Gromow-Witten Invariants and Elliptic Genera
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Citation for published version (APA):

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Chapter 3

Relation with Heterotic Strings

This chapter opens a new part of the thesis, leaving behind the worlds of Gromov-Witten theory or instanton computations and heading towards topics related to the GW potential, both physical and mathematical. The small chapter is devoted to understanding the string theory background of GW theory and its genus $g$ potential. It is purely physical, without derivations, and presents the one-loop integral of the heterotic string amplitude and an attempt at solving it via Jacobi forms. The prerequisites are chapters 2 and appendix A on Jacobi forms; only the new susy index (3.3.2) will appear again later (section 5.2).

The M-theory (or type IIA) approach to Gromov-Witten invariants was very rewarding, as we saw, but another attempt from string theory proves fruitful: this is the calculation in heterotic string theory, following the discovery in 1995 of its description dual to the type IIA theory. Given that the perturbative expansion is governed by the dilaton, that the dilaton lies in a hypermultiplet in type IIA and in a vector multiplet in heterotic, and that the two kinds of multiplets do not mix with each other, the duality then allows us to extract non-perturbative knowledge (instanton corrections,...) in one model from perturbative expansions in the dual model.

3.1 Duality, Moduli and the One-Loop Level

The duality is valid in $4d$ and for $N=2$ susy, and links heterotic theory compactified on $K3 \times T^2$ with type IIA compactified on a CY threefold. Characteristic of this duality is the mapping of the axiodil$ootnote{Our convention for the axiodil or dilaton-axion field is $\langle S \rangle = a + \frac{i}{g_s^2}$ where the string coupling is given by the vev of the dilaton, $g_s = e^{\langle \phi \rangle}$. Hence the limit $S \to \infty$ corresponds to $\phi \to -\infty$ or to $g_s \to 0$, that is, the weak-coupling or perturbative limit.}$ $S$ to one of the Kähler moduli $t$ of CY threefold on the type IIA side. The duality is only valid in the weak-coupling
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limit $S \rightarrow i\infty$, which corresponds to the large-volume limit ($t \rightarrow \infty$) of the type IIA threefold.

In both heterotic and type IIA theories, the prepotential $F_0$ develops logarithmic singularities reminiscent of the Seiberg-Witten analysis: different branches of the enhanced symmetry locus (ESL) collapse in the large moduli limit [AGNT-95]. On the IIA side, this corresponds to the conifold locus in the moduli space of Calabi-Yau 3-folds. On the heterotic side, it corresponds to codimension 1 surfaces of the moduli space where the gauge group $U(1)^{n_v+2}$ is enhanced to $SU(2)$ because two vector multiplets\(^2\) become massless (more details in [G2-04]).

The similar properties of the couplings $F_g$ (or amplitudes $\langle R^4_+ F^{2g-2}_+ \rangle_g$) in heterotic and type IIA theories were seen as a test of this $\mathcal{N}=2$ duality. In order to identify quantities in both theories for a particular configuration of the heterotic gauge group and of the corresponding type IIA compactification, one first needs to specify the map between the moduli, then rewrite the type IIA quantity using heterotic variables and compare it to the result of a heterotic computation in the $S \rightarrow i\infty$ limit.

For instance, for the rank three example with heterotic gauge group Higgsed down to $U(1)^3$ (reviewed in [G2-04]), the massless spectrum has 3 vectors and 129 hypers, corresponding to a CY threefold with $(h^{1,1}, h^{2,1}) = (2, 128)$ for the type IIA theory – say the degree 12 hypersurface in $\mathbb{P}(1,1,2,2,6)$. For this model, the heterotic moduli (dilaton $S$ and torus moduli $T \in O(2, 1)/O(2) \simeq \mathcal{H}$ with duality group $O(2, 1; \mathbb{Z}) \simeq SL(2, \mathbb{Z})$) are mapped to the type IIA Kähler moduli $t_1, t_2$ via $(t_1, t_2) = (T, S/2)$. Three matches between both theories strengthen the duality:

Firstly, the prepotential $F_0$ (or Yukawa couplings $\partial_i \partial_j \partial_k F_0$) can be seen [AGNT-95] to agree with the heterotic prepotential $F_0 = \frac{1}{2} ST + f(T) + \ldots$ in the weak-coupling limit up to the first few powers of $e^{T}$ (for some function $f$ and modulo some instanton corrections to $F_0$). See [G2-04] for more details on this.

Secondly, the $F_g$ satisfy the same holomorphic anomaly equation in the weak-coupling and large-volume limits respectively (i.e. $S \rightarrow i\infty$ and $t_2 \rightarrow \infty$). Indeed, the type IIA (recursive) anomaly equation [BCOV-94], when translated into heterotic context, reads [AGNT-95]:

$$\partial_T F_g = \frac{2\pi i}{(T - T)^2} \left( \partial_T + \frac{2g - 4}{T - T} \right) F_{g-1}.$$  

It can also be checked that the heterotic $F_g$ given below in rather opaque form satisfies this equation.

Thirdly, as another evidence for the duality, one can also check [AGNT-95] that the leading infrared singularity at $T = i$ of both theories agree (that’s where on the type IIA side the two branches of the conifold singularity meet): it is given by $-2\chi(M_g) \left( \frac{2}{\pi i} (T - i) \right)^{2g-2}$, where $\chi(M_g) = B_{2g}/2g(2g - 2)$ is

\(^2\)Note that the rank of the gauge group is the number $n_v$ of vector multiplets (containing the compactification moduli) plus two extra vectors (graviphoton from the sugra multiplet, plus the vector in the vector-tensor multiplet of the dilaton).
the Euler characteristic of the moduli space of genus $g$ Riemann surfaces.

An important difference between the IIA and heterotic calculations is that in the former the $F_g$ are generated at $g$-loop level, while in the latter they all occur at 1-loop level (due to $N=2$ non-renormalisation theorems). The argument is reviewed in [G2-04]: in the context of N=2 Sugra the $F_g$'s have the meaning of moduli-dependent couplings for the low-energy effective action terms $R^2 F_g^{2g-2}$.

These couplings $F_g$ are analytical functions of the N=2 superfields $X^I$, homogeneous of degree $2 - 2g$. The scalar component of $X^0$ is a constrained field and expressible in terms of the Kähler potential $K(Z, \bar{Z})$ for the unconstrained moduli $Z = X/X^0$. Correspondingly, the scalar component of $F_g(X)$ has the following dependence on the string coupling and the Kähler potential [AGNT-93]:

$$F_g(X) = (X^0)^{2-2g} F_g(Z) = \left( e^{K/2} g_s \right)^{2-2g} F_g(Z)$$

Furthermore, the Kähler potential $K$ for vector multiplets depends on the string coupling $g_s$ via the dilaton: this dependence is nil in the case of type IIA (as the dilaton is in a hypermultiplet), and $\log g_s^2$ in the case of heterotic strings. Accordingly, $F_g(X)$ is of order $g_s^{2g-2}$ or $g_s^0$ respectively in the string coupling. Counting string loops, this means $g$-loop or one-loop respectively, as originally claimed.

Let us inspect the heterotic context more closely. We give the one-loop expression for $F_g$ in the next section before presenting the result of its computation in the section afterwards.

3.2 The One-Loop Integral for $F_g$

Let us now present the heterotic expression for the couplings $F_g$ at one-loop, which will be calculated in heterotic theory on $K3 \times T^2$ with susy in the right-moving sector. As in type IIA, they have the meaning of couplings for the $R^2 F_g^{2g-2}$ terms, i.e. amplitudes involving two gravitons and $2g-2$ graviphotons. We briefly sketch the derivation of [AGNT-95].

The $F_g$ are computed using the odd spin structure on the worldsheet torus with insertion of vertex operators for the gravitons $(V_{R+})$ and graviphotons $(V_{F+})$:

$$V_{R+}(p_1) = (\partial Z^2 - i p_1 \chi^1 \chi^2) \bar{\partial} Z^2 e^{i p_1 Z^1} \quad V_{R+}(\bar{p}_2) = (\partial \bar{Z}^1 - i \bar{p}_1 \bar{\chi}^2 \chi^2) \bar{\partial} \bar{Z}^1 e^{i \bar{p}_1 Z^2}$$

$$V_{F+}(p_1) = (\partial X - i p_1 \chi^1 \Psi) \bar{\partial} Z^2 e^{i p_1 Z^1} \quad V_{F+}(\bar{p}_2) = (\partial \bar{X} - i \bar{p}_2 \bar{\chi} \chi) \bar{\partial} \bar{Z}^1 e^{i \bar{p}_2 Z^2},$$

for right-moving space-time bosons $Z$ and their superpartners $\chi$ (in a complex basis), and for right-moving $T^2$ bosons $X$ and their superpartners $\Psi$. The momentum factors $p_i$ bring out a kinematic term in front of the amplitude

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3. Except for $F_0$ and $F_1$ which also carry tree-level contributions
\[ (R^2 F_+^{2g-2})_g = \left\{ V_h(p_1) V_h(p_2) \prod_{i=1}^{g-1} V_F(p_1^{(i)}) \prod_{j=1}^{g-1} V_F(p_2^{(j)}) \right\} = (p_1)^2 (p_2)^2 \prod_{i,j=1}^{g-1} P_1^{(i)} P_2^{(j)} (g)^2 F_g. \]

For \( F_g \), we only mention the ingredients leading to the end result, not expecting the reader to understand fully. The graviton vertices absorb the space-time fermions, while the graviphotons contribute \((p_R/\sqrt{2T_2U_2})^{2g-2}\), which will be summed with \(q^{1/2} |p_L|^2 q^{1/2} |p_R|^2\) over the \( \Gamma_{2,2} \) lattice of the torus. The two left-moving space-time (transverse) bosons and the extra free boson for the left-moving \( U(1) \) current yield \( 1/\eta^3 \), whereas the conformal blocks of the internal SCFT generate the \( K3 \) partition function \( C_{K3}(\tau) = \text{tr}_{r_{RR}} (-1)^F R \eta^{L_0-c/24} \eta^{L_0-c/24} \).

The latter is independent of \( q \) (or \( \bar{\tau} \)), as usual due to Susy for the massive modes and to the absence of instanton contributions: one could always make them vanish by going to large volume of \( K3 \), which only deforms the hypermultiplets (containing the Kähler moduli) and not the vector multiplets on which \( F_g \) depends. Overall [AGNT-95]:

\[ F_g = \frac{1}{2 \pi^2} \frac{1}{(g!)^2} \int_F \frac{d^2 \tau}{\tau_2^3} \frac{1}{\eta^3} \left\{ \prod_{i=1}^{g} \int_{\tau_2} d^2 x_i Z^1 \partial Z^2(x_i) \prod_{j=1}^{g} \int_{\tau_2} d^2 y_j \bar{Z}^2 \partial \bar{Z}^1(y_j) \right\} \times C_{K3}(\tau) \sum_{r_{2,2}} \left( p_R/\sqrt{2T_2U_2} \right)^{2g-2} q^{1/2} |p_L|^2 q^{1/2} |p_R|^2. \]

This can be simplified by evaluating the correlator for space-time bosons and summing \( F_g \) over \( g \) with the auxiliary variable \( \lambda \) to build the full GW potential \( F = \sum_g \lambda^{2g} F_g \) (note the missing factor of \( \lambda^{-2} \) compared to previous versions).

More precisely, the correlator \( \langle \int_{\tau_2} Z \partial Z \rangle \) is summed over \( g \) with \((\frac{\lambda}{\tau_2})^{2g}\) and \( \frac{1}{(g!)^2} \) to yield the function

\[ \left( \frac{2 \pi i \lambda \eta^3}{\vartheta_1(\tau, \lambda)} \right)^2 e^{-\pi \lambda^2/\tau_2} \]

which is modular invariant under \( PSL(2, \mathbb{Z}) \), i.e. under \( \tau \rightarrow \frac{a \tau + b}{c \tau + d} \) and \( \lambda \rightarrow \frac{\lambda}{c \tau + d} \).

The full generating function for the amplitudes at all genera can then be written as

\[ F(\lambda, T, U) = \sum_{g \geq 1} \lambda^{2g} F_g = \frac{1}{2 \pi^2} \int_F \frac{d^2 \tau}{\tau_2} \frac{C_{K3}(\tau)}{\eta^3} \sum_{r_{2,2}} \left( \frac{2 \pi i \lambda \eta^3}{\vartheta_1(\tau, \lambda)} \right)^2 e^{-\pi \lambda^2/\tau_2} q^{1/2} |p_L|^2 q^{1/2} |p_R|^2, \]

(3.2.1)

where \( \lambda := \frac{p_R \tau_2}{\sqrt{2T_2U_2}} \). In [MM-98], the quantity \( C_{K3}/\eta^3 \) takes the value of \( E_1 E_6/\eta^{24} \), in agreement with the \( K3 \) elliptic genus for a \( \mathbb{Z}_2 \) orbifold of heterotic compactification with gauge group \( E_8 \times E_7 \times SU(2) \), as we shall see in (5.2.2).
or \((5.3.8)\). We note that the lattice sum is not simply over \(q^{1/2}|\rho L|^2 q^{1/2}|\rho R|^2\) but also involves the \(p_R\) contained in \(v_1(\tau, \lambda)\).

Note also that \(w.r.t. (\tau, \bar{\tau})\), \(\tau_2\) has weight \((-1, -1)\), \(d^2 \tau \overline{\tau}_2\) has weight \((-1, -1)\), \(E_6 E_6/\eta^{24}\) has weight \((-2, 0)\) while the remaining parts of the integrand have no well-defined weights. It is only upon expanding \(\left(\frac{2\pi i \lambda \gamma_3}{\vartheta_1(\tau, \lambda)}\right)^2 e^{-\pi \lambda^2/\tau_2}\) in even powers of \(\lambda\), say \(\lambda^{2k+2}\), and keeping \(\tilde{p}_R^{2k}\) for the lattice sum, that we obtain a suitable generalised theta function of weight \(1, 1 + 2k\): \(\Theta(T, U) := \sum_{\gamma \in \Gamma_{2,2}} \tilde{p}_R^{2k} q^{1/2}|\rho L|^2 q^{1/2}|\rho R|^2\).

## 3.3 Computing the Integral

We now present the way [MM-98] computed the one-loop integral \((3.2.1)\). They placed themselves in the \textit{rank four example} with heterotic gauge group Higgsed down to \(U(1)^3\) (see [G2-04]). The massless spectrum has 4 vector multiplets and 244 hypers, corresponding to a CY threefold with \((h^{1,1}, h^{2,1}) = (3, 243)\) for the type IIA theory — say the degree 24 hypersurface in \(\mathbb{P}(1, 1, 2, 8, 12)\) (a K3 fibration). For this model, the heterotic moduli are the dilaton \(S\) and the torus moduli \((T, U) =: y\) in the Narain moduli space \(\mathcal{N}^{2,2} = O(2, 2)/O(2) \times O(2) \simeq \mathcal{H} \times \mathcal{H}\) with duality group \(O(2, 2; \mathbb{Z}) \simeq SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \times \mathbb{Z}_2\). These are mapped to the type IIA Kahler moduli via \((t_1, t_2, t_3) = (U, S, T - U)\) [KLM-95], in which we identify \(t_2\) as the modulus for the base \(\mathbb{P}^1\) of the K3 fibration. Thus sending \(S \to i\infty\) means infinite volume of the base of the K3-fibration.

### 3.3.1 Lattice Reduction Technique

The evaluation of the above integral \((3.2.1)\) over the fundamental domain is a \textit{tour de force} [MM-98]. It is based on Borcherds' recursion formula \([B-96]\) for such automorphic integrals or \textit{theta transforms} (as in section 4.2) for a lattice of type \(\Gamma_{s+2,2}\) (even, self-dual, \(8|s\)): at each step the lattice for the theta function is reduced by two dimensions. We spare the reader this cumbersome procedure, but quote again just the results.

Although the results for \(F_g\) look rather messy, their holomorphic limit \(T, U \to \infty\) \((T, U\) fixed\) is quite simple. One does indeed recover the constant map contribution \(F_g^{cl} = \langle 1 \rangle_{g,0}\) (reviewed in [G2-04]). The result for non-constant maps is similar to the \(F_g^{\text{qu}}\) in \((2.3.4)\):

\[
F_g = F_g^{cl} + F_g^{\text{qu}} \simeq -(2g-1) \zeta(2g) \zeta(3-2g) \frac{\chi(X)}{2} + \sum_{r>0} c_{g-1}(\tau^2_2) \text{Li}_{3-2g}(e^{2\pi i r \cdot y}).
\]

The first factor is \((2\pi)^{2g}/2\) times the expected \(\langle 1 \rangle_{g,0}\). In the second factor \((F_g^{\text{qu}}), y = (T, U)\) is the heterotic parameter, \(r = (n, m)\) is a point in the lattice \(\Gamma_{1,1}\) with scalar product \(r \cdot y = nU + mT\) and norm \(r^2 = 2nm\). The condition \(r > 0\) stands for \(n, m \geq 0\) or \((n, m) = (1, -1)\) but not \((0, 0)\). The \(c_{g-1}(n)\) are the
Fourier coefficients of the modular function occurring in (3.2.1):

\[
\frac{E_4 E_6}{\eta^{24}} \left( \frac{2\pi i \lambda \eta^3}{\vartheta_1(\tau, \lambda)} \right)^2 e^{-\pi \lambda^2} = \sum_{g \geq 0 \atop m \geq -1} c_{g-1}(m) \lambda^{2g-2} q^m ,
\]

with \( q := e^{2\pi i \tau} \) and \( E_4 E_6/\eta^{24} \) stands for the \( K3 \) elliptic genus \( C_{K3}/\eta^3 \) for the heterotic \( Z_2 \) orbifold where the gauge group \( E_8 \times E_7 \times SU(2) \times U(1)^4 \) has been Higgsed down to \( U(1)^4 \). That is, we study the rank four example with 244 hypers and 4 vectors (see [G2-04]).

The computation suggests that the result should depend on the region of moduli space, and that a wall-crossing formula will relate different regions. Here, the wall happens to be the codimension one surface \( T_2 = U_2 \). By chance, the wall-crossing behaviour vanishes in the holomorphic limit, as expected from the type II A side where this behaviour reflects the fact that two CY threefolds related by flop transition are birationally equivalent (i.e. same Hodge numbers).

Moreover, by organising the terms in the same way as the type II A result, [MM-98] were able to extract from \( F_2 \) the number of genus 2 curves on a particular CY threefold. To this end, recall from (2.3.4) that

\[
F_2 = \sum_{d > 0} \left( \frac{1}{240} n_d^0 + n_d^2 \right) \text{Li}_{-1}(q^d),
\]

from which we see that \( c_1(A^2) \sim \frac{1}{240} n_d^0 + n_d^2 \). For \( g = 2 \) the \( c_1(A^2) \) can easily be generated in closed form:

\[
g = 2 : \quad \sum c_1(n) q^n = \frac{E_4 E_6}{\eta^{24}} \frac{\pi^4}{90} (5E_2^2 - E_4). \quad (3.3.1)
\]

Hence, if we know the numbers \( n_d^0 \), we can extract the \( n_d^2 \). And indeed they are known:

As mentioned at the beginning of this section, our massless spectrum of 244 hypers and 4 vectors is such that there exists a CY threefold on which to compactify the dual type II A theory. From section 4.4, this is the degree 24 hypersurface in \( \mathbb{P}(1,1,2,8,12) \), which also happens to be a \( K3 \) fibration over \( \mathbb{P}^1 \) where the \( K3 \) is a degree 12 hypersurface in \( \mathbb{P}(1,1,4,6) \) [KLM-95]. Better still, that section tells us that the number \( n_d^2 \) of rational curves is known and equals the coefficients of the (nearly) holomorphic modular form \( E_4 E_6/\eta^{24} \), and so we can easily compute recursively the number \( n_d^2 \) of genus 2 curves.

Since we implicitly work in the limit \( S \to i \infty \) and since \( S \) is mapped to the Kähler modulus of the base of the CY fibration, this base gets infinite volume and all curves we count lie in \( K3 \) fibres. Of course, given the nature of (2.3.4), we could subsequently compute the number \( n_d^3 \) of genus three curves, etc., but unfortunately the \( c_{g-1}(n) \) do not enjoy a closed expression as in (3.3.1) anymore.\(^4\)

\(^4\) It should also be mentioned that [MM-98] took a different form for \( F_2 \) than ours, namely \( \sum_{d > 0} \left( \frac{1}{240} n_d^0 \text{Li}_{-1}(q^d) + n_d^2 q^d \right) \) as in [BCOV-94]. So what they call the 'number of genus two curves' is actually \( n_d^2 \), while we rather associate it with \( n_d^3 \). This can be considered as another petty difference between the mathematical and physical invariants (reviewed in [G2-04]).
3.3.2 Attempt Using Properties of Jacobi Forms

We now present a different approach to hoping to solve the integral in (3.2.1), based on the linear decomposition of Jacobi forms, a trick used in [N-98] to compute one-loop amplitudes in N=2 SCFT, or integrals of elliptic genera. Specifically, the latter author considered a target space $T^2 \times K$ for some CY two-fold $K$, with a Wilson line that will ultimately break this product of target space.

Integrals of elliptic genera over the fundamental domain have also been tackled from a different perspective in [N-98]. Rather than integrating the usual elliptic genus, the author slightly altered that index to obtain the so-called new *supersymmetric index*, and studied its integration. It can be shown that this new index, which is a trace over the left- and right-moving Ramond sectors, conveniently factorises when the compactification space is a product $T^2 \times K$:

$$\text{tr}_{n \times r} (-1)^F J_0 \tilde{J}_0 q^{L_0 - \frac{c}{8}} \tilde{q}^{L_0 - \frac{c}{8}} = \left( \sum_{(pL,pR) \in \Gamma_{2,2}} q^{\frac{1}{2} p_L^2} \tilde{q}^{\frac{1}{2} \tilde{p}_R^2} \right) \left( \text{tr}_K (-1)^F q^{L_0 - \frac{c}{8}} \tilde{q}^{\tilde{L}_0 - \frac{c}{8}} \right),$$

(3.3.2)

where $c$ is the dimension of the target space (3 in our case). The bracketed sum is the famous bosonic partition function for the torus, coming from the trace over the torus part of the threefold $K \times T^2$. The second part comes from the two-fold $K$ and has to do with its elliptic genus at $z = 0$, denoted $\Phi(\tau, 0)$.

Let us briefly comment on the properties of the elliptic genus. Its full expression reads $\Phi(\tau, z) = \text{tr}_{n \times r} (-1)^F q^{L_0 - \frac{c}{8}} y^{J_0}$, see also section 3.4 and (6.3.1), and it is a weak Jacobi form of weight 0 and index $\hat{c}/2$ (see appendix A.3). For two-folds with $SU_2$ holonomy (two-tori, $K3$ surfaces), its expression is well-known [KKY-93]; for $c = 1, 2, 3$ theories, $\Phi(\tau, 0)$ loses its $\tau$-dependence and boils down to the *Witten index* of the theory, i.e. to $\text{tr} (-1)^F = \chi(K)$, the Euler character of the target space. For general models, one recovers the Euler character only upon taking the limit $\lim_{\tau \to i \infty} \Phi(\tau, 0)$.

To make things more interesting, [N-98] introduced a Wilson line, which resulted in $\Phi(\tau, 0)$ not being merely $\chi(K)$ anymore. This inserted an extra compactification modulus (next to $T$ and $U$ for the torus) which prevented from writing the CY threefold as a direct product $K \times T^2$. In that situation, the decomposition property (A.1.2) of Jacobi forms came as a rescuer and helped computing the integral. The property states that a Jacobi form can be written as a finite linear combination of theta functions with modular forms as coefficients. For $z = 0$ the claim reduces to

$$\Phi(\tau, 0) = \sum_{\mu \mod 2m} h_\mu(\tau) \sum_{r \equiv \mu \mod 2m} q^r / 4^m,$$

(3.3.3)

and $m = \frac{c}{2} = 1$. The crux is that the extra summation of $q^r / 4^m$ over $r$ could be joined with $q^{\frac{1}{2} p_L^2}$ in the above (3.3.2), resulting in an overall sum over a $(3, 2)$ lattice rather than just over $\Gamma_{2,2}$. In other words, one simply ended up with a new Siegel theta function of signature $(3, 2)$, something one could integrate over the fundamental domain – using the well-known technique of ‘unfolding’.
Do we stand a chance for a similar trick for the integral in (3.2.1)? Indeed, we recognise $-\eta^6/\vartheta_1(\tau, z)^2$ to be the inverse of the weak Jacobi form $A_{-2,1}$ studied in appendix A.4. Unlike for the elliptic genus above, $z$ is not set to 0 but contains factors of $\lambda$ (string coupling) and $\tau_2$. But this should not discourage us, since these $\tau_2$ factors may be joined with the $\tau_2$ factors in $q^{\frac{1}{2}}p_2^\frac{1}{2}q^{\frac{1}{2}}p_2^\frac{1}{2} = e^{\pi i \tau p^2 - 2\pi \tau_2 p_2^2}$ occurring in the Siegel theta function.

Yet a more serious point flaws our attempt to solve the integral by changing the Siegel theta function into a theta function of signature (3,2), namely the fact that inverses of Jacobi forms do not enjoy the decomposition (A.1.2) into linear combinations of theta functions. This too is explained in appendix A. So the trick of [N-98] cannot be used here.

### 3.4 (0,2) Heterotic Compactn. and Elliptic Genus

We now leave the realm of N=2 heterotic compactifications on $K3 \times T^2$ for a small sidestep of N=1 compactifications on CY manifolds. Although we lose the notion of duality with type IIA theory, we gain a link with the elliptic genus and Jacobi Forms.

Compactification on CY one-, two- or threefolds leaves us with an (0,2) sigma model (reviewed in [G2-04]). To study the elliptic genus, it is even better to consider (0,2) sigma models in generality, i.e. with a $d$-dimensional CY manifold $X$ as target space and a rank $r$ holomorphic vector bundle $E$ on it ($r \geq d$ assumed). The bosonic chiral fields $\Phi_i, \bar{\Phi}_i$ are (anti-) holomorphic coordinates on $X$, while the fermionic chiral fields $\Lambda_i, \bar{\Lambda}_i$ are local (anti-) holomorphic sections of $E(\bar{E})$.

In case $E = T_X$, we recover the (2,2) sigma model (with embedding of the spin connection in the gauge connection if we are in the heterotic context). Anomaly cancellation requires the usual heterotic conditions (if $E$ were the gauge bundle):

$$c_1(E) = c_1(X) = 0, \quad c_2(E) = c_2(X),$$

which are equivalent to the CY condition $c_1(X) = 0$ together with $c_1(E) = 0$ and $\text{ch}_2(E) = \text{ch}_2(X)$.\(^5\)

This section is based on [KaM-94], which presents a generalised notion of elliptic genus, with a dependence on the bundle $E$.

\(^5\)Recall the standard definitions of Chern classes and characters for a bundle $E$ and curvature 2-form $F$ with eigenvalues $\epsilon_i$:

$$c(E) = \det(1 + \frac{F}{2\pi}) = \prod_{i=1}^r (1 + \epsilon_i) = 1 + c_1 + \cdots + c_r, $$

$$\text{ch}(E) = \text{tr} e^{\frac{F}{2\pi}} = \sum_{i=1}^r \epsilon_i = r + c_1 + \cdots + \text{ch}_d$$

$$\text{td}(E) = \prod_{i=1}^r \frac{e^{\epsilon_i}}{1 - e^{-\epsilon_i}} = 1 + \frac{1}{2} c_1 + \frac{1}{12} (c_1^2 + c_2) + \frac{1}{24} c_1 c_2 + \cdots,$$

so that $c_1 = c_1 = \sum_{i=1}^r \epsilon_i$, $c_2 = \sum_{i<j} \epsilon_i \epsilon_j$, $\text{ch}_k = \frac{1}{k!} \sum \text{ch}_k^k$. Note that $c_r = \epsilon_1 \cdots \epsilon_r$ and that $c_k = 0$ for $k > d$ for dimensional reasons. The relation between $\text{ch}_k$ and $c_k$ is given by Newton's formula: $\sum_{k=0}^r (-1)^k k! \text{ch}_k c_{r-k} = 0$. Thus we obtain:

$$\text{ch} = r + c_1 + \frac{1}{2} (c_1^2 - 2c_2) + \frac{1}{3!} (3c_3 - 3c_2c_1 + c_1^3) + \frac{1}{4!} (-4c_4 + 4c_2c_3 + 2c_2^2 - 4c_2c_1^2 + c_1^4) + \cdots$$
Elliptic Genus

The elliptic genus of this sigma model with bundle $E$ is geometrically defined as the index of the Dirac operator for a certain vector bundle $E_{q,y}$:

$$
\Phi_E(\tau, z) = \text{ind } \mathcal{D}_{E_{q,y}} = \int_X \chi(E_{q,y})\, \text{td}(X)
$$

$$
E_{q,y} := (-i)^{r+d}q^{r-d}y^{-\frac{r}{2}} \bigotimes_{n \geq 1} \left( \bigwedge_{-yq^{n-1}} E \otimes \bigwedge_{-y^{-1}q^n} \tilde{E} \otimes S_q^nT_X \otimes S_q^n\tilde{T}_X \right),
$$

(3.4.1)

and $\bigwedge_q E = \bigoplus_{k \geq 0} q^k \bigwedge^k E$, $S_q E = \bigoplus_{k \geq 0} q^k S^k E$. ($\Lambda^k$ and $S^k$ denote the $k$'th exterior and symmetric products.)

We can transform this to a more tractable expression by noting that\(^6\)

\[
\chi \left( \bigotimes_{n \geq 1} \bigwedge_{-yq^{n-1}} E \right) = \prod_{n \geq 1} \prod_{i=1}^r (1 - e^{\psi_i} y q^{n-1})
\]

\[
\chi \left( \bigotimes_{n \geq 1} S_q^n T_X \right) = \prod_{n \geq 1} \prod_{j=1}^d \frac{1}{1 - e^{-x_j} q^n}.
\]

Using also $\prod_{j=1}^d (1 - e^{-x_j}) = (-1)^d \prod_{j=1}^d (1 - e^{-x_j})$ (since $c_1(X) = 0$), we can rewrite our elliptic genus as

$$
\Phi_E(\tau, z) = (-iq^{\frac{1}{2}r-d}y^{-\frac{r}{2}} \int_X \prod_{n \geq 1} \prod_{i=1}^r (1 - e^{\psi_i} y q^{n-1})(1 - e^{-\psi_i} y^{-1} q^n)
\]

\[
\eta^{d-r} \int_X \prod_{j=1}^d \vartheta_1(z + \frac{x_j}{2\pi i}) \prod_{i=1}^r \vartheta_1(\frac{x_i}{2\pi i})
\]

(3.4.2)

In case $c_1 = 0$, the Chern character and Todd class reduce to

$$
\chi = r - c_2 + \frac{1}{2} c_3 + \frac{1}{12} (c_2^2 - 2c_4) + \ldots,
$$

$$
\text{td} = 1 + \frac{1}{12} c_2 + \frac{1}{6!} (3c_2^2 - c_4) + \ldots
$$

Similar formulae hold for the bundle $T_X$, and we denote the eigenvalues of its curvature 2-form by $x_i$.

\(6\)Use $\chi(E \otimes F) = \chi(E) \wedge \chi(F)$ and $\chi(E \oplus F) = \chi(E) + \chi(F)$, which also implies that

$$
\chi(\Lambda^k E) = \text{kth elementary symmetric function of the } e^{\psi_i} = \sum_{i_1 < \ldots < i_k} e^{\psi_{i_1} \ldots e^{\psi_{i_k}}} = (1^k),
\]

$$
\chi(S^k T_X) = \text{kth homogeneous product sum of the } e^{x_i} = h_k(e^{x_j}),
\]

where $h_1 := (1)$, $h_2 := (2) + (1^2)$, $h_3 := (3) + (21) + (1^3)$, etc, and $1 + h_1 + h_2 + \cdots = \prod_{i=1}^d 1/(1 - e^{-x_j})$. The notation here is that of [M-1915], which is a good account of the combinatorics of symmetric functions; most of the elementary formulae already occurred in Girard's founding treatise [G-1629]. In general, $\chi(\Lambda^k E)$ is computed through a product involving $\binom{n}{k}$ terms [H-78]:

$$
c(\Lambda^k E) = \prod_{i_1 < \ldots < i_k} (1 + x_{i_1} + \cdots + x_{i_k})
\]

$$
= 1 + \binom{r-1}{k-1} c_1 E + \left( \binom{r-2}{k-1} c_2 E + \frac{1}{2} \binom{r-1}{k-1} (\binom{r-1}{k-1} - 1) c_1 E^2 \right) + \ldots
\]

For $k = r$, $\Lambda^r E$ is a bundle of rank 1, and so $c(\Lambda^r E) = 1 + c_1$, $\chi(\Lambda^r E) = \sum_{i \geq 0} \frac{1}{i!} c_i$ ($= 1$ if $c_1 = 0$).
which differs by a factor \((-1)^d\) from that of [KaM-94]. Note that \(c_d(X) = x_1 \ldots x_d\) is a class of top dimension, but it will cancel with the denominator since \(\vartheta_1(x)/x = 2\pi y^3 + O(x^2)\). Only the terms of degree \(d\) are picked by the integral.

The topological expression for the elliptic genus will be given in the first line of (6.3.1).

**Landau-Ginsburg and Minimal Models**

In particular, if \(E = T_X\) (for a \((2,2)\) sigma model), we have \(r = d\) and \(e_i = x_j\), and so

\[
\Phi_{T_X} (\tau, z) = \int X \prod_{j=1}^{d} \frac{\vartheta_1(z + \frac{x_j}{2\pi i})}{\vartheta_1(\frac{x_j}{2\pi i})} x_j, \quad \Phi_{T_X} (\tau, 0) = \chi(X),
\]

where at \(z = 0\) we recover the Euler character. The formula for \(\Phi_{T_X} (\tau, z)\) was already suggested in [KKY-93], as a generalisation of the 2- and 3-dimensional cases; the latter case of a CY threefold of Euler character \(\chi\) reads [KKY-93]:

\[
\Phi_{CY_3} (\tau, z) = \frac{\chi}{2} y^{-1/2} \prod_{n \geq 1} \frac{(1 - y^n q^{n-1})(1 - y^{-2} q^n)}{(1 - y^{1/n} q^{n-1})(1 - y^{-2/n} q^n)}.
\]

(3.4.3)

A similar form of elliptic genus is the one for an N=2 LG model with \(n\) chiral superfields of weights \(\omega_1, \ldots, \omega_n\) [W-93]; it exactly matches its counterpart expression in the N=2 minimal model [KKY-93] (rewritten with help of the quintuple identity):

\[
\Phi_{LG} (\tau, z) = \prod_{i=1}^{n} \frac{\vartheta_1((1 - \omega_i)z)}{\vartheta_1(\omega_i z)} = \prod_{i=1}^{n} q^{-1/6} \prod_{n \geq 1} \frac{(1 - y^{2/3} q^{n-1})(1 - y^{-2/3} q^n)}{(1 - y^{1/3} q^{n-1})(1 - y^{-1/3} q^n)} = \Phi_{MM} (\tau, z).
\]

This is of course of utmost significance for the correspondence between LG models and minimal models.

**Transforms like Jacobi Forms**

If we view the curvature of \(E\) only defined up to a multiple, \(i.e.\ e_i = \lambda e_i\), the transformation properties of the elliptic genus are those of a Jacobi form of weight 0 and index \(r/2\) (at least if \(r\) is even – see appendix A):

\[
\Phi_E \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = \epsilon^{r-d} e^{2\pi i \frac{c z^2}{c \tau + d}} \Phi_E (\tau, z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}),
\]

\[
\Phi_E (\tau, z + \lambda \tau + \mu) = (-1)^{r} (\lambda + \mu) e^{2\pi i \frac{c z^2}{c \tau + d}} \Phi_E (\tau, z), \quad \lambda, \mu \in \mathbb{Z},
\]

(3.4.4)

where \(\epsilon\) is the same phase (eighth root of unity) as the one required for the transformation of \(\vartheta_1\).
We now introduce the Hirzebruch genus

$$\chi_y(E) := \sum_{k=0}^{r} (-y)^k \chi(\Lambda^k E) = \sum_{k=0}^{r} (-y)^k \int_X \text{ch}(\Lambda^k E) \td(X),$$

where the second equality is just the RR-Hirzebruch theorem. In terms of it, the elliptic genus has the expansion

$$\Phi_E(\tau, z) = (-1)^{d}(-iz)^{d/2})^{-d}y^{-z/2} \left( \chi_y(E) + q \sum_{k=0}^{r} \left[ (-y)^{k+1} \chi(\Lambda^k E \otimes E) + (-y)^{k} \chi(\Lambda^k E \otimes (T_X \oplus T_X)) \right] + \mathcal{O}(q^2) \right).$$

Thus the $q$-expansion in large brackets starts with the constant term $\chi_y(E)$. It contains the $r$ expressions $\chi(\Lambda^k E)$ in terms of the $r$ classes $c_i(E)$ (and the $c_i(X)$), which can thus be deduced together with the eigenvalues $e_i$. Hence $\Phi_E(\tau, z)$ is fixed if $\chi_y(E)$ and the CY threefold $X$ are known.

If $E = T_X$, i.e. in case of a $(2,2)$ sigma model, $\Phi_E$ is indeed a weak Jacobi form of weight 0 and index $m = r/2 = d/2$ (see appendix A.3). The space of such forms is $m$-dimensional, so $m$ points should be enough to uniquely determine $\Phi_E(\tau, z)$. Also, $y^{-z/2}\chi_y(E)$ is a Laurent polynomial in $y$ of degree at most $m^2$ and its $m^2$ coefficients provide more information than the necessary $m$ points (on the divisor $q = 0$ ($\tau = \infty$)) to fix $\Phi_E$. That is, the constant $q^{r/2}$-term $\chi_y(E)$ together with the transformation properties (3.4.4) determine uniquely the form $\Phi_E(\tau, z)$ if $E = T_X$.

**Hirzebruch Genus**

Let us now compute $\chi_y(E) = \int_X \left( \sum_{k=0}^{r} (-y)^k \text{ch}(\Lambda^k E) \right) \td(X)$ for low dimensions of the CY d-fold, using the formula [F-84]

$$\sum_{k=0}^{r} (-1)^k \text{ch}(\Lambda^k E) = c_r(E) \td^{-1}(E) = \prod_{i=0}^{r} \left( 1 - y e^{-e_i} \right).$$

Inserting the variable $y$, we have

$$\sum_{k=0}^{r} (-y)^k \text{ch}(\Lambda^k E) = \prod_{i=0}^{r} (1 - y e^{-e_i}) = \prod_{i=0}^{r} (1 - y + y(e_i - \frac{e_i^2}{2} + \frac{e_i^3}{3!} + \ldots)).$$

When $c_1 = 0$, this yields (with $c_i := c_i(X)$ by default):

$$\chi_y(E) = \int_X \left( (1 - y)r + (1 - y)^r - 1 y[c_2 + \frac{c_3(E)}{2} - \frac{c_2^2 - 2c_4(E)}{12} + \ldots] \right. \right.$$

$$+ (1 - y)^{r-2} y^2[2c_2 - (c_2c_1 - 3c_3) + (c_3c_1 - 4c_4) + \frac{1}{2}(c_2^2 - 6c_4) + \ldots]$$

$$+ (1 - y)^{r-3} y^3[3c_2 - (c_2c_1 - 3c_3c_1 + 4c_4)] + \ldots$$

$$+ (1 - y)^{r-4} y^4[c_4 + \ldots] + \ldots$$

$$\left. \left. + (1 - y)^{r-5} y^5[5c_2 - (c_2c_1 - 3c_3c_1 + 4c_4)] + \ldots \right) \right) \left( 1 + \frac{1}{12} c_2 + \frac{1}{60}(3c_2^2 - c_4) + \ldots \right),$$
CHAPTER 3. RELATION WITH HETEROTIC STRINGS

or explicitly for the different dimensions $d$:

- $d = 1$: $\chi_y(E) = 0$
- $d = 2$: $\chi_y(E) = (1 - y)^{r-2}(2 + 20y + 2y^2)$
- $d = 3$: $\chi_y(E) = (1 - y)^{r-3}(y + y^2)(-\int \frac{c_3(E)}{2})$
- $d = 4$: $\chi_y(E) = (1 - y)^{r-4}\left(2 + y(-8 + \frac{\int c_4(E)}{6}) + y^2(12 + \frac{\int c_4(E)}{3}) + y^3(-8 + \frac{\int c_4(E)}{6}) + 2y^4\right)$,

etc., where we used the RR-Hirzebruch theorem for the trivial line bundle $O$ (structure sheaf, or $\Omega^0$) on a CY four-fold:

$$2 = h^{00} + h^{04} = \sum_{i=0}^{d} (-1)^i \dim H^i(X, \mathcal{O}) = \chi(O) = \int_X \text{ch}(\mathcal{O}) \text{td}(X) = \int_X \text{td}(X) = \int_X \frac{3c_2^2 - c_4}{6!}$$

(The same trick yields $\chi(O) = \int_X \text{td}(X) = 0, 2, 0$ for $d = 1, 2, 3$.) Similarly, for the general bundle $E$ of rank $r$, the theorem gives:

$$\chi(E) = \int_X \text{ch}(E) \text{td}(X) = \int_X \left(r - c_2 + \frac{c_4(E)}{2} + \frac{c_2^2 - 2c_4(E)}{12} + \ldots\right) \left(1 + \frac{c_2}{6} + \frac{3c_2^2 - c_4}{6} + \ldots\right) = 0, \quad 2r - 24, \quad \frac{1}{2} \int c_3(E), \quad 2r - \frac{1}{6} \int c_4(E) \quad \text{for} \quad d = 1, 2, 3, 4.$$  

The same formula is valid with $\wedge^k E$ instead of $E$ (also in the left bracket, and replace $c_2$ by $c_2(\wedge^k E)$ and $r$ by $\left(\begin{array}{c} r \\ k \end{array}\right)$), since $c_1(\wedge^k E) \sim c_1(E) = 0$. One could then compute $\chi(\wedge^k E)$, multiply it with $(-y)^k$, sum over $k$ and obtain again the above expressions for $\chi_y(E)$.

Restricting to $E = T_X$, with $r = d$, one obtains this way the same result\footnote{For instance, at $d = 4$, one finds $c(\wedge^2 T_X) = (1 + x_1 + x_2)(1 + x_1 + x_3)(1 + x_2 + x_3)(1 + x_2 + x_3)(1 + x_3 + x_4), \text{with} \ c_0 = 1; \ c_2(\wedge^2 T_X) = 2c_2; \ \text{and} \ c_4(\wedge^2 T_X) = 30x_1x_2x_3x_4 + 13x_1^2x_2x_3 + 5x_1^2x_2^2 + 2x_2^3x_3 + \{\text{symm.}\} = 30c_4 + 13c_2c_1 - 11(6c_2c_1 - 24c_4) + \frac{1}{2}(4c_2^2 - 24c_4) = c_2^2 - 4c_4. \ \text{Thus} \ \chi(\wedge^2 T_X) = 0 = \chi(\wedge^4 T_X) = 2, \ \chi(T_X) = \chi(\wedge^2 T_X) = 8 - \frac{\int c_4}{6}, \ \chi(\wedge^3 T_X) = 12 + \frac{2\int c_4}{3}.}$ without the factors $(1 - y)^{r-d}$. In particular, we note that for $d = 2, 3$, $\chi_y(E)$ is simply a multiple of $\chi_y(T_X)$, and so $\Phi_E$ should equally be the same multiple of $\Phi_{T_X}$. For $d = 2$, this multiple does not even depend on $E$ (except through its rank $r$), which is conceivable since the integrand of (3.4.2) contains only $c_2(E) = c_2(T)$ as degree-2 terms; in other words one might as well set $e_1 = x_1, \ e_2 = x_2, \ e_3, \ldots = 0$. For $d = 3$, note that $\Phi_{T_X}|_{\mathcal{O}} = -y^{-3/2}\chi_y(T_X) = -\frac{1}{2}y^{-1/2}(1 + y)$, in accordance with (3.4.3). For $d = 1$, the integrand contains only $c_1(E) = 0$ as degree-1 term; so $\Phi_E = \Phi_{T_X} = 0$. Summing up:

$$d = 1: \quad \Phi_E(\tau, z) = 0$$
$$d = 2: \quad \Phi_E(\tau, z) = \Phi_{T_X}(\tau, z) \left(\frac{1}{\eta} + \frac{1}{\eta^3}\right)^{r-2}$$
$$d = 3: \quad \Phi_E(\tau, z) = \Phi_{T_X}(\tau, z) \left(\frac{1}{\eta} + \frac{1}{\eta^3}\right)^{r-3} \frac{\chi(E)}{\chi(T_X)}.$$  

These forms, as well as the above expressions for $\chi_y(E)$ were already obtained in [KaM-94]. For $d = 4$, $\chi_y(E)$ is not a multiple of $\chi_y(T_X)$ anymore, so nothing can be concluded for the nature of $\Phi_E$. 
