Gromow-Witten Invariants and Elliptic Genera

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Introduction

“When such strings jar, what hope of harmony?” (Shak., King Henry VI)

Always Closer to Unification

String theory, in its modern sense, is the attempt to bring all four forces of nature under one roof, one principle. Although the theory changes rapidly and shows up in a new fashion with every season, its aim is now becoming centennial. It is a rummaging greyhound, but a tireless one.

Ever since the birth of quantum theory a century ago, physicist wondered how it could be incorporated into the known forces governing nature. The electro-magnetic force was very obedient, involving the crucial feature for the coupling to quantum mechanics: the gauge group (in this case Abelian: $U(1)$). This freedom in the choice of phase of an electro-magnetic wave turned out to be a powerful sine-qua-non condition for a theory to be described in terms of the newly-found wave mechanics of the thirties. It is called the gauge principle and is still believed to be the common denominator of all depictions of interactions of matter, a unifying criterion. In the sixties, the weak and strong forces inside the atom’s nucleus were also seen to obey the gauge principle (with gauge group $SU(2)$ and $SU(3)$ resp.), but the gravitational force remained recalcitrant as ever. It still is.

So where are the strings? The unification of all forces of nature is the golden dream of string theory, but the road passes through another unifying goal: to crowd together all animals in the elementary particle zoo into one flock and let each particle consist of the same, tiny fundamental string whose vibrations at different frequencies produce the different masses of the particles. If you wondered how big this string is, beware of disappointment: it’s of “Planck length”, or $10^{-33}$cm long, a distance so small that it will never be probed with any apparatus on earth. No wonder then that the meager predictive power of string theory makes it mala fide in the eyes of many reactionary physicists. And it is good so.1

1“An important scientific innovation rarely makes its way by gradually winning over and converting its opponents.[...] What does happen is that its opponents gradually die out...” (Max Planck)
For elegance rather than precision-matching have always been the hallmark of successful theories. And here, string theory has lots to boast: its solid mathematical foundations, its graceful geometric pillars, stately peristyle and stylish entablature with rich friezes, its refined frontispiece, all converge to shape an elegant work of masonry, a temple for the mind and a feast for the eye. Its connection with several fields of mathematics and stunning contributions make it a favourite to the queen of sciences. Because of its attempt to bring all physical forces under the umbrella of quantum field theory, and because of its god-like status, string theory also goes by the name of “quantising gravity”, or by the pompous “grand unified theory” (GUT), or further by the pretentious “theory of everything” (TOF) – though no one really knows what the final formula should entail, safe perhaps some stuff about strings.

So all matter is now made of fundamental strings. Gone are the four primitive essences (earth, air, fire, water), gone the Pythagoreans’ quintessence, gone Aristotle’s atoms with their flavour, odour and colour, gone the nuclei and electrons, the quarks, leptons, hadrons, etc. They are all of the same breed: stringy substance. What a relief! There is only one building block of the universe, and by swinging on different tunes it does all the job of creation. The language of truth is simple. String theory is all about the same old dream of humanity: about integrating, harmonising, reducing, bridging, uniting, embracing the full extent of cognition, striving towards oneness like the walls of the Gothic heighten towards the keystone in the firmament.

Older “String Theories”

In this sense, string theory is as old as history. Already some 2500 years ago the Greeks were wondering how a single object could entail the secrets of harmony and of divine proportions: they studied the monochord, a single string whose name betrays it was made of guts. The first accounts we have are the contemplations by Pythagoras. He noticed that the frequency of a tune is inversely proportional to the length of a string, that is, halving the string would produce

Pythagoras was no philosopher. According to all presocratic thinkers he was a saint, prophet and founder of a fanatic-religious alliance which would enforce its truths by all political and military means. [...] One has to free oneself from the superficiality of history [...] to see that Pythagoras, Mohammed and Cromwell embody in three cultures one and the same movement [...] puritanism” (Oswald Spengler, Der Untergang des Abendlandes, 1917). Pythagoras’ curriculum vitae promises original views. Not happy with his sunny island of Samos, he fled to Egypt in 535 BC to run for a post of priesthood. Ten years later, the Persian troops ‘brought freedom’ to Egypt and our hero was taken as ‘illegal combatant’ to Babylon, a godsend for someone dreaming to learn Chaldean astronomy under the hanging gardens or the Ishtar gate or on the procession avenue – whose glazed blue bricks would all once carry the frightening name of a fiendish tyrant to come. Nostalgia brought Pythagoras back five years later to his sunny Greek island where he founded his semi-circle, but the local education authority scotched his wheel, so he tried his luck in southern Italy where the public authority was/is conspicuously absent. Nothing is left of Pythagoras’ writings. Only others reported that he viewed the soul as a self-moving number reincarnated in different species until its final purification, and that shape and not matter was the substance of all existing objects.
a double frequency – called an octave because he would subdivide it such that eight successive tones would yield an full octave (corresponding to 12 chromatic semi-tones). He called the tones obtained at 3:2 and 4:3 of the original frequency the quint and the quart respectively (2/3 and 3/4 lengths of string) because they would be 5 or 4 notes away from the ground tone. He then went on to define the unit interval to be the difference between the quint and the fourth. The question for Pythagoras was then: why do a quint and a quart add up to an octave? And why does a quint sound higher than half the octave although its frequency is just at half the octave? He would also define other sounds by the following ratios, rewritten here in the Western way using the eight first letters of the alphabet:

\[
\begin{array}{cccccccc}
C & C\# & D & D\# & E & F & F\# & G & G\# & A & A\# & B & C \\
1/1 & 16/15 & 9/8 & 6/5 & 4/3 & 45/32 & 3/2 & 8/5 & 5/3 & 16/9 & 15/8 & 2/1 \\
\end{array}
\]

Because the scale is not absolute but only relative to the ground tone defining the ratio 1/1, adding two tones in the first line corresponds to multiplying two ratios in the second line. Thus a quint (C-G) and a quart (C-F or G-C) add up to the full octave (C-C), or 3/2·4/3 = 2/1. Likewise, the unit tone (C-D) is the quint minus the quart, or 3/2·3/4 = 9/8. The note F# is exactly in the middle of the octave and should correspond to the ratio closest to \(\sqrt{2}\), i.e. 45/32, and the interval C-F# will sound awkward because it is not a simple ratio as for the quint, quart or major third. Note that only prime numbers up to five are contained in the ratios.

It is quite remarkable that the simpler the ratio, the more beautiful the music to the ear. Somehow harmony is tantamount to simplicity, and our brain is lulled by easy fractions. Moreover, the integral ratios are not just an artificial construct of the ancient Greeks, but are physically reflected in the occurrence of the harmonics (the ground tone of the vibrating string is always accompanied by slight vibration of \(\frac{9}{8}\) of the length of the string: 1/2, 1/3, 1/4, 1/5, etc). In this sense, musical theory was the first subject to unify a good bunch of the Greeks’ logos: mathematics (rational numbers), physics (vibrations of harmonics) and psychology (sensations and harmony). Music made it possible to merge the worlds of ideas, phenomena and emotions. It is flabbergasting that all three – the instrument, the medium and the recipient – have an understanding

\footnote{Strictly speaking, these ratios were put forward by Ptolemy and the scale is called Ptolemaic or just intonation. Pythagoras himself had more complicated ratios, e.g. the tone was E at 81:64 rather than 5:4, corresponding to twice the unit interval, or \(\frac{9}{8}\)^2. Both scales present advantages and shortcomings; the Pythagorean has all whole tones in equal distance of each other (9:8) while the Ptolemaic varies between 9:8 and 10:9; the former has almost all intervals of quints perfect (simple ratio 3:2) but has an awful sounding major third (81:64 instead of 4:3), while the latter has all major thirds perfect (4:3) but has the quint D-A sounding awful (40:27 instead of 3:2). Adding the half-tones (semi-chromatic or “black notes” of the Fiano) adds more dissonances into these two scales, so that a third one was necessary, the mean-tone temperament, where all major thirds are made perfect and the errors of all quints are spread out to keep them minimal. But its distance between chromatic semi-tones is not regular. A fourth scale, the equal temperament, remedies to this; it is defined artificially by equal distances (\(\frac{\sqrt{2}}{2}: 1\), an irrational number! despicable!) between all twelve semi-tones and all keys sound equally good (or bad) – as on the piano.}
for (or preserve) this peculiar feature of rational numbers: that they are ratios of two integers. Much as the compass allows the eye to ‘see’ the number $\pi$ (irrational), a string allows the ear to ‘hear’ rational numbers.

Centuries passed, but man did not give up his striving towards unification and understanding of harmony. Many Greeks\(^4\) delved into the study of music with the intent to coalesce knowledge. After a slumber of a few centuries, the Faustian soul rose again above the magic soul, and the Greek treatises were greedily read and discussed in the divans of the Abbasid Baghdad. The caliph Al-Ma’mun founded the Beit al-Hikma (House of Wisdom) where scholars undertook the Benedictine (or rather jihadic) labour of translating the treasures of Alexandria’s mythical library into the lingua franca of the new empire.

Adding to the ancient body of knowledge, a chain of Arab scholars wrote amply on strings and unison: Al-Kindi, in one of his Seven Treatises, went so far as to bet on cosmological links between the four strings of the Ud and the four seasons of the year, and he fell into raptures at a pathway linking the elements, the humours and stellar constellations. A plethora of authors\(^5\) rivalled on the subject of euphony.

The theories were paralleled by experiments and a wealth of instruments developed: the lute, rebec, guitar, naker, atabal, tymbal, tambourine\(^6\). The abstractions around the rhythm and meter tied the knot with the sentimentality of lyrics and qasidas. They plucked at the heart strings in a variety of ways, including the muwashshah form (in Aleppo)\(^7\). What was originally a sober exercise of the mind now encompassed mystical techniques for mevlevis, whirling dervishes or for the born-again psalmist\(^8\). In the drowsing West, string harmony struggled for its survival (till revived by Marin Mersenne in his Harmonie Universelle, 1627) but was nonetheless kept as one of the four liberal arts of the Quadrivium.

And so, from rational numbers to frequencies, from strings to symmetry, from chordophones to the universe (lit. “turned into one”), human thought strove to unravel the secret passages and shortcuts in the obscure maze of knowledge.

\(^4\)Incl. Aristotle (Problems, De anima), Themistius (Commentaries), Aristoxenus, Euclid (Sectio Canonis, Introduction to Harmony), Nicomachus, Ptolemy (The Harmonics).

\(^5\)Such as Al-Farabi (Kitab Al-Musiqa Al-Kabir, or the Great Book of Music), Ibn Al-Munaggim, Abu Al-Faraj Al-Isfahani (Kitab Al-Aghani), Ibn Sina (or Avicenna, whom Thomas of Aquinas attacked to his own detriment) and Sa’i Ad-Din Al-Urmawi.

\(^6\)All from/via the Arabic: al-‘ud, rabab, qithara (Greek: kithara), naqqara (Persian), attabl, tinbal, tumbur.

\(^7\)Cf. also my tape of Traditional Libyan Songs, or Muwashshahat Libiya – the only one to survive my perambulations :(

\(^8\)Psalterium is from the Hebrew p’samterion, which gave the Arabic al-San‘Teer (or Santoor), a trapezoidal instrument with 72 strings and of – you’ve guessed it – Babylonian style.
The Present Work

Gromov-Witten Theory and Instantons

Our objects counted, maps from Riemann surfaces to a target space, are also referred to as instantons. They first appeared in field theory to characterise relative minima of the euclidean action and were related by Belavin et al (1975) to the non-trivial way in which the gauge potential would transform under an element of the gauge group. They play a growing role in string theory, where the Riemann surface is the string worldsheet; the problem of counting instantons is intimately related to string partition functions. The instantons will be compact holomorphic curves for closed string theory and holomorphic discs for open strings.

For closed string instantons, the physical approach for counting instantons is paralleled by a very well-defined theory developed by algebraic geometers. The latter use moduli spaces of stable maps, on which they integrate pull-backs of differential forms on the target space, to define the Gromov-Witten invariants \( N^g_d \) or \( (1)_{g,\beta} \) (for CY threefold as target spaces). This is the subject of chapter 1.

GW invariants are in general rational numbers and are linearly related to the instanton numbers (or “BPS invariants”) \( n^d_d \). That the latter are integers is a non-trivial fact met in all examples known so far, shadowy for mathematicians but natural for physicists who see in them numbers of D-branes (or “BPS states”) wrapped around particular cycles of the CY threefold. These instanton numbers (or BPS-invariants or Gopakumar-Vafa invariants) are presented in chapter 2.

GW Potential and Heterotic One-Loop Integrals

In chapter 2, we also introduce GW potentials \( F_g \). This is nothing but the genus \( g \) amplitude of topological strings, which is a theory obtained by twisting an N=2 SCFT. The closest string theory among the five 10d theories to describe this model is type IIA, and in this context the \( F_g \) play the role of couplings to the graviphoton field strength in the action. They are exact at genus \( g \), and can thus be computed at strong coupling, also known as the decompactification limit or “M-theory” (11-dimensional).

The full GW potential can be formally rewritten in the form of an infinite product and it is hoped that it has automorphic properties of some sort. The powers occurring in these products contain our instanton numbers.

Another context where the \( F_g \) are exact at one-loop is heterotic string theory, which is related to type IIA by duality. The one-loop integral can be computed via the trick of lattice reduction, and the holomorphic limit agrees with the structure of the constant and non-constant pieces of \( F_g \) in type IIA. This is the subject of chapter 3

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9Non-trivial configurations are those that cannot be deformed continuously to a uniform configuration.
Other prowesses from calculations in heterotic theory involve the full determination of the prepotential, or $F_0$. It goes via an ODE for the gauge coupling that contains integrals of a theta function and a modular form over the fundamental domain $\mathcal{F} = \mathcal{H}/SL(2,\mathbb{Z})$. The latter integrals are resolved by the trick of unfolding the fundamental domain and yield an astounding expression: the logarithm of automorphic products à-la Borcherds. Reading off this result as if we were in type IIA string theory, we could in principle extract our instanton numbers or GW invariants. In other words, there is a salient relationship between GW invariants and automorphic forms – at least for the genus-0 invariants counting rational curves in the target space. Whether this holds for higher-genus invariants was the original task of this thesis and it remains opaque. This is reviewed in chapter 4.

The starting point for the one-loop integrals in the heterotic context are threshold corrections to gauge couplings. These can be computed in the effective field theory and the integrand will be essentially a trace over the internal degrees of freedom. At this stage there is a road junction for the special example of heterotic compactification on $K3 \times T^2$: we can either directly compute the partition function of the model and find the trace over the internal theory to yield $\Gamma_{10,2} E_6/\eta^{24}$, or we can rewrite the abstract integrand using heavy algebra and landing on the so-called “new susy index” (a variant of the elliptic genus). From here, we can anew compute the explicit value of the integrand and we arrive at the same result. It is the author’s merit to have brought these known and scattered results into one common mold and language [Gl-04] in chapter 5.

**Elliptic Genera**

The structures and mathematical tools hidden behind this long chain of dualities, weak- and strong coupling limits and SCFTs are quite enticing. For instance, Borcherds’ construction of automorphic forms goes via lifting of Jacobi forms of index 1 (which are roughly isomorphic to modular forms), and if the latter have zero weight the former admit a product expression. Examples of Jacobi forms include the elliptic genus, a physical-topological object defined by a trace over the Hilbert space of states of an SCFT: $\Phi(q,y) = \text{tr}_{\mathcal{N=4}} (-1)^F q^{L_0 - \frac{c}{24}} y^{J_0}$. It has an explicit expression known for several compactification spaces (CY 1,2,3,4-folds).

For a non-linear sigma model on the complex surface $K3$ (CY two-fold), the above traces (which can be generalised for all combinations of R or NS in the left- and right-moving sectors) yield several topological indices (elliptic genus, Dirac genera, partition functions, $N=4$ characters) that can be written as sums over a finite number of “orbits” of the $N=4$ SCA. The orbits exhibit modular behaviour and can be computed via tensor products of $N=2$ characters from Gepner models (though the ultimate trace should not depend on the model at hand). Herein lies the first major contribution by the author, who computed explicitly the relation between the $N=2$ and $N=4$ characters in several Gepner models. He derived on the way an equation for the cubes of theta functions, and explored [G1-03] the function $\eta^{-1} \sum_{n \in \mathbb{Z}} (-1)^n (6n + 1)^k q^{(6n+1)^2/24}$ for
$k = 1, 2, 3, 4$. This scope is spanned in chapter 6.

**Open String Instantons**

The last chapter closes the loop and comes back to the starting point of this thesis, namely to counting instantons, but for *open strings* (i.e. "disk instantons"). That is, the open holomorphic discs will have their boundaries on a D-brane with the geometry of a special Lagrangian submanifold. According to the model in which we work, this brane will be called the A-brane or B-brane. The instanton numbers are extracted by comparing the superpotentials in both models: for the A-model in the large CY volume limit we obtain the usual instanton sum as corrections to the $4d$ $N=1$ superpotential, while on the B-model side we can compute the superpotential via the Chern-Simons action reduced to the world-volume of the B-brane. Again, the integrality of these disc instantons is striking. In two easy examples of chapter 7, we prove this integrality via methods leading us to derive interesting congruences for binomial coefficients modulo powers of a prime.

The two biggest chapters, 6 and 7, are the crux of the thesis and contain essentially the author's original works. Chapter 5 has its originality lying in its unifying presentation of known calculations, while the first three chapters are introductory material.