (5.12).

**Proof.** From Proposition 5.2 and \( p_n = q_{n+1}, q_n = p_{n+1} \) it follows that

\[
E(|H_{n+1} - H_n|^r | S_n = s_n) = (P(Z_{n+3} = K - s_n))^r E|p_{n+1} A_{n+1} - q_{n+1} B_{n+1}|^r.
\]

Since \( A_n \) and \( B_n \) are disjoint sets and \( E1_{A_n} = p_n \) and \( E1_{B_n} = q_n \), this implies

\[
E(|H_{n+1} - H_n|^r | S_n = s_n) = (P(Z_{n+3} = K - s_n))^r (p_n^{r+1} p_{n+1} + q_n^{r+1} q_{n+1}).
\]

By \( p_n^{r+1} + q_n^{r+1} = \tilde{p}_N^{r+1} + \tilde{q}_N^{r+1} \) this yields equation (5.21).

In the next Lemma we will show that the expected change in the \( r \)-th power of the increments of the hedging strategy, \( E|H_{n+1} - H_n|^r \), is asymptotically proportional to the \( r \)-th power of the increment \( d_N \), where we write \( a_n \sim b_n \) if \( a_n/b_n \to 1 \).

**Lemma 5.2** Assume that in the odd-even binomial model of section 5.2.2 the parameter \( d_N \) is chosen such that \((K - s_0)/d_N \) is an integer. If \( t_n \to t \leq \theta T \), with \( \theta \in (0, 1) \) then the asymptotic behaviour of \( E|H_{n+1} - H_n|^r \) is given by

\[
E|H_{n+1} - H_n|^r \sim d_N c_r N(t), \quad N \to \infty, \tag{5.22}
\]

where \( c_r N(t) \) is given by

\[
c_r N(t) = (\tilde{p}_N^{r+1} + \tilde{q}_N^{r+1}) \frac{\exp \left( -\frac{K^{2r}}{2\sigma^2(T+(r-1)t)} \right)}{\sigma^r \sqrt{(2\pi)^r (T - t)^{r-1}(T + (r - 1)t)}}. \tag{5.23}
\]

Moreover the convergence \( E|H_{n+1} - H_n|^r d_N^{-r} c_{r,N}^{-1}(t) \to 1 \) holds uniformly in \( t \in [0, \theta T] \).
Proof. By the Local Central Limit Theorem (LCLT) for lattice distributions it follows that for $N \to \infty$

$$
\frac{1}{d_N} P(Z_{n+3} = K - s_n) \to \frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp \left(-\frac{(K-s)^2}{2\sigma^2(T-t)}\right), \tag{5.24}
$$

if $s_n \to s$ and $n/N \to t/T$. See Gnedenko and Kolmogorov (1954), page 233: assume $n$ is even. Set $\xi_n = (X_{n-1} + X_n)/d_N$ and $B_n = 1/d_N$ and use that there exists for each $s_n$ a $k$ such that $P(Z_{n+3} = K - s_n) = P(\sum_{i=\frac{1}{2}n}^{\frac{3}{2}n} \xi_2i = K - s_n)$. Convergence holds in $L^1$ and holds for any sequence $s_n \to s$. If $n$ is odd then we write $P(Z_{n+3} = K - s_n) = \tilde{q}_N P(Z_{n+4} = K - s_n - a_n) + \tilde{p}_N P(Z_{n+4} = K - s_n - b_n)$. In view of $s_n - a_n, s_n - b_n \to s$ convergence of the two terms is achieved by the LCLT as described above and hence equation (5.24) does not depend on whether $n$ is odd or even. From Corollary 5.1 it follows that

$$
E[|H_{n+1} - H_n|^r] = \sum_{s_n} E(|H_{n+1} - H_n|^r|S_n = s_n) P(S_n = s_n)
$$

$$
= (\tilde{p}_N^{-1} + \tilde{q}_N^{-1}) E(P(Z_{n+3} = K - S_n|S_n))^r.
$$

Now we use (5.24) and see that

$$
E[|H_{n+1} - H_n|^r] \sim (\tilde{p}_N^{-1} + \tilde{q}_N^{-1}) d_N^r \times \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi (T-t))^r}} \exp \left(-r\frac{(K-s)^2}{2\sigma^2(T-t)}\right) \frac{1}{\sigma \sqrt{2\pi t}} \exp \left(-\frac{s^2}{2\sigma^2 t}\right) ds.
$$

Integrating and simplifying the latter expression one obtains (5.22).

The approximation for $\theta < 1$ holds uniformly on the interval $[0, \theta T]$. \qed

The approximation of expression (5.25) breaks down for $t \to T$. In this case we cannot apply the local Central Limit Theorem. The restriction $\theta < 1$ is essential. If $t_n \to T$, then $Z_{n+3} \to 0$ a.s. Hence we have

$$
E(1_{\{Z_{n+3} = K - s_n\}}|S_N) \to 1_{\{s_T = K\}}.
$$

So the hedge strategy will only vary if the stock price is close to the exercise price near expiration date. Applying the Local Central Limit Theorem to $\frac{1}{d_N} P(S_n = K)$ one can show that

$$
E[|H_{n+1} - H_n|^r] \sim d_N(\tilde{p}_N^{-1} + \tilde{q}_N^{-1}) \frac{1}{\sigma \sqrt{2\pi T}} e^{-\frac{k^2}{2\sigma^2 t}}
$$

The last factor is the same as the last factor in (5.23) with $r = 1$ and $t \to T$. In the remainder of the chapter we shall pay no further attention to the hedging strategy near the expiration date.
The Asymptotic Variation

Let us apply Lemma 5.2 to two examples in order to show how the variation in the hedging strategy depends on the value of \( d_N \).

**Example 5.1** Let \( d_N \sim 2\sigma \sqrt{T/N} \) for \( N \to \infty \) and assume that \((K - s_0)/d_N\) is an integer for all \( N \). The expectation of the quadratic variation of the hedging strategy from \( t = 0 \) to \( t = T/2 \) converges.

**Proof.** Applying Lemma 5.2 we see that
\[
\sum_{n=0}^{N/2} E(H_{n+1} - H_n)^2 \sim 4\sigma^2 \int_0^{T/2} c_{2,N}(t) dt < \infty,
\]
since \( d_N^2 = 4\sigma^2(t_n - t_{n-1}) \) (cf. (5.4)), where \( c_{2,N}(t) \) is defined in (5.23).

In Theorem 5.1 we shall show that the quadratic variation in Example 5.1 converges to the quadratic variation of the continuous hedging strategy given by the SDE of (5.9). There exists a larger class of sequences of \( d_N \), for which the quadratic variation converges. In the next Proposition we provide a condition such that the expectation of the quadratic variation of the hedging strategy in the odd-even binomial model converges.

**Proposition 5.3** Assume the odd-even binomial model of Section 5.2.2. Let \( \theta \in (0,1) \). Let \( m_N \to \infty \) be integers so that \( d_N = (K - s_0)/m_N \) satisfies \( d_N \sim \gamma \sigma \sqrt{T/N} \), for some \( \gamma \in [2, \infty) \). Then the expectation of the quadratic variation of the hedging strategy, over the time interval \([0, \theta T]\), converges:
\[
E[H]_{\theta N} \to (\gamma^2 - 3)E \int_0^{\theta T} (dH_t)^2, \quad N \to \infty. \tag{5.26}
\]

**Proof.** From Lemma 5.2 it follows that the expectation of the quadratic variation converges to
\[
\sum_{n=1}^{\theta N} E(H_{n+1} - H_n)^2 \sim \gamma^2 \sigma^2 \int_0^{\theta T} c_{2,N}(t) dt. \tag{5.27}
\]
From equation (5.5) and \( d_N = (K - s_0)/m_N \) it follows that
\[
\tilde{q}_N \to \frac{1}{2}(1 - \sqrt{(1 - 4/\gamma^2)}), \quad \tilde{q}_N \to \frac{1}{2}(1 + \sqrt{(1 - 4/\gamma^2)}).
\]
After some calculations we see that the right hand side of equation (5.27) can be rewritten as
\[
(\gamma^2 - 3) \int_0^{\theta T} \frac{1}{2\pi \sqrt{(T^2 - t^2)}} \exp \left( -\frac{K^2}{\sigma^2(T + t)} \right) dt. \tag{5.28}
\]
The latter expression is finite for each \( \theta \in [0, 1) \) and is equal to the right hand side of (5.26).

In the next Theorem we present a condition on the sequence \( d_N \) such that the quadratic variation up to time \( t_n \in [0, T) \) of the hedging strategy converges in distribution to the quadratic variation up to time \( t \in [0, T) \) of the continuous hedging strategy. Note that this does not imply convergence of the quadratic variation process.

**Theorem 5.1** Assume the odd-even model of section 5.2.2. Let \( \theta \in (0,1) \). Let \( m_N \to \infty \) be integers and set \( d_N = (K - s_0)/m_N \). The parameter \( d_N \) satisfies \( d_N/(2\sigma \sqrt{T/N}) \to 1 \) if and only if

\[
[H]_{\theta N} \overset{d}{\to} \int_0^{\theta T} (dH_t)^2 \quad N \to \infty
\]

where \( [H] \) and \( \int (dH_t)^2 \) denote the quadratic variation of the discrete and continuous hedging strategy respectively, and \( \overset{d}{\to} \) denotes convergence in distribution.

**Proof.** Let \( Q_N \) denote

\[
Q_N = \sum_{n=1}^{\theta N} \sum_{s_n \in A_n} (P(Z_{n+3} = K - s_n))^2 1_{\{s_{n} = s_{n}\}},
\]

with

\[
A_n = \{s : s = s_0 + \sum_{i=1}^{n} x_i, x_i \in \{b_i, a_i\}\}.
\]

Suppose \( nT/N \to t \) and \( s_n \to s \) for \( N \to \infty \). We may apply the Local Central Limit Theorem to obtain

\[
\left( \frac{1}{d_N} P(Z_{n+3} = K - s_n) \right)^2 \to \frac{1}{2\pi \sigma^2 (T-t)} \exp \left( -\left( \frac{K-s}{\sigma \sqrt{T-t}} \right)^2 \right), N \to \infty.
\]

Set \( \gamma_N = (1/\sigma)d_N \sqrt{N/T} \). From the Martingale Central Limit Theorem it follows that the sequence \( S_n \) converges weakly to \( \sigma B_t \). Applying this, together with (5.31), to expression (5.30) we obtain

\[
Q_N/\gamma_N^2 \overset{d}{\to} \int_0^{\theta T} \frac{1}{2\pi (T-t)} \exp \left( -\left( \frac{B_t - K/\sigma}{\sqrt{T-t}} \right)^2 \right) dt, \quad N \to \infty.
\]
Since \( E[Q_N] \to E[Q] \), where \( Q = \lim_{N \to \infty} Q_N \), the sequence \( Q_N \) is uniformly integrable. Since \( [H]_{\theta N} < Q_N \), the sequence \([H]_{\theta N}\) is uniformly integrable. So convergence (5.29) implies \( E[H]_{\theta N} \to E \int_0^{\theta T} (dH_t)^2 \). From Proposition 5.3 it follows that \( E[H]_{\theta N} \to (\gamma^2 - 3)E \int_0^{\theta T} (dH_t)^2 \), with \( \gamma_N \to \gamma \). Hence the limit relation (5.29) does not hold when \( \gamma \neq 2 \).

Now suppose \( \gamma_N \to 2 \). It follows from equation (5.5) that \( p_{n+1}^2 \to 1/4 \), \( q_{n+1}^2 \to 1/4 \). Hence,

\[
p_{n+1}^2 1_{A_{n+1}} + q_{n+1}^2 1_{B_{n+1}} \to \frac{1}{4}.
\]

From Proposition 5.2 and (5.33) it follows that \([H]_{\theta N} \sim \frac{1}{4} Q_N\). Combining this with (5.32) it follows that

\[
[H]_{\theta N} \xrightarrow{d} \int_0^{\theta T} \frac{1}{2\pi (T - t)} \exp \left( - \left( \frac{B_t - K/\sigma}{\sqrt{T} - t} \right)^2 \right) dt,
\]

which is equal to (5.29) by equation (5.10).

The next Example shows that the quadratic variation may diverge if the sequence \( d_N \) converges to 0 too slowly for \( N \to \infty \).

**Example 5.2** Set \( d_N \sim \frac{\sigma \sqrt{T}}{\sqrt{N}} \). The expectation of the quadratic variation of the hedging strategy from \( t = 0 \) to \( t = T/2 \) diverges to infinity.

**Proof.** From Lemma 5.2, \( d_N^3 \sim \sigma^3 \sqrt{T} (t_n - t_{n-1}) \), and \( \tilde{p}_N^3 + \tilde{q}_N^3 \to 1 \) (cf. (5.5)) it follows that

\[
d_N \sum_{n=0}^{N/2} E(H_{n+1} - H_n)^2 \to \sigma^3 \sqrt{T} \int_0^{T/2} c_{2,N}(t) dt > 0.
\]

Since \( d_N \to 0 \) we see that

\[
\sum_{n=0}^{N/2} E(H_{n+1} - H_n)^2 \to \infty.
\]

In the next Theorem we will show that the limit of the cubic variation of the hedging strategy under the odd-even binomial model defined in Example 5.2 is finite a.s. In fact the asymptotic variation of the hedging strategy may be of any order. First we will formally define the asymptotic order of the variation for a sequence of discrete time processes.
Definition 5.1 Consider a triangular array of random variables \([W_{N,i}, i = 0, \ldots, N, N > 1]\). Assume that \(\sup_{0 \leq i \leq N} W_{N,i} - W_{N,i-1} \to 0\) a.s. for \(N \to \infty\). We say that the asymptotic variation of the array is bounded of order \(r\), \(r \in [1, \infty)\), if
\[
\sum_{i=1}^{N} E|W_{N,i} - W_{N,i-1}|^{r^*} \to \begin{cases} 
\infty & : r^* < r \\
C(r^*) & : r^* = r \\
0 & : r^* > r, \text{ as } N \to \infty.
\end{cases}
\]
This definition is similar to the definition of the \(p-th\) variation in Karatzas and Shreve (1991), page 32. Let us now present the main result.

Theorem 5.2 Let \(r \geq 2\) and \(\theta \in (0,1)\). Let \(H_n\) be the hedging strategy of a European call option \(C(T, K)\) in the odd-even binomial model of section 5.2.2 with price process \(S_n\). Let \(m_N \to \infty\) so that \(d_N = (K - s_0)/m_N \to 0\) satisfies \(d_N \sim \gamma(1/N)^{1/r}\), where \(\gamma\) is a positive constant. Then \(S_n\) converges weakly to a Brownian Motion, the hedging strategy \(H_n\) converges weakly to the continuous process given by the SDE of (5.9), and there exists a finite positive constant \(c\) such that
\[
\sum_{n=1}^{\theta N} E|H_n - H_{n-1}|^{r^*} \to c
\]
holds as \(N \to \infty\).

Proof. From the Martingale Central Limit Theorem it follows that the sequence \(S_n\) converges weakly to a Brownian motion. As concluded in section 5.3.1, the (constant interpolated) hedging strategy converges weakly to the continuous hedging strategy.

Next to show that the asymptotic variation of the hedging strategy under this binomial tree is bounded of order \(r\). Let \(r^* > 0\). From (5.22) it follows that
\[
\sum_{n=1}^{\theta N} E|H_n - H_{n-1}|^{r^*} \sim d_n^{r^*-r} \int_0^\theta c_{r^*, N}(t)dt.
\]
Since \(\theta T < T\), the integral in (5.34) converges to a positive finite constant. Consequently, we have
\[
\sum_{n=1}^{\theta N} E|H_n - H_{n-1}|^{r^*} \to \begin{cases} 
\infty & : r^* < r \\
C(r^*) & : r^* = r \\
0 & : r^* > r, \text{ as } N \to \infty.
\end{cases}
\]
Hence, the asymptotic variation of the hedging strategy is bounded of order \(r\). \(\square\)
Remark 5.1 If we choose \( d_N \sim \log(N)/N \) then \( \sum_{n=1}^{N} E|H_n - H_{n-1}|^r \) diverges to infinity for all \( r \in [2, \infty) \). In this case \( Nd_N \to \infty \) for all \( r \). It follows that \( \sum_{n=1}^{N} E|H_n - H_{n-1}|^r \to \infty \) for any \( r \geq 2 \).

5.4 Conclusion

For a sequence of binomial tree models on a fixed time interval \([0, T]\) the hedging strategy of a European call option converges to the hedging strategy of the limit process as the number of subdivisions goes to infinity and if condition (5.1) holds. The limit process is a continuous martingale and it has bounded quadratic variation a.s. The approximation of the quadratic variation of the limit process converges in probability for any sequence of partitions with width tending to zero. By embedding trinomial trees into a binomial tree setting we are able to construct examples for which the \( r \)-th order variation of the discrete time hedging strategy may be computed explicitly. It is shown that the \( r \)-th order variation may converge or diverge depending on the choice of the parameter \( d_N \) which describes the size of the increments.